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Online Appendix
Signaling Quality to Consumers: The Role of Social Media Marketing

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A. Benchmark Case

A.1. Proof of Proposition 1.

First, we demonstrate that, in a separating equilibrium (if it exists), the high-quality, mid-quality, and low-quality firms will charge H, M, and L, respectively, to maximize their profits; that is, $p^*(q) = q, q \in \{H, M, L\}$. In a separating equilibrium, the firm type is revealed to consumers; therefore, consumer expected valuation equals the firm's quality $q \in \{H, M, L\}$. If the firm charges a price higher than the expected valuation (equal to its quality), no consumer will purchase and the firm cannot make a profit. If the firm charges a price lower than its quality, it can always increase its price to match its quality to increase its profit. In this manner, different firm types charge different prices (equal to its quality), leading to a natural separation of prices.

Second, we demonstrate that if a separating equilibrium exists, the three firm types must spend different levels on SMM from each other. Suppose two types, q and q', where $q > q'$, spend the same amount s. Then the lower type q' has an incentive to increase its price to q to mimic the higher type q. By doing so, the q'-type firm would achieve a higher profit, and the separating equilibrium would collapse.

Third, we show that if a separating equilibrium exists, the low-quality type must spend on SMM at level of $s_L^* = 0$. Under any circumstances, the decision $(s_q, p(E(Q))) = (0, 1)$ remains feasible for all firm types and ensures a positive profit, $r(1 - c_q)$, for the firm of type q. This stems from the fact that consumers, whose expected valuation is at least L, will always purchase the product at price $L = 1$. Therefore, in any equilibrium, the profit of a profit-maximizing firm of type q must be no less than $r(1 - c_q)$, i.e.,

$$\pi(E(Q), q; p(E(Q)), s_q) \geq \pi(L, q; 1, 0) = r(1 - c_q). \tag{A.1}$$

We have characterized the objectives and conditions for the three types in a separating equilibrium (see (4), (5), and (6) in Section 3). The L type firm sets price $p^*(L) = L = 1$, and its objective function (see (4) when $q = L$) becomes $(r + (1 - r)s_L/(1 + s_L)) - s_L$, which is decreasing in s_L. Therefore, the low-quality firm achieves the highest profit (upper bound) r at $s_L = 0$. Meanwhile, condition (A.1) gives the lowest bound of the low-quality type firm profit r. Therefore, the low-quality type must spend on SMM at level of $s_L^* = 0$.

Finally, if a separating equilibrium exists, constraint (6) must be satisfied both when $(q, q') = (L, M)$ and $(q, q') = (M, L)$. This means that the low-quality type has no incentive to mimic the mid-quality type ($\pi(L, L; p(L), s_L) \geq \pi(M, L; p(M), s_M)$), and the mid-quality type has no incentive to mimic the low-quality type ($\pi(M, M; p(M), s_M) \geq \pi(L, M; p(L), s_L)$). Substituting the prices and $s_L = s_L^* = 0$ into the two constraints and extending them, we have

$$\left[r + (1 - r)\frac{s_L}{1 + s_L}\right](L - c_L) \geq \left[r + (1 - r)\frac{s_M}{1 + s_M}\right](M - c_L) - s_M,$$
$$\left[r + (1 - r)\frac{s_M}{1 + s_M}\right](M - c_M) - s_M \geq \left[r + (1 - r)\frac{s_L}{1 + s_L}\right](L - c_M).$$

By summing both sides of the two inequalities, we can simplify the expression to $(1 - r)s_L/(1 + s_L) \geq (1 - r)s_M/(1 + s_M)$. Since $r < 1$, as per the assumption $0 \leq r < 1$, we have $s_L \geq s_M$. Consequently, $s_M = s_L = 0$, leading to a contradiction to a separating equilibrium. Therefore, a separating equilibrium does not exist. □

A.2. Proof of Proposition 2.

We first prove that the partial-pooling equilibrium in which the high- and mid-quality firms pool together (i.e., $\Lambda = \{H, M\}$) does not exist. Second, we prove that the partial-pooling equilibrium where the high- and low-quality firms pool together (i.e., $\Lambda = \{H, L\}$) does not exist either. Finally, we prove the existence of the partial-pooling equilibrium in which the mid- and low-quality firms pool together (i.e., $\Lambda = \{M, L\}$), whereas the high-quality firm separates.

First, suppose a partial-pooling equilibrium with $\Lambda = \{H, M\}$ exists. The high- and mid-quality firms choose the same spending level on SMM and the same product price, denoted respectively by s^*_{HM} and p^*_Λ. In contrast, the low-quality firm separates itself and reveals its type. Using the same argument as in the third part of the proof of Proposition 1, it must spend $s^*_L = 0$ with $p^*(L) = 1$. Furthermore, for this partial-pooling equilibrium, the following two conditions must be satisfied:

$$\pi(L, M; p^*(L), s^*_L) \le \pi(\Lambda, M; p^*_\Lambda, s^*_{HM}), \tag{A.2}$$

$$\pi(\Lambda, L; p^*_\Lambda, s^*_{HM}) \le \pi(L, L; p^*(L), s^*_L), \tag{A.3}$$

in which condition (A.2) guarantees that the mid-quality firm has no incentive to mimic the low-quality firm, and condition (A.3) guarantees that the low-quality firm has no incentive to mimic the pooling firms by spending s^*_{HM} and setting price at p^*_Λ. Note that if the low-quality firm deviates to the spending level s^*_{HM} and price p^*_Λ, consumers are still willing to buy the product since their expected utility is larger than the price. By plugging firm decisions into (A.2) and (A.3), we obtain

$$r(1 - c_M) \le \left[r + (1 - r)\frac{s^*_{HM}}{1 + s^*_{HM}}\right](p^*_\Lambda - c_M) - s^*_{HM}, \tag{A.4}$$

$$\left[r + (1 - r)\frac{s^*_{HM}}{1 + s^*_{HM}}\right](p^*_\Lambda - 0) - s^*_{HM} \le r. \tag{A.5}$$

However, adding both sides of (A.4) and (A.5) gives

$$-rc_M \le -\left[r + (1 - r)\frac{s^*_{HM}}{1 + s^*_{HM}}\right]c_M, \text{ i.e., } r + (1 - r)\frac{s^*_{HM}}{1 + s^*_{HM}} \le r. \tag{A.6}$$

Since $0 \le r < 1$ we have $s^*_{HM} = 0 = s^*_L$. However, in such a partial-pooling equilibrium where the high-quality and mid-quality types pool while the low-quality type separates, we must have $s^*_{HM} \ne s^*_L$, and clearly this contradicts with $s^*_{HM} = 0 = s^*_L$. Therefore, such a partial-pooling equilibrium does not exist.

Second, suppose a partial-pooling equilibrium with $\Lambda = \{H, L\}$ exists. The high- and low-quality firms choose the same spending on SMM and the same product price, denoted respectively by s^*_{HL} and p^*_Λ. In contrast, the mid-quality firm separates itself and reveals its type by choosing s^*_M and $p^*(M)$. For this partial-pooling equilibrium, the following three conditions must be satisfied:

$$\pi(\Lambda, M; p^*_\Lambda, s^*_{HL}) \le \pi(M, M; p^*(M), s^*_M), \tag{A.7}$$

$$\pi(M, L; p^*(M), s^*_M) \le \pi(\Lambda, L; p^*_\Lambda, s^*_{HL}), \tag{A.8}$$

$$\pi(M, H; p^*(M), s^*_M) \le \pi(\Lambda, H; p^*_\Lambda, s^*_{HL}), \tag{A.9}$$

in which condition (A.7) guarantees that the mid-quality firm has no incentive to mimic the pooling firms by spending s^*_{HL} and setting price at p^*_Λ, condition (A.8) guarantees that the low-quality firm has no incentive to mimic the mid-quality firm by spending s^*_M and setting price at $p^*(M)$, and condition (A.9) guarantees that the high-quality firm has no incentive to mimic the mid-quality firm by spending s_M^* and setting price at $p^*(M)$. Note that if the mid-quality firm deviates to the spending level s_{HL}^* and price p_Λ^*, consumers are still willing to buy the product since their expected utility is larger than the price. By plugging firm decisions into (A.7), (A.8), and (A.9), we obtain

$$\left[r+(1-r)\frac{s_{HL}^*}{1+s_{HL}^*}\right](p_\Lambda^*-c_M)-s_{HL}^*\le\left[r+(1-r)\frac{s_M^*}{1+s_M^*}\right](p^*(M)-c_M)-s_M^*, \tag{A.10}$$

$$\left[r+(1-r)\frac{s_M^*}{1+s_M^*}\right](p^*(M)-0)-s_M^*\le\left[r+(1-r)\frac{s_{HL}^*}{1+s_{HL}^*}\right](p_\Lambda^*-0)-s_{HL}^*, \tag{A.11}$$

$$\left[r+(1-r)\frac{s_M^*}{1+s_M^*}\right](p^*(M)-c_H)-s_M^*\le\left[r+(1-r)\frac{s_{HL}^*}{1+s_{HL}^*}\right](p_\Lambda^*-c_H)-s_{HL}^*. \tag{A.12}$$

Then, by adding both sides of (A.10) and (A.11), we obtain $s_M^*\le s_{HL}^*$; by adding both sides of (A.10) and (A.12), we get $s_M^*\ge s_{HL}^*$. Therefore, $s_M^*=s_{HL}^*$. However, in such a partial-pooling equilibrium where the high-quality and low-quality types pool while the mid-quality type separates, we must have $s_{HL}^*\neq s_M^*$, and clearly this contradicts with $s_{HL}^*=s_M^*$. Therefore, such a partial-pooling equilibrium does not exist.

