Online Supplement to “Voluntary Technology Sharing to Rivals”

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A Proofs of the Results for the Main Model

A.1 Proof of Lemma 1

Proof. We distinguish two cases based on the value of q.

(a) $2t < q$: When q is large, all consumers in Submarkets $\bar{1}3$ and $\bar{2}3$ purchase. The firms have incentive to charge a price high enough such that the consumers with the highest misfit cost in Submarkets $\bar{1}3$ and $\bar{2}3$ derive zero utility; that is, $q - t - p_i = 0$, leading to $p_i^* = q - t$, $d_i^* = \frac{1}{2}$, and $\pi_i^* = \frac{1}{2}(q - t)$. We can verify neither firm has incentive to deviate under $2t < q$.

(b) $q \leq 2t$: When q is relatively low, such that some consumers in Submarkets $\bar{1}3$ and $\bar{2}3$ do not purchase but the marginal consumer in Submarket $\bar{1}2$ derives positive utility, the profit functions for the two firms are

$$\begin{cases} \pi_1 = p_1 \left(\frac{1}{3} \frac{t - p_1 + p_2}{2t} + \frac{1}{3} \frac{q - p_1}{t} \right) \\ \pi_2 = p_2 \left(\frac{1}{3} \frac{t - p_2 + p_1}{2t} + \frac{1}{3} \frac{q - p_2}{t} \right) \end{cases} \tag{17}$$

We have $\frac{\partial^2 \pi_1}{\partial p_1^2} = \frac{\partial^2 \pi_2}{\partial p_2^2} = -\frac{1}{t} < 0$. Therefore, the second-order conditions for optimization are satisfied, and the objective functions are well behaved. We drive the first-order conditions as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \frac{t - 2p_1 + p_2}{2t} + \frac{1}{3} \frac{q - 2p_1}{t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{1}{3} \frac{t - 2p_2 + p_1}{2t} + \frac{1}{3} \frac{q - 2p_2}{t} = 0 \end{cases}$$

Solving this system of equations, we can derive $p_i^* = \frac{1}{5}(2q + t)$ and $d_i^* = \frac{1}{10t}(2q + t)$, $i \in \{1, 2\}$. Substituting p_i^* into Equation (17), we have the equilibrium profits as specified in the lemma.

To ensure that some consumers in Submarkets $\bar{1}3$ and $\bar{2}3$ do not purchase requires $q - t - p_i^* \leq 0$, or, equivalently, $q \leq 2t$. To ensure that the marginal consumer in Submarket $\bar{1}2$ derives positive utility requires $q - \frac{1}{2}t - p_i^* \geq 0$, or, equivalently, $q \geq \frac{7}{6}t$ (the assumption imposed in the model). $\square$

A.2 Proof of Lemma 2

Proof. We distinguish four cases based on the value of q.

(a) $\frac{13t}{4+2\theta} < q$: When $0 \leq m_{12}, m_{13}, m_{23} \leq 1$ and the marginal consumers derive positive utilities, the demand functions in Equation (9) are well behaved. We can verify that $\frac{\partial^2 \pi_1}{\partial p_1^2} = -\frac{2}{3t} < 0$ and thus the second-order condition for Firm 1 is satisfied. The Hessian matrix for π_2 is

$$H_{\pi_2}(p_2, p_3) = \begin{bmatrix} \frac{\partial^2 \pi_2}{\partial p_2^2} & \frac{\partial^2 \pi_2}{\partial p_2 \partial p_3} \\ \frac{\partial^2 \pi_2}{\partial p_3 \partial p_2} & \frac{\partial^2 \pi_2}{\partial p_3^2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3t} & \frac{1}{3t} \\ \frac{1}{3t} & -\frac{2}{3t} \end{bmatrix}$$

which is negative definite, satisfying the second-order condition. We derive the first-order conditions of Equation (10) as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1 + p_2 + p_3 + (1-\theta)q + 2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1 - 4p_2 + 2p_3 + (1-\theta)q + 2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1 + 2p_2 - 4p_3 - 2(1-\theta)q + 2t}{6t} = 0 \end{cases} \tag{18}$$

Solving this system of equations, we can derive p_i^* as specified in the lemma. We can verify that $0 \leq m_{12}, m_{13}, m_{23} \leq 1$ within this region, which also ensures that the marginal consumers located at m_{12}, m_{13}, and m_{23} derive positive utility. We can verify that neither firm has profitable deviation.

(b) $\frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta}$: In this case, marginal consumer m_{23} derives zero utility because $q \leq \frac{13t}{4+2\theta}$. When Firm 2 prices the products such that marginal consumer m_{23} derives zero utility, $\frac{q-p_2}{t} + \frac{\theta q - p_3}{t} = 1$, or, equivalently, $p_2 + p_3 = q + \theta q - t$. Using the Lagrange-multiplier method, we have $L(p_2, p_{3,}\lambda) = \pi_2 + \lambda(q + \theta q - t - p_2 - p_3)$. By solving the first-order condition of $L(p_2, p_{3,}\lambda)$, we can derive p_2^* and p_3^* as specified in the lemma, which is independent of p_1. By the first-order condition of π_1 in Equation (10), we can derive p_1^* as specified in the lemma. The condition $\frac{3t}{1+\theta} < q$ ensures that marginal consumer m_{13} derives positive utility. We can verify that marginal consumer m_{12} derives positive utility, $0 \leq m_{12}, m_{13}, m_{23} \leq 1$, and neither firm has incentive to deviate from (p_1^*, p_2^*, p_3^*).

(c) $\frac{14t}{7+5\theta} < q \leq \frac{3t}{1+\theta}$: In this case, marginal consumers m_{13} and m_{23} derive zero utility in equilibrium. Because π_2 shares the same optimization problem as in case (b), p_2^* and p_3^* stay the same as in case (b). When Firm 1 prices its product such that marginal consumer m_{13} derives zero utility, $\frac{q-p_1}{t} + \frac{\theta q - p_3}{t} = 1$, or, equivalently, $p_1 + p_3 = q + \theta q - t$. Therefore, $p_1^* = p_2^*$ in equilibrium. The condition $\frac{14t}{7+5\theta} < q$ ensures that Firm 1 has no incentive to increase the price of Product 1 because the marginal loss from a price increase is greater than the marginal benefit. Meanwhile, Firm 1 has no incentive to decrease the price of Product 1 because the marginal loss of a price decrease is greater than the marginal benefit under $q \leq \frac{3t}{1+\theta}$.

(d) $q \leq \frac{14t}{7+5\theta}$: In this case, the two firms compete in Submarket $\bar{1}2$ but not in Submarket $\bar{1}3$. Consequently, as in case (a) in the proof of Lemma 1, $p_1^* = p_2^* = \frac{1}{5}(2q+t)$. Firm 2 either charges $p_3^* = \frac{1}{5}(-6t + 3q + 5\theta q)$ to cover all of the residual demands in Submarkets $\bar{1}3$ and $\bar{2}3$ or charges a monopoly price $p_3^* = \frac{\theta q}{2}$ for Product 3, whichever is higher. As a result, d_1^*, d_2^*, and π_1^* stay the same as in case (a) of Lemma 1, and π_2^* consists of the revenue from Product 2 (the same as that of Product 1) and the revenue from Product 3. We can verify that neither firm has incentive to deviate.

Substituting p_i^* into Equations (9) and (10), we can derive the equilibrium demands and profits as specified in the lemma. $\square$

A.3 Proof of Proposition 1

Proof. Based on Lemmas 1 and 2, we compare the equilibrium prices and demands before and after the introduction of Product 3 for different cases.

The case with $q \leq 2t$:

When $q \leq \underline{q}(\theta)$, as shown in Lemma 2, the introduction of Product 3 affects neither Product 1's or 2's price or demand. That is, $\hat{p}_i^* = p_i^*$, and $\hat{d}_i^* = d_i^*$, where $i \in \{1, 2\}$.

When $\underline{q}(\theta) < q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{p}_i^* > p_i^*$ and $\hat{d}_i^* < d_i^*$.

When $\frac{3t}{1+\theta} < q$, similarly, we can verify that $\hat{p}_i^* > p_i^*$ and $\hat{d}_i^* < d_i^*$ within the region $q \leq 2t$.

The case with $q > 2t$:

When $q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{p}_i^* > p_i^*$ and $\hat{d}_i^* < d_i^*$.

When $\frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta}$, we can verify that $\hat{p}_2^* > p_2^*$ and $\hat{d}_2^* < d_2^*$. Besides, we have

$$\hat{p}_1^* - p_1^* = \tfrac{1}{4}(t+2q) - (q-t) = \tfrac{5}{4}t - \tfrac{q}{2}$$
$$\hat{d}_1^* - d_1^* = \tfrac{1}{12t}(t+2q) - \tfrac{1}{2} = \tfrac{q}{6t} - \tfrac{5}{12}$$

Therefore, $\hat{p}_1^* \geq p_1^*$ if and only if $q \leq \frac{5t}{2}$, and $\hat{d}_1^* \leq d_1^*$ if and only if $q \leq \frac{5t}{2}$.

When $\frac{13t}{4+2\theta} < q$, we can verify that $\hat{p}_2^* > p_2^*$ and $\hat{d}_2^* < d_2^*$. Besides, we have

$$\hat{p}_1^* - p_1^* = \tfrac{1}{6}(8t + q - \theta q) - (q-t) = \tfrac{1}{6}(14t - 5q - \theta q)$$
$$\hat{d}_1^* - d_1^* = \tfrac{1}{18t}(8t + q - \theta q) - \tfrac{1}{2} = \tfrac{1}{18t}(q - \theta q) - \tfrac{1}{18}$$

Therefore, $\hat{p}_1^* \geq p_1^*$ if and only if $q \leq \frac{14t}{5+\theta}$, and $\hat{d}_1^* \leq d_1^*$ if and only if $q \leq \frac{t}{1-\theta}$.

Altogether, we conclude Proposition 1(a). We note that threshold curves $\frac{5t}{2}$, $\frac{14t}{5+\theta}$, and $\frac{13t}{4+2\theta}$ intersect at $\theta = \frac{3}{5}$. When $\theta \leq \frac{3}{5}$, $\frac{5t}{2} < \frac{14t}{5+\theta} < \frac{13t}{4+2\theta}$ and thus $\hat{p}_1^* \geq p_1^*$ if and only if $q \leq \frac{5t}{2}$; when $\theta > \frac{3}{5}$, $\frac{13t}{4+2\theta} < \frac{14t}{5+\theta} < \frac{5t}{2}$ and thus $\hat{p}_1^* \geq p_1^*$ if and only if $q \leq \frac{14t}{5+\theta}$. Therefore, $\hat{p}_1^* \geq p_1^*$ if and only if $q \leq \min\{\frac{5t}{2}, \frac{14t}{5+\theta}\}$, concluding Product 1's price comparison in Proposition 1(b). We note that threshold curves $\frac{5t}{2}$, $\frac{t}{1-\theta}$, and $\frac{13t}{4+2\theta}$ intersect at $\theta = \frac{3}{5}$. When $\theta \leq \frac{3}{5}$, $\frac{t}{1-\theta} < \frac{5t}{2} < \frac{13t}{4+2\theta}$ and thus $\hat{d}_1^* \leq d_1^*$ if and only if $q \leq \frac{5t}{2}$; when $\theta > \frac{3}{5}$, $\frac{13t}{4+2\theta} < \frac{5t}{2} < \frac{t}{1-\theta}$ and thus $\hat{d}_1^* \leq d_1^*$ if and only if $q \leq \frac{t}{1-\theta}$. Therefore, $\hat{d}_1^* \leq d_1^*$ if and only if $q \leq \max\{\frac{5t}{2}, \frac{t}{1-\theta}\}$, concluding Product 1's demand comparison in Proposition 1(b). □

A.4 Proof of Proposition 3

Proof. Based on Lemmas 1 and 2, we compare the firms' equilibrium profits before and after the introduction of Product 3 for different cases.

The case with $q \leq 2t$:

When $q \leq \underline{q}(\theta)$, as shown in Lemma 2, the introduction of Product 3 affects neither Product 1's nor Product 2's profit. Therefore, $\hat{\pi}_1^* = \pi_1^*$; $\hat{\pi}_2^* > \pi_2^*$ because Firm 2 obtains additional revenue from Product 3.

When $\underline{q}(\theta) < q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{\pi}_i^* > \pi_i^*$, where $i \in \{1, 2\}$.

When $\frac{3t}{1+\theta} < q$, similarly, we can verify that $\hat{\pi}_i^* > \pi_i^*$.

The case with $q > 2t$:

When $q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{\pi}_i^* > \pi_i^*$.

When $\frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta}$, we can verify that $\hat{\pi}_1^* \geq \pi_1^*$. We have

$$\begin{aligned} \hat{\pi}_2^* - \pi_2^* &= \tfrac{1}{48t}[-22t^2 + 2(13+11\theta)qt - (1+10\theta-3\theta^2)q^2] - \tfrac{1}{2}(q-t) \\ &= \tfrac{1}{48t}(-1-10\theta+3\theta^2)(q - \tfrac{1+11\theta-\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t)(q - \tfrac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t) \end{aligned}$$

Because the term in the first bracket is negative and that in the second bracket is positive, $\hat{\pi}_2^* \geq \pi_2^*$ if and only if

$$q \leq \tfrac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t \equiv q_2^*(\theta) \tag{19}$$

When $\frac{13t}{4+2\theta} < q$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{1}{108t}(8t+q-\theta q)^2 - \tfrac{1}{2}(q-t) \\ &= \tfrac{1}{108t}(1-\theta)^2(q - \tfrac{19+8\theta+3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t)(q - \tfrac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t) \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \geq \pi_1^*$ if and only if $q \leq \frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t$.

$$\begin{aligned} \hat{\pi}_2^* - \pi_2^* &= \tfrac{1}{432t}\left[4(10t-q+\theta q)^2 + 27(1-\theta)^2q^2\right] - \tfrac{1}{2}(q-t) \\ &= \tfrac{31}{432t}(1-\theta)^2(q - \tfrac{4(37-10\theta)+6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t)(q - \tfrac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t) \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_2^* \geq \pi_2^*$ if and only if

$$q \leq \tfrac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t \equiv q_2^{**}(\theta) \tag{20}$$

Altogether, we have $\hat{\pi}_1^* \geq \pi_1^*$ if and only if $\underline{q}(\theta) \leq q \leq \frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t$. For Firm 2, we note that threshold curves $q_2^*(\theta)$, $q_2^{**}(\theta)$, and $\frac{13t}{4+2\theta}$ intersect at $\theta = \frac{231+52\sqrt{33}}{1087}$, where $q_2^*(\theta)$ and $q_2^{**}(\theta)$ are defined in Equations (19) and (20), respectively. When θ is below the intersecting point, $q_2^*(\theta) < q_2^{**}(\theta) < \frac{13t}{4+2\theta}$ and thus $\hat{\pi}_2^* \leq \pi_2^*$ if and only if $q \geq q_2^*(\theta)$; when θ is above the intersecting point, $\frac{13t}{4+2\theta} < q_2^{**}(\theta) < q_2^*(\theta)$ and thus $\hat{\pi}_2^* \leq \pi_2^*$ if and only if $q \geq q_2^{**}(\theta)$. Therefore, $\hat{\pi}_2^* \leq \pi_2^*$ if and only if $q \geq \min\{q_2^*(\theta), q_2^{**}(\theta)\}$, concluding the comparison result as in Proposition 3(b). $\square$

A.5 Proof of Proposition 4

Proof. Combining the conditions in Proposition 3 leads to the result in this proposition. Specifically, by the proof of Proposition 3, $q_2^*(\theta)$ and $q_2^{**}(\theta)$ intersect at $\theta = \frac{231+52\sqrt{33}}{1087}$, and $q_2^*(\theta) < q_2^{**}(\theta)$ before the intersection, where $q_2^*(\theta)$ and $q_2^{**}(\theta)$ are defined in Equations (19) and (20), respectively. We can further verify that $q_2^{**}(\theta)$ intersects $\frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t$ at $\theta = \frac{311}{635}$, and $q_2^{**}(\theta) < \frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t$ before the intersection. By noting $\frac{231+52\sqrt{33}}{1087} < \frac{311}{635}$, we conclude $\bar{q}(\theta)$ as in Proposition 4. Moreover, by Equation (14), $\underline{q}(\theta)$ decreases in θ. We can verify that $\bar{q}(\theta)$ increases in θ if $\theta \leq \frac{231+52\sqrt{33}}{1087}$ but decreases otherwise. ☐

A.6 Proof of Corollary 1

Proof. By Lemma 2, if $\underline{q}(\theta) < q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{\pi}_1^*$ increases in θ; If $\frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta}$, $\hat{\pi}_1^*$ is independent of θ; If $\frac{13t}{4+2\theta} < q \leq \bar{q}(\theta)$, $\partial\hat{\pi}_1^*/\partial\theta = -\frac{1}{54t}q(8t+q-\theta q) < 0$. We therefore have the results in Corollary 1. ☐

A.7 Proof of Proposition 5

Proof. In the absence of Product 3, we have

$$sw = \frac{1}{3}\left(\int_0^{m_{12}}(q-xt)\mathrm{d}x + \int_{m_{12}}^1[q-(1-x)t]\mathrm{d}x\right) + \frac{1}{3}\sum_{i=1}^{2}\int_0^{\frac{q-p_i}{t}}(q-xt)\mathrm{d}x, \tag{21}$$

where m_{12} is defined in Equation (2). Substituting the equilibrium prices from Equation (5) into Equation (21), we have

$$sw^* = \begin{cases} q - \frac{5}{12}t & \text{if } 2t < q \\ \frac{1}{300}\left[-29t + 84\left(q + \frac{q^2}{t}\right)\right] & \text{otherwise.} \end{cases}$$

Because $cs^* = sw^* - \pi_1^* - \pi_2^*$, by substituting in the equilibrium profits from Equation (7), we have

$$cs^* = \begin{cases} \frac{7}{12}t & \text{if } 2t < q \\ \frac{1}{300}\left[-41t + 36\left(q + \frac{q^2}{t}\right)\right] & \text{otherwise.} \end{cases}$$

Similarly, substituting the equilibrium prices from Equation (11) into Equation (15), we have

$$\hat{sw}^* = \begin{cases} \frac{-232t^2+101q^2(-1+\theta)^2+32qt(19+8\theta)}{864t} & \text{if } \frac{13t}{4+2\theta} < q \\ \frac{-33t^2+4qt(16+11\theta)+q^2(13+\theta(-26+9\theta))}{96t} & \text{if } \frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta} \\ \frac{-6t^2+8qt(2+\theta)+3q^2(-1+\theta)^2}{24t} & \text{if } \underline{q}(\theta) < q \leq \frac{3t}{1+\theta}. \end{cases}$$

By substituting the equilibrium profits from Equation (13) into $\hat{cs}^* = \hat{sw}^* - \hat{\pi}_1^* - \hat{\pi}_2^*$, we have

$$\hat{cs}^* = \begin{cases} \frac{-1544t^2+31q^2(-1+\theta)^2+32qt(20+7\theta)}{864t} & \text{if } \frac{13t}{4+2\theta} < q \\ \frac{4qt+9t^2+q^2(7+3(-2+\theta)\theta)}{96t} & \text{if } \frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta} \\ \frac{6t^2-2qt(-1+\theta)+q^2(-1+\theta)^2}{24t} & \text{if } \underline{q}(\theta) < q \leq \frac{3t}{1+\theta}. \end{cases}$$

We next compare social welfare and consumer surplus within $\underline{q}(\theta) \leq q \leq \bar{q}(\theta)$.

The case with $q \leq 2t$:

When $\underline{q}(\theta) < q \leq \frac{3t}{1+\theta}$,

$$\begin{aligned} \hat{sw}^* - sw^* &= \frac{-6t^2+8qt(2+\theta)+3q^2(-1+\theta)^2}{24t} - \frac{1}{300}\left[-29t + 84(q + \frac{q^2}{t})\right] \\ &= (\frac{(1-\theta)^2}{8t} - \frac{7}{25t})(q - \frac{2\left(58+50\theta-5\sqrt{49+94\theta+169\theta^2}\right)t}{3(31+50\theta-25\theta^2)})(q - \frac{2\left(58+50\theta+5\sqrt{49+94\theta+169\theta^2}\right)t}{3(31+50\theta-25\theta^2)}) \end{aligned}$$

Because the term in the first bracket is negative and and we can verify that the term in the second bracket is positive, $\hat{sw}^* > sw^*$ if and only if $q < \frac{2\left(58+50\theta+5\sqrt{49+94\theta+169\theta^2}\right)t}{3(31+50\theta-25\theta^2)} \equiv q^*_{sw}(\theta)$.

$$\begin{aligned} \hat{cs}^* - cs^* &= \frac{6t^2-2qt(-1+\theta)+q^2(-1+\theta)^2}{24t} - \frac{1}{300}\left[-41t + 36(q + \frac{q^2}{t})\right] \\ &= (\frac{(1-\theta)^2}{24t} - \frac{3}{25t})(q - \frac{\left(11+25\theta+15\sqrt{49+54\theta-23\theta^2}\right)t}{-47-50\theta+25\theta^2})(q - \frac{\left(11+25\theta-15\sqrt{49+54\theta-23\theta^2}\right)t}{-47-50\theta+25\theta^2}) \end{aligned}$$

Because the term in the first bracket is negative and the term in second bracket is positive, $\hat{cs}^* > cs^*$ if and only if $q < \frac{11+25\theta-15\sqrt{49+54\theta-23\theta^2}}{-47-50\theta+25\theta^2}t$.

When $\frac{3t}{1+\theta} < q$, we can verify that $\hat{cs}^* < cs^*$.

$$\begin{aligned} \hat{sw}^* - sw^* &= \frac{-33t^2+4qt(16+11\theta)+q^2(13+\theta(-26+9\theta))}{96t} - \frac{1}{300}\left[-29t + 84(q + \frac{q^2}{t})\right] \\ &= (\frac{13-26\theta+9\theta^2}{96t} - \frac{7}{25t})(q - \frac{\left(464+550\theta-5\sqrt{381+4998\theta+17437\theta^2}\right)t}{347+650\theta-225\theta^2})(q - \frac{\left(464+550\theta+5\sqrt{381+4998\theta+17437\theta^2}\right)t}{347+650\theta-225\theta^2}) \end{aligned}$$

Because the term in the first bracket is negative and the term in the second bracket is positive, $s\hat{w}^* > sw^*$ if and only if $q < \frac{464+550\theta+5\sqrt{381+4998\theta+174370^2}}{347+650\theta-225\theta^2}t \equiv q_{sw}^{**}(\theta)$.

We can verify that both $q_{sw}^{*}(\theta)$ and $q_{sw}^{**}(\theta)$ increase in θ, and these two threshold curves intersect $\frac{3t}{1+\theta}$ at $\theta = \frac{119+30\sqrt{322}}{1183}$. Further, $q_{sw}^{**}(\theta)$ intersects $2t$ at $\theta = \frac{4+\sqrt{61}}{18}$. Therefore, within the region of $q \leq 2t$, social welfare increases if $q \leq q_{sw}^{*}(\theta)$ when $\theta \leq \frac{119+30\sqrt{322}}{1183}$, and if $q \leq q_{sw}^{**}(\theta)$ when $\frac{119+30\sqrt{322}}{1183} < \theta \leq \frac{4+\sqrt{61}}{18}$, and social welfare always increases when $\frac{4+\sqrt{61}}{18} < \theta$.

The case with $q > 2t$:

When $q \leq \frac{3t}{1+\theta}$, we can verify that $\hat{cs}^* < cs^*$ and $s\hat{w}^* < sw^*$.

When $\frac{3t}{1+\theta} < q \leq \frac{13t}{4+2\theta}$, we can verify that $\hat{cs}^* < cs^*$.

$$\begin{aligned} s\hat{w}^* - sw^* &= \frac{-33t^2+4qt(16+11\theta)+q^2(13+\theta(-26+9\theta))}{96t} - \left(q - \frac{5}{12}t\right) \\ &= \left(\frac{13-26\theta+9\theta^2}{96t}\right)\left(q - \frac{16-22\theta+\sqrt{165-522\theta+421\theta^2}}{13-26\theta+9\theta^2}t\right)\left(q - \frac{16-22\theta-\sqrt{165-522\theta+421\theta^2}}{13-26\theta+9\theta^2}t\right) \end{aligned}$$

If the term in the first bracket is positive, $s\hat{w}^* < sw^*$. If that term is negative, we can verify that the term in the second bracket is positive, and thus $s\hat{w}^* > sw^*$ if and only if $q < \frac{16-22\theta-\sqrt{165-522\theta+421\theta^2}}{13-26\theta+9\theta^2}t \equiv q_{sw}^{***}(\theta)$.

When $\frac{13t}{4+2\theta} < q$, we can verify that $\hat{cs}^* < cs^*$.

$$\begin{aligned} s\hat{w}^* - sw^* &= \frac{-232t^2+101q^2(-1+\theta)^2+32qt(19+8\theta)}{864t} - \left(q - \frac{5}{12}t\right) \\ &= \frac{101(1-\theta)^2}{864t}\left(q - \frac{8(16+3\sqrt{6})t}{101(1-\theta)}\right)\left(q - \frac{8(16-3\sqrt{6})t}{101(1-\theta)}\right) \end{aligned}$$

Because the term in the second bracket is negative, $s\hat{w}^* > sw^*$ if and only if $q < \frac{8(16-3\sqrt{6})t}{101(1-\theta)} \equiv q_{sw}^{****}(\theta)$.

We can verify that both $q_{sw}^{***}(\theta)$ and $q_{sw}^{****}(\theta)$ increase in θ, and these two threshold curves intersect $\frac{13t}{4+2\theta}$ at $\theta = \frac{1413+208\sqrt{6}}{2693}$. Further, $q_{sw}^{***}(\theta)$ intersects $2t$ at $\theta = \frac{4+\sqrt{61}}{18}$. Therefore, within the region of $q > 2t$, social welfare increases if $q \leq q_{sw}^{***}(\theta)$ when $\frac{4+\sqrt{61}}{18} < \theta \leq \frac{1413+208\sqrt{6}}{2693}$, and if $q \leq q_{sw}^{****}(\theta)$ when $\frac{1413+208\sqrt{6}}{2693} < \theta$.

Combining the results from both cases of $q \leq 2t$ and $q > 2t$, we can draw the following conclusions. (a) $\hat{cs}^* \geq cs^*$ if and only if $q \leq \frac{11+25\theta-15\sqrt{49+54\theta-23\theta^2}}{-47-50\theta+25\theta^2}t$ within region $q \leq \min\left\{2t, \frac{3t}{1+\theta}\right\}$. Because $\frac{11+25\theta-15\sqrt{49+54\theta-23\theta^2}}{-47-50\theta+25\theta^2}t < \min\left\{2t, \frac{3t}{1+\theta}\right\}$, we conclude Proposition 5(a). (b) $s\hat{w}^* \leq sw^*$ if and only if $q \leq \bar{q}_{sw}(\theta)$, where $\bar{q}_{sw}(\theta)$ is defined as in Equation (16), concluding Proposition 5(b). $\square$

B Proofs of the Results in Section 6

We first present Lemma 3 with its proof, which shows the equilibrium outcome with Firm 3 introducing Product 3 into the market. This lemma is similar to Lemma 2 in the base model, and is used in the proof of Proposition 6.

Lemma 3. *If Firm 3 introduces Product 3 into the market, the equilibrium prices of the products are*

$$(p_i^*, p_3^*) = \begin{cases} \left(\frac{1}{5}(5t+q-\theta q), \frac{1}{5}(5t-2q+2\theta q)\right) & \text{if } \frac{15t}{6+4\theta} < q \\ \left(\frac{1}{3}(-3t+3q+\theta q), \frac{2\theta q}{3}\right) & \text{if } \frac{18t}{9+5\theta} < q \leq \frac{15t}{6+4\theta} \\ \frac{1}{5}(2q+t), \max\left\{\frac{1}{5}(-6t+3q+5\theta q), \frac{\theta q}{2}\right\} & \text{if } q \leq \frac{18t}{9+5\theta}, \end{cases}$$

the equilibrium demands of the products are

$$(d_i^*, d_3^*) = \begin{cases} \left(\frac{1}{15t}(5t+q-\theta q), \frac{1}{15t}(5t-2q+2\theta q)\right) & \text{if } \frac{15t}{6+4\theta} < q \\ \left(\frac{1}{18t}(9t-2\theta q), \frac{2\theta q}{9t}\right) & \text{if } \frac{18t}{9+5\theta} < q \leq \frac{15t}{6+4\theta} \\ \frac{1}{10t}(2q+t), \min\left\{\frac{1}{5t}(4t-2q), \frac{\theta q}{3t}\right\} & \text{if } q \leq \frac{18t}{9+5\theta}, \end{cases}$$

and the equilibrium profits for the firms are

$$(\pi_i^*, \pi_3^*) = \begin{cases} \left(\frac{1}{75t}(5t+q-\theta q)^2, \frac{1}{75t}(5t-2q+2\theta q)^2\right) & \text{if } \frac{15t}{6+4\theta} < q \\ \left(\frac{1}{54t}(9t-2\theta q)(-3t+3q+\theta q), \frac{4\theta^2 q^2}{27t}\right) & \text{if } \frac{18t}{9+5\theta} < q \leq \frac{15t}{6+4\theta} \\ \frac{1}{50t}(2q+t)^2, \max\left\{\frac{1}{25t}(4t-2q)(-6t+3q+5\theta q), \frac{\theta^2 q^2}{6t}\right\} & \text{if } q \leq \frac{18t}{9+5\theta}, \end{cases}$$

where $i \in \{1, 2\}$.

B.1 Proof of Lemma 3

Proof. We distinguish three cases based on the values of q and θ.

(a) $\frac{15t}{6+4\theta} < q$: When $0 \leq m_{12}, m_{13}, m_{23} \leq 1$ and the marginal consumers all derive positive utilities, the demand functions in Equation (9) are well behaved. We can verify that $\frac{\partial^2 \pi_1}{\partial p_1^2} = \frac{\partial^2 \pi_2}{\partial p_2^2} = \frac{\partial^2 \pi_3}{\partial p_3^2} = -\frac{2}{3t} < 0$. Therefore, the second-order condition for optimization is satisfied. We derive the first-order conditions of the firms' profit functions, $\pi_j = p_j d_j$, where $j \in \{1, 2, 3\}$, as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1 + p_2 + p_3 + (1-\theta)q + 2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1 - 4p_2 + p_3 + (1-\theta)q + 2t}{6t} = 0 \\ \frac{\partial \pi_3}{\partial p_3} = \frac{p_1 + p_2 - 4p_3 - 2(1-\theta)q + 2t}{6t} = 0 \end{cases}$$

Solving this system of equations, we can derive p_j^* as specified in the lemma's first case. We can verify that $0 \leq m_{12}, m_{13}, m_{23} \leq 1$, marginal consumers m_{13} and m_{23} derive positive utilities under the condition $\frac{15t}{6+4\theta} < q$, and marginal consumer m_{12} also derives positive utility.