Finally, we prove the existence of the partial-pooling equilibrium where the mid-quality and low-quality firms pool together ($\Lambda=\{M,L\}$), whereas the high-quality firm separates. Firstly, we prove the "only if" part. Suppose there is such a partial-pooling equilibrium ($\Lambda=\{M,L\}$). The mid- and low-quality firms choose the same spending on SMM and the same product price, denoted respectively by s_{ML}^* and p_Λ^*. After observing (s_{ML}^*,p_Λ^*), consumers are uncertain about whether the firm is of type M or L, and update their prior belief and formulate their posterior belief based on the external signal, and form their expected quality of the product. The pooling price p_Λ^* cannot exceed and must equal the lowest quality expectation

$$\bar{q}_{ML}:=\lambda+(1-\lambda)M.$$

The equilibrium prices and spending levels shall satisfy the constraints of all three types of firms:

$$\pi(H,H;p^*(H),s_H^*)\ge\pi(\Lambda,H;p_\Lambda^*,s_{ML}^*), \tag{A.13}$$

$$\pi(\Lambda,M;p_\Lambda^*,s_{ML}^*)\ge\pi(H,M;p^*(H),s_H^*), \tag{A.14}$$

$$\pi(\Lambda,L;p_\Lambda^*,s_{ML}^*)\ge\pi(H,L;p^*(H),s_H^*), \tag{A.15}$$

$$\pi(\Lambda,L;p_\Lambda^*,s_{ML}^*)\ge\pi(L,L;1,0), \tag{A.16}$$

$$\pi(\Lambda,M;p_\Lambda^*,s_{ML}^*)\ge\pi(L,M;1,0), \tag{A.17}$$

$$\pi(H,H;p^*(H),s_H^*)\ge\pi(L,H;1,0). \tag{A.18}$$

By expanding the profit functions in (A.13), (A.14) and (A.15), we obtain

$$\left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right](H-c_H)-s_H^*\ge\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML}-c_H)-s_{ML}^*, \tag{A.19}$$

$$\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML}-c_M)-s_{ML}^*\ge\left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right](H-c_M)-s_H^*, \tag{A.20}$$

$$\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML}-0)-s_{ML}^*\ge\left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right](H-0)-s_H^*. \tag{A.21}$$

By adding both sides of (A.19) and (A.20), we obtain

$$\left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right](c_M-c_H)\ge\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](c_M-c_H).$$

By assumptions $c_M - c_H < 0$ and $1-r>0$, therefore we have $s_H^* \le s_{ML}^*$. That implies

$$\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right]c_M \ge \left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right]c_M \tag{A.22}$$

By adding both sides of (A.22) and (A.20) we obtain (A.21). Therefore, (A.21) is implied by (A.19) and (A.20) and hence can be omitted. Furthermore, since the high-quality type separates from the set Λ by signaling the spending, we must have

$$s_H^* < s_{ML}^*. \tag{A.23}$$

That means, to separate from the pooling set Λ, the high-quality firm shall choose a lower spending level compared with that of the set Λ.

Furthermore, by expanding the profit functions in (A.16), (A.17) and (A.18), we obtain

$$\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML}-0) - s_{ML}^* \ge r, \tag{A.24}$$

$$\left[r+(1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML}-c_M) - s_{ML}^* \ge r(1-c_M), \tag{A.25}$$

$$\left[r+(1-r)\frac{s_H^*}{1+s_H^*}\right](H-c_H) - s_H^* \ge r(1-c_H). \tag{A.26}$$

It can be checked that (A.24) is implied by (A.25), and hence (A.24) can be disregarded. Therefore, there exists such a partial-pooling equilibrium if and only if there exist s_H^* and s_{ML}^* that satisfy conditions (A.19), (A.20), (A.25), and (A.26).

Next, we consider the high-quality firm's *least-cost separating* in this partial-pooling equilibrium, by analyzing (A.20). Note that $\bar{q}_{ML} < H$ and by (A.23), $s_H^* < s_{ML}^*$. Thus, if

$$\max_{s\ge 0} \pi(\Lambda, M; \bar{q}_{ML}, s) < r(H-c_M),$$

then for $\forall s_{ML}^* \ge 0$, to ensure the least-cost separating, the high-quality firm will choose the SMM spending s_H^* such that the left-hand side of (A.20) is equal to its right-hand side; however, in this case, we always have $s_H^* > s_{ML}^*$, which contradicts the condition (A.23), implying that such a partial-pooling equilibrium does not exist. Alternatively, when

$$\max_{s\ge 0} \pi(\Lambda, M; \bar{q}_{ML}, s) \ge r(H-c_M), \tag{A.27}$$

let s_{ML}^1 and s_{ML}^2 respectively be the smallest and largest s value that satisfies $\pi(\Lambda, M; \bar{q}_{ML}, s) = r(H-c_M)$. Then for each $s_{ML}^* \in [s_{ML}^1, s_{ML}^2]$, there exists a unique $s_H^* < s_{ML}^*$ such that

$$\pi(\Lambda, M; p_\Lambda^*, s_{ML}^*) = \pi(H, M; H, s_H^*). \tag{A.28}$$

Then suppose that (A.27) and (A.28) are satisfied, and that $s_H^* < s_{ML}^*$, we analyze the SMM spending s_{ML}^* in the partial-pooling equilibrium. To do so, we firstly show that (A.13) (or equivalently, (A.19)) is satisfied. By expanding (A.28) and incorporating it into (A.19), we obtain that the latter reduces to $s_H^* \le s_{ML}^*$, which is certainly true by the condition $s_H^* < s_{ML}^*$.

Finally, we analyze the profit function of the mid- and low-quality firms. The low-quality firm's (expected) profit is $\pi(\Lambda, L; \bar{q}_{ML}, s_{ML}^*)$, where $\pi(\Lambda, L; \bar{q}_{ML}, s)$ is given by

$$\pi(\Lambda, L; \bar{q}_{ML}, s) = \left[r+(1-r)\frac{s}{1+s}\right]\left[\lambda+(1-\lambda)M-0\right] - s. \tag{A.29}$$

For the mid-quality firm, its (expected) profit is $\pi(\Lambda, M; \bar{q}_{ML}, s^*_{ML})$, where $\pi(\Lambda, M; \bar{q}_{ML}, s)$ is given by

$$\pi(\Lambda, M; \bar{q}_{ML}, s) = \Big[r + (1-r)\frac{s}{1+s}\Big]\Big[\lambda + (1-\lambda)M - c_M\Big] - s. \tag{A.30}$$

Then by (A.29) and (A.30), we obtain that

$$\frac{\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}, s)}{\mathrm{d}s} = \frac{(1-r)\big(\lambda + (1-\lambda)M\big) - (1+s)^2}{(1+s)^2}, \tag{A.31}$$

$$\frac{\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)}{\mathrm{d}s} = \frac{(1-r)\big(\lambda + (1-\lambda)M - c_M\big) - (1+s)^2}{(1+s)^2}. \tag{A.32}$$

By (A.31) and (A.32), we obtain that $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}, s)/\mathrm{d}s\big|_{s=0} > 0$ if and only if

$$0 \le r < r_1 := \frac{(1-\lambda)(M-1)}{(1-\lambda)M + \lambda}. \tag{A.33}$$

Similarly, we obtain that $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s\big|_{s=0} > 0$ if and only if

$$(M-1)(1-\lambda) > c_M, \text{ and } 0 \le r < \hat{r} := \frac{(1-\lambda)(M-1) - c_M}{\lambda + (1-\lambda)M - c_M}. \tag{A.34}$$

Note that it can be checked that $\hat{r} < r_1$. Therefore, we obtain the following:

(i) $r \in [r_1, 1)$: we have $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}, s)/\mathrm{d}s \le 0$ and $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s < 0$. In this case, for any $s^*_{ML} \ge 0$, there does not exist s^*_H such that both (A.27) and (A.28) are satisfied and that $s^*_H < s^*_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(ii) $r \in [0, r_1)$ and $(M-1)(1-\lambda) \le c_M$: we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s \le 0$. In this case, for any $s^*_{ML} \ge 0$, there does not exist s^*_H such that both (A.27) and (A.28) are satisfied and that $s^*_H < s^*_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(iii) $r \in [\hat{r}, r_1)$ and $(M-1)(1-\lambda) > c_M$: we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s \le 0$. In this case, for any $s^*_{ML} \ge 0$, there does not exist s^*_H such that both (A.27) and (A.28) are satisfied and that $s^*_H < s^*_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(iv) $r \in [0, \hat{r})$ and $(M-1)(1-\lambda) > c_M$: there exist thresholds $\tilde{s}$ and $\hat{s}$ that are respectively (and uniquely) determined by solving $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}, s)/\mathrm{d}s = 0$ and $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s = 0$ with respect to s, and $\tilde{s} > \hat{s}$. Then we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}, s)/\mathrm{d}s \ge 0$ if and only if $s \in [0, \hat{s}]$, and $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}, s)/\mathrm{d}s > 0$ if and only if $s \in [0, \tilde{s}]$. Then by considering (A.27) being satisfied, we obtain that

$$\pi(\Lambda, M; \bar{q}_{ML}, \hat{s}) = \max_{s \ge 0} \pi(\Lambda, M; \bar{q}_{ML}, s) \ge r(H - c_M). \tag{A.35}$$

Thus, when (A.27) (i.e., (A.35)) is satisfied, all conditions (A.16), (A.17) and (A.18) are satisfied. That is, there exists a least-cost partial-pooling equilibrium where $s^*_{ML} \in [s^1_{ML}, s^2_{ML}]$, $s^*_H < s^*_{ML}$ and s^*_H is uniquely determined by (A.28). Notice that it is not difficult to check that conditions $r \in [0, \hat{r})$, $(M-1)(1-\lambda) > c_M$, and (A.35) are equivalent to (A.35), which can be also written as

$$\bar{q}_{ML} - c_M + 1 - 2\sqrt{(1-r)(\bar{q}_{ML} - c_M)} \ge r(H - c_M).$$

Additionally, s^*_H is a function of s^*_{ML} and can be written as $s^*_H(s^*_{ML})$.

The proof of "if" part follows by reversing the previous arguments. □

A.3. Proof of Proposition 3.

In a pooling equilibrium (i.e., $\Lambda = \{H, M, L\}$), the high-, mid-, and low-quality firms choose the same SMM spending, which is denoted by s^*_{HML}. After observing this spending level, consumers update their prior belief and formulate their posterior belief, and form their expected quality. Thus, p^*_Λ cannot exceed and must equal the lowest expected quality $\bar{q}_{HML}$. Identical to Nian and Sundararajan (2022), we specify the off-equilibrium path belief that $\mu(L \mid s, p) = 1$ for $(s, p) \neq (s^*_{HML}, \bar{q}_{HML})$. That means, if any firm deviates by choosing a spending level other than s^*_{HML}, consumers will believe it to be of low-quality type. Therefore, the spending level s^*_{HML} only needs to satisfy condition (A.1), i.e.,

$$\pi(\Lambda, L; p^*_\Lambda, s^*_{HML}) \geq \pi(L, L; 1, 0), \tag{A.36}$$

$$\pi(\Lambda, M; p^*_\Lambda, s^*_{HML}) \geq \pi(L, M; 1, 0), \tag{A.37}$$

$$\pi(\Lambda, H; p^*_\Lambda, s^*_{HML}) \geq \pi(L, H; 1, 0). \tag{A.38}$$

By solving (A.36), (A.37), and (A.38) with respect to s^*_{HML}, we obtain that $s^*_{HML} \in [0, \min\{\bar{s}_L, \bar{s}_M, \bar{s}_H\}]$, in which, $\bar{s}_L$, $\bar{s}_M$, and $\bar{s}_H$ are, respectively, the unique positive root of (A.36), (A.37), and (A.38) replacing "$\leq$" by "$=$", i.e.,

$$\left[r + (1-r)\frac{\bar{s}_q}{1+\bar{s}_q}\right](\bar{q}_{HML} - c_q) - \bar{s}_q - r(1-c_q) = 0, \; q \in \{L, M, H\}. \tag{A.39}$$

Since the left-hand side of (A.39) is decreasing in c_q and concave in $\bar{s}_q$, we obtain that $\bar{s}_H < \bar{s}_M < \bar{s}_L$. Therefore, for any $\check{s} \in [0, \bar{s}_H]$, where

$$\bar{s}_H = \left(\bar{q}_{HML} - 1 - c_H - r(1-c_H) + \sqrt{\left(\bar{q}_{HML} - 1 - c_H - r(1-c_H)\right)^2 + 4r(\bar{q}_{HML} - 1)}\right)/2, \tag{A.40}$$

there exists a pooling equilibrium, in which the high-quality, mid-quality, and low-quality firms pool together with the same spending $s^*_{HML} = \check{s}$ and price $p^*_\Lambda = \bar{q}_{HML}$. $\square$

A.4. Proof of Proposition 4

From Propositions 1 to 3 we know that only two types of equilibria can exist: (i) three types pool together ($\Lambda = \{H, M, L\}$), and (ii) the mid-quality and low-quality firms pool, whereas the high-quality firm separates ($\Lambda = \{M, L\}$). Then we use the intuitive criterion, which is given below and contains two steps, to test each type of equilibria separately:

Step 1: For each off-equilibrium spending (message) $s \in S$, form the set $J(s)$ consisting of all types q such that the firm's payoff in equilibrium, i.e., $v^*(q)$, satisfies

$$J(s) = \{q \mid v^*(q) > \pi(H, q; p(H), s)\}.$$

Step 2: A pure strategy PBE fails the intuitive criterion if and only if

$$v^*(q) < \pi(\min\{Q \backslash J(s)\}, q; p(\min\{Q \backslash J(s)\}), s), \tag{A.41}$$

for any $q \in Q \backslash J(s) \neq \emptyset$.

Part (i). We apply the intuitive criterion to test the partial-pooling equilibrium in Proposition 2, following the two steps above.

In the partial-pooling equilibrium with $\Lambda = \{M, L\}$, for any equilibrium spendings s^*_{ML} and s^*_H as characterized in Proposition 2, define $\tilde{s}_L$ as the smallest root of $\pi(\Lambda, L; \bar{q}_{ML}, s^*_{ML}) = \pi(H, L; H, s)$ with respect to s. It is not difficult to check that $s_H^* < \tilde{s}_L < s_{ML}^*$. Then consider the deviation s. For $s \in [0, s_H^*)$, by Proposition 2, $J(s) = \{H, M, L\}$, and thus $Q \backslash J(s) = \emptyset$. For $s \geq \tilde{s}_L$, we have either $L \in Q \backslash J(s)$ or $Q \backslash J(s) = \emptyset$. For $s \in [s_H^*, \tilde{s}_L)$, by the definition of $\tilde{s}_L$ and Proposition 2, we have $J(s) = L$, and thus $Q \backslash J(s) = \{H, M\}$. Therefore, in Step 2, the worst case is that consumers believe that the deviation is from the mid-quality firm. For the mid-quality firm, it obtains profit $v^*(M) = \pi(\Lambda, M; \bar{q}_{ML}, s_{ML}^*)$ in equilibrium, and the worst profit $\pi(M, M; M, s)$ in deviation. For the high-quality firm, it obtains profit $v^*(H) = \pi(H, H; H, s_H^*)$ in equilibrium, and the worst profit $\pi(M, H; M, s)$ in deviation. Therefore, this partial-pooling equilibrium survives the intuitive criterion if and only if for $\forall s \in [s_H^*, \tilde{s}_L)$, both $\pi(\Lambda, M; \bar{q}_{ML}, s_{ML}^*) \geq \pi(M, M; M, s)$ and $\pi(H, H; H, s_H^*) \geq \pi(M, H; M, s)$ hold true, i.e.,

$$\left[r + (1-r)\frac{s_{ML}^*}{1+s_{ML}^*}\right](\bar{q}_{ML} - c_M) - s_{ML}^* \geq \max_{s \in [s_H^*, \tilde{s}_L]} \left\{\left[r + (1-r)\frac{s}{1+s}\right](M - c_M) - s\right\}, \tag{A.42}$$

$$\left[r + (1-r)\frac{s_H^*}{1+s_H^*}\right](H - c_H) - s_H^* \geq \max_{s \in [s_H^*, \tilde{s}_L]} \left\{\left[r + (1-r)\frac{s}{1+s}\right](M - c_H) - s\right\}. \tag{A.43}$$

Note that both (A.42) and (A.43) can hold true given the parameter space.

Part (ii). As a preparation, for $\Lambda = \{H, M, L\}$, we define $s_p := \arg\max_{s \geq 0} \pi(\Lambda, L; \bar{q}_{HML}, s)$, i.e.,

$$s_p = \begin{cases} \sqrt{(1-r)\bar{q}_{HML}} - 1, & \text{if } (1-r)\bar{q}_{HML} > 1 \\ 0, & \text{if } (1-r)\bar{q}_{HML} \leq 1. \end{cases} \tag{A.44}$$

Then we analyze the following two cases.

Case (a): $rH < \bar{q}_{HML} + 1 - 2\sqrt{(1-r)\bar{q}_{HML}}$, which implies $(1-r)\bar{q}_{HML} > 1$.

Define $\ddot{s}$ as the smallest root of $rH = \pi(\Lambda, L; \bar{q}_{HML}, s)$ with respect to s, i.e.,

$$\ddot{s} := \left(\bar{q}_{HML} - 1 - rH - \sqrt{(rH + 1 - \bar{q}_{HML})^2 - 4r(H - \bar{q}_{HML})}\right)/2. \tag{A.45}$$

Consider the pooling equilibria characterized in Proposition 3 with spending $\check{s} \in [0, \min\{\ddot{s}, \bar{s}_H\}]$. We then show that all such pooling equilibria with $\check{s} \in [0, \min\{\ddot{s}, \bar{s}_H\}]$ survive the intuitive criterion. Note that either $\ddot{s} < \bar{s}_H$ or $\ddot{s} > \bar{s}_H$ can happen. Then we apply the intuitive criterion following the above two steps.

Recall from Appendix A.3 that we have $v^*(q) = \pi(\Lambda, q; \bar{q}_{HML}, \check{s}) \geq \pi(L, q; L, 0)$. Thus, for $Q \backslash J(s) \neq \emptyset$, if $L \in Q \backslash J(s)$, i.e., $L \notin J(s)$, then condition (A.41) cannot be satisfied, and hence the pooling equilibrium survives the intuitive criterion. Therefore, to show that the pooling equilibria survive the intuitive criterion, we only need to show that $L \notin J(s)$ when $J(s) \neq \{H, M, L\}$.

In preparation, define s_1, s_2, and s_3 as the largest value of s that, respectively, solves the equations:

$$\pi(H, H; H, s) = \pi(\Lambda, H; \bar{q}_{HML}, \check{s}),$$

$$\pi(H, M; H, s) = \pi(\Lambda, M; \bar{q}_{HML}, \check{s}),$$

$$\pi(H, L; H, s) = \pi(\Lambda, L; \bar{q}_{HML}, \check{s}).$$

Since $\pi(H, q; H, s) - \pi(\Lambda, q; \bar{q}_{HML}, \check{s})$ is decreasing in c_q for $c_q \geq 0$ and $s > \check{s}$, we have $0 < s_1 < s_2 < s_3$. Clearly, based on the last equality above, $\pi(H, L; H, s_3) - \pi(\Lambda, L; \bar{q}_{HML}, \check{s}) = 0$, and $\pi(H, L; H, s)$ is decreasing in s around s_3. Thus, we obtain that $\pi(H, M; H, s_3) - \pi(\Lambda, M; \bar{q}_{HML}, \check{s}) < 0$, and $\pi(H, M; H, s)$ is decreasing in s in the neighborhood of s_3. Therefore, for s_2 such that $\pi(H, M; H, s_2) - \pi(\Lambda, M; \bar{q}_{HML}, \check{s}) = 0$, we have $s_2 < s_3$. By the same reasoning process, we have $0 < s_1 < s_2$, i.e., $0 \leq \check{s} < s_1 < s_2 < s_3$.

For $0 \leq s \leq \check{s}$, as $v^*(L) \leq \pi(H,L;H,s)$, we have $L \notin J(s)$. Therefore, the pooling equilibrium must survive the intuitive criterion test when $0 \leq s \leq \check{s}$. Similarly, for $\check{s} < s \leq s_3$, we also have $L \notin J(s)$; and for $s > s_3$, we have $J(s) = \{H,M,L\}$, and thus $Q \backslash J(s) = \emptyset$. Therefore, we conclude that the pooling equilibrium survives the intuitive criterion.

Case (b): $rH \geq \bar{q}_{HML} + 1 - 2\sqrt{(1-r)\bar{q}_{HML}}$.

For any $\check{s} \in [0, \bar{s}_H]$, the spending and price pair $(s^*_\Lambda, p^*_\Lambda) = (\check{s}, \bar{q}_{HML})$, satisfies the conditions for a pooling equilibrium. Then consider the deviation s. For $0 \leq s \leq s_3$, as $v^*(L) \leq \pi(H,L;H,s)$, we have $L \notin J(S)$. For $s > s_3$, we have $J(S) = \{H,M,L\}$, and thus $Q \backslash J(S) = \emptyset$. Therefore, we conclude that all the pooling equilibria survive the intuitive criterion. $\square$

B. Information-Revelation Role of Social Media Marketing

B.1. Proof of Proposition 5

In a separating equilibrium (if any), since product quality is fully revealed to consumers, the information-revelation of SMM does not play a role. Therefore, the proof is the same as that of Proposition 1. □

B.2. Proof of Proposition 6

We prove each part of the proposition separately.