(b) $\frac{18t}{9+5\theta} < q \leq \frac{15t}{6+4\theta}$: In this case, marginal consumers m_{13} and m_{23} derive zero utility because $q \leq \frac{15t}{6+4\theta}$. When Firms 1 and 2 price their products such that marginal consumers m_{13} and m_{23} derive zero utilities, $\frac{q-p_1}{t} + \frac{\theta q - p_3}{t} = 1$ and $\frac{q-p_2}{t} + \frac{\theta q - p_3}{t} = 1$, or, equivalently, $p_1 + p_3 = q + \theta q - t$ and $p_2 + p_3 = q + \theta q - t$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda) = \pi_1 + \lambda_1(q + \theta q - t - p_1 - p_3)$ and $L_2(p_2, \lambda) = \pi_2 + \lambda_2(q + \theta q - t - p_2 - p_3)$. By solving the following system of equations,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1 + p_2 + p_3 + (1-\theta)q + 2t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1 - 4p_2 + p_3 + (1-\theta)q + 2t}{6t} - \lambda_2 = 0 \\ \frac{\partial \pi_3}{\partial p_3} = \frac{p_1 + p_2 - 4p_3 - 2(1-\theta)q + 2t}{6t} = 0 \\ q + \theta q - t - p_1 - p_3 = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma's second case. The condition $\frac{18t}{9+5\theta} < q$ ensures that neither Firm 1 nor 2 has incentive to increase product prices, and condition $q \leq \frac{15t}{6+4\theta}$ ensures that neither Firm 1 nor 2 has incentive to decrease product prices. We can verify that marginal consumer m_{12} derives positive utility, $0 \leq m_{12}, m_{13}, m_{23} \leq 1$, and Firm 3 has no incentive to deviate from p_3^* under these two conditions.

(c) $q \leq \frac{18t}{9+5\theta}$: In this case, Firms 1 and 2 compete with each other but they do not compete against Firm 3. Consequently, as in case (a) in the proof of Lemma 1, $p_1^* = p_2^* = \frac{1}{5}(2q + t)$. Firm 3 either charges $p_3^* = \frac{1}{5}(-6t + 3q + 5\theta q)$ to cover all of the residual demands in Submarkets $1\bar{3}$ and $2\bar{3}$ or charges a monopoly price $p_3^* = \frac{\theta q}{2}$ for Product 3, whichever is higher. We can verify that no firm has incentive to deviate.

Substituting p_j^* into the demand functions and the firms' profit functions, we can derive the equilibrium demands and profits as specified in the lemma. $\square$

We next prove Proposition 6. We use regular notations (e.g., p_1^*) for the equilibrium outcome in the absence of the new product and the notations with hats and primes (e.g., $\hat{p}_1'^*$) for the equilibrium outcome in the presence of the new product.

B.2 Proof of Proposition 6

Proof. Based on Lemmas 1 and 3, we compare the equilibrium profit of Firm 1 before and after the introduction of Product 3.

(a) The case with $q \leq 2t$:

(a.1) When $q \leq \underline{q}'(\theta) = \frac{18t}{9+5\theta}$, as shown in Lemma 3, the introduction of Product 3 does not affect Product 1's profit. Therefore, $\hat{\pi}_1'^* = \pi_1^*$.

(a.2) When $\underline{q}'(\theta) < q \leq \frac{15t}{6+4\theta}$, we can verify that $\hat{\pi}_1'^* \geq \pi_1^*$.

(a.3) When $\frac{15t}{6+4\theta} < q$,

$$\begin{aligned}\hat{\pi}_1'^* - \pi_1^* &= \tfrac{1}{75t}(5t + q - \theta q)^2 - \tfrac{1}{50t}(2q + t)^2 \\ &= \left[\tfrac{1}{75t}(1-\theta)^2 - \tfrac{2}{25t}\right](q - \tfrac{(4-10\theta-\sqrt{6}(9+\theta))t}{2(5+2\theta-\theta^2)})(q - \tfrac{(4-10\theta+\sqrt{6}(9+\theta))t}{2(5+2\theta-\theta^2)})\end{aligned}$$

Because the term in the first bracket is negative and we can verify that the term in the second bracket is positive, $\hat{\pi}_1'^* \geq \pi_1^*$ if and only if $q \leq \frac{4-10\theta+\sqrt{6}(9+\theta)}{2(5+2\theta-\theta^2)}t$.

(b) The case with $q > 2t$:

$$\begin{aligned}\hat{\pi}_1'^* - \pi_1^* &= \tfrac{1}{75t}(5t + q - \theta q)^2 - \tfrac{1}{2}(q - t) \\ &= \tfrac{1}{75t}(1-\theta)^2(q - \tfrac{\left(55+20\theta+5\sqrt{3(27+56\theta-8\theta^2)}\right)t}{4(1-\theta)^2})(q - \tfrac{\left(55+20\theta-5\sqrt{3(27+56\theta-8\theta^2)}\right)t}{4(1-\theta)^2})\end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1'^* \geq \pi_1^*$ if and only if $q \leq \frac{55+20\theta-5\sqrt{3(27+56\theta-8\theta^2)}}{4(1-\theta)^2}t$.

We can verify that both $\frac{55+20\theta-5\sqrt{3(27+56\theta-8\theta^2)}}{4(1-\theta)^2}t$ and $\frac{4-10\theta+\sqrt{6}(9+\theta)}{2(5+2\theta-\theta^2)}t$ decrease in θ. Further, these two curves intersect $2t$ at $\theta = \frac{1}{4}\left(14 - 5\sqrt{6}\right)$. Before this intersecting point, $2t < \frac{55+20\theta-5\sqrt{3(27+56\theta-8\theta^2)}}{4(1-\theta)^2}t < \frac{4-10\theta+\sqrt{6}(9+\theta)}{2(5+2\theta-\theta^2)}t$. After the point, $\frac{4-10\theta+\sqrt{6}(9+\theta)}{2(5+2\theta-\theta^2)}t$ is the lowest. Altogether, we have the results as in Proposition 6. $\square$

B.3 Proof of Proposition 7

Proof. Combining the conditions in Propositions 4 and 6 leads to the results in this proposition. □

C Proof of the Results for Benchmark Cases

We first present the equilibrium results for the two benchmarks and then prove Proposition 2.

C.1 Equilibrium for Benchmark with Cannibalization Only

Lemma 4. *In the benchmark with cannibalization only, the equilibrium prices of the products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases} \left(q-t, \frac{1}{10}\left(7q-4t+2\theta q\right), \frac{1}{10}\left(3q-6t+8\theta q\right)\right) & \text{if } \frac{66t}{33-2\theta} < q \\ \left(\frac{1}{59}\left(26q+7t+2\theta q\right), \frac{1}{59}\left(38q-17t+12\theta q\right), \frac{1}{59}\left(21q-42t+47\theta q\right)\right) & \text{if } \max\{\frac{12t}{6+5\theta}, \frac{49t}{54-14\theta}\} < q \leq \frac{66t}{33-2\theta} \\ \left(\frac{4}{7}q, \frac{1}{7}\left(10q-7t\right), \frac{1}{7}\left(-3q+7\theta q\right)\right) & \text{if } q \leq \frac{49t}{54-14\theta} \end{cases}$$

the equilibrium demands of the products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases} \left(\frac{1}{60t}\left(-3q+36t+2\theta q\right), \frac{1}{20t}\left(3q+4t-2\theta q\right), \frac{1}{30t}\left(-3q+6t+2\theta q\right)\right) & \textit{if } \frac{66t}{33-2\theta} < q \\ \left(\frac{1}{118t}\left(26q+7t+2\theta q\right), \frac{1}{354t}\left(30q+117t-34\theta q\right), \frac{1}{59t}\left(-7q+14t+4\theta q\right)\right) & \textit{if } \max\{\frac{12t}{6+5\theta}, \frac{49t}{54-14\theta}\} < q \leq \frac{66t}{33-2\theta} \\ \left(\frac{2q}{7t}, \frac{2}{3}-\frac{2q}{7t}, \frac{q}{7t}\right) & \textit{if } q \leq \frac{49t}{54-14\theta} \end{cases}$$

and the equilibrium profits for the two firms (π_1^*, π_2^*) *are*

$$\begin{cases} \left(\frac{(q-t)(-3q+36t+2\theta q)}{60t}, \frac{-24t^2+q^2(3-2\theta)^2+24qt(1+\theta)}{120t}\right) & \textit{if } \frac{66t}{33-2\theta} < q \\ \left(\frac{(26q+7t+2\theta q)^2}{6962t}, \frac{-5517t^2+2q^2(-1+9\theta)(-129+40\theta)+2qt(3732+2461\theta)}{20886t}\right) & \textit{if } \max\{\frac{12t}{6+5\theta}, \frac{49t}{54-14\theta}\} < q \leq \frac{66t}{33-2\theta} \\ \left(\frac{8q^2}{49t}, \frac{1274qt-686t^2+21q^2(-23+7\theta)}{1029t}\right) & \textit{if } q \leq \frac{49t}{54-14\theta} \end{cases}$$

Proof. When Product 3 serves only Submarket $\bar{2}3$ (but not Submarket $\bar{1}3$), because no competition exists between Products 1 and 3, Firm 2 has incentive to charge high prices for Products 2 and 3 such that either the marginal consumer m_{23} derives zero utility or Submarket $\bar{2}3$ is not fully covered. We distinguish four cases for the interfirm competition. (We can verify that the second-order conditions for profit maximization are satisfied.)

(a) $\frac{66t}{33-2\theta} < q$: When q is large, all consumers in Submarket $\bar{1}3$ purchase. Firm 1 has incentive to charge a price high enough such that the consumer with the highest misfit cost in Submarket $\bar{1}3$ derive zero utility (i.e., $p_1^* = q - t$), while Firm 2 chooses the optimal prices for Products 2 and 3 to compete against Product 1 and to just fully serve Submarket $\bar{23}$. When $0 \leq m_{12}, m_{23} \leq 1$ and marginal consumer m_{12} derives positive utility, the demand functions for the products are

$$\begin{cases} d_1 = \frac{1}{3} m_{12} + \frac{1}{3} \\ d_2 = \frac{1}{3}(1 - m_{12}) + \frac{1}{3} m_{23} \\ d_3 = \frac{1}{3}(1 - m_{23}) \end{cases}$$

Because marginal consumer m_{23} derives zero utility, $\frac{q-p_2}{t} + \frac{\theta q - p_3}{t} = 1$, or, equivalently, $p_2 + p_3 = q + \theta q - t$. Using the Lagrange-multiplier method, we have $L_2(p_2, p_3, \lambda) = \pi_2 + \lambda(q + \theta q - t - p_2 - p_3)$—in which $\pi_2 = p_2 d_2 + p_3 d_3$—and $\pi_1 = p_1^* d_1$. By the first-order conditions for $L_2(p_2, p_3, \lambda)$, we have

$$\begin{cases} \frac{\partial L_2}{\partial p_2} = \frac{-4p_2 + p_1^* + 2p_3 + (1-\theta)q + 2t}{6t} - \lambda = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{2p_2 - 2p_3 - (1-\theta)q + t}{6t} - \lambda = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \end{cases}$$

Solving the above system of equations, we have $p_2^* = \frac{1}{10}(7q - 4t + 2\theta q)$ and $p_3^* = \frac{1}{10}(3q - 6t + 8\theta q)$. We can verify that $0 \leq m_{12}, m_{23} \leq 1$ within this region, which also ensures that the marginal consumer located at m_{12} derives positive utility. We can verify that neither firm has profitable deviation.

(b) $\max\{\frac{12t}{6+5\theta}, \frac{49t}{54-14\theta}\} < q \leq \frac{66t}{33-2\theta}$: In this case, because the value of q is not large enough (i.e., $q \leq \frac{66t}{33-2\theta}$), Firm 1 has incentive to charge a price higher than $q - t$ to only partially serve Submarket $\bar{13}$. Therefore, the demand functions for the three products are

$$\begin{cases} d_1 = \frac{1}{3} m_{12} + \frac{1}{3} \frac{q-p_1}{t} \\ d_2 = \frac{1}{3}(1 - m_{12}) + \frac{1}{3} m_{23} \\ d_3 = \frac{1}{3}(1 - m_{23}) \end{cases} \tag{22}$$

By the first-order conditions for $\pi_1 = p_1 d_1$ and $L_2(p_2, p_3, \lambda)$, we have

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{t-6p_1+p_2+2q}{6t} = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{-4p_2+p_1+2p_3+(1-\theta)q+2t}{6t} - \lambda = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{2p_2-2p_3-(1-\theta)q+t}{6t} - \lambda = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \end{cases}$$

Solving this system of equations, we can derive $p_1^* = \frac{1}{59}(26q + 7t + 2\theta q)$, $p_2^* = \frac{1}{59}(38q - 17t + 12\theta q)$, and $p_3^* = \frac{1}{59}(21q - 42t + 47\theta q)$. We can verify that $0 \leq m_{12}, m_{23} \leq 1$ and $0 \leq \frac{q-p_1^*}{t} \leq 1$ within this region. The condition $\frac{49t}{54-14\theta} < q$ ensures that the marginal consumer located at m_{12} derives positive utility. We can verify that the condition $\frac{12t}{6+5\theta} < q$ ensures that Firm 2 has no incentive to increase the price of Product 2 or Product 3 to have Submarket $\bar{2}3$ not fully covered.

(c) $q \leq \frac{49t}{54-14\theta}$: Because $q \leq \frac{49t}{54-14\theta}$, marginal consumer m_{12} derives zero utility, $\frac{q-p_1}{t} + \frac{q-p_2}{t} = 1$, or, equivalently, $p_1+p_2 = 2q-t$. Using the Lagrange-multiplier method, we have $L_2(p_2, p_3, \lambda_1, \lambda_2) = \pi_2 + \lambda_1(2q - t - p_1 - p_2) + \lambda_2(q + \theta q - t - p_2 - p_3)$ in which $\pi_2 = p_2 d_2 + p_3 d_3$, and $\pi_1 = p_1 d_1$, with d_1, d_2, and d_3 specified as in Equation (22). By solving the following first-order conditions for π_1 and $L_2(p_2, p_3, \lambda_1, \lambda_2)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{t-6p_1+p_2+2q}{6t} = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{-4p_2+p_1+2p_3+(1-\theta)q+2t}{6t} - \lambda_1 - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{2p_2-2p_3-(1-\theta)q+t}{6t} - \lambda_2 = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \\ 2q - t - p_1 - p_2 = 0 \end{cases}$$

we can derive $p_1^* = \frac{4}{7}q$, $p_2^* = \frac{1}{7}(10q - 7t)$, and $p_3^* = \frac{1}{7}(-3q + 7\theta q)$. We can verify that $0 \leq m_{12}, m_{23} \leq 1$ and $0 \leq \frac{q-p_1^*}{t} \leq 1$ within this region, and neither firm has profitable deviation.

(d) $q \leq \frac{12t}{6+5\theta}$: In this case, Firm 2 charge prices for Product 2 and Product 3 such that Submarket $\bar{2}3$ is not fully covered, and thus Submarket $\bar{2}3$ is no longer competitive.

Substituting the equilibrium prices into the corresponding demand and profit functions, we can derive the equilibrium demands and profits as in the lemma. $\square$

C.2 Equilibrium for Benchmark with Additional Competition Only

Lemma 5. *In the benchmark with additional competition only, the equilibrium prices of the products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases} \left(\frac{1}{7}\left(3q+3t-\theta q\right), q-t, \frac{1}{7}\left(-2q+5t+3\theta q\right)\right) & \textit{if } \frac{52t}{25+\theta} < q \\ \left(\frac{1}{20}\left(5q+16t-3\theta q\right), \frac{1}{40}\left(15q+12t-\theta q\right), \frac{1}{40}\left(-15q+36t+17\theta q\right)\right) & \textit{if } \frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta} \\ \left(\frac{1}{3}\left(3q-3t+\theta q\right), \frac{1}{18}\left(9q+\theta q\right), \frac{2}{3}\theta q\right) & \textit{if } \frac{18t}{9+5\theta} < q \leq \frac{108t}{45+29\theta} \end{cases}$$

the equilibrium demands of the products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases} \left(\frac{1}{21t}\left(3q+3t-\theta q\right), \frac{1}{42t}\left(31t-4q-\theta q\right), \frac{1}{42t}\left(-2q+5t+3\theta q\right)\right) & \textit{if } \frac{52t}{25+\theta} < q \\ \left(\frac{1}{60t}\left(5q+16t-3\theta q\right), \frac{1}{80t}\left(15q+12t-\theta q\right), \frac{1}{240t}\left(-15q+36t+17\theta q\right)\right) & \textit{if } \frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta} \\ \left(\frac{1}{108t}\left(-9q+72t-17\theta q\right), \frac{1}{36t}\left(9q+\theta q\right), \frac{1}{9t}\theta q\right) & \textit{if } \frac{18t}{9+5\theta} < q \leq \frac{108t}{45+29\theta} \end{cases}$$

and the equilibrium profits for the two firms (π_1^*, π_2^*) *are*

$$\begin{cases} \left(\frac{1}{147t}\left(3q+3t-\theta q\right)^2, \frac{-192t^2+q^2(-3+\theta)(8+9\theta)+qt(225+37\theta)}{294t}\right) & \textit{if } \frac{52t}{25+\theta} < q \\ \left(\frac{1}{1200t}\left(5q+16t-3\theta q\right)^2, \frac{432t^2+q^2(225+\theta(-150+73\theta))+288qt\theta}{2400t}\right) & \textit{if } \frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta} \\ \left(\frac{(-3q+3t-\theta q)(-72t+9q+17\theta q)}{324t}, \frac{q^2(81+18\theta+49\theta^2)}{648t}\right) & \textit{if } \frac{18t}{9+5\theta} < q \leq \frac{108t}{45+29\theta} \end{cases}$$

Proof. We distinguish four cases for the firms' competition. (We can verify that the second-order conditions for profit maximization are satisfied.)

(a) $\frac{52t}{25+\theta} < q$: When q is large, all consumers in Submarket $\bar{2}3$ purchase. Firm 2 has incentive to charge a price high enough for Product 2 such that the consumer with the highest misfit cost in Submarket $\bar{2}3$ derives zero utility; that is, $p_2^* = q - t$. When $0 \leq m_{12}, m_{13} \leq 1$ and the marginal consumers m_{12} and m_{13} derive positive utilities, the demand functions for the three products are

$$\begin{cases} d_1 = \frac{1}{3}m_{12} + \frac{1}{3}m_{13} \\ d_2 = \frac{1}{3}(1-m_{12}) + \frac{1}{3} \\ d_3 = \frac{1}{3}(1-m_{13}) \end{cases}$$

By the first-order conditions for the firms' profit functions (i.e., $\pi_1 = p_1 d_1$ and $\pi_2 = p_2 d_2 + p_3 d_3$), we get

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2^*+p_3+(1-\theta)q+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1-2p_3-(1-\theta)q+t}{6t} = 0 \end{cases}$$

Solving this system of equations, we can derive $p_1^* = \frac{1}{7}\left(3q+3t-\theta q\right)$ and $p_3^* = \frac{1}{7}\left(-2q+5t+3\theta q\right)$. We can verify that $0 \leq m_{12}, m_{13} \leq 1$ within this region, which also ensures that the marginal consumers located at m_{12} and m_{13} both derive positive utility. We can verify that neither firm has profitable deviation.

(b) $\frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta}$: In this case, the value of q is not large enough, and Firm 2 has incentive to charge a price higher than $q-t$ for Product 2 to only partially serve Submarket $2\bar{3}$. Therefore, the demand functions for the three products are

$$\begin{cases} d_1 = \frac{1}{3}m_{12} + \frac{1}{3}m_{13} \\ d_2 = \frac{1}{3}(1-m_{12}) + \frac{1}{3}\frac{q-p_2}{t} \\ d_3 = \frac{1}{3}(1-m_{13}) \end{cases} \tag{23}$$

By the first-order conditions for the firms' profit functions, we have

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1-\theta)q+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{t-6p_2+p_1+2q}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1-2p_3-(1-\theta)q+t}{6t} = 0 \end{cases}$$

Solving this system of equations, we can derive $p_1^* = \frac{1}{20}\left(5q+16t-3\theta q\right)$, $p_2^* = \frac{1}{40}\left(15q+12t-\theta q\right)$, and $p_3^* = \frac{1}{40}\left(-15q+36t+17\theta q\right)$. We can verify that $0 \leq m_{12}, m_{13} \leq 1$ and $0 \leq \frac{q-p_2^*}{t} \leq 1$ within this region, which also ensures that the marginal consumers located at m_{12} and m_{13} both derive positive utility. We can verify that neither firm has profitable deviation.

(c) $\frac{18t}{9+5\theta} < q \leq \frac{108t}{45+29\theta}$: Because $q \leq \frac{108t}{45+29\theta}$, marginal consumer m_{13} derives zero utility, $\frac{q-p_1}{t} + \frac{\theta q-p_3}{t} = 1$, or, equivalently, $p_1+p_3 = q+\theta q-t$. Using the Lagrange-multiplier method, we have $L_1(p_1,\lambda) = \pi_1 + \lambda(q+\theta q-t-p_1-p_3)$ in which $\pi_1 = p_1 d_1$, and $\pi_2 = p_2 d_2 + p_3 d_3$, with d_1, d_2, and d_3 specified as in Equation (23). By solving the the first-order conditions for $L_1(p_1, \lambda)$ and π_2,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1-\theta)q+2t}{6t} - \lambda = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{t-6p_2+p_1+2q}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1-2p_3-(1-\theta)q+t}{6t} = 0 \\ q + \theta q - t - p_1 - p_3 = 0 \end{cases}$$

we can derive $p_1^* = \frac{1}{3}(3q - 3t + \theta q)$, $p_2^* = \frac{1}{18}(9q + \theta q)$, and $p_3^* = \frac{2}{3}\theta q$. We can verify that $0 \leq m_{12}, m_{13} \leq 1$ and $0 \leq \frac{q-p_2^*}{t} \leq 1$ within this region, which also ensures that the marginal consumer located at m_{12} derives positive utility. We can verify that neither firm has profitable deviation.

(d) $q \leq \frac{18t}{9+5\theta}$: In this case, the two firms compete in Submarket $1\bar{2}$, but Submarket $1\bar{3}$ is no longer competitive.

Substituting the equilibrium prices into the corresponding demand and profit functions, we can derive the equilibrium demands and profits as specified in the lemma. □

C.3 Proof of Proposition 2

Proof. (a) Based on Lemmas 1 and 4, we can verify that both Product 2's price and Firm 2's profit become higher after the introduction of Product 3.

(b) Based on Lemmas 1 and 5, when $q \leq 2t$, after the introduction of Product 3, if $\frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta}$, we can verify that Product 1's price increases but Firm 1's profit decreases. If $\frac{18t}{9+5\theta} < q \leq \frac{108t}{45+29\theta}$, we can verify that Product 1's price increases. The difference between Firm 1's profits with and without Product 3's entry is

$$\frac{(-3q+3t-\theta q)(-72t+9q+17\theta q)}{324t} - \frac{1}{50t}(2q+t)^2 = \left[\frac{(-3-\theta)(9+17\theta)}{324t} - \frac{4}{50t}\right]\left(q - \frac{18t}{9+5\theta}\right)\left(q - \frac{309t}{147+85\theta}\right)$$

Because the term in the first bracket is negative and the term in the second bracket is positive, Firm 1's profit increases with the entry of Product 3 if and only if $q \leq \frac{309t}{147+85\theta}$.

When $q > 2t$, if $\frac{108t}{45+29\theta} < q \leq \frac{52t}{25+\theta}$, we can verify that the price of Product 1 increases but Firm 1's profit decreases. If $q > \frac{52t}{25+\theta}$, the difference between the prices of Product 1 with and without Product 3 is

$$\frac{1}{7}(3q + 3t - \theta q) - (q - t) = \frac{1}{7}(-4q + 10t - \theta q).$$

Therefore, the price of Product 1 increases if and only if $q \leq \frac{10t}{4+\theta}$. Moreover, we can verify that Firm 1's profit decreases.

Altogether, we have the comparison results as in Proposition 2. $\square$

D An Extended Model with a General θ

In the baseline model, we assume the technology-transfer rate θ to be $\frac{43}{100} \leq \theta \leq 1$. We consider $\theta \leq 1$ because the new product is produced by the shared technology, and therefore, its valuation should not exceed that of the existing product produced under the original technology. In practice, when Firm 2 can combine the shared technology with its existing technology or further innovate based on the shared technology, it is possible that the new product might be even superior to the existing product in valuation. As such, we consider a model extension in which θ might be greater than 1. With a little abuse of terminology, we continue to call θ the technology-transfer rate. In this case, θ measures the ratio of the new product's valuation to the existing product's valuation. For ease of exposition, we consider θ to be not too large (e.g., $\theta \leq 2$); otherwise, Firm 1 should have no incentive to share, as we shall illustrate below and also be consistent with our intuition. In this extension, we also relax the assumption on the lower bound on θ by considering $\theta \geq 0$. Everything else remains the same as the baseline model.

Corresponding to Lemma 2 under the baseline model, we first derive the equilibrium prices, demands, and profits for the firms with the new product. We then illustrate the conditions under which the new product is introduced in equilibrium.

Lemma 6. *Under* $0 \leq \theta \leq 2$, *if Firm 2 serves the market with both Products 2 and 3, the equilibrium prices of the products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases} \left(\frac{1}{6}(8t+q-\theta q), \frac{1}{12}(20t+q-\theta q), \frac{5}{12}(4t-q+\theta q)\right) & \text{if } \max\{\frac{13t}{4+2\theta}, \frac{16t}{7+\theta}\} < q \leq \frac{2t}{\theta-1} \text{ (Case I)} \\ \left(\frac{1}{4}(t+2q), \frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+q+3\theta q)\right) & \text{if } \max\{\frac{3t}{1+\theta}, \frac{3t}{3-\theta}\} < q \leq \min\{\mathring{q}, \frac{13t}{4+2\theta}\} \text{ (Case II)} \\ \left(\frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+q+3\theta q)\right) & \text{if } \frac{14t}{7+5\theta} < q \leq \min\{\frac{3t}{1+\theta}, \frac{2t}{1-\theta}\} \text{ and } \theta \leq 1 \text{ (Case III)} \\ \left(\frac{1}{5}(t+2q), \frac{1}{5}(t+2q), \max\{\frac{1}{5}(-6t+3q+5\theta q), \frac{\theta q}{2}\}\right) & \text{if } q \leq \frac{14t}{7+5\theta} \text{ (Case IV)} \\ \text{Mixed-Strategy Equilibrium} & \text{if } \mathring{q} < q \leq \frac{2t}{1-\theta} \text{ (Case V)} \\ (q-t, q-t, \theta q) & \text{if } \frac{2t}{1-\theta} < q \text{ (Case VI)} \\ \left(\frac{1}{6}(8t+q-\theta q), \frac{1}{6}(13t+2q-2\theta q), \frac{1}{6}(7t-4q+4\theta q)\right) & \text{if } \frac{2t}{-1+\theta} < q \text{ (Case VII)} \\ \left(\frac{4}{3}(-t+q), \frac{1}{3}(t+2q), \frac{1}{6}(2t+q+3\theta q)\right) & \text{if } \max\{\frac{56t}{29+3\theta}, \frac{10t}{1+3\theta}\} < q \leq \frac{16t}{7+\theta} \text{ (Case VIII)} \\ \left(\frac{1}{4}(-2t+5q-\theta q), \frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+q+3\theta q)\right) & \text{if } \frac{14t}{17-5\theta} < q \leq \min\{\frac{10t}{1+3\theta}, \frac{3t}{3-\theta}\} \text{ and } 1 < \theta \text{ (Case IX)} \\ \left(\frac{2}{25}(2t+7q-\theta q), \frac{1}{25}(-29t+36q+2\theta q), \frac{1}{25}(-t+9q+13\theta q)\right) & \text{if } \frac{197t}{123+11\theta} < q \leq \min\{\frac{t}{4-2\theta}, \frac{56t}{29+3\theta}\} \text{ (Case X)} \\ \left(\frac{1}{5}(t+2q), \frac{2}{5}(-3t+4q), \frac{1}{5}(t-3q+5\theta q)\right) & \text{if } \max\{\frac{48t}{59-15\theta}, \frac{t}{4-2\theta}\} < q \leq \frac{14t}{17-5\theta} \text{(Case XI)} \\ \left(\frac{1}{57}(17t+27q-5\theta q), \frac{1}{114}(49t+51q-\theta q), \frac{3}{19}(5t-q+3\theta q)\right) & \text{if } \frac{253t}{81+61\theta} < q \leq \frac{197t}{123+11\theta} \text{ (Case XII)} \\ \left(\frac{1}{83}(7t+45q-3\theta q), \frac{2}{83}(-21t+31q+9\theta q), \frac{1}{83}(-41t+21q+65\theta q)\right) & \text{if } q \leq \min\{\frac{48t}{59-15\theta}, \frac{253t}{81+61\theta}\} \text{ (Case XIII)} \end{cases} \tag{24}$$

in which

$$\mathring{q} = \begin{cases} \frac{104t+23\sqrt{30}t}{44+10\sqrt{30}-28\theta} & \text{if } \frac{309+58\sqrt{30}}{1813} < \theta \\ \frac{\left(6-11\theta-\sqrt{-12\theta+85\theta^2}\right)t}{3-10\theta+3\theta^2} & \text{otherwise} \end{cases} \tag{25}$$

and the mixed-strategy equilibrium involves Firm 1 taking prices p_1^{1*} *and* p_1^{2*} *with probabilities* α_1 *and* $1-\alpha_1$*, respectively, and Firm 2 taking the prices* (p_2^{1*}, p_3^{1*}) *and* (p_2^{2*}, p_3^{2*}) *with probabilities* α_2 *and* $1-\alpha_2$*, respectively, where*

$$\begin{aligned} (p_1^{1*}, p_2^{1*}, p_3^{1*}) &= \begin{cases} \left(\frac{1}{4}(t+2q), \frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+q+3\theta q)\right) & \text{if } q < \frac{7t}{3-\theta} \\ \left(\frac{1}{4}(-6t+5q-\theta q), \frac{1}{4}(-2t+3q+\theta q), \frac{1}{4}(-2t+q+3\theta q)\right) & \text{otherwise} \end{cases} \\ (p_1^{2*}, p_2^{2*}, p_3^{2*}) &= \begin{cases} \left(\frac{1}{40}(-69t+54q-8\theta q), \frac{1}{40}(-11t+26q+8\theta q), \frac{1}{40}(-29t+14q+32\theta q)\right) & \text{if } q < \frac{29t}{14-8\theta} \\ (q-t, q-t, \theta q) & \text{otherwise} \end{cases} \\ (\alpha_1, \alpha_2) &= \begin{cases} \left(\frac{6(9t-4q-2\theta q)^2}{(79t-34q+8\theta q)(29t-14q-32\theta q)}, \frac{84t-24q+8\theta q}{5(2q+t)}\right) & \text{if } \frac{7t}{3-\theta} < q \\ \left(\frac{3(2t-q+\theta q)^2}{2q(2q-5t)\theta}, 11 - \frac{24q}{2q+t}\right) & \text{if } \frac{29t}{14-8\theta} < q \leq \frac{7t}{3-\theta} \\ \left(\frac{3(2t-q+\theta q)}{2q\theta}, \frac{12t}{q(5-\theta)-6t} - 1\right) & \text{otherwise} \end{cases} \end{aligned} \tag{26}$$

The equilibrium (expected) demands of the products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases} \left(\frac{1}{18t}(8t+q-\theta q), \frac{1}{72t}(20t+7q-7\theta q), \frac{1}{72t}(20t-11q+11\theta q)\right) & \text{Case I} \\ \left(\frac{1}{12t}(t+2q), \frac{1}{24t}(11t+q-3\theta q), \frac{1}{24t}(11t-5q+3\theta q)\right) & \text{Case II} \\ \left(\frac{1}{12t}(4t+q-\theta q), \frac{1}{12t}(4t+q-\theta q), \frac{1}{6t}(2t-q+\theta q)\right) & \text{Case III} \\ \left(\frac{1}{10t}(2q+t), \frac{1}{10t}(2q+t), \min\left\{\frac{1}{5t}(4t-2q), \frac{\theta q}{3t}\right\}\right) & \text{Case IV} \\ \begin{cases} \left(\frac{-12qt+12t^2+q^2(1+\theta)(3+\theta)}{4q[-6t+q(5-\theta)]\theta}, 2+\frac{6(q-t)}{6t+q(-5+\theta)}-\frac{(q-2t)^2}{16qt\theta}+\frac{q\theta}{16t}, \right. \\ \left. \frac{[18t+q(-5+\theta)][2t+q(-1+\theta)](q-2t+q\theta)}{16qt[-6t+q(5-\theta)]\theta}\right) & \text{if } \frac{7t}{3-\theta}<q \\ \left(\frac{3(q-2t)^2-2q(q-7t)\theta+3q^2\theta^2}{4q(2q+t)\theta}, \right. \\ \left. \frac{-(q-2t)^2(2q+t)+4\theta q(q-t)t\theta+q^2(2q-23t)\theta^2}{16qt(2q+t)\theta}, \frac{(2q-11t)[-2t+q(1-\theta)](q-2t+q\theta)}{16qt(2q+t)\theta}\right) & \text{if } \frac{29t}{14-8\theta}<q\leq\frac{7t}{3-\theta} \\ \left(\frac{(89t+2q(-7+4\theta))\left[49t^2+2qt(-109+13\theta)+2q^2(47+\theta(92+3\theta))\right]}{300t(2q+t)[-29t+2q(7+16\theta)]}, \right. \\ \frac{34391t^3-6qt^2(5673+7274\theta)+4q^2t[4932+\theta(4932+1633\theta)]+8q^3[-682+\theta(-1398+\theta(601+96\theta))]}{600t(2q+t)[29t-2q(7+16\theta)]}, & \text{otherwise} \\ \left. \frac{8269t^3-6qt^2(3377+4976\theta)+4q^2t[3423+2\theta(3699+631\theta)]+8q^3[-353+2\theta(-471+\theta(127+42\theta))]}{600t(2q+t)[29t-2q(7+16\theta)]}\right) \end{cases} & \text{Case V} \\ \left(\frac{1}{2}, \frac{1}{2}, 0\right) & \text{Case VI} \\ \left(\frac{1}{18t}\left(q+8t-q\theta\right), \frac{1}{36t}\left(t-q+\theta q\right), \frac{1}{36t}\left(19t-q+\theta q\right)\right) & \text{Case VII} \\ \left(\frac{1}{36t}\left(-5q+32t-3q\theta\right), \frac{1}{36t}\left(7q+2t-3q\theta\right), \frac{1}{18t}\left(t+q(-1+3\theta)\right)\right) & \text{Case VIII} \\ \left(\frac{1}{12}\left(4+\frac{q(-1+\theta)}{t}\right), \frac{1}{6t}\left(q+2t-q\theta\right), \frac{1}{12}\left(4+\frac{q(-1+\theta)}{t}\right)\right) & \text{Case IX} \\ \left(\frac{1}{25t}\left(7q+2t-q\theta\right), \frac{1}{50t}\left(-8q+37t-6q\theta\right), \frac{1}{50t}\left(9t+q(-6+8\theta)\right)\right) & \text{Case X} \\ \left(\frac{1}{10t}\left(2q+t\right), \frac{1}{5t}\left(-2q+4t\right), \frac{1}{10t}\left(2q+t\right)\right) & \text{Case XI} \\ \left(\frac{1}{114t}\left(27q+17t-5q\theta\right), \frac{1}{12t}\left(3q+t-q\theta\right), \frac{1}{684t}\left(131t+q(-87+109\theta)\right)\right) & \text{Case XII} \\ \left(\frac{1}{166t}\left(45q+7t-3q\theta\right), \frac{2}{83t}\left(14t+q(7-6\theta)\right), \frac{1}{498t}\left(213t+q(-101+51\theta)\right)\right) & \text{Case XIII} \end{cases} \tag{27}$$

and the equilibrium (expected) profits for the two firms (π_1^*, π_2^*) *are*

$$
\begin{cases}
\left(\frac{1}{108t}(8t+q-\theta q)^2, \frac{1}{432t}[4(10t-q+\theta q)^2+27(1-\theta)^2q^2]\right) & \text{Case I} \\
\left(\frac{1}{48t}(t+2q)^2, \frac{1}{48t}[-22t^2+2(13+11\theta)qt-(1+10\theta-3\theta^2)q^2]\right) & \text{Case II} \\
\left(\frac{1}{48t}(4t+q-\theta q)(-2t+3q+\theta q), \frac{1}{48t}[-16t^2+2(9+7\theta)qt+(1-\theta)(1-5\theta)q^2]\right) & \text{Case III} \\
\left(\frac{1}{50t}(2q+t)^2, \frac{1}{50t}(2q+t)^2+\max\{\frac{\theta^2q^2}{6t}, \frac{1}{25t}(4t-2q)(-6t+3q+5\theta q)\}\right) & \text{Case IV} \\
\begin{cases}
\left(\frac{1}{2}(q-t), \frac{(q-t)\left(-4t^2-q^2(-1+\theta)^2+4qt(1+\theta)\right)}{16qt\theta}\right) & \text{if } \frac{7t}{3-\theta}<q \\
\left(\frac{1}{2}(q-t), \frac{(q-t)\left(-4t^2-q^2(-1+\theta)^2+4qt(1+\theta)\right)}{16qt\theta}\right) & \text{if } \frac{29t}{14-8\theta}<q\leq\frac{7t}{3-\theta} \\
\left(\begin{array}{l} \frac{(-89t+14q-8\theta q)(69t-54q+8\theta q)}{4800t}, \frac{1}{48t(29t-2q(7+16\theta))}\left[-277t^3+qt^2(369+565\theta)\right. \\ \left.+q^2t(15-\theta(506+257\theta))-2q^3(31+\theta(4\theta+\theta(-79+20\theta)))\right] \end{array}\right) & \text{otherwise}
\end{cases} & \text{Case V} \\
\left(\frac{1}{2}(q-t), \frac{1}{2}(q-t)\right) & \text{Case VI} \\
\left(\frac{1}{108t}(q+8t-q\theta)^2, \frac{1}{108t}\left[73t^2+47qt(-1+\theta)+q^2(-1+\theta)^2\right]\right) & \text{Case VII} \\
\left(\frac{1}{27t}(q-t)[32t-q(5+3\theta)], \frac{1}{108t}\left[4t^2+2qt(5+3\theta)+q^2\left(13-6\theta+9\theta^2\right)\right]\right) & \text{Case VIII} \\
\left(\frac{1}{48\mathrm{t}}(2\mathrm{t}-5q+\theta q)(4\mathrm{t}-q+\theta q), \frac{1}{48t}\left[-16t^2+q^2(-5+\theta)(-1+\theta)+2qt(7+9\theta)\right]\right) & \text{Case IX} \\
\left(\frac{1}{625\mathrm{t}}2(-2\mathrm{t}-7q+\theta q)^2, \frac{1}{1250t}\left[-1082t^2+qt(1651+357\theta)+q^2\left(-342-238\theta+92\theta^2\right)\right]\right) & \text{Case X} \\
\left(\frac{1}{50t}(2q+t)^2, \frac{1}{50t}\left[-47t^2+2q^2(-19+5\theta)+qt(87+5\theta)\right]\right) & \text{Case XI} \\
\left(\frac{1}{6498t}(17t+27q-5\theta q)^2, \frac{1}{25992t}\left[4861t^2+2qt(183+2339\theta)+q^2(3429-3246\theta+1981\theta^2)\right]\right) & \text{Case XII} \\
\left(\frac{1}{13778t}(7t+45q-3\theta q)^2, \frac{1}{41334t}\left[-15789t^2+46qt(337+387\theta)+q^2(3087-8446\theta+2019\theta^2)\right]\right) & \text{Case XIII}
\end{cases}
\tag{28}
$$

Proof. We distinguish thirteen cases based on the value of q. For all these cases, we can verify that the second-order conditions for profit maximization are satisfied.

Case I where $\max\{\frac{13t}{4+2\theta}, \frac{16t}{7+\theta}\} < q \leq \frac{2t}{\theta-1}$: This case is the same as Case (a) in the proof of Lemma 2. In this case, Condition $\frac{13t}{4+2\theta} < q$ ensures that marginal consumer m_{23} derives positive utility, Condition $\frac{16t}{7+\theta} < q$ ensures that marginal consumer m_{12} derives positive utility, and Condition $q \leq \frac{2t}{\theta-1}$ ensures $0 \leq m_{23}$.

Case II where $\max\{\frac{3t}{1+\theta}, \frac{3t}{3-\theta}\} < q \leq \min\{\mathring{q}, \frac{13t}{4+2\theta}\}$: This case is the same as Case (b) in the proof of Lemma 2, in which marginal consumer m_{23} derives zero utility because $q \leq \frac{13t}{4+2\theta}$. Condition $\frac{3t}{1+\theta} < q$ ensures that marginal consumer m_{13} derives positive utility, and Condition $\frac{3t}{3-\theta} < q$ ensures that marginal consumer m_{12} derives positive utility. Moreover, Condition $q \leq \mathring{q}$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing the price of Product 3 to completely cede Submarket $\bar{1}3$ to Firm 1 while decreasing the price of Product 2 to keep the consumers in its exclusive Submarket $\bar{2}3$ just fully covered. Given $p_1^* = \frac{1}{4}(t+2q)$, we can solve the optimal deviation prices for Firm 2. Specifically, because no demand exists for Product 3 in Submarket $\bar{1}3$, the profit function for Firm 2 changes as follows:

$$\pi_2 = \frac{p_2}{3}\left(\frac{t - p_2 + p_1^*}{2t} + \frac{t - p_2 + p_3 + q - \theta q}{2t}\right) + \frac{p_3}{3}\left(1 - \frac{t - p_2 + p_3 + q - \theta q}{2t}\right) \tag{29}$$

Because marginal consumer m_{23} derives zero utility, using the Lagrange-multiplier method, we have $L(p_2, p_{3,}\lambda) = \pi_2 + \lambda(q + \theta q - t - p_2 - p_3)$. By solving the following system of equations,

$$\begin{cases} \frac{\partial L}{\partial p_2} = 0 \\ \frac{\partial L}{\partial p_3} = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \end{cases}$$

we can derive $p_2^{2*} = \frac{1}{40}(-11t + 26q + 8\theta q)$ and $p_3^{2*} = \frac{1}{40}(-29t + 14q + 32\theta q)$ when $q < \frac{29t}{14-8\theta}$. When $\frac{29t}{14-8\theta} \leq q$, the value of $\frac{1}{40}(-29t + 14q + 32\theta q)$ is higher than Product 3's valuation, θq, and thus we have $p_2^{2*} = q - t$ and $p_3^{2*} = \theta q$ instead. Substituting p_2^{2*} and p_3^{2*} into Equation (29), we can derive Firm 2's optimal profit from the deviation as

$$\pi_2^{2*} = \begin{cases} \frac{-519t^2+4qt(177+116\theta)+4q^2(9+8\theta(-7+2\theta)}{1920t} & \text{if } q < \frac{29t}{14-8\theta} \\ \frac{(17t-2q)(q-t)}{24t} & \text{otherwise} \end{cases}$$

Comparing π_2^{2*} against π_2^*, we have $\pi_2^{2*} > \pi_2^*$ if $\frac{104t+23\sqrt{30}t}{44+10\sqrt{30}-28\theta} < q$ when $q < \frac{29t}{14-8\theta}$, and if $\frac{\left(6-11\theta-\sqrt{-12\theta+85\theta^2}\right)t}{3-10\theta+3\theta^2} < q$ when $q \geq \frac{29t}{14-8\theta}$. We notice that the threshold curves $\frac{104t+23\sqrt{30}t}{44+10\sqrt{30}-28\theta}$, $\frac{\left(6-11\theta-\sqrt{-12\theta+85\theta^2}\right)t}{3-10\theta+3\theta^2}$, and $\frac{29t}{14-8\theta}$ intersect at $\theta = \frac{309+58\sqrt{30}}{1813}$. Therefore, $\pi_2^{2*} \geq \pi_2^*$ if and only if $\mathring{q} \leq q$, where $\mathring{q}$ is defined in Equation (25). In other words, Condition $q \leq \mathring{q}$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*).

Case III where $\frac{14t}{7+5\theta} < q \leq \min\{\frac{3t}{1+\theta}, \frac{2t}{1-\theta}\}$ and $\theta \leq 1$: This case is the same as Case (c) in the proof of Lemma 2, in which marginal consumers m_{13} and m_{23} derive zero utility in equilibrium because $q \leq \frac{3t}{1+\theta}$. Moreover, Condition $\frac{14t}{7+5\theta} < q$ ensures that Firm 1 (Firm 2) has no incentive to increase the price of Product 1 (Product 2) because the marginal loss from a price increase is greater than the marginal benefit. Condition $q \leq \frac{2t}{1-\theta}$ ensures that $m_{13} \leq 1$ and $m_{23} \leq 1$, and Condition

$\theta \leq 1$ ensures that marginal consumer m_{12} derives positive utility.

Case IV where $q \leq \frac{14t}{7+5\theta}$: This case is the same as Case (d) in the proof of Lemma 2, in which the two firms compete in Submarket $\bar{1}2$ but not in Submarket $\bar{1}3$. Consequently, as in case (a) in the proof of Lemma 1, $p_1^* = p_2^* = \frac{1}{5}(2q+t)$. Firm 2 either charges $p_3^* = \frac{1}{5}(-6t+3q+5\theta q)$ to cover all of the residual demands in Submarkets $\bar{1}3$ and $\bar{2}3$ or charges a monopoly price $p_3^* = \frac{\theta q}{2}$ for Product 3, whichever is higher.

In Cases I, II, III, and IV, substituting p_i^* from Equation (24) into Equations (9) and (10), we can derive the equilibrium demands and profits as specified in the lemma.

Case V where $\mathring{q} < q \leq \frac{2t}{1-\theta}$: Because $\mathring{q} < q$, Firm 2 can profitably deviate from the price setting $p_2^{1*} = \frac{1}{4}(-2t + 3q + \theta q)$ and $p_3^{1*} = \frac{1}{4}(-2t + q + 3\theta q)$ in Case II, via pulling product 3 out of the inter-product competition. Three different scenarios might arise:

(V.1) $q < \frac{29t}{14-8\theta}$: In this case, Firm 2 would like to increase the price of Product 3 to completely cede Submarket $\bar{1}3$ to Firm 1 while decreasing the price of Product 2 to keep the consumers in Submarket $\bar{2}3$ just fully covered by Products 2 and 3. As show in Case II, the prices $p_2^{2*} = \frac{1}{40}(-11t + 26q + 8\theta q)$ and $p_3^{2*} = \frac{1}{40}(-29t + 14q + 32\theta q)$ are the optimal deviation prices for Firm 2, given $p_1^{1*} = \frac{1}{4}(t + 2q)$. In response, Firm 1 increases the price of Product 1, from $p_1^{1*} = \frac{1}{4}(t + 2q)$ to $p_1^{2*} = \frac{1}{40}(-69t + 54q - 8\theta q)$, to make the submarket $\bar{1}3$ just covered by its product, by solving $q - p_1^{2*} - t = \theta q - p_3^{2*}$. However, observing the increased price of Product 1, Firm 2 would like to switch back to the competitive pricing strategy for Submarket $\bar{1}3$, followed by the decrease in the price of Product 1 as well. Therefore, neither $(p_1^{1*}, p_2^{1*}, p_3^{1*})$ nor $(p_1^{2*}, p_2^{2*}, p_3^{2*})$ can be sustained as a pure-strategy equilibrium, and the two firms compete under a mixed strategy in equilibrium.

(V.2) $\frac{29t}{14-8\theta} < q \leq \frac{7t}{3-\theta}$: When $\frac{29t}{14-8\theta} < q$, the value of $\frac{1}{40}(-29t + 14q + 32\theta q)$ is greater than θq. Therefore, in the deviation from $(p_1^{1*}, p_2^{1*}, p_3^{1*})$, Firm 2 would set the price of Product 3 as $p_3^{2*} = \theta q$ such that no consumer in Submarket $\bar{1}3$ purchases Product 3, and, accordingly, $p_2^{2*} = q - t$ to make Submarket $\bar{2}3$ just fully covered. In response, Firm 1 would increase the price of Product 1 from p_1^{1*} to $p_1^{2*} = q - t$. Nonetheless, observing the increased price of Product 1, Firm 2 would like to switch back to the competitive pricing strategy for Submarket $\bar{1}3$, followed by the change in the price of Product 1 as well. Therefore, neither $(p_1^{1*}, p_2^{1*}, p_3^{1*})$ nor $(p_1^{2*}, p_2^{2*}, p_3^{2*})$ can be sustained as a pure-strategy equilibrium, and the two firms compete under a mixed strategy in equilibrium.

(V.3) $\frac{7t}{3-\theta} < q$: When $\frac{7t}{3-\theta} < q$, we have $p_1^{1*} = q - t - (\theta q - p_3^{1*}) = \frac{1}{4}(-6t + 5q - \theta q)$, instead of

$p_1^{1*} = \frac{1}{4}(t+2q)$, because Product 3 has significant valuation disadvantage such that Product 1 has already covered the entire Submarket $\bar{1}3$ before the deviation by Firm 2. In other words, Firms 1 and 2 would mix between $(p_1^{1*} = \frac{1}{4}(-6t+5q-\theta q), p_2^{1*} = \frac{1}{4}(-2t+3q+\theta q), p_3^{1*} = \frac{1}{4}(-2t+q+3\theta q))$ and $(p_1^{2*} = q-t, p_2^{2*} = q-t, p_3^{2*} = \theta q)$.

Supposing that Firm 1 takes p_1^{1*} with probability α_1 and p_1^{2*} with probability $1-\alpha_1$ and Firm 2 takes (p_2^{1*}, p_3^{1*}) with probability α_2 and (p_2^{2*}, p_3^{2*}) with probability $1-\alpha_2$, we have the constant-profit conditions as follows:

$$\begin{cases} p_1^{1*}\left(\frac{t-p_1^{1*}+p_2^{1*}}{2t} + \frac{t-p_1^{1*}+p_3^{1*}+q-\theta q}{2t}\right)\alpha_2 + p_1^{1*}\left(\frac{t-p_1^{1*}+p_2^{2*}}{2t}+1\right)(1-\alpha_2) \\ = p_1^{2*}\left(\frac{t-p_1^{2*}+p_2^{1*}}{2t} + \frac{t-p_1^{2*}+p_3^{1*}+q-\theta q}{2t}\right)\alpha_2 + p_1^{2*}\left(\frac{t-p_1^{2*}+p_2^{2*}}{2t}+1\right)(1-\alpha_2) \\ \left[p_2^{1*}\left(\frac{t-p_2^{1*}+p_1^{1*}}{2t} + \frac{q-p_2^{1*}}{t}\right) + p_3^{1*}\left(\frac{t-p_3^{1*}+p_1^{1*}++q-\theta q}{2t} + \frac{\theta q-p_3^{1*}}{t}\right)\right]\alpha_1 \\ \quad + \left[p_2^{1*}\left(\frac{t-p_2^{1*}+p_1^{2*}}{2t} + \frac{q-p_2^{1*}}{t}\right) + p_3^{1*}\left(\frac{t-p_3^{1*}+p_1^{2*}+q-\theta q}{2t} + \frac{\theta q-p_3^{1*}}{t}\right)\right](1-\alpha_1) \\ = \left[p_2^{2*}\left(\frac{t-p_2^{2*}+p_1^{1*}}{2t} + \frac{q-p_2^{2*}}{t}\right) + p_3^{2*}\left(\frac{\theta q-p_3^{2*}}{t}\right)\right]\alpha_1 + \left[p_2^{2*}\left(\frac{t-p_2^{2*}+p_1^{2*}}{2t} + \frac{q-p_2^{2*}}{t}\right) + p_3^{2*}\left(\frac{\theta q-p_3^{2*}}{t}\right)\right](1-\alpha_1) \end{cases} \tag{30}$$

Substituting the values of the prices $(p_1^{1*}, p_2^{1*}, p_3^{1*})$ and $(p_1^{2*}, p_2^{2*}, p_3^{2*})$ into Equation (30) for (V.1), (V.2), and (V.3), respectively, we can solve the equation system and derive α_1 and α_2 in Equation (26). We can verify that neither firm has incentive to deviate from the obtained mixed-strategy equilibrium.

Substituting $(p_1^{1*}, p_2^{1*}, p_3^{1*})$, $(p_1^{2*}, p_2^{2*}, p_3^{2*})$, and α_1 and α_2 from Equation (26) into the demand and profit functions,

$$\begin{cases} d_1^* = \frac{1}{3}\alpha_1 \times \left[\left(\frac{t-p_1^{1*}+p_2^{1*}}{2t} + \frac{t-p_1^{1*}+p_3^{1*}+q-\theta q}{2t}\right)\alpha_2 + \left(\frac{t-p_1^{1*}+p_2^{2*}}{2t}+1\right)(1-\alpha_2)\right] \\ \qquad +\frac{1}{3}(1-\alpha_1)\left[\left(\frac{t-p_1^{2*}+p_2^{1*}}{2t} + \frac{t-p_1^{2*}+p_3^{1*}+q-\theta q}{2t}\right)\alpha_2 + \left(\frac{t-p_1^{2*}+p_2^{2*}}{2t}+1\right)(1-\alpha_2)\right] \\ d_2^* = \frac{1}{3}\alpha_2\left[\left(\frac{t-p_2^{1*}+p_1^{1*}}{2t} + \frac{q-p_2^{1*}}{t}\right)\alpha_1 + \left(\frac{t-p_2^{1*}+p_1^{2*}}{2t} + \frac{q-p_2^{1*}}{t}\right)(1-\alpha_1)\right] \\ \qquad +\frac{1}{3}(1-\alpha_2)\left[\left(\frac{t-p_2^{2*}+p_1^{1*}}{2t} + \frac{q-p_2^{2*}}{t}\right)\alpha_1 + \left(\frac{t-p_2^{2*}+p_1^{2*}}{2t} + \frac{q-p_2^{2*}}{t}\right)(1-\alpha_1)\right] \\ d_3^* = \frac{1}{3}\alpha_2\left[\left(\frac{t-p_3^{1*}+p_1^{1*}++q-\theta q}{2t} + \frac{\theta q-p_3^{1*}}{t}\right)\alpha_1 + \left(\frac{t-p_3^{1*}+p_1^{2*}+q-\theta q}{2t} + \frac{\theta q-p_3^{1*}}{t}\right)(1-\alpha_1)\right] \\ \qquad +\frac{1}{3}(1-\alpha_2)\left[\left(\frac{\theta q-p_3^{2*}}{t}\right)\alpha_1 + \left(\frac{\theta q-p_3^{2*}}{t}\right)(1-\alpha_1)\right] \end{cases}$$

and

$$\begin{cases} \pi_1^* & = & \frac{1}{3}\left[p_1^{1*}\left(\frac{t-p_1^{1*}+p_2^{1*}}{2t}+\frac{t-p_1^{1*}+p_3^{1*}+q-\theta q}{2t}\right)\alpha_2+p_1^{1*}\left(\frac{t-p_1^{1*}+p_2^{2*}}{2t}+1\right)(1-\alpha_2)\right] \\ \pi_2^* & = & \frac{1}{3}\left[p_2^{1*}\left(\frac{t-p_2^{1*}+p_1^{1*}}{2t}+\frac{q-p_2^{1*}}{t}\right)+p_3^{1*}\left(\frac{t-p_3^{1*}+p_1^{1*}++q-\theta q}{2t}+\frac{\theta q-p_3^{1*}}{t}\right)\right]\alpha_1 \\ & & +\frac{1}{3}\left[p_2^{1*}\left(\frac{t-p_2^{1*}+p_1^{2*}}{2t}+\frac{q-p_2^{1*}}{t}\right)+p_3^{1*}\left(\frac{t-p_3^{1*}+p_1^{2*}+q-\theta q}{2t}+\frac{\theta q-p_3^{1*}}{t}\right)\right](1-\alpha_1) \end{cases}$$

we can derive the equilibrium expected demands and profits as specified in the lemma.

Case VI where $\frac{2t}{1-\theta} < q$: When $\frac{2t}{1-\theta} < q$, the prices p_1^{1*} and p_1^{2*} in Scenario (V.3) satisfies $p_1^{1*} > p_1^{2*}$, and hence Firm 1 has no incentive to deviate from p_1^{2*} to p_1^{1*}, given $p_2^{2*} = q - t$ and $p_3^{2*} = \theta q$. As such, the two firms compete with $p_1^{2*} = q - t$, $p_2^{2*} = q - t$, and $p_3^{2*} = \theta q$ in equilibrium (a pure-strategy equilibrium).

Case VII where $\frac{2t}{\theta-1} < q$: When $\frac{2t}{\theta-1} < q$, $m_{23} = 0$. When Firm 2 prices the products such that marginal consumer m_{23} is indifferent between Products 2 and 3, $\theta q - p_3 - t = q - p_2$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(\theta q - p_3 - t - q + p_2)$. By solving the first-order condition of (p_2, p_3, λ) and by the first-order condition of π_1 in Equation (10),

$$\begin{cases} \frac{\partial L}{\partial p_2} = 0 \\ \frac{\partial L}{\partial p_3} = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \\ \frac{\partial \pi_1}{\partial p_1} = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate from (p_1^*, p_2^*, p_3^*).

Case VIII where $\max\{\frac{56t}{29+3\theta}, \frac{10t}{1+3\theta}\} < q \leq \frac{16t}{7+\theta}$: When $q \leq \frac{16t}{7+\theta}$, Firm 1 strategically prices its product such that marginal consumer m_{12} derives zero utility, leading to $q + q - t - p_1 - p_2 = 0$. Using the Lagrange-multiplier method, we have $L(p_1, \lambda) = \pi_1 + \lambda(q + q - t - p_1 - p_2)$. By solving the first-order condition of $L(p_1, \lambda)$ and by the first-order condition of π_2 in Equation (10),

$$\begin{cases} \frac{\partial L}{\partial p_1} = 0 \\ q + q - t - p_1 - p_2 = 0 \\ \frac{\partial \pi_2}{\partial p_2} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate from (p_1^*, p_2^*, p_3^*). Condition $\frac{10t}{1+3\theta} < q$ ensures that marginal consumer m_{23} derives positive utility, and Condition $\frac{56t}{29+3\theta} < q$ ensures that Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price.

Case IX where $\frac{14t}{17-5\theta} < q \leq \min\{\frac{10t}{1+3\theta}, \frac{3t}{3-\theta}\}$ and $1 < \theta$: When $q \leq \frac{3t}{3-\theta}$, marginal consumer m_{12} derives zero utility, and when $q \leq \frac{10t}{1+3\theta}$, marginal consumer m_{23} derives zero utility as well. When Firm 1 prices its product such that marginal consumer m_{12} derives zero utility, $q+q-t-p_1-p_2 = 0$. When Firm 2 prices the products such that marginal consumer m_{23} derives zero utility, $q+\theta q-t-p_2-p_3 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda_1) = \pi_1 + \lambda_1(q+q-t-p_1-p_2)$ and $L_2(p_2, p_3, \lambda) = \pi_2 + \lambda_2(q+\theta q - t - p_2 - p_3)$. By solving the following system of equations,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = 0 \\ \frac{\partial L_2}{\partial p_2} = 0 \\ \frac{\partial L_2}{\partial p_3} = 0 \\ q + q - t - p_1 - p_2 = 0 \\ q + \theta q - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{14t}{17-5\theta} < q$ ensures that Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price. Condition $1 < \theta$ ensures that m_{13} derives positive utility.

In Cases VI, VII, VIII, and IX, substituting p_i^* from Equation (24) into Equations (9) and (10), we can derive the equilibrium demands and profits as specified in the lemma.

Case X where $\frac{197t}{123+11\theta} < q \leq \min\{\frac{t}{4-2\theta}, \frac{56t}{29+3\theta}\}$: When $q \leq \frac{56t}{29+3\theta}$, Firm 1 would like to increase its product price as if it competes with only Product 3 but not with Product 2, and Firm 2 chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{12}$ while Product 3 competes against Product 1 in Submarket $\bar{13}$. While Firm 2's profit function stays the same as that in Equation (10), Firm 1's demand and profit functions change into

$$\begin{cases} d_1 = \frac{1}{3} \times \left(\frac{q-p_1}{t} + m_{13}\right) \\ \pi_1 = p_1 \times d_1 \end{cases} \tag{31}$$

When Firm 2 prices its product such that marginal consumer m_{12} derives zero utility, $q+q-t-p_1-p_2=0$. Using the Lagrange-multiplier method, we have $L(p_2, p_{3,}, \lambda) = \pi_2 + \lambda(q+q-t-p_1-p_2)$. By solving the first-order conditions of π_1 (in Equation (31)) and $L(p_2, p_{3,}, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = 0 \\ \frac{\partial L}{\partial p_2} = 0 \\ \frac{\partial L}{\partial p_3} = 0 \\ q+q-t-p_1-p_2=0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{197t}{123+11\theta} < q$ ensures that Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 2' price. Condition $q \leq \frac{t}{4-2\theta}$ ensures that m_{23} derives positive utility.