Part (i): We show that there does not exist a partial-pooling equilibrium in which the high-quality firm pools with either the low- or mid-quality firm.

Firstly, suppose that there exists a partial-pooling equilibrium in which the high- and low-quality firms pool together, whereas the mid-quality firm separates, i.e., $\Lambda = \{H, L\}$. Then the following conditions must be satisfied:

$$\pi^\star(\Lambda, L; p^\star_\Lambda, s^\star_\Lambda) \geq \pi^\star(M, L; p^\star(M), s^\star_M), \tag{B.1}$$

$$\pi^\star(M, M; p^\star(M), s^\star_M) \geq \pi^\star(\Lambda, M; p^\star_\Lambda, s^\star_\Lambda), \tag{B.2}$$

$$\pi^\star(\Lambda, H; p^\star_\Lambda, s^\star_\Lambda) \geq \pi^\star(M, H; p^\star(M), s^\star_M), \tag{B.3}$$

in which (B.1) and (B.3) ensure that the low-quality and high-quality firms do not have an incentive to mimic the mid-quality firm, and (B.2) ensures that the mid-quality firm has no incentive to mimic the pooling types. Note that if the mid-quality firm deviates to the spending level $s^\star_{HL}$ and price $p^\star_\Lambda$, consumers are still willing to buy the product since their expected utility is larger than the price. Expanding (B.1), (B.2), and (B.3) gives

$$\left[r + (1-r)\frac{s^\star_\Lambda}{1+s^\star_\Lambda}\right] p^\star_\Lambda - s^\star_\Lambda \geq \left[r + (1-r)\frac{s^\star_M}{1+s^\star_M}\right] p^\star(M) - s^\star_M, \tag{B.4}$$

$$\left[r + (1-r)\frac{s^\star_M}{1+s^\star_M}\right] (p^\star(M) - c_M) - s^\star_M \geq \left[r + (1-r)\frac{s^\star_\Lambda}{1+s^\star_\Lambda}\right] (p^\star_\Lambda - c_M) - s^\star_\Lambda, \tag{B.5}$$

$$\left[r + (1-r)\frac{s^\star_\Lambda}{1+s^\star_\Lambda}\right] (p^\star_\Lambda - c_H) - s^\star_\Lambda \geq \left[r + (1-r)\frac{s^\star_M}{1+s^\star_M}\right] (p^\star(M) - c_H) - s^\star_M. \tag{B.6}$$

Then, adding both sides of (B.4) and (B.5) gives

$$s^\star_M \leq s^\star_\Lambda.$$

Adding both sides of (B.5) and (B.6) gives

$$s^\star_M \geq s^\star_\Lambda.$$

Therefore $s^\star_M = s^\star_\Lambda$. This contradicts $s^\star_M \neq s^\star_\Lambda$ in such a partial-pooling equilibrium. Therefore, the partial-pooling equilibrium with $\Lambda = \{H, L\}$ does not exist.

Secondly, suppose a partial-pooling equilibrium with $\Lambda = \{H, M\}$ exists. The high- and mid-quality firms choose the same spending level on SMM and the same product price, denoted respectively by $s^\star_{HM}$ and $p^\star_\Lambda$. In contrast, the low-quality firm separates itself and reveals its type. Using the same argument as in the third part of the proof of Proposition 1, it must spend $s^\star_L = 0$ with $p^\star(L) = 1$. Furthermore, for this partial-pooling equilibrium, the following two conditions must be satisfied:

$$\pi(L, M; p^\star(L), s^\star_L) \leq \pi(\Lambda, M; p^\star_\Lambda, s^\star_{HM}), \tag{B.7}$$

$$\pi(\Lambda, L; p^\star_\Lambda, s^\star_{HM}) \leq \pi(L, L; p^\star(L), s^\star_L), \tag{B.8}$$

in which condition (B.7) guarantees that the mid-quality firm has no incentive to mimic the low-quality firm, and condition (B.8) guarantees that the low-quality firm has no incentive to mimic the pooling firms by spending $s^\star_{HM}$ and setting price at $p^\star_\Lambda$. Note that if the low-quality firm deviates to the spending level $s^\star_{HM}$ and price $p^\star_\Lambda$, consumers are still willing to buy the product since their expected utility is larger than the price. By plugging firm decisions into (B.7) and (B.8), we obtain

$$r(1-c_M) \le \left[r+(1-r)\frac{s^\star_{HM}}{1+s^\star_{HM}}\right](p^\star_\Lambda - c_M) - s^\star_{HM}, \tag{B.9}$$

$$\left[r+(1-r)\frac{s^\star_{HM}}{1+s^\star_{HM}}\right](p^\star_\Lambda - 0) - s^\star_{HM} \le r. \tag{B.10}$$

However, adding both sides of (B.9) and (B.10) gives

$$-rc_M \le -\left[r+(1-r)\frac{s^\star_{HM}}{1+s^\star_{HM}}\right]c_M, \text{ i.e., } r+(1-r)\frac{s^\star_{HM}}{1+s^\star_{HM}} \le r. \tag{B.11}$$

Since $0 \le r < 1$, we have $s^\star_{HM} = 0 = s^\star_L$. However, in such a partial-pooling equilibrium where the high-quality and mid-quality types pool while the low-quality type separates, we must have $s^\star_{HM} \neq s^\star_L$, and clearly this contradicts with $s^\star_{HM} = 0 = s^\star_L$. Therefore, such a partial-pooling equilibrium does not exist.

Part (ii): we prove the existence of the partial-pooling equilibrium with $\Lambda = \{M, L\}$.

Firstly, we prove the "only if" part. Suppose that there is a partial-pooling equilibrium, where the mid-quality and low-quality firms pool together ($\Lambda = \{M, L\}$), whereas the high-quality firm separates. The mid- and low-quality firms choose the same spending on SMM and the same product price, denoted respectively by $s^\star_{ML}$ and $p^\star_\Lambda$. After observing $(s^\star_{ML}, p^\star_\Lambda)$, consumers are uncertain about whether the firm is of type M or L, and update their prior belief and formulate their posterior belief based on the external signal, and form their expected quality of the product. The pooling price $p^\star_\Lambda$ cannot exceed and must equal the lowest quality expectation

$$\bar{q}_{ML}(s^\star_{ML}) := \frac{\lambda + s^\star_{ML}}{1+s^\star_{ML}} + \left(1 - \frac{\lambda + s^\star_{ML}}{1+s^\star_{ML}}\right)M.$$

The equilibrium prices and spending levels shall satisfy the constraints of all three types of firms:

$$\pi(H,H;p^\star(H),s^\star_H) \ge \pi(\Lambda,H;p^\star_\Lambda,s^\star_{ML}), \tag{B.12}$$

$$\pi(\Lambda,M;p^\star_\Lambda,s^\star_{ML}) \ge \pi(H,M;p^\star(H),s^\star_H), \tag{B.13}$$

$$\pi(\Lambda,L;p^\star_\Lambda,s^\star_{ML}) \ge \pi(H,L;p^\star(H),s^\star_H), \tag{B.14}$$

$$\pi(\Lambda,L;p^\star_\Lambda,s^\star_{ML}) \ge \pi(L,L;1,0), \tag{B.15}$$

$$\pi(\Lambda,M;p^\star_\Lambda,s^\star_{ML}) \ge \pi(L,M;1,0), \tag{B.16}$$

$$\pi(H,H;p^\star(H),s^\star_H) \ge \pi(L,H;1,0). \tag{B.17}$$

By expanding the profit functions in (B.12), (B.13) and (B.14), we obtain

$$\left[r+(1-r)\frac{s^\star_H}{1+s^\star_H}\right](H-c_H) - s^\star_H \ge \left[r+(1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right](\bar{q}_{ML}(s^\star_{ML}) - c_H) - s^\star_{ML}, \tag{B.18}$$

$$\left[r+(1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right](\bar{q}_{ML}(s^\star_{ML}) - c_M) - s^\star_{ML} \ge \left[r+(1-r)\frac{s^\star_H}{1+s^\star_H}\right](H-c_M) - s^\star_H, \tag{B.19}$$

$$\left[r+(1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right](\bar{q}_{ML}(s^\star_{ML}) - 0) - s^\star_{ML} \ge \left[r+(1-r)\frac{s^\star_H}{1+s^\star_H}\right](H-0) - s^\star_H. \tag{B.20}$$

By adding both sides of (B.18) and (B.19), we obtain

$$\left[r+(1-r)\frac{s_H^\star}{1+s_H^\star}\right](c_M-c_H)\geq\left[r+(1-r)\frac{s_{ML}^\star}{1+s_{ML}^\star}\right](c_M-c_H).$$

By assumptions $c_M-c_H<0$ and $1-r>0$, therefore we have $s_H^\star\leq s_{ML}^\star$. That implies

$$\left[r+(1-r)\frac{s_{ML}^\star}{1+s_{ML}^\star}\right]c_M\geq\left[r+(1-r)\frac{s_H^\star}{1+s_H^\star}\right]c_M \tag{B.21}$$

By adding both sides of (B.21) and (B.19) we obtain (B.20). Therefore, (B.20) is implied by (B.18) and (B.19) and hence can be omitted. Furthermore, since the high-quality type separates from set Λ by signaling the spending, we must have

$$s_H^\star<s_{ML}^\star. \tag{B.22}$$

That means, to separate from the pooling set Λ, the high-quality firm shall choose a lower spending level compared with the pooling spending level.

Furthermore, by expanding the profit functions in (B.15), (B.16) and (B.17), we obtain

$$\left[r+(1-r)\frac{s_{ML}^\star}{1+s_{ML}^\star}\right](\bar{q}_{ML}(s_{ML}^\star)-0)-s_{ML}^\star\geq r, \tag{B.23}$$

$$\left[r+(1-r)\frac{s_{ML}^\star}{1+s_{ML}^\star}\right](\bar{q}_{ML}(s_{ML}^\star)-c_M)-s_{ML}^\star\geq r(1-c_M), \tag{B.24}$$

$$\left[r+(1-r)\frac{s_H^\star}{1+s_H^\star}\right](H-c_H)-s_H^\star\geq r(1-c_H). \tag{B.25}$$

It can be checked that (B.23) is implied by (B.24), and hence (B.23) can be disregarded. Therefore, there exists such a partial-pooling equilibrium if and only if there exist $s_H^\star$ and $s_{ML}^\star$ that satisfy conditions (B.18), (B.19), (B.24), and (B.25).