Case XI where $\max\{\frac{48t}{59-15\theta}, \frac{t}{4-2\theta}\} < q \leq \frac{14t}{17-5\theta}$: When $q \leq \frac{14t}{17-5\theta}$, Firm 1 would like to increase its product price as if it competes with only Product 3 but not with Product 2, and Firm 2 chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{12}$ while Product 3 competes against Product 1 in Submarket $\bar{13}$. As in Case X, Firm 1's demand and profit functions are as in Equation (31). When Firm 2 prices its product such that marginal consumer m_{12} derives zero utility, $q+q-t-p_1-p_2=0$. Moreover, because $\frac{t}{4-2\theta} < q$, m_{23} derives zero utility, and hence $q+\theta q-t-p_2-p_3=0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda_1, \lambda_2) = \pi_2 + \lambda_1(q+q-t-p_1-p_2) + \lambda_2(q+\theta q-t-p_2-p_3)$. By solving the first-order condition of π_1 (in

Equation (31)) and $L(p_2, p_3, \lambda_1, \lambda_2)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = 0 \\ \frac{\partial L}{\partial p_2} = 0 \\ \frac{\partial L}{\partial p_3} = 0 \\ q + q - t - p_1 - p_2 = 0 \\ q + \theta q - t - p_2 - p_3 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{48t}{59-15\theta} < q$ ensures that Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 2's price.

In Cases X and XI, substituting p_i^* from Equation (24) into Equation (31), we can derive the equilibrium demand and profit for Firm 1 as specified in the lemma. Substituting p_i^* from Equation (24) into Firm 2's demand and profit functions in Equations (9) and (10), we can derive the equilibrium demand and profit for Firm 2 as specified in the lemma.

Case XII where $\frac{253t}{81+61\theta} < q \leq \frac{197t}{123+11\theta}$: When $q \leq \frac{197t}{123+11\theta}$, Firm1 and Firm 2 do not compete in Submarket $\bar{1}2$ but in Submarket $\bar{1}3$. Therefore, the two firms' demand and profit functions become

$$\begin{cases} d_1 = \frac{1}{3} \times \left(\frac{q-p_1}{t} + m_{13}\right) \\ d_2 = \frac{1}{3} \times \left(\frac{q-p_2}{t} + m_{23}\right) \\ d_3 = \frac{1}{3} \times (1 - m_{13} + 1 - m_{23}) \end{cases} \tag{32}$$

and

$$\begin{cases} \pi_1 = p_1 \times d_1 \\ \pi_2 = p_2 \times d_2 + p_3 \times d_3 \end{cases} \tag{33}$$

By solving the first-order conditions of π_1 and π_2 in Equation (33), we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{253t}{81+61\theta} < q$ ensures that marginal consumer m_{23} derives positive utility.

Case XIII where $q \leq \min\{\frac{48t}{59-15\theta}, \frac{253t}{81+61\theta}\}$: When $q \leq \frac{48t}{59-15\theta}$, Firm1 and Firm 2 do not compete in Submarket $\bar{1}2$ but in Submarket $\bar{1}3$. Therefore, the two firms' demand and profit functions

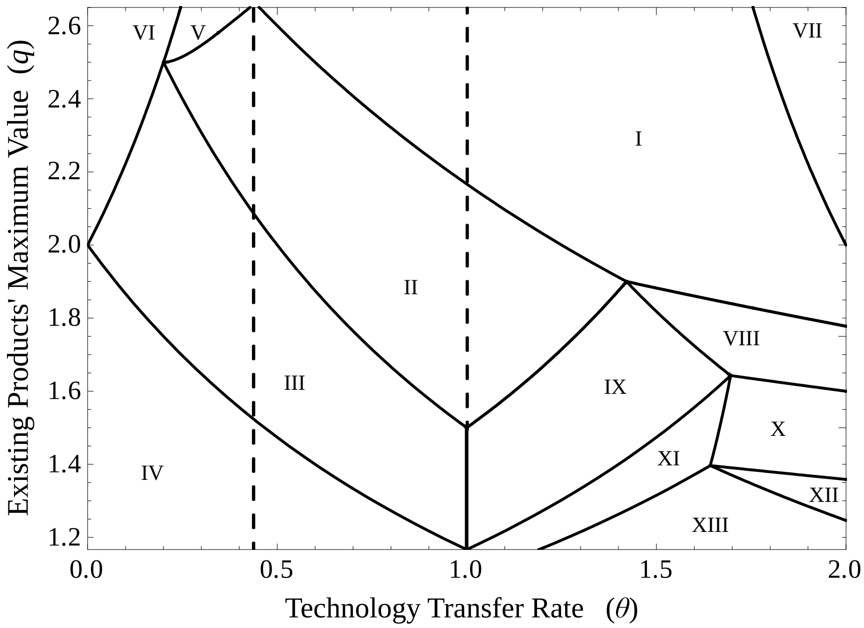

Figure 9: Equilibrium Outcome after the Introduction of Product 3 by Firm 2 ($t = 1$)

change to those in Equations (32) and (33) as well. Moreover, because $q \leq \frac{253t}{81+61\theta}$, marginal consumer m_{23} derives zero utility. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(q + \theta q - t - p_2 - p_3)$. By solving the first-order condition of π_1 and $L(p_2, p_3, \lambda)$, we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma.

In Cases XII and XIII, substituting p_i^* from Equation (24) into Equations (32) and (33), we can derive the equilibrium demands and profits for the two firms as specified in the lemma. $\square$

Figure 9 illustrates the equilibrium outcome, with the region between the two dashed lines representing the results for the baseline model (i.e., with $\frac{43}{100} \leq \theta \leq 1$). Regions I, II, III, and IV in Figure 9 correspond to Cases I, II, III, and IV in Equation (24), respectively, which represent the four pricing patterns discussed in Section 4.2. Specifically, in Region I, both q and θ are high such that the existing products offer high value and the new product offers comparable value to consumers. Consequently, Firms 1 and 2 compete aggressively with each other in Submarkets $\bar{12}$ and $\bar{13}$, which shapes the equilibrium prices. In Region II, in which either q or θ is not high, Firm 2 optimally sets prices for its two product to just cover all the consumers in its exclusive Submarket $\bar{23}$ and extracts all the marginal consumer's surplus. Firm 1 reacts by optimally choosing its price for Product 1 to compete against both Products 2 and 3. In Region III, due to the reduction in the product valuation, Firm 1 finds it is optimal to set the highest price to serve the entire residual demand from Product 3 in Submarket $1\bar{3}$ such that Submarket $1\bar{3}$ just fully covered. In Region IV, the value of the existing products is low and the new product's value is even lower. Therefore, Products 1 and 2 compete directly as in the absence of Product 3. Firm 2 uses Product 3 to serve the residual demands in Submarkets $1\bar{3}$ and $2\bar{3}$.

Extending the range of θ from $\frac{43}{100} \leq \theta \leq 1$ to $0 \leq \theta \leq 2$ results in the new equilibrium Regions V∼XIII. In the following, for brevity, we emphasize only on the change in equilibrium when moving from one region to another, with Region II or Region I as the origin.

Moving northwest from Region II, in Region V (corresponding to Case V in Equation (24)), q is high but θ is low. Due to the relatively low valuation of the new product compared to existing products, Firm 2 may consider deviating its prices in this region by withdrawing the new product from inter-firm competition. Specifically, Firm 2 can increase the price of Product 3 to cede Submarket $1\bar{3}$ to Firm 1, while simultaneously decreasing the price of Product 2 to fully serve consumers in Submarket $2\bar{3}$. In response, Firm 1 may increase the price of Product 1 to cover Submarket $1\bar{3}$ as well. However, if the price of Product 1 increases, Firm 2 would decrease the price of Product 3 and increase the price of Product 2 back to optimally compete against Firm 1, similar to its strategy in Region II. Such a reaction would induce Firm 1 to decrease the price of Product 1, further leading to a re-deviation by Firm 2. Consequently, Firms 1 and 2 enter a loop of changing their prices, resulting in only a mixed-strategy equilibrium in this case. The changes in their prices subtly depend on the values of q and θ, as summarized in Equation (26).

On the other hand, when the value of θ becomes lower and (q, θ) ranges in Region VI (corresponding to Case VI in Equation (24)), Firm 2 chooses to compete against Firm 1 with only Product 2, whose valuation is significantly higher than that of Product 3. In particular, it optimally sets the price of Product 3 exactly equal to its valuation. Thus, no consumers gain positive utility, nor are they willing to purchase Product 3 in Region VI. As a result, the prices of Products 1 and 2 are set to just fully serve $1\bar{3}$ and $2\bar{3}$, respectively, as in the absence of Product 3.

In contrast to Regions V and VI, moving northeast from Region I, in Region VII (corresponding to Case VII in Equation (24)), the existing products' valuation is high but the new product's valuation is even much higher. In this case, Firm 2 chooses to cover its exclusive Submarket $2\bar{3}$ by only the new product, and Firm 1 reacts to its rival's prices by optimally choosing its price for

Product 1 to compete against both Products 2 and 3.

Moving south from Region I to Region VIII (corresponding to Case VIII in Equation (24)), or moving southeast from Region II to Region IX (corresponding to Case IX in Equation (24)), due to the reduction in the value of the existing products, Firm 1 chooses to price Product 1 such that it just serves the residual demand from Product 2 in Submarket $\bar{1}2$ while it competes against Product 3 in Submarket $\bar{1}3$. On the other hand, moving south further from Region VIII to Region X (corresponding to Case X in Equation (24)), or moving southeast further from Region IX to Region XI (corresponding to Case XI in Equation (24)), due to the further reduction in the value of the existing products, Firm 1 chooses to price Product 1 as if it competes with only Product 3 but not with Product 2; in reverse, Firm 2 optimally chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{1}2$ while Product 3 competes against Product 1 in Submarket $\bar{1}3$.

Finally, moving further south from Region X to Region XII (corresponding to Case XII in Equation (24)), or moving southeast further from Region XI to Region XIII (corresponding to Case XIII in Equation (24)), the valuation of the existing products is too low, preventing Submarket $\bar{1}2$ from being fully covered. This contrasts with the situation in the absence of the new product, where Submarket $\bar{1}2$ is fully covered under the two-firm competition. In the presence of the new product, due to cannibalization concerns, Firm 2 raises the price of Product 2 compared to the case in the absence of the new product. Consequently, some consumers in Submarket $\bar{1}2$ with high degrees of misfit do not purchase the low-valuation products.

By comparing the equilibrium profits with and without the introduction of Product 3, we can derive the conditions under which the new product would be voluntarily introduced in equilibrium.

Proposition 8. *In equilibrium, the new product would be introduced to the market if and only if* $\underline{q}(\theta) \leq q \leq \bar{q}(\theta)$, *where*

$$\underline{q}(\theta) = \begin{cases} \frac{14t}{7+5\theta} & \text{if } \theta \leq 1 \\ \frac{14t}{17-5\theta} & \text{otherwise,} \end{cases}$$

and

$$\bar{q}(\theta)=\begin{cases}\frac{2t}{1-\theta} & \text{if } \theta\leq\frac{1}{5}\\ \frac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t & \text{if } \frac{1}{5}<\theta\leq\frac{231+52\sqrt{33}}{1087}\\ \frac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t & \text{if } \frac{231+52\sqrt{33}}{1087}<\theta\leq\frac{311}{635}\\ \frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t & \text{if } \frac{311}{635}<\theta\leq\frac{1}{2}\left(10-3\sqrt{6}\right)\\ \frac{15\sqrt{6}(15+\theta)+4(23-50\theta)}{191+50\theta-25\theta^2}t & \text{if } \frac{1}{2}\left(10-3\sqrt{6}\right)<\theta\leq\frac{1}{773}\left(8253-2880\sqrt{6}\right)\\ \frac{871+75\theta+15\sqrt{783-504\theta+25\theta^2}}{2(179+75\theta)}t & \text{otherwise,}\end{cases} \tag{34}$$

Proof. Based on Lemmas 1 and 6, we compare the firms' equilibrium profits before and after the introduction of Product 3 for different cases.

The case with $q\leq 2t$:

Case I of Lemma 6: When $\max\{\frac{13t}{4+2\theta},\frac{16t}{7+\theta}\}<q\leq\frac{2t}{\theta-1}$, we have

$$\begin{aligned}\hat{\pi}_1^*-\pi_1^* &= \tfrac{1}{108t}(8t+q-\theta q)^2-\tfrac{1}{50t}(2q+t)^2\\ &= \tfrac{1}{2}\left(\tfrac{(1-\theta)^2}{54t}-\tfrac{4}{25t}\right)\left(q-\tfrac{4(-23+50\theta)+15\sqrt{6}(15+\theta)}{-191-50\theta+25\theta^2}t\right)\left(q-\tfrac{4(-23+50\theta)-15\sqrt{6}(15+\theta)}{-191-50\theta+25\theta^2}t\right)\end{aligned}$$

Because the term in the first bracket is negative and that in the second bracket is positive, $\hat{\pi}_1^*\geq\pi_1^*$ if and only if $q\leq\frac{15\sqrt{6}(15+\theta)+4(23-50\theta)}{191+50\theta-25\theta^2}t$. We can verify that $\hat{\pi}_2^*>\pi_2^*$.

Case II of Lemma 6: When $\max\{\frac{3t}{1+\theta},\frac{3t}{3-\theta}\}<q\leq\min\{\mathring{q},\frac{13t}{4+2\theta}\}$, we can verify that $\hat{\pi}_1^*>\pi_1^*$ and $\hat{\pi}_2^*>\pi_2^*$.

Case III of Lemma 6: When $\frac{14t}{7+5\theta}<q\leq\min\{\frac{3t}{1+\theta},\frac{2t}{1-\theta}\}$ and $\theta\leq 1$, we can verify that $\hat{\pi}_1^*>\pi_1^*$ and $\hat{\pi}_2^*>\pi_2^*$.

Case IV of Lemma 6: When $q\leq\frac{14t}{7+5\theta}$, as shown in Lemma 6, the introduction of Product 3 affects neither Product 1's nor Product 2's profit. Therefore, $\hat{\pi}_1^*=\pi_1^*$; $\hat{\pi}_2^*>\pi_2^*$ because Firm 2 obtains additional revenue from Product 3.

Case VIII of Lemma 6: When $\max\{\frac{56t}{29+3\theta},\frac{10t}{1+3\theta}\}<q\leq\frac{16t}{7+\theta}$, we have

$$\begin{aligned}\hat{\pi}_1^*-\pi_1^* &= \tfrac{1}{27t}(q-t)\left[32t-q(5+3\theta)\right]-\tfrac{1}{50t}(2q+t)^2\\ &= -\tfrac{1}{2}\left(\tfrac{(10+6\theta)}{27t}+\tfrac{4}{25t}\right)\left(q-\tfrac{871+75\theta+15\sqrt{783-504\theta+25\theta^2}}{2(179+75\theta)}t\right)\left(q-\tfrac{871+75\theta-15\sqrt{783-504\theta+25\theta^2}}{2(179+75\theta)}t\right)\end{aligned}$$

Because we can verify that the term in the third bracket is positive, $\hat{\pi}_1^*\geq\pi_1^*$ if and only if $q\leq\frac{871+75\theta+15\sqrt{783-504\theta+25\theta^2}}{2(179+75\theta)}t$. We can verify that $\hat{\pi}_2^*>\pi_2^*$.

Case IX of Lemma 6: When $\frac{14t}{17-5\theta} < q \leq \min\{\frac{10t}{1+3\theta}, \frac{3t}{3-\theta}\}$ and $1 < \theta$, we can verify that $\hat{\pi}_1^* \geq \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

Case X of Lemma 6: When $\frac{197t}{123+11\theta} < q \leq \min\{\frac{t}{4-2\theta}, \frac{56t}{29+3\theta}\}$, we can verify that $\hat{\pi}_1^* < \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

Case XI of Lemma 6: When $\max\{\frac{48t}{59-15\theta}, \frac{t}{4-2\theta}\} < q \leq \frac{14t}{17-5\theta}$, we can verify that $\hat{\pi}_1^* < \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

Case XII of Lemma 6: When $\frac{253t}{81+61\theta} < q \leq \frac{197t}{123+11\theta}$, we can verify that $\hat{\pi}_1^* < \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

Case XIII of Lemma 6: When $q \leq \min\{\frac{48t}{59-15\theta}, \frac{253t}{81+61\theta}\}$, we can verify that $\hat{\pi}_1^* < \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

The case with $q > 2t$:

Case I of Lemma 6: When $\max\{\frac{13t}{4+2\theta}, \frac{16t}{7+\theta}\} < q \leq \frac{2t}{\theta-1}$, we have

$$\begin{aligned}\hat{\pi}_1^* - \pi_1^* &= \tfrac{1}{108t}(8t + q - \theta q)^2 - \tfrac{1}{2}(q - t) \\ &= \tfrac{1}{108t}(1 - \theta)^2\left(q - \tfrac{19+8\theta+3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t\right)\left(q - \tfrac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t\right)\end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \geq \pi_1^*$ if and only if $q \leq \frac{19+8\theta-3\sqrt{3(9+20\theta-2\theta^2)}}{(1-\theta)^2}t$.

$$\begin{aligned}\hat{\pi}_2^* - \pi_2^* &= \tfrac{1}{432t}\left[4(10t - q + \theta q)^2 + 27(1 - \theta)^2 q^2\right] - \tfrac{1}{2}(q - t) \\ &= \tfrac{31}{432t}(1 - \theta)^2\left(q - \tfrac{4(37-10\theta)+6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t\right)\left(q - \tfrac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t\right)\end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_2^* \geq \pi_2^*$ if and only if $q \leq \frac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2}t$.

Case II of Lemma 6: When $\max\{\frac{3t}{1+\theta}, \frac{3t}{3-\theta}\} < q \leq \min\{\mathring{q}, \frac{13t}{4+2\theta}\}$, we can verify that $\hat{\pi}_1^* \geq \pi_1^*$. We have

$$\begin{aligned}\hat{\pi}_2^* - \pi_2^* &= \tfrac{1}{48t}[-22t^2 + 2(13 + 11\theta)qt - (1 + 10\theta - 3\theta^2)q^2] - \tfrac{1}{2}(q - t) \\ &= -\tfrac{1}{48t}(1 + 10\theta - 3\theta^2)\left(q - \tfrac{1+11\theta-\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t\right)\left(q - \tfrac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t\right)\end{aligned}$$

Because the terms in the first and second brackets are positive, $\hat{\pi}_2^* \geq \pi_2^*$ if and only if $q \leq \frac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2}t$.

Case III of Lemma 6: When $\frac{14t}{7+5\theta} < q \leq \min\{\frac{3t}{1+\theta}, \frac{2t}{1-\theta}\}$ and $\theta \leq 1$, we can verify that $\hat{\pi}_1^* \geq \pi_1^*$ and $\hat{\pi}_2^* \geq \pi_2^*$.

Case V of Lemma 6: When $\mathring{q} < q \leq \frac{2t}{1-\theta}$, we can verify that $\hat{\pi}_1^* \geq \pi_1^*$ and $\hat{\pi}_2^* \leq \pi_2^*$.

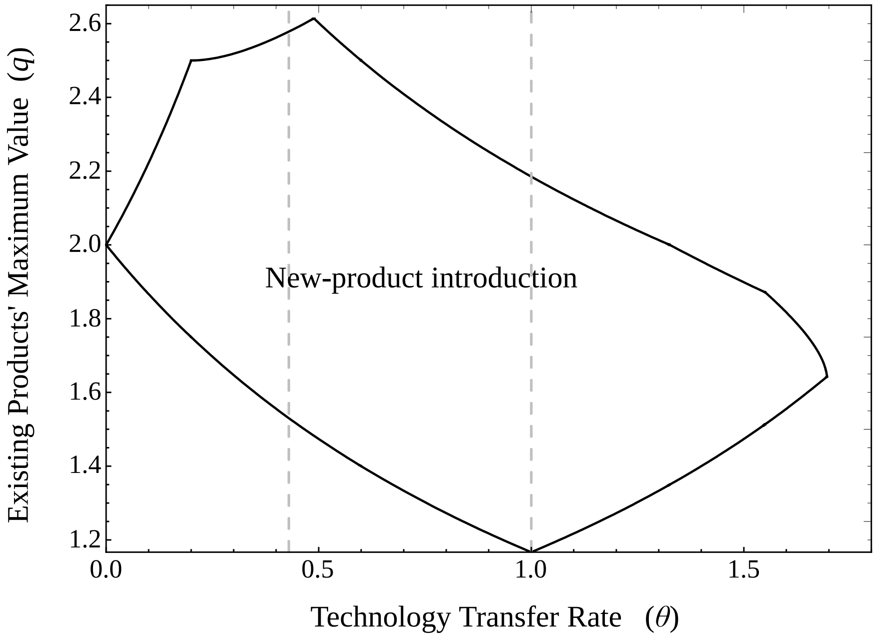

Figure 10: Conditions for New-Product Introduction ($t = 1$)

Case VI of Lemma 6: When $\frac{2t}{1-\theta} < q$, $\hat{\pi}_1^* = \pi_1^*$ and $\hat{\pi}_2^* = \pi_2^*$.

Case VII of Lemma 6: When $\frac{2t}{-1+\theta} < q$, we can verify that $\hat{\pi}_1^* < \pi_1^*$ and $\hat{\pi}_2^* > \pi_2^*$.

Combining the results from both cases with $q \leq 2t$ and with $q > 2t$, we obtain the comparison results as stated in Proposition 8. $\square$

Figure 10 illustrates the equilibrium results. The area between the dashed lines represents the results for the baseline model (i.e., with $\frac{43}{100} \leq \theta \leq 1$). Evidently, the results we derive under the baseline model continue to hold under this extended model. Furthermore, even if the new product offers a higher valuation, Firm 1 might continue to have an incentive to share its technology. Thus, the new product would be introduced to the market as long as θ is not too large. More importantly, the insights from our baseline model remain consistent—the focal firm would like to share its technology if the entry of the new product changes the two firms' competition landscape. In particular, the focal firm has an incentive to share its technology when the new product causes a cannibalization consideration in the rival's multiproduct pricing, imposing a positive externality on the focal firm, and meanwhile, the new product does not create too much additional competition pressure on the focal firm.

E The Case with Endogenous Valuation of the New Product

In the main model, we assume the valuation of the new product is exogenous. In this appendix, we examine the case in which the valuation of the new product is endogenously determined. Note that, on one hand, the valuation of the new product can be influenced by how much of the technology the focal firm can share to a rival. On the other hand, the valuation of the new product can also be influenced by how well the rival can use the technology, which depends on the rival's existing technology, facilities, or even disposable capital. In this appendix, we consider Firm 1 chooses the optimal θ (how much to share) in the first place. Then, under Firm 1's optimal sharing (i.e., θ^*), we examine whether Firm 2 makes full use of the shared technology; that is, Firm 2 can choose to use θ^* or any θ below, without or with consideration of associated development cost for the new product. We allow θ to range from 0 to 2 (i.e., $0 \leq \theta \leq 2$).

E.1 Optimal θ Chosen by Firm 1

We first examine Firm 1's optimal sharing level. Proposition 9 summarizes the result.

Proposition 9. *The optimal technology sharing rate that Firm 1 chooses is as follows:*

$$\theta^* = \begin{cases} \frac{31q^2-40qt+6\sqrt{6}\sqrt{31q^3t-81q^2t^2}}{31q^2} & \textit{if } \frac{\left(185-4\sqrt{33}\right)}{62}t < q \\ \frac{5q^2-11qt+\sqrt{28q^4-116q^3t+115q^2t^2}}{3q^2} & \textit{if } \frac{5}{2}t < q \leq \frac{\left(185-4\sqrt{33}\right)}{62}t \\ \frac{3t-q}{q} & \textit{if } \frac{3}{2}t < q \leq \frac{5}{2}t \\ 1 & \textit{otherwise} \end{cases} \tag{35}$$

Proof. We derive the optimal θ for Firm 1 by examining the first-order derivative of π_1^* presented in Lemma 6, within the region where Firm 1 has an incentive to share and Firm 2 has an incentive to adopt (i.e., $\underline{q}(\theta) \leq q \leq \bar{q}(\theta)$, as prescribed in Proposition 8). According to the proof of Proposition 8, this region only involves five cases of Lemma 6: Cases I, II, III, VIII, and IX.

Case I: When $\max\{\frac{13t}{4+2\theta}, \frac{16t}{7+\theta}\} < q \leq \frac{2t}{\theta-1}$, we have

$$\frac{\partial\pi_1^*}{\partial\theta} = -\frac{8t+q-\theta q}{54t}q < -\frac{8t-\theta q}{54t}q < 0$$

Case II: When $\max\{\frac{3t}{1+\theta}, \frac{3t}{3-\theta}\} < q \leq \min\{\mathring{q}, \frac{13t}{4+2\theta}\}$, we have $\frac{\partial\pi_1^*}{\partial\theta} = 0$.

Case III: When $\frac{14t}{7+5\theta} < q \leq \min\{\frac{3t}{1+\theta}, \frac{2t}{1-\theta}\}$ and $\theta \leq 1$, we have

$$\frac{\partial \pi_1^*}{\partial \theta} = -\frac{q-3t+\theta q}{24t} q \geq 0$$

because $q \leq \frac{3t}{1+\theta}$.

Case VIII: When $\max\{\frac{56t}{29+3\theta}, \frac{10t}{1+3\theta}\} < q \leq \frac{16t}{7+\theta}$, we have $\frac{\partial \pi_1^*}{\partial \theta} = \frac{-q+t}{9t} q < 0$.

Case IX: When $\frac{14t}{17-5\theta} < q \leq \min\{\frac{10t}{1+3\theta}, \frac{3t}{3-\theta}\}$ and $1 < \theta$, we have

$$\frac{\partial \pi_1^*}{\partial \theta} = -\frac{-3q+3t+\theta q}{24t} q < 0$$

because $q \leq \frac{3t}{3-\theta}$.

Therefore, Firm 1's profit increases in θ only in Case III and (weakly) decreases in θ in all other cases. Consequently, the upper bound of θ in Case III must be Firm 1's optimal choice whenever possible, and the lower bound of θ, defined by $\bar{q}(\theta)$ in Equation (34), must be Firm 1's optimal choice when q is above the maximum q in Case III. The upper bound of θ in Case III is defined by $q = \frac{3t}{1+\theta}$ and $\theta = 1$: When $q \leq \frac{3t}{2}$, the upper bound is $\theta^* = 1$; When $q > \frac{3t}{2}$, the upper bound is $\theta^* = \frac{3t-q}{q}$.

When q is above the maximum q in Case III (which is $q = \frac{5}{2}$, determined by the boundary curves of Case III, $q = \frac{3t}{1+\theta}$ and $q = \frac{2t}{1-\theta}$), θ^* takes the value of the lower bound of $\bar{q}(\theta)$. Based on $q = \frac{4(37-10\theta)-6\sqrt{6(13+122\theta-81\theta^2)}}{31(1-\theta)^2} t$ and $q = \frac{1+11\theta+\sqrt{3+42\theta+115\theta^2}}{1+10\theta-3\theta^2} t$ (the third and second cases in Equation (34)), we can derive θ^* as in Equation (35). These two curves intersect at $q = \frac{\left(185-4\sqrt{33}\right)}{62} t$. The former equation occurs when $q > \frac{\left(185-4\sqrt{33}\right)}{62} t$, and the latter occurs when $\frac{5t}{2} < q \leq \frac{\left(185-4\sqrt{33}\right)}{62} t$. □

Figure 11 illustrates the optimal technology sharing rate for Firm 1. We find that θ^* first decreases and then increases in q. The intuition is largely consistent and implied by our discussion in the main model. As discussed in the main model, the focal firm would prefer to share its technology if and only if the entry of the new product induces a cannibalization consideration in the rival's multiproduct pricing, thereby imposing a positive externality on the focal firm. At the same time, the new product must not create excessive additional competition pressure on the focal firm. The optimal θ is largely determined by this tradeoff. When q is low (i.e., when $q \leq \frac{3}{2}t$), the focal firm optimally choose a large θ (i.e., $\theta^* = 1$), primarily enjoying the benefit of positive externality

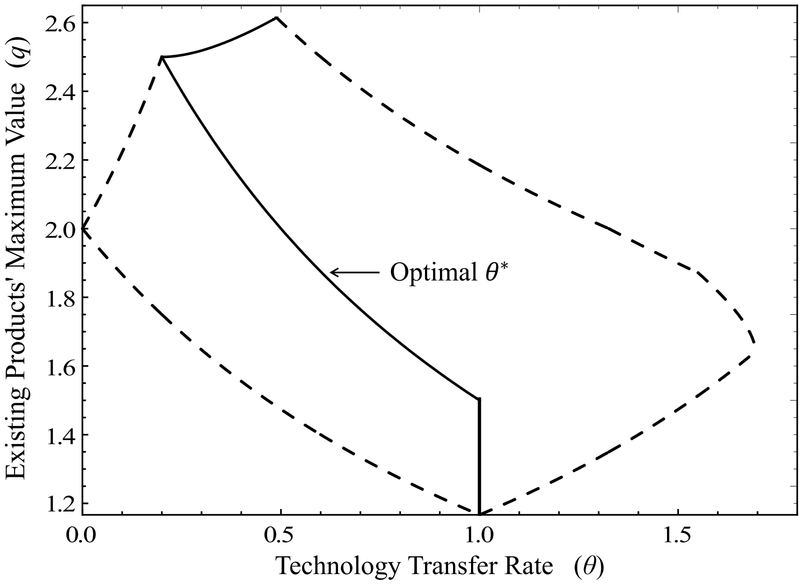

Figure 11: Optimal Technology Sharing Rate by Firm 1 ($t = 1$)

because, in this case, additional competition is not a big concern. When q increases, the focal firm's product serves more consumers in Submarket $1\bar{3}$ in the absence of the new product, making the additional competition a concern. The focal firm then balances the benefit of positive externality and the negative effect of additional competition. The optimal θ is shaped by this tradeoff, and it decreases in q as the negative effect of additional competition becomes more pronounced when q increases.

When q is high (i.e., when $q > \frac{5}{2}t$), although Firm 1 has a similar tradeoff between the positive externality and the negative effect of additional competition, it cannot choose a very low θ. Otherwise, Firm 2 has no incentive to adopt the technology. As a result, the optimal θ takes the threshold value under which Firm 2 just has an incentive to adopt the technology. That explains the kink at $q = \frac{5}{2}t$ and the opposite trends before and after the kink.

E.2 Optimal θ Chosen by Firm 2 under Firm 1's Optimal Sharing

Next, given Firm 1's optimal sharing (i.e., θ^*), we examine Firm 2's choice of the valuation of the new product. Specifically, we explore whether Firm 2 would like to make full use of the shared technology; in other words, Firm 2 can choose to use θ^* or any θ below, or equivalently, the valuation of the new product (θq) subject to the constraint $\theta q \leq \theta^* q$. We first consider the scenario where choosing the valuation is cost-free to examine the strategic implications of the technology-transfer rate for Firm 2. Subsequently, we investigate the case in which a higher valuation incurs a higher development cost to examine the effects of the cost.

The Case with Cost-Free Choice

When there is no cost involved in Firm 2's choice, we find that Firm 2 will always make full use of the shared technology; in other words, Firm 2 will always set its new product's valuation at the highest possible level. Intuitively, this is because a higher valuation for the new product results in a greater competitive advantage. Technically, when$q > \frac{5}{2}t$, Firm 2 will make full use of the shared technology because θ^* is chosen in a way that gives Firm 2 just enough incentive to adopt the technology. Any lower value of θ could make Firm 2 worse off and provide Firm 2 with a disincentive to adopt the technology.

When $q \leq \frac{5}{2}t$, we can verify that Firm 2's profit increases in θ for all $\theta \leq \theta^*$, provided that Firm 1 has an incentive to share. In particular, in this case, by the analysis in Online Appendix D, we have

$$\pi_2 = \frac{1}{48t}[-16t^2 + 2(9+7\theta)qt + (1-\theta)(1-5\theta)q^2]$$

We can derive its first-order derivative as

$$\frac{\partial \pi_2}{\partial \theta} = \frac{q}{24t}(7t - 3q + 5\theta q)$$

When $q \leq 2t$,

$$\frac{\partial \pi_2}{\partial \theta} = \frac{q}{24t}(7t - 3q + 5\theta q) > \frac{q}{24t}(\frac{7q}{2} - 3q + 5\theta q) > 0$$

When $2t < q$, the thresholds for Firm 1's incentive to share is $\theta > \frac{q-2t}{q}$, and thus

$$\frac{\partial \pi_2}{\partial \theta} = \frac{q}{24t}(7t - 3q + 5\theta q) > \frac{q}{24t}(7t - 3q + 5\frac{q-2t}{q}q) = \frac{q}{24t}(2q - 3t) > 0$$

Therefore, Firm 2's profit monotonically increases in θ. Consequently, Firm 2 always makes full use of the shared technology with Firm 1's optimally chosen level and develops the new product with the valuation of $\theta^* q$.

The Case with Convex Cost

We assume that the cost for Firm 2 to develop Product 3 is $c(\theta q)^2$, a convex function of the new product's valuation, where c measures the cost efficiency. As such, the optimal choice of θ should depend on the value of c. We consider c is not too low (e.g., $c > \frac{5}{48t}$) so that the profit function of Firm 2, $\pi_2 - c(\theta q)^2$, is well-behaved and concave in θ. Then, we can derive the optimal choice of θ for Firm 2, which might take any value up to Firm 1's optimal sharing, θ^*, where θ^* is defined in Equation (35).

In particular, in this case, we have Firm 2's profit function as follows:

$$\pi_2(\theta) - c(\theta q)^2 = \frac{1}{48t}[-16t^2 + 2(9+7\theta)qt + (1-\theta)(1-5\theta)q^2] - c(\theta q)^2$$

By the first-order condition (without considering the constraints), we have

$$\theta^*_{foc} = \frac{-3q+7t}{q(-5+48ct)}$$

Notice the thresholds for Firm 1's incentive to share are $\theta \geq \frac{q-2t}{q}$ when $2t < q$, and $\theta \geq \frac{7(2t-q)}{5q}$ when $q \leq 2t$. Because of the concavity of Firm 2's profit function, Firm 2's optimal choice of θ should be either θ^*_{foc}, the lower bound of Firm 1's sharing incentive, or the upper bound of Firm 1's optimal sharing (i.e., θ^*), depending on the relative value of θ^*_{foc}. When θ^*_{foc} is between the lower and upper bounds, θ^*_{foc} is optimal (as the interior solution). When θ^*_{foc} is below (above) the lower (upper) bound, the lower (upper) bound is optimal (as the corner solution). Therefore, if Firm 2 introduces the new product, the optimal choice of θ is

$$\begin{cases} \theta^*_{c1} = \max\left\{\frac{q-2t}{q}, \min\left\{\theta^*, \theta^*_{foc}\right\}\right\} & \text{when } 2t < q \\ \theta^*_{c2} = \max\left\{\frac{7(2t-q)}{5q}, \min\left\{\theta^*, \theta^*_{foc}\right\}\right\} & \text{when } q \leq 2t \end{cases} \tag{36}$$

Furthermore, we have to check whether Firm 2 is better off introducing a new product with the optimally chosen valuation. Firm 2 would introduce the new product if and only if the new product can bring in a higher profit. Specifically, when $2t < q$, if

$$\pi_2(\theta^*_{c1}) - c(\theta^*_{c1} q)^2 > \frac{1}{2}(q-t),$$

Firm 2 introduces the new product and the optimal θ is $\theta_c^* = \theta_{c1}^*$, where $\frac{1}{2}(q-t)$ is Firm 2's profit without introducing the new product (by Lemma 1 in the manuscript). When $q \leq 2t$, if

$$\pi_2(\theta_{c2}^*) - c(\theta_{c2}^* q)^2 > \frac{1}{50t}(2q+t)^2,$$

Firm 2 introduces the new product and the optimal θ is $\theta_c^* = \theta_{c2}^*$, where $\frac{1}{50t}(2q+t)^2$ is Firm 2's profit without introducing the new product (by Lemma 1 in the manuscript).

Altogether, we have the following result regarding Firm 2's optimal choice of θ.

Proposition 10. *Given the new-product development cost being $c(\theta q)^2$, Firm 2's optimal choice of the new product's valuation is $\theta_c^* q$, where*[1]

$$\theta_c^* = \begin{cases} \begin{cases} \theta_{c1}^* & \textit{if } \left(\pi_2 - c(\theta q)^2\right) |_{\theta=\theta_{c1}^*} > \frac{1}{2}(q-t) \\ 0 & \textit{otherwise} \end{cases} & \textit{if } 2t < q \\ \begin{cases} \theta_{c2}^* & \textit{if } \left(\pi_2 - c(\theta q)^2\right) |_{\theta=\theta_{c2}^*} > \frac{1}{50t}(2q+t)^2 \\ 0 & \textit{otherwise} \end{cases} & \textit{if } q \leq 2t \end{cases}$$

and θ_{c1}^* and θ_{c2}^* are defined in Equation (36).

Figures 12(a) and 12(b) illustrate the optimal choice of θ by Firm 2 with $c = 0.15$ and $c = 0.12$, respectively. Intuitively, a lower cost leads to a higher value of θ_c^*. While a higher θ_c^* might bring in some competitive advantage in the new product, it also leads to a higher development cost. Firm 2 must balance the gain from the competition and the development cost of the new product to make its choice. As long as the cost efficiency c is not too high, the new product can still benefit Firm 2 and Firm 1 via cannibalization and the cannibalization externality, respectively. In this case, Firm 1 has an incentive to share its technology, and our result that cannibalization externality promotes technology sharing under the main model continues to hold.

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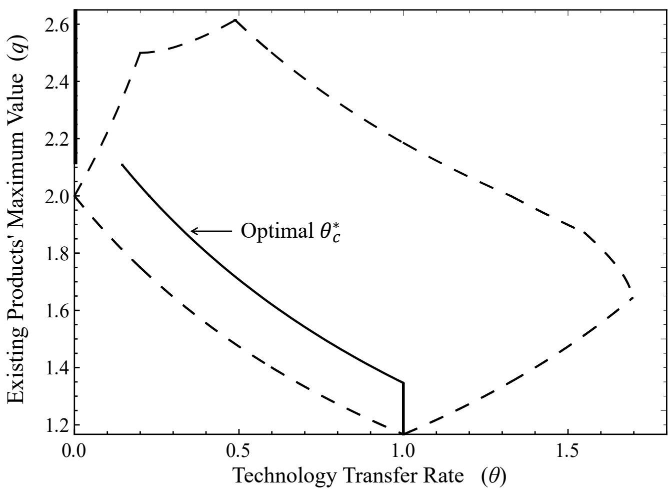
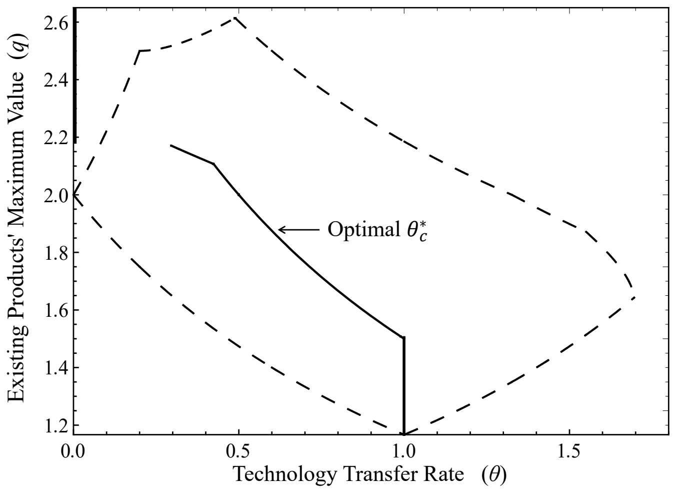
[1] When $\theta_c^* = 0$, it means Firm 2 has no incentive to adopt the technology.



(a) $c = 0.15$



(b) $c = 0.12$

Figure 12: Optimal Choice of θ by Firm 2 ($t = 1$)

In summary, the analysis of the case with endogenous valuation for the new product demonstrates the robustness of our main findings. Specifically, the cannibalization consideration in the rival's multiproduct pricing imposes an externality on the focal firm, thereby giving rise to the focal firm's incentive to share technology.

F The Case with Endogenous Location of the New Product

In the main model, we use the spokes model as the framework to examine the competition among the products. Based on the standard three-spoke model, we assume that each product is located at the origin of a spoke, and consumers in Submarket $\bar{ij}$ $(i, j = 1, 2, 3; i \neq j)$ consider Products i or j as their possible product choices. This standard three-spoke model is equivalent to the triangle model illustrated in Figure 13. Each submarket (e.g., Submarket $\bar{12}$) is represented as a Hotelling line in this figure, functioning similarly to the Hotelling model. In this appendix, we consider that the location of the new product is endogenous. Specifically, Firm 2 can position the new product at any position on the left side or the bottom side of the triangle. For tractability, we assume $q_1 = q_2 = q_3 = q$. Everything else remains the same as in the main model.

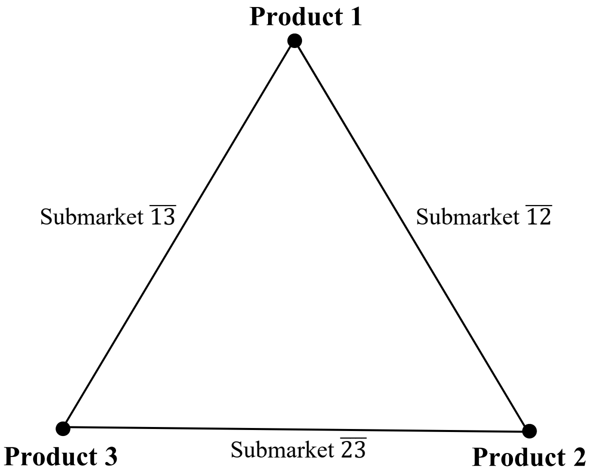


Figure 13: Equivalence between a Three-Spoke Model and a Triangle Model

We consider a revised game as follows: In Stage 1, Firm 1 decides whether to share its technology with Firm 2. If shared, Firm 2 decides whether to adopt the technology to introduce the new product into the market and, if adopted, determines the location of the new product. In Stage 2, the two firms engage in price competition by simultaneously setting their respective product prices. In Stage 3, consumers make their purchase decisions.

We consider that if Product 3 is to be introduced, it is located with a distance of x away from Product 1 along the left side of the triangle. Accordingly, Product 3 is $2 - x$ away from Product 2, as shown in Figure 14. As a result, Products 1 and 3 compete exclusively in a Submarket $\bar{13}$ with the mass of consumers equal to $\frac{1}{3}x$, while Products 2 and 3 compete exclusively in a Submarket

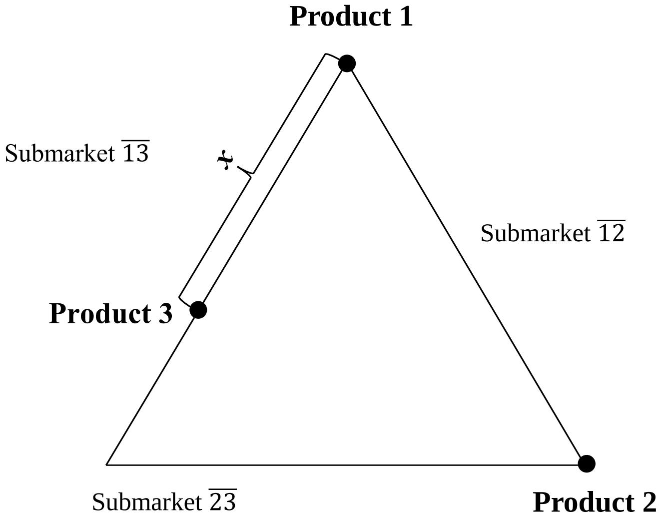


Figure 14: The Model with Endogenous Location for Product 3 ($q_1 = q_2 = q_3 = q$)

$\bar{2}3$ with the mass of consumers equal to $\frac{1}{3}(2-x)$. For ease of exposition, we present cases with $\frac{1}{2} \leq x \leq \frac{3}{2}$.

Following the same approach of backward induction as in the main model, we solve this game. In the absence of the new product, the subgame remains the same as in the main model, and the equilibrium outcome is summarized in Lemma 1. When the new product is introduced, given its location x, we can derive the subgame equilibrium in Stage 2.

As in the main model, if Submarket $\bar{ij}$ is fully covered, we have the marginal consumer m_{ij} who is indifferent to purchasing Products i and j. By letting $u_i = u_j$, we can derive

$$\begin{cases} m_{12} = \frac{t-p_1+p_2}{2t} \\ m_{13} = \frac{x \cdot t-p_1+p_3}{2t} \\ m_{23} = \frac{(2-x)t-p_2+p_3}{2t} \end{cases} . \tag{37}$$

Each product's demand, which is the sum of demands from the two submarkets, can be derived as

$$\begin{cases} d_1 = \frac{1}{3}\min\left\{\frac{q-p_1}{t}, m_{12}\right\} + \frac{1}{3}\min\left\{\frac{q-p_1}{t}, m_{13}\right\} \\ d_2 = \frac{1}{3}\min\left\{\frac{q-p_2}{t}, 1-m_{12}\right\} + \frac{1}{3}\min\left\{\frac{q-p_2}{t}, m_{23}\right\} \\ d_3 = \frac{1}{3}\min\left\{\frac{q-p_3}{t}, x-m_{13}\right\} + \frac{1}{3}\min\left\{\frac{q-p_3}{t}, 2-x-m_{23}\right\} \end{cases} \tag{38}$$

We use the function min because we allow possibilities of both complete and incomplete market coverage in Submarket $\bar{ij}$. If $q - p_i - m_{ij}t > 0$, Submarket $\bar{ij}$ is fully covered, and consumers located between Product i and m_{ij} purchase Product i, and those located between m_{ij} and Product j purchase Product j. Otherwise, the market is not fully covered, and consumers buy Product i (or Product j) only when $u_i \geq 0$ (or $u_j \geq 0$), leading to the demand $\frac{q-p_i}{t}$ (or $\frac{q-p_j}{t}$) instead. The two firms' profits can be formulated as

$$\pi_1 = p_1 d_1 \text{ and } \pi_2 = p_2 d_2 + p_3 d_3. \tag{39}$$

Each firm chooses its price(s) to maximize its profit. Similar to Lemma 2 of the manuscript, Lemma 7 summarizes the equilibrium results.

Lemma 7. *If Firm 2 serves the market with both Products 2 and 3, given the location (x) of the new product, the equilibrium prices of the three products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases}
\left(\frac{1}{6}(7t+tx), \frac{1}{12}(23t-3tx), \frac{1}{12}(21t-tx)\right) & \textit{if } \frac{-6q+17t}{4t} < x \leq \frac{24q-35t}{13t} \\
\left(\frac{1}{4}(2q+t(-1+2x)), q+\frac{1}{12}t(-11+5x), q+\frac{1}{12}t(-13+7x)\right) & \textit{if } \frac{-14q+25t}{8t} < x \leq \min\{\frac{6q+2t}{11t}, \frac{6q+16t}{25t}, \frac{-6q+17t}{4t}\} \\
\left(q-\frac{1}{12}t(1+5x), q+\frac{1}{12}t(-11+5x), q+\frac{1}{12}t(-13+7x)\right) & \textit{if } \max\{\frac{6q+2t}{11t}, \frac{-36q+77t}{35t}\} < x \leq 1 \\
\left(\frac{1}{22}(4q+7t+7tx), \frac{1}{132}(48q+29t+7tx), \frac{1}{132}(48q+7t+29tx)\right) & \textit{if } x \leq \min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\} \\
\left(\frac{1}{7}(10q-7t), \frac{4q}{7}, \frac{1}{42}(24q-7t+7tx)\right) & \textit{if } \frac{192q-203t}{49t} < x \leq \frac{-36q+77t}{35t} \\
\left(\frac{4q}{3}-\frac{1}{9}t(7+5x), \frac{1}{18}(12q+t(17-11x)), \frac{1}{9}(6q+t(7-4x))\right) & \textit{if } \max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\} < x \leq \frac{96q-119t}{49t} \\
\left(\frac{1}{25}(12q+7t-3tx), \frac{1}{25}(22q+17t-18tx), \frac{1}{25}(38q-7t-22tx)\right) & \textit{if } \max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\} < x \leq \frac{134q-126t}{71t} \\
\left(\frac{1}{57}(22q+21t-4tx), \frac{1}{19}(6q+23t-8tx), \frac{1}{114}(50q+84t-35tx)\right) & \textit{if } \max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\} < x \\
\left(\frac{1}{13}(18q-21t+4tx), \frac{4}{13}(2q+2t-tx), \frac{7}{26}(2q+2t-tx)\right) & \textit{if } \max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\} < x \leq \frac{-74q+147t}{28t} \\
\left(\frac{q}{2}, 2q-t-\frac{q}{2}, \frac{1}{6}(5q-tx)\right) & \textit{if } \frac{2q+6t}{7t} < x \leq \frac{-23q+42t}{8t} \\
\left(q+\frac{1}{12}t(13-19x), q+\frac{1}{12}t(-11+5x), q+\frac{1}{12}t(-13+7x)\right) & \textit{if } \max\{\frac{6q+16t}{25t}, 1\} < x \leq \min\{\frac{36q+77t}{119t}, \frac{-12q+67t}{37t}\} \\
\left(\frac{1}{5}(2q-t+2tx), \frac{1}{5}(2q-11t+12tx), \frac{1}{5}(8q+t-7tx)\right) & \textit{if } \frac{36q+77t}{119t} < x \leq \min\{\frac{2(q+6t)}{13t}, \frac{22q+39t}{63t}, \frac{6q+7t}{14t}\} \\
\left(\frac{1}{83}(42q-t+8tx), \frac{1}{83}(86q-89t+48tx), \frac{1}{83}(80q-77t+35tx)\right) & \textit{if } \frac{22q+39t}{63t} < x \leq \min\{\frac{-142q+450t}{197t}, \frac{38q+7t}{56t}\} \\
\left(\frac{1}{15}(14q+t-8tx), \frac{8}{15}(2q-2t+tx), \frac{7}{15}(2q-2t+tx)\right) & \textit{if } \max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\} < x \leq \min\{\frac{13q+2t}{16t}, \frac{-22q+82t}{41t}\} \\
\left(\frac{q}{2}, \frac{3q}{2}-t, \frac{q}{2}+t(-1+x)\right) & \textit{if } \max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\} < x \leq \frac{2q+6t}{7t} \\
\left(\frac{1}{2}(2q+t-2tx), \frac{1}{2}(2q-3t+2tx), \frac{1}{2}(2q-t)\right) & \textit{if } \frac{6q+7t}{14t} < x \leq \min\{\frac{q+t}{2t}, \frac{2q+13t}{14t}\}
\end{cases} \tag{40}$$

the equilibrium demands of the three products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases}
\left(\frac{7+x}{18}, -\frac{5}{72}(-5+x), \frac{19+x}{72}\right) & \textit{if } \frac{-6q+17t}{4t} < x \leq \frac{24q-35t}{13t} \\
\left(\frac{1}{12}\left(-1+\frac{2q}{t}+2x\right), \frac{1}{24}\left(14-\frac{2q}{t}-3x\right), \frac{1}{24}\left(12-\frac{2q}{t}-x\right)\right) & \textit{if } \frac{-14q+25t}{8t} < x \leq \min\{\frac{6q+2t}{11t}, \frac{6q+16t}{25t}, \frac{-6q+17t}{4t}\} \\
\left(\frac{1}{36}(-5+17x), \frac{1}{18}(11-5x), \frac{1}{36}(19-7x)\right) & \textit{if } \max\{\frac{6q+2t}{11t}, \frac{-36q+77t}{35t}\} < x \leq 1 \\
\left(\frac{4q+7t(1+x)}{66t}, \frac{48q+29t+7tx}{264t}, \frac{48q+7t+29tx}{264t}\right) & \textit{if } x \leq \min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\} \\
\left(-\frac{2q}{7t}+\frac{1}{36}(17+7x), \frac{2q}{7t}, \frac{2q}{7t}+\frac{1}{12}(-1+x)\right) & \textit{if } \frac{192q-203t}{49t} < x \leq \frac{-36q+77t}{35t} \\
\left(\frac{1}{108}\left(77-\frac{24q}{t}+19x\right), \frac{1}{54}\left(10+\frac{6q}{t}-7x\right), \frac{1}{108}\left(11+\frac{12q}{t}-5x\right)\right) & \textit{if } \max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\} < x \leq \frac{96q-119t}{49t} \\
\left(\frac{1}{50}\left(7+\frac{12q}{t}-3x\right), \frac{1}{150}\left(41+\frac{6q}{t}-14x\right), \frac{1}{150}\left(88-\frac{42q}{t}+23x\right)\right) & \textit{if } \max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\} < x \leq \frac{134q-126t}{71t} \\
\left(\frac{1}{114}\left(21+\frac{22q}{t}-4x\right), \frac{22q+192t-61tx}{684t}, \frac{1}{12}\left(2+\frac{2q}{t}-x\right)\right) & \textit{if } \max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\} < x \\
\left(-\frac{2(5q+t(-21+4x))}{39t}, \frac{6(q+t)-3tx}{52t}, \frac{1}{12}\left(2+\frac{2q}{t}-x\right)\right) & \textit{if } \max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\} < x \leq \frac{-74q+147t}{28t} \\
\left(\frac{q}{3t}, \frac{1}{36}\left(30-\frac{10q}{t}-7x\right), \frac{(-5q+tx)(-2(q+t)+tx)}{72t}\right) & \textit{if } \frac{2q+6t}{7t} < x \leq \frac{-23q+42t}{8t} \\
\left(\frac{1}{36}(-19+31x), \frac{1}{36}(29-17x), \frac{1}{18}(13-7x)\right) & \textit{if } \max\{\frac{6q+16t}{25t}, 1\} < x \leq \min\{\frac{36q+77t}{119t}, \frac{-12q+67t}{37t}\} \\
\left(\frac{1}{10}\left(-1+\frac{2q}{t}+2x\right), \frac{1}{30}\left(37+\frac{6q}{t}-34x\right), -\frac{2(3q+t-7tx)}{15t}\right) & \textit{if } \frac{36q+77t}{119t} < x \leq \min\{\frac{2(q+6t)}{13t}, \frac{22q+39t}{63t}, \frac{6q+7t}{14t}\} \\
\left(\frac{42q-t+8tx}{166t}, -\frac{50q-349t+136tx}{498t}, \frac{2(3q+7t(11-5x))}{249t}\right) & \textit{if } \frac{22q+39t}{63t} < x \leq \min\{\frac{-142q+450t}{197t}, \frac{38q+7t}{56t}\} \\
\left(\frac{2(q+t(-1+8x))}{45t}, -\frac{2(q+8t(-2+x))}{45t}, \frac{2(q-7t(-2+x))}{45t}\right) & \textit{if } \max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\} < x \leq \min\{\frac{13q+2t}{16t}, \frac{-22q+82t}{41t}\} \\
\left(\frac{q}{3t}, \frac{2}{3}-\frac{q}{3t}, \frac{q+2t-2tx}{3t}\right) & \textit{if } \max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\} < x \leq \frac{2q+6t}{7t} \\
\left(\frac{1}{3}(-1+2x), 1-\frac{2x}{3}, \frac{1}{3}\right) & \textit{if } \frac{6q+7t}{14t} < x \leq \min\{\frac{q+t}{2t}, \frac{2q+13t}{14t}\}
\end{cases} \tag{41}$$

and the equilibrium profits for the two firms (π_1^*, π_2^*) *are*

$$\begin{cases}
\left(\frac{1}{108}\mathrm{t}(7+\mathrm{x})^2, \frac{1}{432}\mathrm{t}(487+\mathrm{x}(-94+7\mathrm{x}))\right) & \textit{if } \frac{-6q+17t}{4t} < x \leq \frac{24q-35t}{13t} \\
\left(\frac{(-2q+t-2tx)^2}{48t}, \frac{1}{144}\left(-\frac{24q^2}{t}-36q(-5+x)+t(-155+(100-11x)x)\right)\right) & \textit{if } \frac{-14q+25t}{8t} < x \leq \min\left[\frac{6q+2t}{11t}, \frac{6q+16t}{25t}, \frac{-6q+17t}{4t}\right] \\
\left(-\frac{1}{432}(-5+17x)(-12q+t+5tx), \frac{1}{144}(q(164-68x)+t(-163+(148-33x)x))\right) & \textit{if } \max\{\frac{6q+2t}{11t}, \frac{-36q+77t}{35t}\} < x \leq 1 \\
\left(\frac{(4q+7t(1+x))^2}{1452t}, \frac{2304q^2+1728qt(1+x)+t^2(445+x(406+445x))}{17424t}\right) & \textit{if } x \leq \min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\} \\
\left(-\frac{(10q-7t)(72q-7t(17+7x))}{1764t}, \frac{16q^2}{49t}+\frac{2}{21}q(-1+x)+\frac{1}{72}t(-1+x)^2\right) & \textit{if } \frac{192q-203t}{49t} < x \leq \frac{-36q+77t}{35t} \\
\left(\frac{(12q-t(7+5x))(t(77+19x)-24q)}{972t}, \frac{\left(\frac{144q^2}{t}+q(372-228x)+t(247+x(97x-308))\right)}{972}\right) & \textit{if } \max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\} < x \leq \frac{96q-119t}{49t} \\
\left(\frac{(12q+t(7-3x))^2}{1250t}, \frac{-1464q^2+2qt(2321+691x)+t^2(81-x(3073+254x))}{3750t}\right) & \textit{if } \max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\} < x \leq \frac{134q-126t}{71t} \\
\left(\frac{(22q+t(21-4x))^2}{6498t}, \frac{2164q^2+4qt(2102-841x)+t^2(12024+x(-8804+1641x))}{25992t}\right) & \textit{if } \max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\} < x \\
\left(-\frac{2(5q+t(-21+4x))(18q+t(-21+4x))}{507t}, \frac{163(-2(q+t)+tx)^2}{4056t}\right) & \textit{if } \max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\} < x \leq \frac{-74q+147t}{28t} \\
\left(\frac{q^2}{6t}, \frac{-20q^2+4qt(30-7x)+t^2(-60+x(12+x))}{72t}\right) & \textit{if } \frac{2q+6t}{7t} < x \leq \frac{-23q+42t}{8t} \\
\left(\frac{1}{432}(12q+t(13-19x))(-19+31x), \frac{1}{144}(4q(55-31x)+t(-219+(232-61x)x))\right) & \textit{if } \max\{\frac{6q+16t}{25t}, 1\} < x \leq \min\{\frac{36q+77t}{119t}, \frac{-12q+67t}{37t}\} \\
\left(\frac{(-2q+t-2tx)^2}{50t}, \frac{-84q^2+12qt(-3+26x)+t^2\left(-411+874x-604x^2\right)}{150t}\right) & \textit{if } \frac{36q+77t}{119t} < x \leq \min\{\frac{2(q+6t)}{13t}, \frac{22q+39t}{63t}, \frac{6q+7t}{14t}\} \\
\left(\frac{(-42q+t-8tx)^2}{13778t}, \frac{-3340q^2+4qt(14545-6219x)+t^2(-54777+4(12604-2857x)x)}{41334t}\right) & \textit{if } \frac{22q+39t}{63t} < x \leq \min\{\frac{-142q+450t}{197t}, \frac{38q+7t}{56t}\} \\
\left(\frac{2(14q+t-8tx)(q+t(-1+8x))}{675t}, -\frac{2(2q+t(-2+x))(q+113t(-2+x))}{675t}\right) & \textit{if } \max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\} < x \leq \min\{\frac{13q+2t}{16t}, \frac{-22q+82t}{41t}\} \\
\left(\frac{q^2}{6t}, \frac{1}{3}\left(4q-\frac{q^2}{t}-2t(2+(-2+x)x)\right)\right) & \textit{if } \max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\} < x \leq \frac{2q+6t}{7t} \\
\left(\frac{1}{6}(-1+2x)(2q+t-2tx), \frac{1}{3}(-2q(-2+x)+t(-5-2(-3+x)x))\right) & \textit{if } \frac{6q+7t}{14t} < x \leq \min\{\frac{q+t}{2t}, \frac{2q+13t}{14t}\}
\end{cases} \tag{42}$$

Proof. We distinguish sixteen cases based on the value of x, as illustrated in Figure 15. (We can

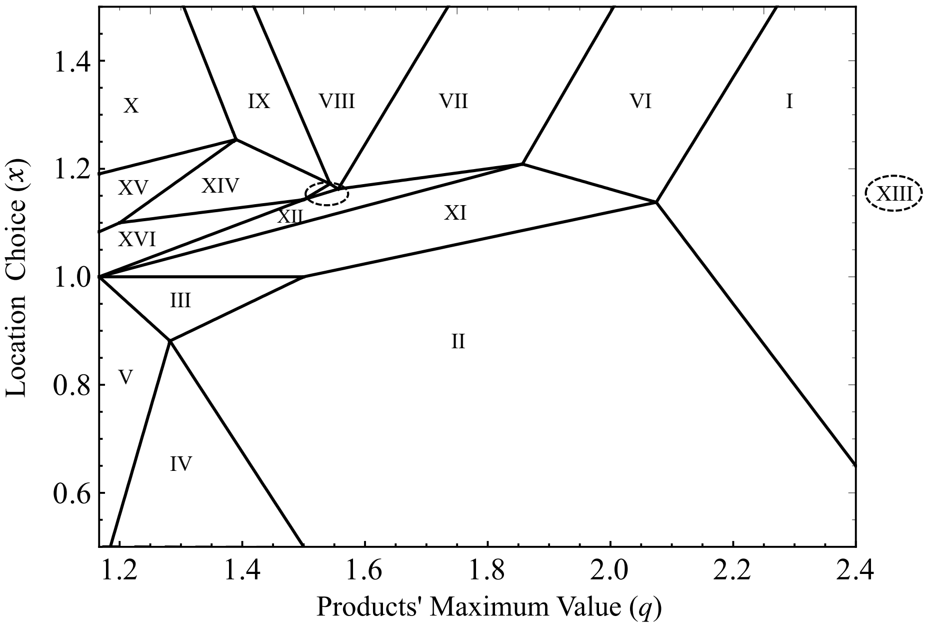


Figure 15: Equilibrium Outcome Given the Location of Product 3 ($t = 1$)

verify that the second-order conditions for profit maximization are satisfied.)