Next, we consider the high-quality firm's *least-cost separating* in this partial-pooling equilibrium, by analyzing (B.19). Note that $\bar{q}_{ML}(s_{ML}^\star)<H$ and by (B.22), $s_H^\star<s_{ML}^\star$. Thus, if

$$\max_{s\geq 0}\pi(\Lambda,M;\bar{q}_{ML}(s),s)<r(H-c_M),$$

then for $\forall s_{ML}^\star\geq 0$, to ensure the least-cost separating, the high-quality firm will choose the SMM spending $s_H^\star$ such that the left-hand side of (B.19) is equal to its right-hand side; however, in this case, we always have $s_H^\star>s_{ML}^\star$, which contradicts the condition (B.22), implying that such a partial-pooling equilibrium does not exist. Alternatively, when

$$\max_{s\geq 0}\pi(\Lambda,M;\bar{q}_{ML}(s),s)\geq r(H-c_M), \tag{B.26}$$

let s_{ML}^3 and s_{ML}^4 respectively be the smallest and largest s value that satisfies $\pi(\Lambda,M;\bar{q}_{ML}(s),s)=r(H-c_M)$. Then for each $s_{ML}^\star\in[s_{ML}^3,s_{ML}^4]$, there exists a unique $s_H^\star<s_{ML}^\star$ such that

$$\pi(\Lambda,M;\bar{q}_{ML}(s_{ML}^\star),s_{ML}^\star)=\pi(H,M;H,s_H^\star). \tag{B.27}$$

Then suppose that (B.26) and (B.27) are satisfied, and that $s_H^\star<s_{ML}^\star$, we analyze the SMM spending $s_{ML}^\star$ in the partial-pooling equilibrium. To do so, we firstly show that (B.12) (or equivalently, (B.18)) is satisfied. By expanding (B.27) and incorporating it into (B.18), we obtain that the latter is reduced to $s_H^\star\leq s_{ML}^\star$, which is certainly true by the condition $s_H^\star<s_{ML}^\star$.

Finally, we analyze the profit functions of the mid- and low-quality firms. The low-quality firm's (expected) profit is $\pi(\Lambda, L; \bar{q}_{ML}(s^\star_{ML}), s^\star_{ML})$, where $\pi(\Lambda, L; \bar{q}_{ML}(s), s)$ is given by

$$\pi(\Lambda, L; \bar{q}_{ML}(s), s) = \left[r + (1-r)\frac{s}{1+s}\right]\left[\frac{\lambda+s}{1+s} + \left(1 - \frac{\lambda+s}{1+s}\right)M - 0\right] - s. \tag{B.28}$$

For the mid-quality firm, its (expected) profit is $\pi(\Lambda, M; \bar{q}_{ML}(s^\star_{ML}), s^\star_{ML})$, where $\pi(\Lambda, M; \bar{q}_{ML}(s), s)$ is given by

$$\pi(\Lambda, M; \bar{q}_{ML}(s), s) = \left[r + (1-r)\frac{s}{1+s}\right]\left[\frac{\lambda+s}{1+s} + \left(1 - \frac{\lambda+s}{1+s}\right)M - c_M\right] - s. \tag{B.29}$$

Then by (B.28) and (B.29), we obtain that

$$\frac{\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}(s), s)}{\mathrm{d}s} = \frac{f_L(s)}{(1+s)^3}, \tag{B.30}$$

$$\frac{\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)}{\mathrm{d}s} = \frac{f_M(s)}{(1+s)^3}, \tag{B.31}$$

where $f_M(s) = f_L(s) - c_M(1-r)(1+s)$ and

$$f_L(s) = -1 + M(1-2r-s)(1-\lambda) + \lambda - s\big[1 + s(3+s) + \lambda\big] + r(1-s-2\lambda). \tag{B.32}$$

By (B.32), we further obtain that

$$f_L'(s) = -1 - r - 3s(2+s) - M(1-\lambda) - \lambda < 0, \tag{B.33}$$

and hence $f_L(s)$ is decreasing in $s \geq 0$. Then,

$$f_L(0) = (M-1)(1-\lambda) + r\big[1 - 2\big(M(1-\lambda) + \lambda\big)\big],$$

and hence $f_L(0) > 0$ if and only if

$$0 \leq r < r_2 := \frac{(M-1)(1-\lambda)}{2\big(M(1-\lambda)+\lambda\big) - 1}. \tag{B.34}$$

Similarly, we obtain that

$$f_M'(s) = -1 - r - 3s(2+s) - M(1-\lambda) - \lambda - c_M(1-r) < 0, \tag{B.35}$$

and hence $f_M(s)$ is decreasing in $s \geq 0$. Then,

$$f_M(0) = (M-1)(1-\lambda) - c_M + r\big[1 + c_M - 2\big(M(1-\lambda)+\lambda\big)\big],$$

and hence $f_M(0) > 0$ if and only if

$$(M-1)(1-\lambda) > c_M, \text{ and } 0 \leq r < \bar{r} := \frac{(M-1)(1-\lambda) - c_M}{2\big(M(1-\lambda)+\lambda\big) - (1+c_M)}. \tag{B.36}$$

Note that it can be checked that $\bar{r} < r_2$.

Therefore, we obtain the following:

(i) $r \in [r_2, 1)$: we have $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}(s), s)/\mathrm{d}s \leq 0$ and $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)/\mathrm{d}s < 0$. In this case, for any $s^\star_{ML} \geq 0$, there does not exist $s^\star_H$ such that both (B.26) and (B.27) are satisfied and that $s^\star_H < s^\star_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(ii) $r \in [0, r_2)$ and $(M-1)(1-\lambda) \leq c_M$: we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)/\mathrm{d}s \leq 0$. In this case, for any $s^\star_{ML} \geq 0$, there does not exist $s^\star_H$ such that both (B.26) and (B.27) are satisfied and that $s^\star_H < s^\star_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(iii) $r \in [\bar{r}, r_2)$ and $(M-1)(1-\lambda) > c_M$: we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)/\mathrm{d}s \le 0$. In this case, for any $s^\star_{ML} \ge 0$, there does not exist $s^\star_H$ such that both (B.26) and (B.27) are satisfied and that $s^\star_H < s^\star_{ML}$. Thus, this partial-pooling equilibrium does not exist.

(iv) $r \in [0, \bar{r})$ and $(M-1)(1-\lambda) > c_M$: there exist thresholds $\bar{\bar{s}}$ and $\bar{s}$ that are respectively (and uniquely) determined by solving $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}(s), s)/\mathrm{d}s = 0$ and $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)/\mathrm{d}s = 0$ with respect to s, and $\bar{\bar{s}} > \bar{s}$. Then we have $\mathrm{d}\pi(\Lambda, M; \bar{q}_{ML}(s), s)/\mathrm{d}s \ge 0$ if and only if $s \in [0, \bar{s}]$, and $\mathrm{d}\pi(\Lambda, L; \bar{q}_{ML}(s), s)/\mathrm{d}s \ge 0$ if and only if $s \in [0, \bar{\bar{s}}]$. Then by considering (B.26) being satisfied, we obtain that

$$\pi(\Lambda, M; \bar{q}_{ML}(\bar{s}), \bar{s}) = \max_{s \ge 0} \pi(\Lambda, M; \bar{q}_{ML}(s), s) \ge r(H - c_M). \tag{B.37}$$

Thus, when (B.26) (i.e., (B.37)) is satisfied, all (B.15), (B.16), and (B.17) are satisfied. That is, there exists a least-cost partial-pooling equilibrium where $s^\star_{ML} \in [s^3_{ML}, s^4_{ML}]$, $s^\star_H < s^\star_{ML}$ and $s^\star_H$ is uniquely determined by (B.27). Notice that it is not difficult to check that conditions $r \in [0, \bar{r})$, $(M-1)(1-\lambda) > c_M$, and (B.37) are equivalent to (B.37). Additionally, $s^\star_H$ is a function of $s^\star_{ML}$ and can be written as $s^\star_H(s^\star_{ML})$.

The proof of "if" part follows by reversing the previous arguments. □

B.3. Proof of Corollary 1

The formal statement of Corollary 1 is given by the following Corollary B.1. Thus, to prove Corollary 1, we only need to prove Corollary B.1.

COROLLARY B.1. *Compared to the benchmark case, under SMM when it has the information revelation role, the partial-pooling equilibrium where the low- and mid-quality firms pool while the high-quality firm separates (i.e., $\Lambda = \{M, L\}$) emerges in fewer cases. Formally, this partial-pooling equilibrium can occur under both the traditional marketing and SMM if and only if*

$$H \le \pi(\Lambda, M; \bar{q}_{ML}(\bar{s}), \bar{s})/r + c_M, \tag{B.38}$$

but can only occur under the traditional marketing scenario (i.e., as characterized in Proposition 2(ii)) if and only if

$$\pi(\Lambda, M; \bar{q}_{ML}(\bar{s}), \bar{s})/r + c_M < H \le \pi(\Lambda, M; \bar{q}_{ML}, \hat{s})/r + c_M. \tag{B.39}$$

Next, we prove the corollary. By (A.35) and (B.37), if

$$H \le \pi(\Lambda, M; \bar{q}_{ML}(\bar{s}), \bar{s})/r + c_M,$$

this partial-pooling equilibrium can happen regardless of the traditional marketing or SMM scenario. If

$$\pi(\Lambda, M; \bar{q}_{ML}(\bar{s}), \bar{s})/r + c_M < H \le \pi(\Lambda, M; \bar{q}_{ML}, \hat{s})/r + c_M,$$

under the traditional marketing, this partial-pooling equilibrium exists as characterized by Proposition 2(ii); whereas under SMM with its information revelation role, this partial-pooling equilibrium does not exist. Alternatively, if

$$H > \pi(\Lambda, M; \bar{q}_{ML}, \hat{s})/r + c_M,$$

this partial-pooling equilibrium will never occur, regardless of the traditional marketing or SMM scenario. Furthermore, by reversing the previous arguments, we complete the proof of the corollary. Therefore, we obtain that when there is SMM with its information revelation role, this partial-pooling equilibrium emerges in fewer cases. □

B.4. Proof of Proposition 7

In a pooling equilibrium $\Lambda = \{H, M, L\}$, the high-quality, mid-quality, and low-quality firms choose the same spending on SMM, which is denoted by $s^\star_{HML}$. After observing this spending level, consumers update their prior belief and formulate their posterior belief, and form their expected quality. Thus, the pooling price $p^\star_\Lambda$ cannot exceed and must equal the lowest expected quality $\bar{q}_{HML}(s^\star_{HML})$, which is given by

$$\bar{q}_{HML}(s^\star_{HML}) := \frac{\lambda + s^\star_{HML}}{1 + s^\star_{HML}} + \left(1 - \frac{\lambda + s^\star_{HML}}{1 + s^\star_{HML}}\right)\frac{H + M}{2}.$$

Identical to Nian and Sundararajan (2022), we specify the off-equilibrium path belief that $\mu(L \mid s, p) = 1$ for $(s, p) \neq (s^\star_{HML}, \bar{q}_{HML}(s^\star_{HML}))$. That means, if any firm deviates by choosing a spending level other than $s^\star_{HML}$, consumers will believe it to be of the low-quality type. Therefore, the spending level $s^\star_{HML}$ only needs to satisfy the following conditions, i.e.,

$$\pi(\Lambda, L; \bar{q}_{HML}(s^\star_{HML}), s^\star_{HML}) \geq \pi(L, L; 1, 0), \tag{B.40}$$

$$\pi(\Lambda, M; \bar{q}_{HML}(s^\star_{HML}), s^\star_{HML}) \geq \pi(L, M; 1, 0), \tag{B.41}$$

$$\pi(\Lambda, H; \bar{q}_{HML}(s^\star_{HML}), s^\star_{HML}) \geq \pi(L, H; 1, 0). \tag{B.42}$$

By solving (B.40), (B.41), and (B.42) with respect to $s^\star_{HML}$, we obtain that $s^\star_{HML} \in [0, \min\{\bar{\bar{s}}_L, \bar{\bar{s}}_M, \bar{\bar{s}}_H\}]$, in which, $\bar{\bar{s}}_L$, $\bar{\bar{s}}_M$, and $\bar{\bar{s}}_H$ are, respectively, the unique positive root of (B.40), (B.41), and (B.42) replacing "$\leq$" by "$=$", i.e.,

$$\left[r + (1 - r)\frac{\bar{\bar{s}}_q}{1 + \bar{\bar{s}}_q}\right](\bar{q}_{HML}(\bar{\bar{s}}_q) - c_q) - \bar{\bar{s}}_q - r(1 - c_q) = 0, \; q \in \{L, M, H\}. \tag{B.43}$$

Note that the left-hand side of (B.43) is decreasing in c_q, and is either decreasing in $\bar{\bar{s}}_q$, or firstly increasing and then decreasing in $\bar{\bar{s}}_q$. Hence we obtain that $\bar{\bar{s}}_H < \bar{\bar{s}}_M < \bar{\bar{s}}_L$. Therefore, for any $\check{s} \in [0, \bar{\bar{s}}_H]$, where $\bar{\bar{s}}_H$ is uniquely determined by (B.43) for $q = H$, there exists a pooling equilibrium, in which the high-quality, mid-quality, and low-quality firms pool together with the same spending $s^\star_{HML} = \check{s}$ and price $p^\star_\Lambda = \bar{q}_{HML}(\check{s})$. □

B.5. Proof of Corollary 2

By Propositions 3 and 7, we can check that $\bar{s}_H > \bar{\bar{s}}_H$, which completes the proof. □

B.6. Proof of Proposition 8

We prove each part of the proposition separately.

Part (i): We apply the intuitive criterion to test the partial-pooling equilibrium in Proposition 6(ii), following the two steps as described in Appendix A.4.

In the partial-pooling equilibrium with $\Lambda = \{M, L\}$, for any equilibrium spendings $s^\star_{ML}$ and $s^\star_H(s^\star_{ML})$ as characterized in Proposition 6(ii), define $\check{s}_L$ as the smallest root of $\pi(\Lambda, L; \bar{q}_{ML}(s^\star_{ML}), s^\star_{ML}) = \pi(H, L; H, s)$ with respect to s. For simplicity, write $s^\star_H(s^\star_{ML})$ as $s^\star_H$. Note that $s^\star_H < \check{s}_L < s^\star_{ML}$. Then consider the deviation s. For $s \in [0, s^\star_H)$, by Proposition 6(ii), $J(s) = \{H, M, L\}$, and thus $Q \backslash J(s) = \emptyset$. For $s \geq \check{s}_L$, we have either $L \in Q \backslash J(s)$ or $Q \backslash J(s) = \emptyset$.

For $s \in [s^\star_H, \check{s}_L)$, by the definition of $\check{s}_L$ and Proposition 6(ii), we have $J(s) = \{L\}$, and thus $Q \backslash J(s) = \{H, M\}$. Therefore, in Step 2, the worst case is that consumers believe the deviation is from the mid-quality firm. For the mid-quality firm, it obtains profit $v^*(M) = \pi(\Lambda, M; \bar{q}_{ML}(s^\star_{ML}), s^\star_{ML})$ in equilibrium, and the worst profit $\pi(M, M; M, s)$ in deviation. For the high-quality firm, it obtains profit $v^*(H) = \pi(H, H; H, s^\star_H)$ in equilibrium, and the worst profit

$\pi(M,H;M,s)$ in deviation. Therefore, this partial-pooling equilibrium survives the intuitive criterion if and only if for $\forall s \in [s_H^\star, \check{s}_L)$, both $\pi(\Lambda, M; \bar{q}_{ML}(s_{ML}^\star), s_{ML}^\star) \geq \pi(M,M;M,s)$ and $\pi(H,H;H,s_H^\star) \geq \pi(M,H;M,s)$ hold true, i.e.,

$$\Big[r+(1-r)\frac{s_{ML}^\star}{1+s_{ML}^\star}\Big](\bar{q}_{ML}(s_{ML}^\star)-c_M)-s_{ML}^\star \geq \max_{s\in[s_H^\star,\check{s}_L]}\Big\{\Big[r+(1-r)\frac{s}{1+s}\Big](M-c_M)-s\Big\}, \tag{B.44}$$

$$\Big[r+(1-r)\frac{s_H^\star}{1+s_H^\star}\Big](H-c_H)-s_H^\star \geq \max_{s\in[s_H^\star,\check{s}_L]}\Big\{\Big[r+(1-r)\frac{s}{1+s}\Big](M-c_H)-s\Big\}. \tag{B.45}$$

Part (ii): We apply the intuitive criterion to the pooling equilibrium. As a preparation, for $\Lambda = \{H,M,L\}$, we define $s_{Lp} := \arg\max_s \pi(\Lambda, L; \bar{q}_{HML}(s), s)$. Then we analyze the following two cases.

Case (a): $rH < \pi(\Lambda, L; \bar{q}_{HML}(s_{Lp}), s_{Lp})$.

Define $\check{s}_L$ as the smallest one of two real roots of $rH = \pi(\Lambda, L; \bar{q}_{HML}(s), s)$ with respect to s. Consider the pooling equilibria as characterized in Proposition 7 with spending $\check{s} \in [0, \min\{\check{s}_L, \bar{\bar{s}}_H\}]$. We then show that all such pooling equilibria with $\check{s} \in [0, \min\{\check{s}_L, \bar{\bar{s}}_H\}]$ survive the intuitive criterion.

Recall from Appendix B.4, we have $v^*(q) = \pi(\Lambda, q; \bar{q}_{HML}(\check{s}), \check{s}) \geq \pi(L,q;L,0)$. Thus, for $Q\backslash J(s) \neq \emptyset$, if $L \in Q\backslash J(s)$, i.e., $L \notin J(s)$, then (A.41) cannot be satisfied, i.e., the pooling equilibrium survives the intuitive criterion. Therefore, to show that the pooling equilibria survive the intuitive criterion, we only need to show that $L \notin J(s)$ when $J(s) \neq \{H,M,L\}$.

In preparation, define s_4, s_5, and s_6 as the largest value of s that, respectively, solves the equations:

$$\pi(H,H;H,s) = \pi(\Lambda, H; \bar{q}_{HML}(\check{s}), \check{s}),$$
$$\pi(H,M;H,s) = \pi(\Lambda, M; \bar{q}_{HML}(\check{s}), \check{s}),$$
$$\pi(H,L;H,s) = \pi(\Lambda, L; \bar{q}_{HML}(\check{s}), \check{s}).$$

Note that by the fact that $\pi(H,q;H,s) - \pi(\Lambda, q; \bar{q}_{HML}(\check{s}), \check{s})$ is decreasing in c_q for $c_q \geq 0$ and $s > \check{s}$, we can obtain that $0 < s_4 < s_5 < s_6$. We observe that, based on the last equality above, $\pi(H,L;H,s_6) - \pi(\Lambda, L; \bar{q}_{HML}(\check{s}), \check{s}) = 0$, and $\pi(H,L;H,s)$ is decreasing in s around s_6. Therefore, we obtain that $\pi(H,M;H,s_6) - \pi(\Lambda, M; \bar{q}_{HML}(\check{s}), \check{s}) < 0$, and $\pi(H,M;H,s)$ is decreasing in s in the neighborhood of s_6. Consequently, for s_5 such that $\pi(H,M;H,s_5) - \pi(\Lambda, M; \bar{q}_{HML}(\check{s}), \check{s}) = 0$, we have $s_5 < s_6$. By the same reasoning process, we obtain that $0 < s_4 < s_5$, i.e., $0 \leq \check{s} < s_4 < s_5 < s_6$.

For $0 \leq s \leq \check{s}$, as $v^*(L) \leq \pi(H,L;H,s)$, we have $L \notin J(s)$. Therefore, the pooling equilibrium must survive the intuitive criterion when $0 \leq s \leq \check{s}$. Similarly, for $\check{s} < s \leq s_6$, we also have $L \notin J(s)$; and for $s > s_6$, we have $J(s) = \{H,M,L\}$, and thus $Q\backslash J(s) = \emptyset$. Therefore, we conclude that the pooling equilibrium survives the intuitive criterion.