(I) $\frac{-6q+17t}{4t} < x \leq \frac{24q-35t}{13t}$: When $0 \leq m_{12}, m_{13}, m_{23} \leq 1$ and the marginal consumers derive positive utilities, the demand functions in Equation (38) become

$$\begin{cases} d_1 = \frac{1}{3}m_{12} + \frac{1}{3}m_{13} \\ d_2 = \frac{1}{3}\left(1 - m_{12}\right) + \frac{1}{3}m_{23} \\ d_3 = \frac{1}{3}\left(x - m_{13}\right) + \frac{1}{3}\left(2 - x - m_{23}\right) \end{cases} \tag{43}$$

We can derive the first-order conditions of the firms' profit functions as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} = 0 \end{cases} \tag{44}$$

Solving this system of equations, we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq \frac{24q-35t}{13t}$ ensures marginal consumer m_{13} derives positive utility, and Condition $\frac{-6q+17t}{4t} < x$ ensures marginal consumer m_{23} derives positive utility. We can verify that neither firm has profitable deviation.

(II) $\frac{-14q+25t}{8t} < x \leq \min\{\frac{6q+2t}{11t}, \frac{6q+16t}{25t}, \frac{-6q+17t}{4t}\}$: In this case, marginal consumer m_{23} derives zero utility because $x \leq \frac{-6q+17t}{4t}$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - (2-x)\,t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda\,[2q - (2-x)\,t - p_2 - p_3]$. By solving the first-order condition of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} - \lambda = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq \frac{6q+16t}{25t}$ ensures marginal consumer m_{13} derives positive utility. Condition $x \leq \frac{6q+2t}{11t}$ ensures marginal consumer m_{12} derives positive utility. Condition $\frac{-14q+25t}{8t} < x$ ensures Firm 2 has no incentive to increase its product prices such that its exclusive Submarket $\bar{23}$ is not fully covered.

(III) $\max\{\frac{6q+2t}{11t}, \frac{-36q+77t}{35t}\} < x \leq 1$: Because $\frac{6q+2t}{11t} < x$, both marginal consumers m_{12} and m_{23} derive zero utility. When Firm 1 prices it product such that marginal consumer m_{12} derives zero utility, $2q - t - p_1 - p_2 = 0$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - (2-x)\,t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda_1) = \pi_1 + \lambda_1(2q - t - p_1 - p_2)$ and $L_2(p_2, p_3, \lambda_2) = \pi_2 + \lambda_2\,[2q - (2-x)\,t - p_2 - p_3]$. By solving the first-order condition of $L_1(p_1, \lambda_1)$ and $L_2(p_2, p_3, \lambda_2)$, i.e., solving the following system of equations

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} - \lambda_2 = 0 \\ 2q - t - p_1 - p_2 = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq 1$ ensures marginal consumer m_{13} derives positive utility. Condition $\frac{-36q+77t}{35t} < x$ ensures Firm 2 has no incentive to increase its product prices such that its exclusive Submarket $\bar{23}$ is not fully covered.

(IV) $x \leq \min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\}$: Because $x \leq \frac{-14q+25t}{8t}$, Firm 2 increase its product prices such that its exclusive Submarket $\bar{23}$ is not fully covered. While Product 1's demand function stays as in Equation (43), Products 2's and 3's demands become

$$\begin{cases} d_2 = \frac{1}{3}\left(1 - m_{12}\right) + \frac{1}{3}\frac{q-p_2}{t} \\ d_3 = \frac{1}{3}\left(x - m_{13}\right) + \frac{1}{3}\frac{q-p_3}{t} \end{cases} \tag{45}$$

Substituting d_2 and d_3 in Equation (45) into $\pi_2 = p_2 d_2 + p_3 d_3$, and solving the first-order conditions of π_1 and π_2,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-6p_2+2q+t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1-6p_3+2q+tx}{6t} = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq \frac{192q-203t}{49t}$ ensures marginal consumer m_{12} derives positive utility.

(V) $\frac{192q-203t}{49t} < x \leq \frac{-36q+77t}{35t}$: In this case, Product 1's demand function stays as in Equation (43), and Products 2's and 3's demands are as in Equation (45). Because $x \leq \frac{-36q+77t}{35t}$, Firm 2 prices its product prices such that its exclusive Submarket $\bar{23}$ is not fully covered. Moreover, because $\frac{192q-203t}{49t} < x$, marginal consumer m_{12} derives zero utility. Thus, $2q - t - p_1 - p_2 = 0$. Using the Lagrange-multiplier method, we have $L(p_1, \lambda) = \pi_1 + \lambda(2q - t - p_1 - p_2)$. By solving the first-order condition of $L(p_1, \lambda)$ and π_2,

$$\begin{cases} \frac{\partial L}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} - \lambda = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-6p_2+2q+t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1-6p_3+2q+tx}{6t} = 0 \\ 2q - t - p_1 - p_2 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate.

(VI) $\max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\} < x \leq \frac{96q-119t}{49t}$: In this case, the product demands stay as in Equation (43). Because $\frac{24q-35t}{13t} < x$, marginal consumer m_{13} derives zero utility. Thus, $2q - tx - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_1, \lambda) = \pi_1 + \lambda(2q - tx - p_1 - p_3)$.

By solving the first-order condition of $L(p_1, \lambda)$ and π_2,

$$\begin{cases} \frac{\partial L}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} - \lambda = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} = 0 \\ 2q - tx - p_1 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{-12q+67t}{37t} < x$ ensures marginal consumer m_{23} derives positive utility. Condition $x \leq \frac{96q-119t}{49t}$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price.

(VII) $\max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\} < x \leq \frac{134q-126t}{71t}$: Because $\frac{96q-119t}{49t} < x$, Firm 1 would like to increase its product price as if it competes with only Product 2 but not with Product 3, and Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{1}3$ while Product 2 competes against Product 1 in Submarket $\bar{1}2$. Firm 2's demand function is as in Equation (43), and Firm 1's demand function becomes

$$d_1 = \frac{1}{3}m_{12} + \frac{1}{3}\frac{q-p_1}{t} \tag{46}$$

When Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{1}3$, $2q - tx - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(2q - tx - p_1 - p_3)$. Substituting $d_1$in Equation (46) into $\pi_1 = p_1 d_1$, and solving the first-order condition of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} - \lambda = 0 \\ 2q - tx - p_1 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{2(q+6t)}{13t} < x$ ensures marginal consumer m_{23} derives positive utility. Condition $x \leq \frac{134q-126t}{71t}$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 3's price.

(VIII) $\max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\} < x$: Because $\frac{134q-126t}{71t} < x$, Firm 1 and Firm 2 do not compete in Submarket $\bar{1}3$ but in Submarket $\bar{1}2$. Product 1's demand function is as in Equation (46), Product 2's demand function is as in Equation (43), and Product 3's demand function becomes

$$d_3 = \frac{1}{3}\frac{q - p_3}{t} + \frac{1}{3}\left(2 - x - m_{23}\right) \tag{47}$$

Substituting d_1 in Equation (46) into $\pi_1 = p_1 d_1$, substituting d_2 in Equation (43) and d_3 in Equation (47) into $\pi_2 = p_2 d_2 + p_3 d_3$, and solving the following system of equations

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{2p_2-6p_3+2q+(2-x)t}{6t} = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{-74q+147t}{28t} < x$ ensures marginal consumer m_{12} derives positive utility. Condition $\frac{-142q+450t}{197t} < x$ ensures marginal consumer m_{23} derives positive utility.

(IX) $\max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\} < x \leq \frac{-74q+147t}{28t}$: In this case, the firms' profit functions are the same as in (VIII). Because $x \leq \frac{-74q+147t}{28t}$, marginal consumers m_{12} derive zero utility. Thus, $2q-t-p_1-p_2 = 0$. Using the Lagrange-multiplier method, we have $L(p_1, \lambda) = \pi_1 + \lambda(2q-t-p_1-p_2)$. By solving the first-order conditions of $L(p_1, \lambda)$ and π_2,

$$\begin{cases} \frac{\partial L}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} - \lambda = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{2p_2-6p_3+2q+(2-x)t}{6t} = 0 \\ 2q - t - p_1 - p_2 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{-23q+42t}{8t} < x$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price. Condition $\frac{-22q+82t}{41t} < x$ ensures marginal consumer m_{23} derives positive utility.

(X) $\frac{2q+6t}{7t} < x \leq \frac{-23q+42t}{8t}$: Because $x \leq \frac{-23q+42t}{8t}$, Firm 1 would like to increase its product price as if it does not compete with Firm 2, either in Submarket $\bar{1}2$ or in Submarket $\bar{1}3$, while Firm 2 chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{1}2$. As a result, while Firm 2's product demand functions are as in (IX), Firm 1's product demand becomes

$$d_1 = \frac{1}{3}\frac{q-p_1}{t} + \frac{1}{3}\frac{q-p_1}{t} \tag{48}$$

When Firm 2 chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{1}2$, $2q - t - p_1 - p_2 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(2q - t - p_1 - p_2)$. Substituting d_1 in Equation (48) into $\pi_1 = p_1 d_1$, and solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{2}{3}\frac{q-2p_1}{t} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{2p_2-6p_3+2q+(2-x)t}{6t} = 0 \\ 2q - t - p_1 - p_2 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{2q+6t}{7t} < x$ ensures marginal consumer m_{23} derives positive utility.

(XI) $\max\{\frac{6q+16t}{25t}, 1\} < x \leq \min\{\frac{36q+77t}{119t}, \frac{-12q+67t}{37t}\}$: In this case, the firms' demand and profit functions are as in (I). Because $\frac{6q+16t}{25t} < x$, marginal consumer m_{13} derives zero utility, and because $x \leq \frac{-12q+67t}{37t}$, marginal consumer m_{23} derives zero utility. When Firm 1 prices its product such that marginal consumer m_{13} derives zero utility, $2q - tx - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda) = \pi_1 + \lambda_1(2q - tx - p_1 - p_3)$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - (2-x)\,t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_2(p_2, p_3, \lambda) = \pi_2 + \lambda_2\,[2q - (2-x)\,t - p_2 - p_3]$. By solving the first-order condition of $L_1(p_1, \lambda)$ and $L_2(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+(1+x)t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} - \lambda_2 = 0 \\ 2q - tx - p_1 - p_3 = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $1 < x$ ensures marginal consumer m_{12} derives positive utility. Condition $x \leq \frac{36q+77t}{119t}$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price.

(XII) $\frac{36q+77t}{119t} < x \leq \min\{\frac{2(q+6t)}{13t}, \frac{22q+39t}{63t}, \frac{6q+7t}{14t}\}$: Because $\frac{36q+77t}{119t} < x$, Firm 1 would like to increase its product price as if it competes with only Product 2 but not with Product 3, and Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $1\bar{3}$ while Product 2 competes against Product 1 in Submarket $1\bar{2}$. Firm 2's demand function is as in Equation (43), and Firm 1's demand function is as in Equation (46). Moreover, because $x \leq \frac{2(q+6t)}{13t}$, marginal consumer m_{23} derives zero utility. When Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $1\bar{3}$, $2q - tx - p_1 - p_3 = 0$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - (2-x)\,t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda_1, \lambda_2) = \pi_2 + \lambda_1(2q - tx - p_1 - p_3) + \lambda_2\left[2q - (2-x)\,t - p_2 - p_3\right]$. Solving the first-order condition of π_1 and $L(p_2, p_3, \lambda_1, \lambda_2)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+2t}{6t} - \lambda_1 - \lambda_2 = 0 \\ 2q - tx - p_1 - p_3 = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq \frac{6q+7t}{14t}$ ensures marginal consumer m_{12} derives positive utility. Condition $x \leq \frac{22q+39t}{63t}$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 3's price.

(XIII) $\frac{22q+39t}{63t} < x \leq \min\{\frac{-142q+450t}{197t}, \frac{38q+7t}{56t}\}$: Because $x \leq \frac{22q+39t}{63t}$, Firm 1 and Firm 2 do not compete in Submarket $1\bar{3}$ but in Submarket $1\bar{2}$. Product 1's demand function is as in Equation (46), Product 2's demand function isas in Equation (43), and Product 3's demand function is as Equation (47). Moreover, because $x \leq \frac{-142q+450t}{197t}$, marginal consumer m_{23} derives zero utility. Substituting d_2 in Equation (43) and d_3 in Equation (47) into $\pi_2 = p_2 d_2 + p_3 d_3$, and using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda\left[2q - (2-x)\,t - p_2 - p_3\right]$. Substituting

d_1 in Equation (46) into $\pi_1 = p_1 d_1$, and solving the following system of equations

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{2p_2-6p_3+2q+(2-x)t}{6t} - \lambda = 0 \\ 2q-(2-x)\,t-p_2-p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $x \leq \frac{38q+7t}{56t}$ ensures marginal consumer m_{12} derives positive utility.

(XIV) $\max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\} < x \leq \min\{\frac{13q+2t}{16t}, \frac{-22q+82t}{41t}\}$: In this case, Firm 1 and Firm 2 do not compete in Submarket $\bar{1}3$, and the firms' demand function are as in (XIII). Because $\frac{38q+7t}{56t} < x$, marginal consumer m_{12} derives zero utility. When Firm 1 prices its product such that marginal consumer m_{12} derives zero utility, $2q-t-p_1-p_2=0$. Using the Lagrange-multiplier method, we have $L_1(p_1,\lambda) = \pi_1 + \lambda_1(2q-t-p_1-p_2)$. Moreover, because $x \leq \frac{-22q+82t}{41t}$, marginal consumer m_{23} derives zero utility. Using the Lagrange-multiplier method, we have $L_2(p_2,p_3,\lambda) = \pi_2 + \lambda_2\left[2q-(2-x)\,t-p_2-p_3\right]$. Solving the first-order condition of $L_1(p_1,\lambda)$ and $L_2(p_2,p_3,\lambda)$,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-6p_1+p_2+2q+t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+(3-x)t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{2p_2-6p_3+2q+(2-x)t}{6t} - \lambda_2 = 0 \\ 2q-t-p_1-p_2 = 0 \\ 2q-(2-x)\,t-p_2-p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{2q+13t}{14t} < x$ ensures Firm 1 and Firm 2 do not compete in Submarket $\bar{1}3$. Condition $x \leq \frac{13q+2t}{16t}$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price.

(XV) $\max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\} < x \leq \frac{2q+6t}{7t}$: Because $\frac{13q+2t}{16t} < x$, Firm 1 would like to increase its product price as if it does not compete with Firm 2, either in Submarket $\bar{1}2$ or in Submarket $\bar{1}3$, while Firm 2 chooses to use Product 2 to just serve the residual demand from Product 1 in Submarket $\bar{1}2$. As a result, Firm 2's product demand functions are as in (XIV), and Firm 1's product demand is as in Equation (48). When Firm 2 chooses to use Product 2 to just serve the residual demand from

Product 1 in Submarket $\bar{12}$, $2q - t - p_1 - p_2 = 0$. Moreover, because $x \leq \frac{2q+6t}{7t}$, marginal consumer m_{23} derives zero utility, and thus $2q - (2-x)\,t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda_1, \lambda_2) = \pi_2 + \lambda_1(2q - t - p_1 - p_2) + \lambda_2\left[2q - (2-x)\,t - p_2 - p_3\right]$. Solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda_1, \lambda_2)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{2}{3}\frac{q-2p_1}{t} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1 - 4p_2 + 2p_3 + (3-x)t}{6t} - \lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial p_3} = \frac{2p_2 - 6p_3 + 2q + (2-x)t}{6t} - \lambda_2 = 0 \\ 2q - t - p_1 - p_2 = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{q+t}{2t} < x$ ensures Firm 1 and Firm 2 do not compete in Submarket $\bar{13}$.

(XVI) $\frac{6q+7t}{14t} < x \leq \min\{\frac{q+t}{2t}, \frac{2q+13t}{14t}\}$: In this case, the firms' demand and profit functions are as in (I). We have marginal consumers m_{12}, m_{13}, and m_{23} all derives zero utility. Therefore, solving the following system of equations

$$\begin{cases} 2q - t - p_1 - p_2 = 0 \\ 2q - tx - p_1 - p_3 = 0 \\ 2q - (2-x)\,t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate.

Substituting p_i^* into Equations (38) and (39), we can derive the equilibrium demands and profits as specified in the lemma. □

Because of the asymmetry in the product locations, the equilibrium result in this case is much more complicated than that in the main model, even though we assume that $q_1 = q_2 = q_3 = q$. Fortunately, we are able to derive the closed-form result for this subgame, which enables us to analyze the optimal location choice. By examining the optimal location choice in each region and considering the optimal choice across different regions, we can derive Firm 2's optimal choice for the location of Product 3. Proposition 11 summarizes the result.

Proposition 11. *Given Firm 1 shares its technology with Firm 2, if Firm 2 adopts the technology and introduces the new product, Firm 2's optimal choice of location for the new product* (x^*) *is*

$$x^* = \begin{cases} \frac{-18q+50t}{11t} & \text{if } \frac{13}{6}t < q \\ \frac{-6q+17t}{4t} & \text{if } \frac{361}{174}t < q \leq \frac{13}{6}t \\ \frac{6q+16t}{25t} & \text{if } \frac{3}{2}t < q \leq \frac{361}{174}t \\ \frac{6q+2t}{11t} & \text{if } \frac{17}{13}t < q \leq \frac{3}{2}t \\ \frac{-34q+74t}{33t} & \text{if } \frac{41}{34}t < q \leq \frac{17}{13}t \\ 1 & \text{if } q \leq \frac{41}{34}t \end{cases} \tag{49}$$

Proof. We first distinguish sixteen cases and examine the optimal location choice for each case in Lemma 7. We then consider the optimal choice across different cases and conclude. (We can verify that the second-order conditions for the profit maximization are satisfied.)

(I) $\frac{-6q+17t}{4t} < x \leq \frac{24q-35t}{13t}$: We have $\frac{\partial \pi_2}{\partial x} = \frac{t}{216}\left(-47+7x\right) < 0$, and thus the optimal location is at $\frac{-6q+17t}{4t}$ within this case.

(II) $\frac{-14q+25t}{8t} < x \leq \min\{\frac{6q+2t}{11t}, \frac{6q+16t}{25t}, \frac{-6q+17t}{4t}\}$: By the first-order condition,

$$\frac{\partial \pi_2}{\partial x} = \frac{1}{72}\left[-18q + \left(50 - 11x\right)t\right] = 0$$

we have the first-order solution at $x_{foc} = \frac{-18q+50t}{11t}$. We note that Curve $\frac{-18q+50t}{11t}$ crosses Curve $\frac{-6q+17t}{4t}$ at $\frac{13}{6}t$ from the above, $\frac{-6q+17t}{4t}$ and $\frac{6q+16t}{25t}$ intersect at $\frac{361}{174}t$, and $\frac{6q+16t}{25t}$ and $\frac{6q+2t}{11t}$ at $\frac{3}{2}t$. Because of the concavity of the objective function, the optimal location is as follows:

$$x^* = \begin{cases} \frac{-18q+50t}{11t} & \text{if } \frac{13}{6}t < q \\ \frac{-6q+17t}{4t} & \text{if } \frac{361}{174}t < q \leq \frac{13}{6}t \\ \frac{6q+16t}{25t} & \text{if } \frac{3}{2}t < q \leq \frac{361}{174}t \\ \frac{6q+2t}{11t} & \text{if } \frac{259}{202}t < q \leq \frac{3}{2}t \end{cases} \tag{50}$$

where $\frac{259}{202}t$ is the lowest possible q within this region.

(III) $\max\{\frac{6q+2t}{11t}, \frac{-36q+77t}{35t}\} < x \leq 1$: By the first-order condition,

$$\frac{\partial \pi_2}{\partial x} = \frac{1}{72}\left[-34q + \left(74 - 33x\right)t\right] = 0$$

we have the first-order solution at $x_{foc} = \frac{-34q+74t}{33t}$. We note that Curve $x_{foc} = \frac{-34q+74t}{33t}$ crosses 1 at $\frac{41t}{34}$ from the above, and intersects $\frac{6q+2t}{11t}$ at $\frac{17t}{13}$. Because of the concavity of the objective function, with this region, the optimal location is at follows:

$$x^* = \begin{cases} \frac{6q+2t}{11t} & \text{if } \frac{17}{13}t < q \leq \frac{3}{2}t \\ \frac{-34q+74t}{33t} & \text{if } \frac{41}{34}t < q \leq \frac{17}{13}t \\ 1 & \text{if } q \leq \frac{41}{34}t \end{cases} \tag{51}$$

where $\frac{3}{2}t$ is the highest possible q within this region.

(IV) $x \leq \min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\}$: We have $\frac{\partial \pi_2}{\partial x} = \frac{1}{8712}\left[864q + (203+445x)\,t\right] > 0$, and thus the optimal location is at $\min\{\frac{192q-203t}{49t}, \frac{-14q+25t}{8t}\}$.

(V) $\frac{192q-203t}{49t} < x \leq \frac{-36q+77t}{35t}$: We have

$$\frac{\partial \pi_2}{\partial x} = \frac{2q}{21} + \frac{-1+x}{36}t > \frac{2}{21} \times \frac{7}{6}t + \frac{-1+\frac{1}{2}}{36}t = \frac{56}{504}t + \frac{-7}{504}t > 0$$

Therefore, the optimal location is at $\frac{-36q+77t}{35t}$.

(VI) $\max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\} < x \leq \frac{96q-119t}{49t}$: We have $\frac{\partial \pi_2}{\partial x} = \frac{1}{486}\left[-114q + t\,(-154+97x)\right] < 0$. Therefore, the optimal location is at $\max\{\frac{-12q+67t}{37t}, \frac{24q-35t}{13t}\}$.

(VII) $\max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\} < x \leq \frac{134q-126t}{71t}$: We have

$$\begin{aligned} \frac{\partial \pi_2}{\partial x} &= \frac{1}{3750}\left[1382q - t\,(3073+508x)\right] \\ &< \frac{1}{3750}\left[1382 \times \frac{12}{5}t - t\left(3073 + 508 \times \frac{1}{2}\right)\right] = \frac{t}{3750}\left[\frac{16584}{5} - \frac{16635}{5}\right] < 0 \end{aligned}$$

Therefore, the optimal location is at $\max\{\frac{96q-119t}{49t}, \frac{2(q+6t)}{13t}\}$.

(VIII) $\max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\} < x$: We have

$$\frac{\partial \pi_2}{\partial x} = \frac{1}{12996}\left[-1682q + t\,(-4402+1641x)\right] < 0$$

Therefore, the optimal location is at $\max\{\frac{-74q+147t}{28t}, \frac{-142q+450t}{197t}, \frac{134q-126t}{71t}\}$.

(IX) $\max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\} < x \leq \frac{-74q+147t}{28t}$: We have $\frac{\partial \pi_2}{\partial x} = \frac{163}{2028}\left[-2q + t\,(-2+x)\right] < 0$. Therefore, the optimal location is at $\max\{\frac{-23q+42t}{8t}, \frac{-22q+82t}{41t}\}$.

(X) $\frac{2q+6t}{7t} < x \leq \frac{-23q+42t}{8t}$: We have $\frac{\partial \pi_2}{\partial x} = \frac{1}{36}\left[-14q + t\,(6+x)\right] < 0$. Therefore, the optimal location is at $\frac{2q+6t}{7t}$.

(XI) $\max\{\frac{6q+16t}{25t}, 1\} < x \leq \min\{\frac{36q+77t}{119t}, \frac{-12q+67t}{37t}\}$: Because $1 < x$, we have

$$\begin{aligned} \frac{\partial \pi_2}{\partial x} &= \tfrac{1}{72}[-62q + t(116 - 61x)] \\ &< \tfrac{1}{72}[-62q + t(116 - 61)] = \tfrac{1}{72}(-62q + 55t) < 0 \end{aligned}$$

Therefore, the optimal location is at $\max\{\frac{6q+16t}{25t}, 1\}$.

(XII) $\frac{36q+77t}{119t} < x \leq \min\{\frac{2(q+6t)}{13t}, \frac{22q+39t}{63t}, \frac{6q+7t}{14t}\}$: By the first-order condition,

$$\frac{\partial \pi_2}{\partial x} = \tfrac{1}{75}\left[156q + t\left(437 - 604x\right)\right] = 0$$

we have the first-order solution at $x_{foc} = \frac{156q+437t}{604t}$. Because of the concavity of the objective function, the optimal location is at $\min\{\max\{\frac{156q+437t}{604t}, \frac{36q+77t}{119t}\}, \frac{6q+7t}{14t}\}$.

(XIII) $\frac{22q+39t}{63t} < x \leq \min\{\frac{-142q+450t}{197t}, \frac{38q+7t}{56t}\}$: Because $1 < x$ in this case, we have

$$\begin{aligned} \frac{\partial \pi_2}{\partial x} &= \tfrac{-2}{20667}\left[6219q + 2t\left(-6302 + 2857x\right)\right] \\ &< \tfrac{-2}{20667}\left[6219q + 2t\left(-6302 + 2857\right)\right] = \tfrac{-2}{20667}\left[6219q - 6890t\right] \\ &< \tfrac{-2}{20667}\left[6219 \times \tfrac{7}{6}t - 6890t\right] = \tfrac{-2}{20667} \times \tfrac{2193}{6}t < 0 \end{aligned}$$

Therefore, the optimal location is at $\frac{22q+39t}{63t}$.

(XIV) $\max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\} < x \leq \min\{\frac{13q+2t}{16t}, \frac{-22q+82t}{41t}\}$: Because $1 < x$ in this case, we have

$$\begin{aligned} \frac{\partial \pi_2}{\partial x} &= \tfrac{-2}{675}\left[227q + 226t\left(-2 + x\right)\right] \\ &< \tfrac{-2}{675}\left[227q + 226t\left(-2 + 1\right)\right] = \tfrac{-2}{675}\left[227q - 226t\right] < 0 \end{aligned}$$

Therefore, the optimal location is at $\max\{\frac{2q+13t}{14t}, \frac{38q+7t}{56t}\}$.

(XV) $\max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\} < x \leq \frac{2q+6t}{7t}$: Because $1 < x$ in this case, $\frac{\partial \pi_2}{\partial x} = -\frac{4}{3}t\left(-1 + x\right) < 0$. Therefore, the optimal location is at $\max\{\frac{q+t}{2t}, \frac{13q+2t}{16t}\}$.

(XVI) $\frac{6q+7t}{14t} < x \leq \min\{\frac{q+t}{2t}, \frac{2q+13t}{14t}\}$: Because $1 < x$ in this case, we have

$$\begin{aligned} \frac{\partial \pi_2}{\partial x} &= -\tfrac{2}{3}\left[q + t\left(-3 + 2x\right)\right] \\ &< -\tfrac{2}{3}\left[q + t\left(-3 + 2\right)\right] = -\tfrac{2}{3}\left(q - t\right) < 0 \end{aligned}$$

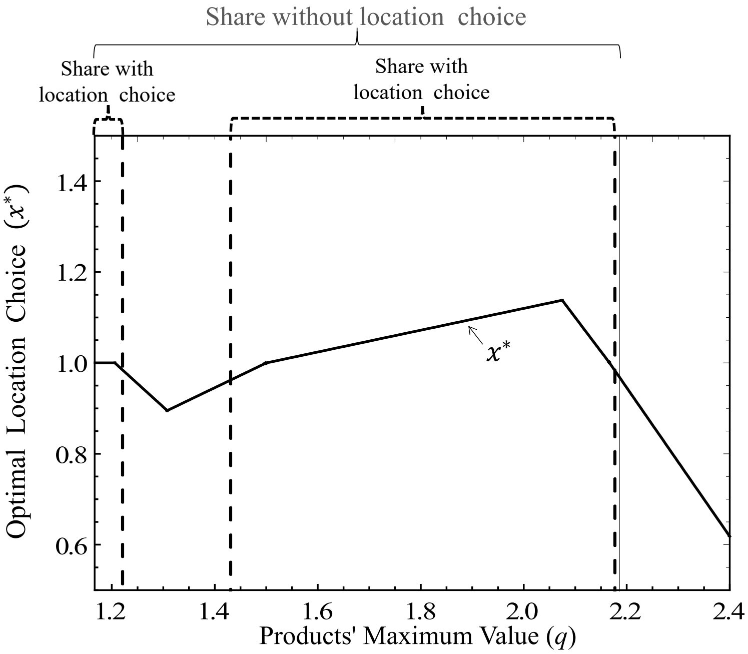


Figure 16: Firm 1's Incentive for Sharing with Endogenous Location of Product 3 ($t = 1$)

Therefore, the optimal location is at $\frac{6q+7t}{14t}$.

Next, we compare the optimal location across different regions. First, we note that π_2 is monotonic in x in all regions except in Regions II, III, and XII. Due to the monotonicity within each region and continuity across regions (as illustrated in the Figure 15), the optimal location must be located in Regions II, III, and XII. Second, we can verify that π_2 under the optimal location in Region XII is not as good as the optimal choice in Regions III or Region II. Therefore, the optimal location choice must from x^* derived in Equations (50) and (51). Note that Equations (50) and (51) overlap over the region $\frac{259}{202}t < q < \frac{17}{13}$. By the concavity of π_2 in Region III, the location in Equation (51) is optimal. Combining Equations (50) and (51) then leads to x^* in Equation (49). $\square$

Figure 16 illustrates the optimal location. As shown in Figure 16, when Firm 2 can strategically choose the location of Product 3, to a large extent, the firm chooses its location around the bottom-left angle of the triangle (i.e., x^* is close to 1). Similar to the traditional Hotelling model, on one hand, Firm 2 has an incentive to choose the location of Product 3 close to Product 1, which increases its demand in Submarket $\bar{2}3$. On the other hand, staying close to Product 1 could intensify the competition, providing the firm with an incentive to stay away from Product 1. The former is called the "demand effect," and the latter is called the "strategic effect." The optimal location is dictated by the balance between these two effects.

The tradeoff between these two effects also determines the nuance in the change in the optimal location—the optimal location x^* first decreases, then increases, and finally decreases in q. For example, when q is low, the competition against Firm 1 is soft. As q increases, Firm 2 tends to increase its product prices and earn higher profits from its exclusive submarket, while not significantly sacrificing the profit to competition. Therefore, in this case, as q increases, Firm 2 would like to position the new product to be closer to Product 1.

Anticipating Firm 2's optimal location choice, Firm 1 decides whether to share its technology. Proposition 12 summarizes the equilibrium outcome regarding Firm 1's sharing incentive.