Case (b): $rH \geq \pi(\Lambda, L; \bar{q}_{HML}(s_{Lp}), s_{Lp})$.

By Proposition 7, for any $\check{s} \in [0, \bar{\bar{s}}_H]$, there exists a pooling equilibrium with the spending and price pair $(s_\Lambda^\star, p_\Lambda^\star) = (\check{s}, \bar{q}_{HML}(\check{s}))$, where $\Lambda = \{H,M,L\}$. Then consider the deviation of SMM spending to s. For $0 \leq s \leq s_6$, as $v^*(L) \leq \pi(H,L;H,s)$, we have $L \notin J(S)$. For $s > s_6$, we have $J(S) = \{H,M,L\}$, and thus $Q\backslash J(S) = \emptyset$. Therefore, we conclude that all the pooling equilibria survive the intuitive criterion. □

B.7. Proof of Corollary 3

The formal statement of Corollary 3 is given by the following Corollary B.2. Therefore, to prove Corollary 3, we only need to prove the Corollary B.2 below.

COROLLARY B.2. *Consider the partial-pooling equilibrium with $\Lambda = \{M, L\}$. For each partial-pooling equilibrium surviving the intuitive criterion with spendings $(s^\star_{ML}, s^\star_H(s^\star_{ML}))$ under the SMM scenario with its information revelation role, there must exist a partial-pooling equilibrium with spendings $(s^\star_{ML}, s^*_H(s^\star_{ML}))$ that also survives the intuitive criterion under the traditional marketing scenario. That is, compared with the traditional marketing, there are fewer cases of partial-pooling equilibria with $\Lambda = \{M, L\}$ that survive the intuitive criterion under the SMM with information revelation role.*

Next, we prove the corollary. Consider the partial-pooling equilibrium with $\Lambda = \{M, L\}$. Recall from Proposition 4 that under the traditional marketing, the partial-pooling equilibrium survives the intuitive criterion if and only if conditions (A.42) and (A.43) are satisfied. Additionally, by Proposition 8, under the SMM, the partial-pooling equilibrium survives the intuitive criterion if and only if conditions (18) and (19) are satisfied. Note that under the traditional marketing, the equilibrium spending for the mid- and low-quality firms is $s^*_{ML} \in [s^1_{ML}, s^2_{ML}]$; and under the SMM, the equilibrium spending for the mid- and low-quality firms is $s^\star_{ML} \in [s^3_{ML}, s^4_{ML}]$. We observe that $[s^3_{ML}, s^4_{ML}] \subset [s^1_{ML}, s^2_{ML}]$. Therefore, for each equilibrium with spending pair $(s^\star_{ML}, s^\star_H(s^\star_{ML}))$ under the SMM, there exists an equilibrium with the spending pair $(s^\star_{ML}, s^*_H(s^\star_{ML}))$ under the traditional marketing.

Then to prove that compared with the traditional marketing, there are fewer cases of partial-pooling equilibria that survive the intuitive criterion under the SMM with its information revelation role, we only need to show that for each partial-pooling equilibrium surviving the intuitive criterion with spendings $(s^\star_{ML}, s^\star_H(s^\star_{ML}))$ under the SMM (i.e., conditions (18) and (19) are satisfied), there must exist a partial-pooling equilibrium with spendings $(s^\star_{ML}, s^*_H(s^\star_{ML}))$ that also survives the intuitive criterion under the traditional marketing (i.e., conditions (A.42) and (A.43) are satisfied).

First, we show that for each $(s^\star_{ML}, s^\star_H(s^\star_{ML}))$ under the SMM with conditions (18) and (19) being satisfied (i.e., (B.44) and (B.45) being satisfied), the condition (A.42) is satisfied for $(s^\star_{ML}, s^\star_H(s^\star_{ML}))$ under the traditional marketing. Define $\bar{s}^1_M$ as the smallest root of solving the equation below with respect to s:

$$\left[r + (1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right](\bar{q}_{ML}(s^\star_{ML}) - c_M) - s^\star_{ML} = \left[r + (1-r)\frac{s}{1+s}\right](M - c_M) - s. \tag{B.46}$$

Similarly, define $\bar{s}^2_M$ as the smallest root of solving the equation below with respect to s:

$$\left[r + (1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right](\bar{q}_{ML} - c_M) - s^\star_{ML} = \left[r + (1-r)\frac{s}{1+s}\right](M - c_M) - s. \tag{B.47}$$

Then it is not difficult to obtain that $s^\star_H(s^\star_{ML}) < \bar{s}^1_M < s^\star_{ML}$, and condition (B.44) being satisfied implies $\bar{s}^1_M \geq \check{s}_L$. Next, we derive that $\bar{s}^1_M$ is a function of $\check{s}_L$. By the definition of $\check{s}_L$, it satisfies

$$\left[r + (1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right]\bar{q}_{ML}(s^\star_{ML}) - s^\star_{ML} = \left[r + (1-r)\frac{\check{s}_L}{1+\check{s}_L}\right]H - \check{s}_L. \tag{B.48}$$

Since $\bar{s}^1_M$ satisfies (B.46), by comparing (B.46) and (B.48), we obtain that

$$\left[r + (1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\right]c_M + \left[r + (1-r)\frac{\bar{s}^1_M}{1+\bar{s}^1_M}\right](M - c_M) - \bar{s}^1_M = \left[r + (1-r)\frac{\check{s}_L}{1+\check{s}_L}\right]H - \check{s}_L. \tag{B.49}$$

Similarly, we can also obtain that

$$\Big[r+(1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\Big]c_M+\Big[r+(1-r)\frac{\bar{s}^2_M}{1+\bar{s}^2_M}\Big](M-c_M)-\bar{s}^2_M=\Big[r+(1-r)\frac{\tilde{s}_L}{1+\tilde{s}_L}\Big]H-\tilde{s}_L. \tag{B.50}$$

Note that the left-hand side of (B.49) is increasing in $\bar{s}^1_M$, and the right-hand side of (B.49) is increasing in $\check{s}_L$. Then we, by the implicit function theorem, obtain

$$\frac{\mathrm{d}\bar{s}^1_M}{\mathrm{d}\check{s}_L}=\frac{(1-r)H}{(1+\check{s}_L)^2}-1\Bigg/\frac{(1-r)(M-c_M)}{(1+\bar{s}^1_M)^2}-1>1, \tag{B.51}$$

where the inequality follows from $H>M$ and $\bar{s}^1_M>\check{s}_L$. Therefore, when $\check{s}_L$ increases by $\tilde{s}_L-\check{s}_L$ to $\tilde{s}_L$, $\bar{s}^1_M$ increases by more than $\tilde{s}_L-\check{s}_L$ to $\bar{s}^2_M$. Thus, we obtain that $\bar{s}^2_M>\bar{s}^1_M+\tilde{s}_L-\check{s}_L\geq\tilde{s}_L$, which implies that the condition (A.42) is satisfied for $(s^\star_{ML},s^*_H(s^\star_{ML}))$ under the traditional marketing.

Second, we show that for each $(s^\star_{ML},s^\star_H(s^\star_{ML}))$ under the SMM with conditions (18) and (19) being satisfied, the condition (A.43) is also satisfied for $(s^\star_{ML},s^*_H(s^\star_{ML}))$ under the traditional marketing. In preparation, define $\bar{s}^1_H$ as the smallest root of solving the equation below with respect to s:

$$\Big[r+(1-r)\frac{s^*_H(s^\star_{ML})}{1+s^*_H(s^\star_{ML})}\Big](H-c_H)-s^*_H(s^\star_{ML})=\Big[r+(1-r)\frac{s}{1+s}\Big](M-c_H)-s. \tag{B.52}$$

Note that if $\bar{s}^1_H$ does not exist, the condition (A.43) must be satisfied. Hence, we only focus on the situation where $\bar{s}^1_H$ exists. Then since we already showed that $\bar{s}^2_M>\tilde{s}_L$, to prove that the condition (A.43) is satisfied for $(s^\star_{ML},s^*_H(s^\star_{ML}))$ under the traditional marketing, it is sufficient to show that $\bar{s}^1_H>\bar{s}^2_M$. Recall from the definition of $s^*_H(s^\star_{ML})$ that

$$\Big[r+(1-r)\frac{s^\star_{ML}}{1+s^\star_{ML}}\Big](\bar{q}_{ML}-c_M)-s^\star_{ML}=\Big[r+(1-r)\frac{s^*_H(s^\star_{ML})}{1+s^*_H(s^\star_{ML})}\Big](H-c_M)-s^*_H(s^\star_{ML}). \tag{B.53}$$

Then by (B.47) and (B.53), we obtain

$$\Big[r+(1-r)\frac{s^*_H(s^\star_{ML})}{1+s^*_H(s^\star_{ML})}\Big](H-c_M)-s^*_H(s^\star_{ML})=\Big[r+(1-r)\frac{\bar{s}^2_M}{1+\bar{s}^2_M}\Big](M-c_M)-\bar{s}^2_M. \tag{B.54}$$

Next, since $\bar{s}^1_H$ satisfies (B.52), by plugging $\bar{s}^1_H$ into (B.52), we rewrite (B.52) as

$$\begin{aligned}\Big[r+(1-r)\frac{s^*_H(s^\star_{ML})}{1+s^*_H(s^\star_{ML})}\Big](H-c_M)-s^*_H(s^\star_{ML})=&\Big[r+(1-r)\frac{\bar{s}^1_H}{1+\bar{s}^1_H}\Big](M-c_M)-\bar{s}^1_H\\&+(1-r)(c_H-c_M)\Big(\frac{s^*_H(s^\star_{ML})}{1+s^*_H(s^\star_{ML})}-\frac{\bar{s}^1_H}{1+\bar{s}^1_H}\Big).\end{aligned} \tag{B.55}$$

Since $s^*_H(s^\star_{ML})<\bar{s}^1_H$, $s^*_H(s^\star_{ML})<\bar{s}^2_M$, $\bar{s}^1_H$ is only in the domain of s such that $\Big[r+(1-r)\frac{s}{1+s}\Big](M-c_M)-s$ is increasing in s, and $\Big[r+(1-r)\frac{s}{1+s}\Big](M-c_M)-s$ is increasing in s around $s=\bar{s}^2_M$, we by (B.54) and (B.55) obtain that $\bar{s}^1_H>\bar{s}^2_M$. Therefore, the condition (A.43) is satisfied for $(s^\star_{ML},s^*_H(s^\star_{ML}))$ under the traditional marketing. $\square$

C. A Continuum of Quality Types

In the base model in §4 of the paper, consistent with Nian and Sundararajan (2022) (referred to as NS, hereafter), we consider the role of SMM in information revelation and assume that the firm can be one of three types (i.e., high-, mid-, and low-quality types). Following NS, we further extend this model by considering that the firm has a continuum of quality types. In this extension, we adopt the same model and assumptions as in their paper. For brevity, and in line with §5.1 in NS, we focus only on the case where SMM has the information revelation role. Specifically, the firm's quality types are assumed to be uniformly distributed over the interval $[\underline{q}, \bar{q}]$, and the quality q is unobservable to consumers before purchase. The unit production cost for the type-q firm is $c(q) = aq$. In contrast to Proposition 10 in NS, we obtain Proposition C.1 below.