Proposition 12. *When Firm 2 can strategically choose the location of Product 3, Firm 1 has an incentive to share its technology if and only if* $q < \left(\frac{5087101}{4559798} + \frac{495\sqrt{240913}}{2279899}\right) t$ *or* $\left(\frac{5879}{8642} + \frac{1320\sqrt{6}}{4321}\right) t < q < \left(\frac{1349}{98} - \frac{66}{49}\sqrt{74}\right) t$.

Proof. Based on the results in Proposition 11, we compare Firm 1's equilibrium profits before and after the introduction of Product 3.

When $\frac{13}{6}t < q$,

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \frac{(-2q+t-2t\frac{-18q+50t}{11t})^2}{48t} - \frac{1}{2}(q-t) \\ &= \frac{49}{1452t}\left[q - \left(\frac{1349}{98} - \frac{66}{49}\sqrt{74}\right)t\right]\left[q - \left(\frac{1349}{98} + \frac{66}{49}\sqrt{74}\right)t\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* > \pi_1^*$ if and only if $q < \left(\frac{1349}{98} - \frac{66}{49}\sqrt{74}\right) t$.

When $\frac{361}{174}t < q \leq \frac{13}{6}t$, $\hat{\pi}_1^* - \pi_1^* = \frac{(-2q+t-2t\frac{-6q+17t}{4t})^2}{48t} - \frac{1}{2}(q-t)$. We can verify that $\hat{\pi}_1^* > \pi_1^*$.

When $\frac{3}{2}t < q \leq \frac{361}{174}t$, if $2t < q$, $\hat{\pi}_1^* - \pi_1^* = \frac{(-2q+t-2t\frac{6q+16t}{25t})^2}{48t} - \frac{1}{2}(q-t)$, and otherwise$\hat{\pi}_1^* - \pi_1^* = \frac{(-2q+t-2t\frac{6q+16t}{25t})^2}{48t} - \frac{1}{50t}(2q+t)^2$. We can verify that $\hat{\pi}_1^* > \pi_1^*$ in both scenarios.

When $\frac{17}{13}t < q \leq \frac{3}{2}t$,

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \frac{(-2q+t-2t\frac{6q+2t}{11t})^2}{48t} - \frac{1}{50t}(2q+t)^2 \\ &= \frac{4321}{36300t}\left[q - \left(\frac{5879}{8642} + \frac{1320\sqrt{6}}{4321}\right)t\right]\left[q - \left(\frac{5879}{8642} - \frac{1320\sqrt{6}}{4321}\right)t\right] \end{aligned}$$

Because the term in the second bracket is positive, $\hat{\pi}_1^* > \pi_1^*$ if and only if $q > \left(\frac{5879}{8642} + \frac{1320\sqrt{6}}{4321}\right) t$.

When $\frac{41}{34}t < q \leq \frac{17}{13}t$,

$$\begin{aligned}\hat{\pi}_1^* - \pi_1^* &= -\tfrac{1}{432}(-5 + 17\tfrac{-34q+74t}{33t})(-12q + t + 5t\tfrac{-34q+74t}{33t}) - \tfrac{1}{50t}(2q+t)^2 \\ &= -\tfrac{2279899}{2940300t}\left[q - \left(\tfrac{5087101}{4559798} + \tfrac{495\sqrt{240913}}{2279899}\right)t\right]\left[q - \left(\tfrac{5087101}{4559798} - \tfrac{495\sqrt{240913}}{2279899}\right)t\right]\end{aligned}$$

Because the term in the second bracket is positive, $\hat{\pi}_1^* > \pi_1^*$ if and only if $q < \left(\frac{5087101}{4559798} + \frac{495\sqrt{240913}}{2279899}\right)t$.

When $q \leq \frac{41}{34}t$, $\hat{\pi}_1^* - \pi_1^* = -\frac{1}{432}(-5+17)(-12q+t+5t) - \frac{1}{50t}(2q+t)^2$. We can verify that $\hat{\pi}_1^* > \pi_1^*$.

All together, we have the results as in Proposition 12. □

Two observations are worth highlighting about the equilibrium outcome. First, as shown in Figure 16, we can see that Firm 1 continues to have an incentive to share its technology even when Firm 2 optimally chooses the location of Product 3. Therefore, our main conclusion that the cannibalization externality promotes technology sharing continues to hold in this case, demonstrating the robustness of our main insights. Second, as shown in Figure 16, when Firm 1 has an incentive to share its technology, the optimal location of the new product turns out to be close to $x = 1$, which is exactly the exogenous location of the new product in our main model. We observe that x^* is a little distant from 1 when q is large. However, in the presence of a large q, the focal firm has no incentive to share its technology anyway, even when the location of Product 3 is exogenous.

Of course, quantitatively, this variant has some minor differences. We observe that the range in which the focal firm is willing to share its technology becomes smaller than in the main model. Such a quantitative difference is well expected because, in this case, we generally give Firm 2 an additional layer of strategy (optimization, in technical terms) in using the new product to compete with the focal firm. This will negatively affect the focal firm's profit under technology sharing, and consequently, the focal firm has less incentive to share than before.

In particular, we find that Firm 1's sharing incentive becomes lower with the endogenous locations when the existing products' valuation is low. This is because, in this case, Firm 2 has an incentive to position the new product closer to Product 1. Such a position leads to lower differentiation from Firm 1's product and imposes high competition pressure on Firm 1, discouraging Firm 1 from sharing. The only exception is with a rather low valuation in the existing products (as shown by q approaching $\frac{7}{6}t$ in Figure 16). This is because, when q is rather low, the introduction of the new product changes Product 1's demand elasticity, which also accounts for Firm 1's incentive to share its technology. This demand-elasticity-decreasing effect works independently of the cannibalization externality, as also explained in the manuscript.

In summary, the main results in the main model are robust even when we endogenize the location of Product 3.

G The Model Variant with $q_1 = q_3 \neq q_2$

In the main model, we consider a technology-transfer rate (θ) and assume $q_1 = q_2 = q$ and $q_3 = \theta q$, where $\theta \leq 1$. Because the new product (Product 3) and the focal firm's product (Product 1) are produced under the same technology, one might argue that these two should have a comparable valuation. As such, in this appendix, we consider a model variant in which $q_1 = q_3$. We do not impose restrictions on q_2, which can be smaller or greater than q_1. For instance, when $q_1 = q_3 > q_2$, the shared technology gives Firm 2 the opportunity to launch a new product with a valuation higher than its existing product, increasing the firm's competitive advantage. Everything else remains the same as in the main model.

Similar to the main model, we first derive the equilibrium without new-product entry. As in the main model, in Submarket $\bar{12}$, by letting $u_1 = u_2$ in Equation (1), we can obtain the marginal consumer who is indifferent to purchasing Products 1 and 2 located at m_{12}, the distance from Product 1:

$$m_{12} = \frac{t - p_1 + p_2 + q_1 - q_2}{2t}. \tag{52}$$

Consumers located between Product 1 and m_{12} purchase Product 1, and those located between m_{12} and Product 2 purchase Product 2. In Submarket $\bar{i3}$ ($i = 1, 2$), consumers buy Product i when $u_i \geq 0$. Accordingly, we can formulate Product 1's and 2's demands as

$$\begin{cases} d_1 = \frac{1}{3} m_{12} + \frac{1}{3} \min\left\{\frac{q_1 - p_1}{t}, 1\right\} \\ d_2 = \frac{1}{3}(1 - m_{12}) + \frac{1}{3} \min\left\{\frac{q_2 - p_2}{t}, 1\right\} \end{cases} \tag{53}$$

Their profit functions take the same form as in Equation (4).

Each firm chooses its price to maximize profit. By solving the first-order conditions of the profit functions in Equation (4), we can derive each firm's best response to its rival. Based on these best responses, we can derive the equilibrium outcome as summarized in Lemma 8.

Lemma 8. *In the case without new-product entry, the equilibrium prices of Products 1 and 2 are*

$$(p_1^*, p_2^*) = \begin{cases} (q_1 - t, q_2 - t) & \text{if } 2t < q_1 \text{ and } 2t < q_2 \\ \left(q_1 - t, \frac{1}{2}q_2\right) & \text{if } 14t - 6q_1 < q_2 \leq 2t \\ \left(\frac{1}{2}q_1, q_2 - t\right) & \text{if } 14t - 6q_2 < q_1 \leq 2t \\ \left(\frac{1}{35}\left(7t + 17q_1 - 3q_2\right), \frac{1}{35}\left(7t - 3q_1 + 17q_2\right)\right) & \text{if } \max\{q_1 + 6q_2, q_2 + 6q_1\} \leq 14t \end{cases} \tag{54}$$

the equilibrium demands of Products 1 and 2 are

$$(d_1^*, d_2^*) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } 2t < q_1 \text{ and } 2t < q_2 \\ \left(\frac{2}{3} - \frac{q_2}{12t}, \frac{q_2}{4t}\right) & \text{if } 14t - 6q_1 < q_2 \leq 2t \\ \left(\frac{q_1}{4t}, \frac{2}{3} - \frac{q_1}{12t}\right) & \text{if } 14t - 6q_2 < q_1 \leq 2t \\ \left(\frac{1}{70t}\left(7t + 17q_1 - 3q_2\right), \frac{1}{70t}\left(7t - 3q_1 + 17q_2\right)\right) & \text{if } \max\{q_1 + 6q_2, q_2 + 6q_1\} \leq 14t \end{cases} \tag{55}$$

and the equilibrium profits for the two firms are

$$(\pi_1^*, \pi_2^*) = \begin{cases} \left(\frac{1}{2}\left(q_1 - t\right), \frac{1}{2}\left(q_2 - t\right)\right) & \text{if } 2t < q_1 \text{ and } 2t < q_2 \\ \left(\frac{1}{12t}\left(8t - q_2\right)\left(q_1 - t\right), \frac{1}{8t}q_2^2\right) & \text{if } 14t - 6q_1 < q_2 \leq 2t \\ \left(\frac{1}{8t}q_1^2, \frac{1}{12t}\left(8t - q_1\right)\left(q_2 - t\right)\right) & \text{if } 14t - 6q_2 < q_1 \leq 2t \\ \left(\frac{1}{2450t}\left(7t + 17q_1 - 3q_2\right)^2, \frac{1}{2450t}\left(7t - 3q_1 + 17q_2\right)^2\right) & \text{if } \max\{q_1 + 6q_2, q_2 + 6q_1\} \leq 14t \end{cases} \tag{56}$$

Proof. We distinguish four cases based on the value of q. (We can verify that the second-order conditions for profit maximization are satisfied.)

(a) $2t < q_1$ and $2t < q_2$: When q is large, all consumers in Submarkets $1\bar{3}$ and $2\bar{3}$ purchase. The firms have incentive to charge a price high enough such that the consumers with the highest misfit cost in Submarkets $1\bar{3}$ and $2\bar{3}$ derive zero utility; that is, $q_i - p_i - t = 0$. We can verify neither firm has incentive to deviate under $2t < q_1$ and $2t < q_2$.

(b) $14t - 6q_1 < q_2 \leq 2t$: Because $14t - 6q_1 < 2t$, $2t < q_1$. Firm 1 has incentive to charge a price high enough such that the consumer with the highest misfit cost in Submarkets $1\bar{3}$ derive zero utility; that is, $q_1 - p_1 - t = 0$, leading to $p_1^* = q_1 - t$. On the other hand, When $q_2 \leq 2t$ such that some consumers in Submarket $2\bar{3}$ do not purchase, the profit function for Firm 2 is

$$\pi_2 = p_2\left(\frac{1}{3}\frac{t - p_2 + p_1 + q_2 - q_1}{2t} + \frac{1}{3}\frac{q_2 - p_2}{t}\right) \tag{57}$$

Substituting p_1^* into Equation (57), by the first-order condition for π_2, we have $p_2^* = \frac{1}{2}q_2$. We can verify neither firm has incentive to deviate.

(c) $14t - 6q_2 < q_1 \leq 2t$: This case is symmetric to Case (b), so is the equilibrium outcome.

(d) $\max\{q_1 + 6q_2, q_2 + 6q_1\} \leq 14t$: In this case, some consumers in Submarkets $\bar{13}$ and $\bar{23}$ do not purchase. The profit function for Firm 2 is in Equation (57). Similarly, the profit function for Firm 1 is

$$\pi_1 = p_1 \left(\frac{1}{3} \frac{t - p_1 + p_2 + q_1 - q_2}{2t} + \frac{1}{3} \frac{q_1 - p_1}{t} \right) \tag{58}$$

Solving the first-order conditions for the two firms simultaneously,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \frac{t - 2p_1 + p_2 + q_1 - q_2}{2t} + \frac{1}{3} \frac{q_1 - 2p_1}{t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{1}{3} \frac{t - 2p_2 + p_1 + q_2 - q_1}{2t} + \frac{1}{3} \frac{q_2 - 2p_2}{t} = 0 \end{cases}$$

we can derive $p_1^* = \frac{1}{35}(7t + 17q_1 - 3q_2)$ and $p_2^* = \frac{1}{35}(7t - 3q_1 + 17q_2)$.

Substituting p_i^* into Equations (53) and (4), respectively, we have the equilibrium demands and profits as specified in the lemma. □

In general, Firm 1 and Firm 2 consider both the competitive segment and their respective monopoly segment to determine their product prices, and the optimal prices reflect their trade-offs in both segments (as in the fourth case in Equation (54)). However, when one of the products' value is sufficiently high (as in the second or third case in Equation (54)), the firm with the high product valuation prices its product as if it focuses only on its monopoly segment because its profit in the monopoly segment dominates that in the competitive segment. Meanwhile, in reaction, the firm with a relatively low product value still considers both the competitive segment and its monopoly segment to optimally choose its price. When both Product 1's and Product 2's values are sufficiently high, both firms price their products as if they consider only their respective monopoly segment (as in the first case in Equation (54)).

Similarly, we can derive the equilibrium with Firm 2 introducing the new product. As in the main model, if Firm 2 serves the market with both Products 2 and 3, all consumers' two preferred products are both available in the market. Similar to Submarket $\bar{12}$, if Submarket $\bar{i3}$, $i \in \{1, 2\}$, is fully covered, we have the marginal consumer m_{i3} who is indifferent to purchasing Products i and

3. Similarly to m_{12} in Equation (52), by letting $u_i = u_3$, we can derive

$$m_{i3} = \frac{t - p_i + p_3 + q_i - q_3}{2t}. \tag{59}$$

Each product's demand, which is the sum of demands from the two submarkets, can be derived as

$$\begin{cases} d_1 = \frac{1}{3}\min\left\{\frac{q_1-p_1}{t}, m_{12}\right\} + \frac{1}{3}\min\left\{\frac{q_1-p_1}{t}, m_{13}\right\} \\ d_2 = \frac{1}{3}\min\left\{\frac{q_2-p_2}{t}, 1 - m_{12}\right\} + \frac{1}{3}\min\left\{\frac{q_2-p_2}{t}, m_{23}\right\} \\ d_3 = \frac{1}{3}\min\left\{\frac{q_3-p_3}{t}, 1 - m_{13}\right\} + \frac{1}{3}\min\left\{\frac{q_3-p_3}{t}, 1 - m_{23}\right\} \end{cases} \tag{60}$$

We use the function min because we allow possibilities of both complete and incomplete market coverage in Submarket $\bar{ij}$. When $q_i - p_i - m_{ij}t > 0$, Submarket $\bar{ij}$ is fully covered; Consumers located between Product i and m_{ij} purchase Product i, and those located between m_{ij} and Product j purchase Product j. When Submarket $\bar{ij}$ is not fully covered, consumers buy Product i (Product j) only when $u_i \geq 0$ ($u_j \geq 0$), which leads to the demand $\frac{q_i-p_i}{t}$ ($\frac{q_j-p_j}{t}$) instead. The two firms' profits take the same form as in Equation (10). Following the same approach as in the main model, we can derive the equilibrium outcome as summarized in Lemma 9.

Lemma 9. *If Firm 2 serves the market with both Products 2 and 3, the equilibrium prices of the three products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases} \left(\frac{1}{6}(8t + q_1 - q_2), \frac{5}{12}(4t - q_1 + q_2), \frac{1}{12}(20t + q_1 - q_2)\right) & \text{if } \max\{16t - 7q_1, \frac{13t-4q_1}{2}\} < q_2 \\ \left(\frac{4}{3}(-t + q_1), \frac{1}{6}(2t + q_1 + 3q_2), \frac{1}{3}(t + 2q_1)\right) & \text{if } \max\{\frac{10t-q_1}{3}, \frac{56t-29q_1}{3}\} < q_2 \leq 16t - 7q_1 \\ \left(\frac{2}{25}(2t + 7q_1 - q_2), \frac{1}{25}(-t + 9q_1 + 13q_2), \frac{1}{25}(-29t + 36q_1 + 2q_2)\right) & \text{if } \max\{\frac{-t+4q_1}{2}, \frac{197t-123q_1}{11}\} < q_2 \leq \frac{56t-29q_1}{3} \\ \left(\frac{1}{57}(17t + 27q_1 - 5q_2), \frac{3}{19}(5t - q_1 + 3q_2), \frac{1}{114}(49t + 51q_1 - q_2)\right) & \text{if } \frac{253t-81q_1}{61} < q_2 \leq \frac{197t-123q_1}{11} \\ \left(\frac{1}{4}(t + 2q_1), \frac{1}{4}(-2t + q_1 + 3q_2), \frac{1}{4}(-2t + 3q_1 + q_2)\right) & \text{if } 3t - q_1 < q_2 \leq \min\{-3t + 3q_1, \frac{13t-4q_1}{2}\} \\ \left(\frac{1}{4}(-2t + 5q_1 - q_2), \frac{1}{4}(-2t + q_1 + 3q_2), \frac{1}{4}(-2t + 3q_1 + q_2)\right) & \text{if } \max\{q_1, -3t + 3q_1\} < q_2 \leq \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\} \\ \left(\frac{1}{5}(t + 2q_1), \frac{1}{5}(t - 3q_1 + 5q_2), \frac{1}{5}(-6t + 8q_1)\right) & \text{if } \frac{-14t+17q_1}{5} < q_2 \leq \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\} \\ \left(\frac{1}{83}(7t + 45q_1 - 3q_2), \frac{1}{83}(-41t + 21q_1 + 65q_2), \frac{2}{83}(-21t + 31q_1 + 9q_2)\right) & \text{if } \frac{-48t+59q_1}{15} < q_2 \leq \frac{253t-81q_1}{61} \\ \left(\frac{1}{4}(-2t + 3q_1 + q_2), \frac{1}{4}(-2t + q_1 + 3q_2), \frac{1}{4}(-2t + 3q_1 + q_2)\right) & \text{if } q_2 \leq \min\{q_1, 3t - q_1\} \end{cases} \tag{61}$$

the equilibrium demands of the three products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases} \left(\frac{1}{18t}\left(8t+q_1-q_2\right), \frac{1}{72t}\left(20t-11q_1+11q_2\right), \frac{1}{72t}\left(20t+7q_1-7q_2\right)\right) & \text{if } \max\{16t-7q_1, \frac{13t-4q_1}{2}\} < q_2 \\ \left(\frac{1}{36t}\left(32t-5q_1-3q_2\right), \frac{1}{18t}\left(t-q_1+3q_2\right), \frac{1}{36t}\left(2t+7q_1-3q_2\right)\right) & \text{if } \max\{\frac{10t-q_1}{3}, \frac{56t-29q_1}{3}\} < q_2 \leq 16t-7q_1 \\ \left(\frac{1}{25t}\left(2t+7q_1-q_2\right), \frac{1}{50t}\left(9t-6q_1+8q_2\right), \frac{1}{50t}\left(37t-8q_1-6q_2\right)\right) & \text{if } \max\{\frac{-t+4q_1}{2}, \frac{197t-123q_1}{11}\} < q_2 \leq \frac{56t-29q_1}{3} \\ \left(\frac{1}{114t}\left(17t+27q_1-5q_2\right), \frac{1}{684t}\left(131t-87q_1+109q_2\right), \frac{1}{12t}\left(t+3q_1-q_2\right)\right) & \text{if } \frac{253t-81q_1}{61} < q_2 \leq \frac{197t-123q_1}{11} \\ \left(\frac{1}{12t}\left(t+2q_1\right), \frac{1}{24t}\left(11t-5q_1+3q_2\right), \frac{1}{24t}\left(11t+q_1-3q_2\right)\right) & \text{if } 3t-q_1 < q_2 \leq \min\{-3t+3q_1, \frac{13t-4q_1}{2}\} \\ \left(\frac{1}{12t}\left(4t-q_1+q_2\right), \frac{1}{12t}\left(4t-q_1+q_2\right), \frac{1}{6t}\left(2t+q_1-q_2\right)\right) & \text{if } \max\{q_1, -3t+3q_1\} < q_2 \leq \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\} \\ \left(\frac{1}{10t}\left(t+2q_1\right), \frac{1}{10t}\left(t+2q_1\right), \frac{1}{5t}\left(4t-2q_1\right)\right) & \text{if } \frac{-14t+17q_1}{5} < q_2 \leq \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\} \\ \left(\frac{1}{166t}\left(7t+45q_1-3q_2\right), \frac{1}{498t}\left(213t-101q_1+51q_2\right), \frac{2}{83t}\left(14t+7q_1-6q_2\right)\right) & \text{if } \frac{-48t+59q_1}{15} < q_2 \leq \frac{253t-81q_1}{61} \\ \left(\frac{1}{12t}\left(4t+q_1-q_2\right), \frac{1}{6t}\left(2t-q_1+q_2\right), \frac{1}{12t}\left(4t+q_1-q_2\right)\right) & \text{if } q_2 \leq \min\{q_1, 3t-q_1\} \end{cases} \tag{62}$$

and the equilibrium profits for the two firms, (π_1^*, π_2^*), *are*

$$\begin{cases} \left(\frac{(8t+q_1-q_2)^2}{108t}, \frac{[400t-(q_1-q_2)(80t-31q_1+31q_2)]}{432t}\right) & \textit{if } \max\{16t-7q_1, \frac{13t-4q_1}{2}\} < q_2 \\ \left(\frac{(-t+q_1)(32t-5q_1-3q_2)}{27t}, \frac{\left[4t^2+10tq_1+13q_1^2+6(t-q_1)q_2+9q_2^2\right]}{108t}\right) & \textit{if } \max\{\frac{10t-q_1}{3}, \frac{56t-29q_1}{3}\} < q_2 \leq 16t-7q_1 \\ \left(\frac{(2t+7q_1-q_2)^2}{625t}, \frac{\left[-1082t^2+1651tq_1-342q_1^2+119(3t-2q_1)q_2+92q_2^2\right]}{1250t}\right) & \textit{if } \max\{\frac{-t+4q_1}{2}, \frac{197t-123q_1}{11}\} < q_2 \leq \frac{56t-29q_1}{3} \\ \left(\frac{(17t+27q_1-5q_2)^2}{6498t}, \frac{\left[4861t^2+3429q_1^2+6q_1(61t-541q_2)+4678tq_2+1981q_2^2\right]}{25992t}\right) & \textit{if } \frac{253t-81q_1}{61} < q_2 \leq \frac{197t-123q_1}{11} \\ \left(\frac{(t+2q_1)^2}{48t}, \frac{\left[-22t^2-q_1^2+22tq_2+3q_2^2-q_1(-26t+10q_2)\right]}{48t}\right) & \textit{if } 3t-q_1 < q_2 \leq \min\{-3t+3q_1, \frac{13t-4q_1}{2}\} \\ \left(\frac{(-2t+5q_1-q_2)(4t-q_1+q_2)}{48t}, \frac{\left[-16t^2+q_1(14t+5q_1)-6(-3t+q_1)q_2+q_2^2\right]}{48t}\right) & \textit{if } \max\{q_1, -3t+3q_1\} < q_2 \leq \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\} \\ \left(\frac{(t+2q_1)^2}{50t}, \frac{\left[-47t^2+87tq_1-38q_1^2+5(t+2q_1)q_2\right]}{50t}\right) & \textit{if } \frac{-14t+17q_1}{5} < q_2 \leq \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\} \\ \left(\frac{(7t+45q_1-3q_2)^2}{13778t}, \frac{\left[-15789t^2+3087q_1^2+2q_1(7751t-4223q_2)+3q_2(5934t+673q_2)\right]}{41334t}\right) & \textit{if } \frac{-48t+59q_1}{15} < q_2 \leq \frac{253t-81q_1}{61} \\ \left(\frac{(-2t+3q_1+q_2)(4t+q_1-q_2)}{48t}, \frac{\left[-16t^2+q_1^2+6q_1(3t-q_2)+14tq_2+5q_2^2\right]}{48t}\right) & \textit{if } q_2 \leq \min\{q_1, 3t-q_1\} \end{cases} \tag{63}$$

Proof. We distinguish night cases based on the value of q_2. (We can verify that the second-order conditions for profit maximization are satisfied.)

(I) $\max\{16t-7q_1, \frac{13t-4q_1}{2}\} < q_2$: When $0 \leq m_{12}, m_{13}, m_{23} \leq 1$ and the marginal consumers derive positive utilities, the demand functions in Equation (60) are as follows:

$$\begin{cases} d_1 = \frac{1}{3}m_{12} + \frac{1}{3}m_{13} \\ d_2 = \frac{1}{3}\left(1-m_{12}\right) + \frac{1}{3}m_{23} \\ d_3 = \frac{1}{3}\left(1-m_{13}\right) + \frac{1}{3}\left(1-m_{23}\right) \end{cases} \tag{64}$$

We can derive the first-order conditions of the firms' profit functions as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+q_1-q_2+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3-2q_1+2q_2+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+q_1-q_2+2t}{6t} = 0 \end{cases} \tag{65}$$

Solving this system of equations, we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $16t - 7q_1 < q_2$ ensures marginal consumer m_{13} derives positive utility, and Condition $\frac{13t-4q_1}{2} < q_2$ ensures marginal consumer m_{23} derives positive utility. We can verify that neither firm has profitable deviation.

(II) $\max\{\frac{10t-q_1}{3}, \frac{56t-29q_1}{3}\} < q_2 \leq 16t - 7q_1$: In this case, marginal consumer m_{13} derives zero utility because $q_2 \leq 16t - 7q_1$. When Firm 1 prices its product such that marginal consumer m_{13} derives zero utility, $2q_1 - t - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_1, \lambda) = \pi_1 + \lambda(2q_1 - t - p_1 - p_3)$. By solving the first-order conditions of $L(p_1, \lambda)$ and π_2,

$$\begin{cases} \frac{\partial L}{\partial p_1} = \frac{-4p_1+p_2+p_3+q_1-q_2+2t}{6t} - \lambda = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3-2q_1+2q_2+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+q_1-q_2+2t}{6t} = 0 \\ 2q_1 - t - p_1 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{10t-q_1}{3} < q_2$ ensures marginal consumer m_{23} derives positive utility, and Condition $\frac{56t-29q_1}{3} < q_2$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price.

(III) $\max\{\frac{-t+4q_1}{2}, \frac{197t-123q_1}{11}\} < q_2 \leq \frac{56t-29q_1}{3}$: Because $q_2 \leq \frac{56t-29q_1}{3}$, Firm 1 would like to increase its product price as if it competes with only Product 2 but not with Product 3, and Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{1}\bar{3}$ while Product 2 competes against Product 1 in Submarket $\bar{1}\bar{2}$. While Firm 2's demand function remains the same as in Equation (64), Firm 1's demand function function becomes

$$d_1 = \frac{1}{3} \times \left(\frac{q_1 - p_1}{t} + m_{12} \right) \tag{66}$$

When Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{1}3$, $2q_1 - t - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(2q_1 - t - p_1 - p_3)$. Substituting d_1 in Equation (66) into $\pi_1 = p_1 d_1$, and solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \times \left(\frac{q_1 - 2p_1}{t} + \frac{t - 2p_1 + p_2 + q_1 - q_2}{2t} \right) = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1 - 4p_2 + 2p_3 - 2q_1 + 2q_2 + 2t}{6t} = 0 \\ \frac{\partial L}{\partial p_3} = \frac{p_1 + 2p_2 - 4p_3 + q_1 - q_2 + 2t}{6t} - \lambda = 0 \\ 2q_1 - t - p_1 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{-t+4q_1}{2} < q_2$ ensures marginal consumer m_{23} derives positive utility, and Condition $\frac{197t-123q_1}{11} < q_2$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 3' price.

(IV) $\frac{253t-81q_1}{61} < q_2 \leq \frac{197t-123q_1}{11}$: Because $q_2 \leq \frac{197t-123q_1}{11}$, Firm 1 and Firm 2 do not compete in Submarket $\bar{1}3$ but in Submarket $\bar{1}2$. Similar to Product 1's demand function in Equation (66), Product 3's demand function becomes

$$d_3 = \frac{1}{3} \times \left(\frac{q_1 - p_3}{t} + 1 - m_{23} \right) \tag{67}$$

Substituting d_3 in Equation (67) into $\pi_2 = p_2 d_2 + p_3 d_3$, and solving the first-order conditions of π_1 and π_2,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \times \left(\frac{q_1 - 2p_1}{t} + \frac{t - 2p_1 + p_2 + q_1 - q_2}{2t} \right) = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{1}{3} \times \left(1 - \frac{t - p_1 + 2p_2 + q_1 - q_2}{2t} + \frac{t - 2p_2 + p_3 + q_2 - q_1}{2t} \right) = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{1}{3} \times \left(\frac{q_1 - 2p_3}{t} + 1 - \frac{t - p_2 + 2p_3 + q_2 - q_1}{2t} \right) = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{253t-81q_1}{61} < q_2$ ensures marginal consumer m_{23} derives positive utility.

(V) $3t - q_1 < q_2 \leq \min\{-3t + 3q_1, \frac{13t-4q_1}{2}\}$: In this case, the demand functions are as in Equation (64). Because $q_2 \leq \frac{13t-4q_1}{2}$, marginal consumer m_{23} derives zero utility. Thus, $q_1 + q_2 - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(q_1 + q_2 - t - p_2 - p_3)$. By solving the first-order condition of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+q_1-q_2+2t}{6t} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1-4p_2+2p_3-2q_1+2q_2+2t}{6t} - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{p_1+2p_2-4p_3+q_1-q_2+2t}{6t} - \lambda = 0 \\ q_1 + q_2 - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $q_2 < -3t + 3q_1$ ensures marginal consumer m_{13} derives positive utility. Condition $3t - q_1 < q_2$ ensures marginal consumer m_{12} derives positive utility. We can verify that neither firm has incentive to deviate.