PROPOSITION C.1. *When the firm has a continuum of quality that is uniformly distributed over the interval $[\underline{q}, \bar{q}]$, and the SMM has an information revelation role:*

(i) The separating perfect Bayesian equilibrium does not exist if $\underline{q}(1-a)(1-r) \le 1$ or

$$\bar{q} > \left(1 + \underline{q}(1-a) - 2\sqrt{\underline{q}(1-a)(1-r)}\right) \Big/ r + a\underline{q}. \tag{C.1}$$

(ii) The partial-pooling equilibrium as stated in Proposition 10 of NS, in which firm types $q \in [\underline{q}, q_1) \cup (q_2, \bar{q}]$ choose $(0, p_\Lambda^*)$ and firm types $q \in [q_1, q_2]$ choose (s_q^*, q), does not exist, for $\forall \underline{q} \le q_1 \le q_2 \le \bar{q}$, $p_\Lambda^* > 0$, and $s_q^* > 0$.

Proposition C.1(i) demonstrates that a separating equilibrium, where firms with different quality types choose different prices and SMM spending to reveal their types, does not exist in certain cases. To prevent a lower-quality firm from mimicking itself, a higher-quality firm must spend less on SMM than the lower-quality firm to preserve a small market base. When the quality $\underline{q}$ of the lowest-type firm is sufficiently low (i.e., $\underline{q}(1-a)(1-r) \le 1$), it will spend nothing on SMM in a separating equilibrium. Since choosing a negative SMM spending is infeasible for higher-quality firms, separating from the type $\underline{q}$ is impossible. In other words, in such cases, the type $\underline{q}$ always has incentives to mimic some higher-quality firms. On the other hand, if the quality $\bar{q}$ of the highest-quality firm is sufficiently high (i.e., if $\bar{q}$ satisfies (C.1)), then the lowest type $\underline{q}$ always benefits from mimicking the highest type $\bar{q}$ due to the sufficiently high price $\bar{q}$ can charge. Therefore, in stark contrast to Proposition 10(1) in NS, which claims that there always exists a unique separating equilibrium, we show that a separating equilibrium *does not* always exist. Note that, in line with §5.1 of NS, the unit production cost of the lowest-quality type in this extension is $c(\underline{q}) = a\underline{q} \neq 0$ for $\underline{q} > 0$, while it is $c_L = 0$ for the lowest-quality type in the base model. Therefore, the base model and the extension with a continuum of firm types in NS are not consistent. This is one of the reasons why a separating equilibrium may exist when the firm has a continuum of quality types, as shown in Proposition C.1(i).

In contrast to Proposition 10(2) in NS, Proposition C.1(ii) above shows that there *does not* exist a partial-pooling equilibrium where high- and low-quality firms pool together by choosing the same price and spending nothing on SMM, while mid-quality firms separate and reveal their types by choosing distinct prices and SMM spendings. The intuition is the same as that in the base model. Therefore, in line with our findings in the base model, there *does not* exist an inverted U-shaped relationship between SMM spending and firm qualities when the firm has a continuum of quality types and SMM has the information revelation role.

C.1. Proof of Proposition C.1

We prove each part of the proposition separately.

Part (i). Firstly, we show the SMM spending of the lowest-type firm in a separating equilibrium. As we have analyzed in Section A.1, in the separating equilibrium, the lowest-type firm will charge the same SMM spending as that under complete information and obtain the same profits. The profit of the lowest-type firm under complete information is given by

$$\pi(\underline{q},\underline{q};p(\underline{q}),s)=\left[r+(1-r)\frac{s}{1+s}\right](\underline{q}-a\underline{q})-s,$$

and by maximizing $\pi(\underline{q},\underline{q};p(\underline{q}),s)$ with respect to s, we obtain the optimal SMM spending $s^*_{\underline{q}}$, which is given by

$$s^*_{\underline{q}}:=\max\{0,s^c_{\underline{q}}\},\ \text{where } s^c_{\underline{q}}=\sqrt{\underline{q}(1-a)(1-r)}-1, \tag{C.2}$$

with the resulting profit being

$$\pi(\underline{q},\underline{q};\underline{q},s^c_{\underline{q}})=1+\underline{q}(1-a)-2\sqrt{\underline{q}(1-a)(1-r)}. \tag{C.3}$$

Second, we show that in a separating equilibrium, the equilibrium spendings $s^*_{q'}>s^*_q$, for $\forall \underline{q}\le q'<q\le\bar{q}$. Suppose that there is a separating equilibrium. The following conditions must be satisfied:

$$\pi(q,q;q,s^*_q)\ge\pi(q',q;q',s^*_{q'}), \tag{C.4}$$

$$\pi(q',q';q',s^*_{q'})\ge\pi(q,q';q,s^*_q), \tag{C.5}$$

in which (C.4) guarantees that the firm type q does not have incentives to mimic the type q', and (C.5) guarantees that the firm type q' does not have incentives to mimic the type q. By expanding (C.4) and (C.5), we obtain

$$\left[r+(1-r)\frac{s^*_q}{1+s^*_q}\right](q-aq)-s^*_q\ge\left[r+(1-r)\frac{s^*_{q'}}{1+s^*_{q'}}\right](q'-aq)-s^*_{q'}, \tag{C.6}$$

$$\left[r+(1-r)\frac{s^*_{q'}}{1+s^*_{q'}}\right](q'-aq')-s^*_{q'}\ge\left[r+(1-r)\frac{s^*_q}{1+s^*_q}\right](q-aq')-s^*_q. \tag{C.7}$$

By adding both sides of (C.6) and (C.7), we obtain that $s^*_q\le s^*_{q'}$. Since $s^*_q\ne s^*_{q'}$ in a separating equilibrium, we then must have $s^*_q<s^*_{q'}$ for $\forall \underline{q}\le q'<q\le\bar{q}$, implying that $s^*_q<s^*_{\underline{q}}$ for $\forall \underline{q}<q\le\bar{q}$.

Finally, by (C.2) and $s^*_q<s^*_{\underline{q}}$ for $\forall\underline{q}<q\le\bar{q}$, we obtain that the separating equilibrium does not exist if $\underline{q}(1-a)(1-r)\le 1$ (i.e., if $s^*_{\underline{q}}=0$). Furthermore, given $\underline{q}(1-a)(1-r)>1$, it can be verified that $\pi(q,q;q,s)$ is increasing in $s\in[0,s^c_q]$, for $\forall\underline{q}\le q\le\bar{q}$. Then if $\pi(\bar{q},\underline{q};\bar{q},0)>\pi(\underline{q},\underline{q};\underline{q},s^c_{\underline{q}})$, i.e.,

$$\bar{q}>\left(1+\underline{q}(1-a)-2\sqrt{\underline{q}(1-a)(1-r)}\right)\Big/r+a\underline{q},$$

there must exist $\varepsilon>0$, such that mimicking type $q\in(\bar{q}-\varepsilon,\bar{q}]$ is more profitable for the lowest type $\underline{q}$, which implies that the separating equilibrium collapses.

Part (ii). Suppose there exists a partial-pooling equilibrium in which firm types $q\in[\underline{q},q_1)\cup(q_2,\bar{q}]$ choose $(0,p^*_\Lambda)$ and firm types $q\in[q_1,q_2]$ choose (s^*_q,q). Let $q_L\in[\underline{q},q_1)$, $q_M\in[q_1,q_2]$, and $q_H\in(q_2,\bar{q}]$. In this partial-pooling equilibrium, type q_L does not have incentives to mimic type q_M, and thus the condition below must be satisfied:

$$r(p^*_\Lambda-aq_L)\ge\left[r+(1-r)\frac{s^*_{q_M}}{1+s^*_{q_M}}\right](q_M-aq_L)-s^*_{q_M}; \tag{C.8}$$

similarly, the type q_H does not have incentives to mimic type q_M, and hence the condition below must be satisfied:

$$r(p_\Lambda^* - aq_H) \geq \left[r + (1-r)\frac{s_{q_M}^*}{1+s_{q_M}^*}\right](q_M - aq_H) - s_{q_M}^*; \tag{C.9}$$

the type q_M does not have incentives to mimic type Λ, and hence the condition below must be satisfied:

$$\left[r + (1-r)\frac{s_{q_M}^*}{1+s_{q_M}^*}\right](q_M - aq_M) - s_{q_M}^* \geq r(p_\Lambda^* - aq_M). \tag{C.10}$$

By adding both sides of (C.8) and (C.10), we obtain that $s_{q_M}^* \leq 0$; and by adding both sides of (C.9) and (C.10), we obtain that $s_{q_M}^* \geq 0$. Thus, we obtain that $s_{q_M}^* = 0 = s_\Lambda^*$, which contradicts the fact $s_{q_M}^* > s_\Lambda^* = 0$ as specified in this partial-pooling equilibrium in Proposition 10(2) of Nian and Sundararajan (2022).

Furthermore, applying the same approach, we show that any partial-pooling equilibrium in which firm types $q \in [\underline{q}, q_1) \cup (q_2, \bar{q}]$ choose $(s_\Lambda^*, p_\Lambda^*)$ to pool together and firm types $q \in [q_1, q_2]$ choose (s_q^*, q) with $s_q^* > s_\Lambda^* \geq 0$ to reveal their types, does not exist. □

References

Nian T, Sundararajan A (2022) Social media marketing, quality signaling, and the goldilocks principle. *Information Systems Research* 33(2):540–556.