(VI) $\max\{q_1, -3t + 3q_1\} < q_2 \leq \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\}$: Because $-3t + 3q_1 < q_2$, marginal consumer m_{13} derives zero utility. Thus, $2q_1 - t - p_1 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda_1) = \pi_1 + \lambda_1(2q_1 - t - p_1 - p_3)$. Because $q_2 \leq \frac{10t-q_1}{3}$, marginal consumer m_{23} derives zero utility. Thus, $q_1 + q_2 - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_2(p_2, p_3, \lambda_2) = \pi_2 + \lambda_2(q_1 + q_2 - t - p_2 - p_3)$. By solving the first-order condition of $L_1(p_1, \lambda_1)$ and $L_2(p_2, p_3, \lambda_2)$,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+q_1-q_2+2t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3-2q_1+2q_2+2t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+q_1-q_2+2t}{6t} - \lambda_2 = 0 \\ 2q_1 - t - p_1 - p_3 = 0 \\ q_1 + q_2 - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $q_2 \leq \frac{-14t+17q_1}{5}$ ensures Firm 1 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing its product price. Condition $q_1 < q_2$ ensures marginal consumer m_{12} derives positive utility.

(VII) $\frac{-14t+17q_1}{5} < q_2 \leq \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\}$: Because $\frac{-14t+17q_1}{5} < q_2$, Firm 1 would like to increase its product price as if it competes with only Product 2 but not with Product 3, and Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{13}$ while Product 2 competes against Product 1 in Submarket $\bar{12}$. In this case, Firm 2's demand function is as in Equation (64), and Firm 1's demand function is as in Equation (66). When Firm 2 chooses to use Product 3 to just serve the residual demand from Product 1 in Submarket $\bar{13}$,

$2q_1 - t - p_1 - p_3 = 0$. Moreover, because $q_2 \leq \frac{-t+4q_1}{2}$, marginal consumer m_{23} derives zero utility. Thus, $q_1 + q_2 - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda_1, \lambda_2) = \pi_2 + \lambda_1(2q_1 - t - p_1 - p_3) + \lambda_2(q_1 + q_2 - t - p_2 - p_3)$. Substituting d_1 in Equation (66) into $\pi_1 = p_1 d_1$, and solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda_1, \lambda_2)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \times \left(\frac{q_1 - 2p_1}{t} + \frac{t - 2p_1 + p_2 + q_1 - q_2}{2t} \right) = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1 - 4p_2 + 2p_3 - 2q_1 + 2q_2 + 2t}{6t} - \lambda_2 = 0 \\ \frac{\partial L}{\partial p_3} = \frac{p_1 + 2p_2 - 4p_3 + q_1 - q_2 + 2t}{6t} - \lambda_1 - \lambda_2 = 0 \\ 2q_1 - t - p_1 - p_3 = 0 \\ q_1 + q_2 - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $q_2 \leq \frac{-48t+59q_1}{15}$ ensures Firm 2 has no incentive to deviate from (p_1^*, p_2^*, p_3^*) by increasing Product 3' price.

(VIII) $\frac{-48t+59q_1}{15} < q_2 \leq \frac{253t-81q_1}{61}$: Because $\frac{-48t+59q_1}{15} < q_2$, Firm 1 and Firm 2 do not compete in Submarket $\bar{13}$ but in Submarket $\bar{12}$. Product 1's demand function is as in Equation (66), and Product 3's demand function is as in Equation (67). Moreover, because $q_2 \leq \frac{253t-81q_1}{61}$, marginal consumer m_{23} derives zero utility. Thus, $q_1 + q_2 - t - p_2 - p_3 = 0$. Substituting d_3 into $\pi_2 = p_2 d_2 + p_3 d_3$, and using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(q_1 + q_2 - t - p_2 - p_3)$. Substituting d_1 in Equation (66) into $\pi_1 = p_1 d_1$, and solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3} \times \left(\frac{q_1 - 2p_1}{t} + \frac{t - 2p_1 + p_2 + q_1 - q_2}{2t} \right) = 0 \\ \frac{\partial L}{\partial p_2} = \frac{1}{3} \times \left(1 - \frac{t - p_1 + 2p_2 + q_1 - q_2}{2t} + \frac{t - 2p_2 + p_3 + q_2 - q_1}{2t} \right) - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{1}{3} \times \left(\frac{q_1 - 2p_3}{t} + 1 - \frac{t - p_2 + 2p_3 + q_2 - q_1}{2t} \right) - \lambda = 0 \\ q_1 + q_2 - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate.

(IX) $q_2 \leq \min\{q_1, 3t - q_1\}$: In this case, the demand functions are as in Equation (64). Moreover, because $q_2 \leq 3t - q_1$, both marginal consumers m_{12} and m_{23} derive zero utility. Thus, $q_1 + q_2 - t - p_1 - p_2 = 0$ and $q_1 + q_2 - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda_1) = \pi_1 + \lambda_1(q_1 + q_2 - t - p_1 - p_2)$ and $L_2(p_2, p_3, \lambda_2) = \pi_2 + \lambda_2(q_1 + q_2 - t - p_2 - p_3)$. By

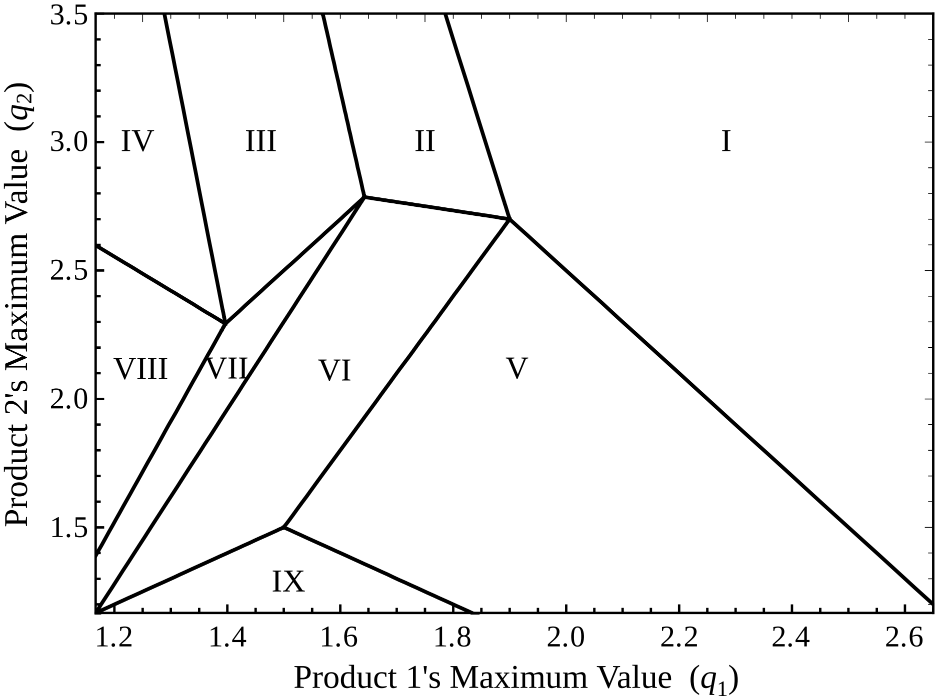


Figure 17: Equilibrium Outcome with Introduction of Product 3 by Firm 2 ($q_1 = q_3 \neq q_2, t = 1$)

solving the first-order conditions of $L_1(p_1, \lambda_1)$ and $L_2(p_2, p_3, \lambda_2)$,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{-4p_1+p_2+p_3+q_1-q_2+2t}{6t} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3-2q_1+2q_2+2t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{p_1+2p_2-4p_3+q_1-q_2+2t}{6t} - \lambda_2 = 0 \\ q_1 + q_2 - t - p_1 - p_2 = 0 \\ q_1 + q_2 - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate. The condition $q_2 \leq q_1$ensures marginal consumer m_{13} derives positive utility.

Substituting p_i^* into Equations (60) and (10), we can derive the equilibrium demands and profits as specified in the lemma. □

The equilibrium outcome is illustrated in Figure 17. In Region I in Figure 17, both q_1 and q_2 are high so that the three products offer comparable values to consumers. Thus, firms 1 and 2 compete aggressively with each other in both Submarkets $\bar{12}$ and $\bar{13}$, which shapes the equilibrium prices. Moving west from Region I, due to the reduction in the value of Product 1 and Product 3, in equilibrium, the two firms find it is optimal to set the highest prices for Product 1 and Product 3 so that Submarket $\bar{13}$ is just fully covered. In other words, all the marginal consumer's surplus is extracted in Submarket $\bar{13}$. Specifically, in Region II, corresponding to the second case in Equation (61), Firm 1 set the highest price for Product 1 so that it just covers the residual demand from Product 3 in Submarket $\bar{13}$, albeit it still competes aggressively with Product 2 in submarket $\bar{12}$. In reverse, moving west further from Region II to Region III, corresponding to the third case in Equation (61), Firm 2 sets the highest price for Product 3 so that Product 3 just fully covers the residual demand from Product 1 in the Submarket $\bar{13}$, albeit Product 2 still competes aggressively with Product 1 in Submarket $\bar{12}$. Moving west further from Region III to Region IV, corresponding to the fourth case in Equation (61), due to the too low valuation in Product 1 and in Product 3, Submarket $\bar{13}$ cannot be fully covered in equilibrium.

Moving south from Region I to Region V, corresponding to the fifth case in Equation (61), in which the value of q_2 is low, Firm 2 chooses the optimal prices for its two products to just cover all the consumers in its exclusive Submarket $\bar{23}$ and extracts all the marginal consumer's surplus. Firm 1 reacts to its rival's prices by optimally choosing its price for Product 1 to compete against both Products 2 and 3. Moving west from Region V, due to the reduction in the value of Product 1 and Product 3, in equilibrium, the two firms find it is optimal to set the highest prices for Product 1 and Product 3 such that submarket $\bar{13}$ is just fully covered as well. Specifically, in Region VI, corresponding to the sixth case in Equation (61), Firm 1 set the highest price for Product 1 such that it just covers the residual demand from Product 3 in Submarket $\bar{13}$, albeit it still competes aggressively with Product 2 in Submarket $\bar{12}$. In reverse, moving west further from Region VI to Region VII, corresponding to the seventh case in Equation (61), Firm 2 sets the highest price for Product 3 such that Product 3 just fully covers the residual demand from Product 1 in Submarket $\bar{13}$, albeit Product 2 still competes aggressively with Product 1 in Submarket $\bar{12}$. Moving west further from Region VII to Region VIII, corresponding to the eighth case in Equation (61), due to the too low value in Product 1 and in Product 3, the submarket $\bar{13}$ cannot be fully covered in equilibrium.

Note that the difference in equilibrium between Regions I and V, between Regions II and VI, between Regions III and VII, and between Regions IV and VIII all lies in the fact that Product 2 and Product 3 competes in Submarket $\bar{23}$ so that the marginal consumer's surplus is positive in the former regions but is zero in the latter due to the reduction in the value of Product 2. When both the values of q_2 and q_1 become lower, as shown in Region IX, corresponding to the ninth case in Equation (61), the two firms set prices to extract all the surplus of marginal consumer in Submarket $\bar{1}2$, in addition to that of marginal consumer in Submarket $\bar{2}3$.

By comparing the equilibrium profits with and without the introduction of Product 3, we can derive the conditions under which the focal firm would prefer to share its technology in equilibrium. Under this model variant, because Product 3 offers value comparable to Product 1, we can show that Firm 2 always has incentive to adopt the technology if shared. Therefore, the conditions for the focal firm to share its technology are the same as the conditions for the new-product introduction in equilibrium. We summarize the conditions in the following proposition.

Proposition 13. *In equilibrium, the new product would be introduced to the market if and only if* $\underline{q}'' \leq q_2 \leq \bar{q}''$, *where*

$$\underline{q}'' = \begin{cases} q_1 & \text{if } q_1 \leq \frac{3}{2}t \\ 3t - q_1 & \text{if } \frac{3}{2}t < q_1 \leq \left(\frac{341}{470} + \frac{84}{235}\sqrt{6}\right)t \\ \frac{1}{3}\left(7t + 17q_1\right) - \frac{35\sqrt{t^2+4tq_1+4q_1^2}}{6\sqrt{6}} & \text{if } \left(\frac{341}{470} + \frac{84}{235}\sqrt{6}\right)t < q_1 \leq \left(\frac{13}{10} + \frac{3}{5}\sqrt{\frac{3}{2}}\right)t \\ \frac{33t^2-28tq_1+4q_1^2}{4t-4q_1} & \text{otherwise} \end{cases}$$

and

$$\bar{q}'' = \begin{cases} \frac{1}{10}\left(34t - 3q_1\right) & \text{if } q_1 \leq \frac{4}{3}t \\ \frac{1}{4}\left(8t + 3q_1\right) & \text{if } \frac{4}{3}t < q_1 \leq \frac{8}{5}t \\ \frac{256t^2-296tq_1+67q_1^2}{24t-24q_1} & \text{if } \frac{8}{5}t < q_1 \leq \left(\frac{128}{101} + \frac{24}{101}\sqrt{6}\right)t \\ 8t + q_1 - 3\sqrt{\frac{3}{2}}q_1 & \text{if } \left(\frac{128}{101} + \frac{24}{101}\sqrt{6}\right)t < q_1 \leq 2t \\ 8t + q_1 - 3\sqrt{6}\sqrt{-t^2 + tq_1} & \text{if } 2t < q_1 \leq \left(21 - 3\sqrt{39}\right)t \\ \frac{1}{2}\left(25t - 7q_1\right) - \frac{3}{2}\sqrt{9t^2 - 14tq_1 + 5q_1^2} & \text{otherwise} \end{cases}$$

Proof. Because the conditions for the focal firm to share its technology are the same as the conditions for new-product introduction in equilibrium, we compare Firm 1's equilibrium profits before and after the introduction of Product 3 for different cases, based on Lemmas 8 and 9.

The case with $\max\{q_1 + 6q_2, q_2 + 6q_1\} \leq 14t$**:** This case occurs in the region with both low q_1 and low q_2 of Lemma 8, which overlaps with Regions V, VI, VII, VIII, and IX of Lemma 9.

Region V: When $3t - q_1 < q_2 \le \min\{-3t + 3q_1, \frac{13t-4q_1}{2}\}$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(t+2q_1)^2}{48t} - \tfrac{1}{2450t}\left(7t + 17q_1 - 3q_2\right)^2 \\ &= -\tfrac{9}{2450t}\left[q_2 - \left(\tfrac{1}{3}\left(7t + 17q_1\right) - \tfrac{35\sqrt{t^2+4tq_1+4q_1^2}}{6\sqrt{6}}\right)\right]\left[q_2 - \left(\tfrac{1}{3}\left(7t + 17q_1\right) + \tfrac{35\sqrt{t^2+4tq_1+4q_1^2}}{6\sqrt{6}}\right)\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \ge \frac{1}{3}\left(7t + 17q_1\right) - \frac{35\sqrt{t^2+4tq_1+4q_1^2}}{6\sqrt{6}}$.

Region VI: When $\max\{q_1, -3t+3q_1\} < q_2 \le \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\}$, we can verify that $\hat{\pi}_1^* \ge \pi_1^*$.

Region VII: When $\frac{-14t+17q_1}{5} < q_2 \le \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

Region VIII: When $\frac{-48t+59q_1}{15} < q_2 \le \frac{253t-81q_1}{61}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

Region IX: When $q_2 \le \min\{q_1, 3t - q_1\}$, we can verify that $\hat{\pi}_1^* \le \pi_1^*$.

The case with $14t - 6q_2 < q_1 \le 2t$: This case occurs in the region with low q_1 and high q_2 of Lemma 8, which overlaps with Regions I, II, III, IV, V, VI, VII, and VIII of Lemma 9.

Case I: When $\max\{16t - 7q_1, \frac{13t-4q_1}{2}\} < q_2$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(8t+q_1-q_2)^2}{108t} - \tfrac{1}{8t}q_1^2 \\ &= \tfrac{1}{108t}\left[q_2 - \left(8t + q_1 - 3\sqrt{\tfrac{3}{2}}q_1\right)\right]\left[q_2 - \left(8t + q_1 + 3\sqrt{\tfrac{3}{2}}q_1\right)\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \le 8t + q_1 - 3\sqrt{\frac{3}{2}}q_1$.

Case II: When $\max\{\frac{10t-q_1}{3}, \frac{56t-29q_1}{3}\} < q_2 \le 16t - 7q_1$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(-t+q_1)(32t-5q_1-3q_2)}{27t} - \tfrac{1}{8t}q_1^2 \\ &= -\tfrac{q_1-t}{9t}\left[q_2 - \tfrac{256t^2-296tq_1+67q_1^2}{24t-24q_1}\right] \end{aligned}$$

Therefore, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \le \frac{256t^2-296tq_1+67q_1^2}{24t-24q_1}$.

Case III: When $\max\{\frac{-t+4q_1}{2}, \frac{197t-123q_1}{11}\} < q_2 \le \frac{56t-29q_1}{3}$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(2t+7q_1-q_2)^2}{625t} - \tfrac{1}{8t}q_1^2 \\ &= \tfrac{4}{1250t}\left[q_2 - \tfrac{1}{4}\left(8t + 3q_1\right)\right]\left[q_2 - \tfrac{1}{4}\left(8t + 53q_1\right)\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \le \frac{1}{4}\left(8t + 3q_1\right)$.

Case IV: When $\frac{253t-81q_1}{61} < q_2 \le \frac{197t-123q_1}{11}$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(17t+27q_1-5q_2)^2}{6498t} - \tfrac{1}{8t}q_1^2 \\ &= \tfrac{25}{6498t}\left[q_2 - \tfrac{1}{10}(34t-3q_1)\right]\left[q_2 - \tfrac{1}{10}(34t+111q_1)\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \le \frac{1}{10}(34t-3q_1)$.

Case V: When $3t-q_1 < q_2 \le \min\{-3t+3q_1, \frac{13t-4q_1}{2}\}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

Case VI: When $\max\{q_1, -3t+3q_1\} < q_2 \le \min\{\frac{-14t+17q_1}{5}, \frac{10t-q_1}{3}\}$, we can verify that $\hat{\pi}_1^* \ge \pi_1^*$.

Case VII: When $\frac{-14t+17q_1}{5} < q_2 \le \min\{\frac{-48t+59q_1}{15}, \frac{-t+4q_1}{2}\}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

Case VIII: When $\frac{-48t+59q_1}{15} < q_2 \le \frac{253t-81q_1}{61}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

The case with $14t-6q_1 < q_2 \le 2t$: This case occurs in the region with high q_1 and low q_2 of Lemma 8, which overlaps with Regions I and V of Lemma 9.

Case I: When $\max\{16t-7q_1, \frac{13t-4q_1}{2}\} < q_2$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(8t+q_1-q_2)^2}{108t} - \tfrac{1}{12t}(8t-q_2)(q_1-t) \\ &= \tfrac{1}{108t}\left[q_2 - \left(\tfrac{1}{2}(25t-7q_1) - \tfrac{3}{2}\sqrt{9t^2-14tq_1+5q_1^2}\right)\right]\left[q_2 - \left(\tfrac{1}{2}(25t-7q_1) + \tfrac{3}{2}\sqrt{9t^2-14tq_1+5q_1^2}\right)\right] \end{aligned}$$

Because the term in the second bracket is negative, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \le \frac{1}{2}(25t-7q_1) - \frac{3}{2}\sqrt{9t^2-14tq_1+5q_1^2}$.

Case V: When $3t-q_1 < q_2 \le \min\{-3t+3q_1, \frac{13t-4q_1}{2}\}$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(t+2q_1)^2}{48t} - \tfrac{1}{12t}(8t-q_2)(q_1-t) \\ &= \tfrac{q_1-t}{12t}\left[q_2 - \tfrac{33t^2-28tq_1+4q_1^2}{4t-4q_1}\right] \end{aligned}$$

Therefore, $\hat{\pi}_1^* \ge \pi_1^*$ if and only if $q_2 \ge \frac{33t^2-28tq_1+4q_1^2}{4t-4q_1}$.

The case with $2t < q_1$ and $2t < q_2$: This case occurs in the region with high q_1 and high q_2 of Lemma 8, which overlaps with Regions I and V of Lemma 9.

Case I: When $\max\{16t-7q_1, \frac{13t-4q_1}{2}\} < q_2$, we have

$$\begin{aligned} \hat{\pi}_1^* - \pi_1^* &= \tfrac{(8t+q_1-q_2)^2}{108t} - \tfrac{1}{2}(q_1-t) \\ &= \tfrac{1}{108t}\left[q_2 - \left(8t+q_1-3\sqrt{6}\sqrt{-t^2+tq_1}\right)\right]\left[q_2 - \left(8t+q_1+3\sqrt{6}\sqrt{-t^2+tq_1}\right)\right] \end{aligned}$$

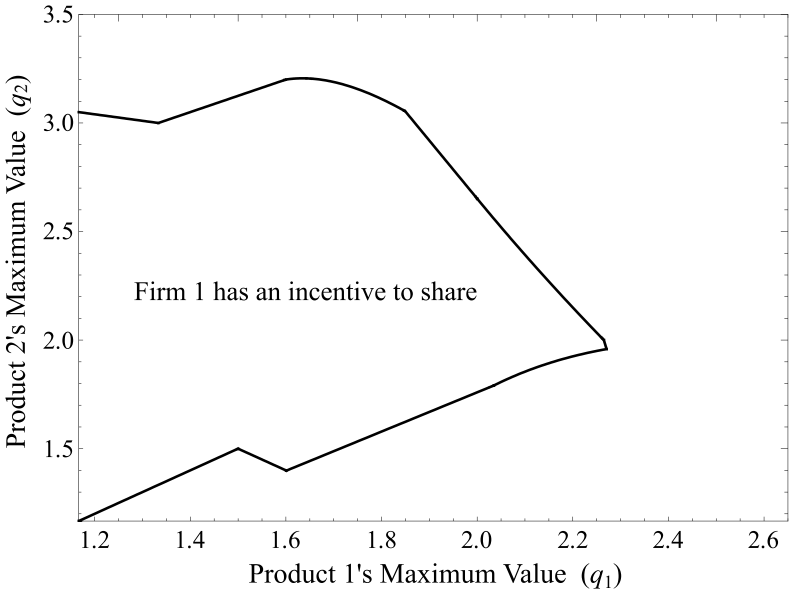


Figure 18: Conditions for Firm 1 to Share Technology ($q_1 = q_3 \neq q_2; t = 1$)

Because the term in the second bracket is negative, $\hat{\pi}_1^* \geq \pi_1^*$ if and only if $q_2 \leq 8t + q_1 - 3\sqrt{6}\sqrt{-t^2 + tq_1}$.

Case V: When $3t - q_1 < q_2 \leq \min\{-3t + 3q_1, \frac{13t-4q_1}{2}\}$, we can verify that $\hat{\pi}_1^* > \pi_1^*$.

Altogether, we have the comparison results as in Proposition 13. □

The conditions are illustrated in Figure 18. In this case, because Product 3 offers value comparable to Product 1, if Firm 2 offers both the new product and the existing product, the entry of the new product can impact the two firms' competition and exert a cannibalization externality on the focal firm. As in the main model, the focal firm would prefer to share its technology only when q_1 (q_3) is not too high and q_2 is neither high nor low (i.e., $\underline{q}'' \leq q_2 \leq \bar{q}''$) such that the new product impose limited additional competition pressure on the focal firm.

It is worth noting that when Product 3's valuation is higher than Product 2's, Firm 2 might drop Product 2 when it introduces Product 3. We examine this possibility and find that it can happen but only under very specific and strict conditions. These conditions are illustrated by Figure 19—Firm 2 drops Product 2 only in the small region at the bottom right. Specifically, this occurs only when the valuation of Products 1 and 3 are high but the valuation of Product 2 is very low, similar to the scenario with high q and low θ in the main model. In fact, whether Firm 2 has incentive to drop Product 2 under this model variant is similar to, or is a mirror problem of, whether Firm

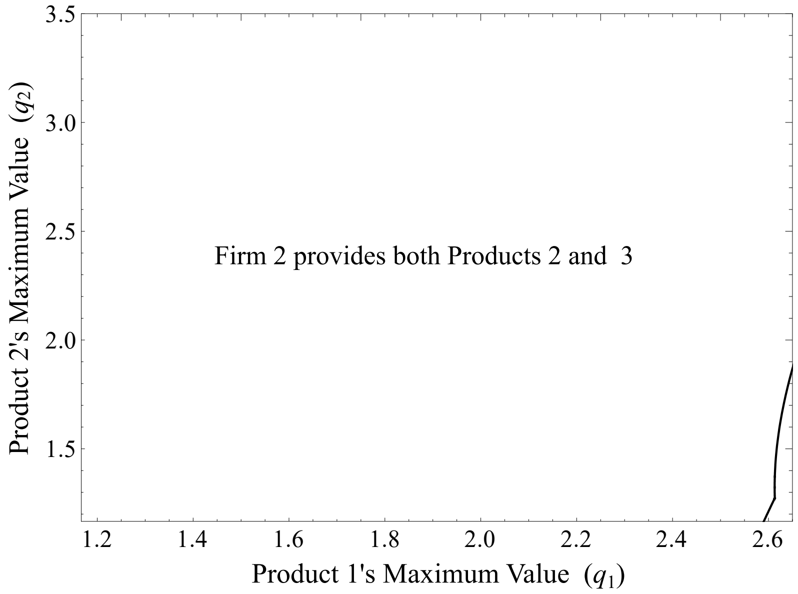


Figure 19: Conditions for Firm 2 to Provide both Product 2 and Product 3 ($q_1 = q_3 \neq q_2; t = 1$)

2 has incentive to adopt the shared technology and introduce a new product in the main model because both involve Firm 2 having a product with valuation comparable to Firm 1's product and both concern Firm 2's incentive to make the other product available to the market. This explains why Figure 19 is inherently the same as Figure 4(b) in the paper; that is, they are identical after some rotation and relabeling the axes.

Nonetheless, in this model variant, the region in which Firm 2 prefers to drop Product 2 has no overlap with the region in which Firm 1 has incentive to share its technology. That is, when Firm 2 prefers to drop Product 2 if the new technology is shared, Firm 1 has no incentive to share its technology in the first place. As a result, the conditions under which Product 3 is introduced, together with its existing product, are the same as the conditions under which Firm 1 has an incentive to share its technology. Altogether, when the valuation of the focal firm's product is comparable to the competitor's one product (either existing or new), the focal firm has an incentive to share its technology if and only if the valuation of the competitor's other product is neither too low nor too high. This insight holds consistently for both the main model and this model variant.

H The Model Variant with $t_{13} < t$ and $q_1 = q_2 = q_3$

In the main model, we assume that Products 1, 2, and 3 are equally horizontally differentiated (i.e., $t_{12} = t_{13} = t_{23} = t$). However, because the new product (Product 3) and the focal firm's product (Product 1) are produced using the same technology, one could argue that these two should compete more intensely. In other words, the substitutability between Products 1 and 3 is higher than that between Products 1 and 2, as well as between Products 2 and 3.

In this appendix, we explore this possibility with a model variant in which $t_{13} < t_{12} = t_{23} = t$. Because $t_{13} < t$, the shared technology enables Firm 2 to introduce a new product (i.e., Product 3) that serves as a closer substitute to Firm 1's product (i.e., Product 1), thereby intensifying competition between the firms. This raises an interesting question: would Firm 1 still be willing to share its technology with Firm 2?

To focus on the effects of the asymmetry in differentiation across different submarkets, we assume $q_1 = q_2 = q_3 = q$. Everything else remains the same as in the main model. To avoid trivial cases, we further assume that t_{13} is not too small (i.e., $\frac{3}{5}t \leq t_{13} < t$); otherwise, the little differentiation and strong competition would discourage Firm 1 from sharing.

Similar to the main model, we first derive the equilibrium without new-product entry. Similar to Equation (3) in the main model, we can formulate Product 1's and 2's demands as

$$\begin{cases} d_1 = \frac{1}{3}m_{12} + \frac{1}{3}\min\left\{\frac{q_1 - p_1}{t_{13}}, 1\right\} \\ d_2 = \frac{1}{3}(1 - m_{12}) + \frac{1}{3}\min\left\{\frac{q_2 - p_2}{t}, 1\right\} \end{cases} \tag{68}$$

Their profit functions take the same form as in Equation (4).

Each firm chooses its price to maximize profit. By solving the first-order conditions of the profit functions in Equation (4), we can derive each firm's best response to its rival. Based on these best responses, we can derive the equilibrium outcome as summarized in Lemma 10.

Lemma 10. *In the case without new-product entry, the equilibrium prices of Products 1 and 2* (p_1^*, p_2^*) *are*

$$\begin{cases} (q - t_{13}, q - t) & \textit{if } \frac{7t - t_{13}}{3} < q \\ \left(q - t_{13}, \frac{1}{6}\left(3q + t - t_{13}\right)\right) & \textit{if } \frac{31tt_{13} + 11t_{13}^2}{12t + 9t_{13}} < q \leq \frac{7t - t_{13}}{3} \\ \left(\frac{12qt + 2qt_{13} + 7tt_{13}}{24t + 11t_{13}}, \frac{10qt + 4t^2 + 4qt_{13} + 3tt_{13}}{24t + 11t_{13}}\right) & \textit{otherwise,} \end{cases} \tag{69}$$

the equilibrium demands of the two products (d_1^*, d_2^*) *are*

$$\begin{cases} \left(\frac{1}{6}\left(2+\frac{t_{13}}{t}\right), \frac{1}{6}\left(4-\frac{t_{13}}{t}\right)\right) & \textit{if } \frac{7t-t_{13}}{3} < q \\ \left(\frac{-3q+19t+5t_{13}}{36t}, \frac{3q+t-t_{13}}{12t}\right) & \textit{if } \frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}} < q \leq \frac{7t-t_{13}}{3} \\ \left(\frac{(2t+t_{13})[12qt+(2q+7t)t_{13}]}{6tt_{13}(24t+11t_{13})}, \frac{10qt+4t^2+4qt_{13}+3tt_{13}}{48t^2+22tt_{13}}\right) & \textit{otherwise,} \end{cases} \tag{70}$$

and the equilibrium profits for the two firms (π_1^*, π_2^*) *are*

$$\begin{cases} \left(\frac{(q-t_{13})(2t+t_{13})}{6t}, \frac{(q-t)(4t-t_{13})}{6t}\right) & \textit{if } \frac{7t-t_{13}}{3} < q \\ \left(\dfrac{(q-t_{13})(-3q+19t+5t_{13})}{36t}, \dfrac{(3q+t-t_{13})^2}{72t}\right) & \textit{if } \frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}} < q \leq \frac{7t-t_{13}}{3} \\ \left(\frac{(2t+t_{13})[12qt+(2q+7t)t_{13}]^2}{6tt_{13}(24t+11t_{13})^2}, \frac{[2t(5q+2t)+(4q+3t)t_{13}]^2}{2t(24t+11t_{13})^2}\right) & \textit{otherwise.} \end{cases} \tag{71}$$

Proof. We distinguish three cases based on the value of q. (We can verify that the second-order conditions for profit maximization are satisfied.)

(a) $\frac{7t-t_{13}}{3} < q$: When q is large, all consumers in Submarkets $\bar{1}3$ and $\bar{2}3$ purchase. The firms have incentive to charge a price high enough such that the consumers with the highest misfit cost in Submarkets $\bar{1}3$ and $\bar{2}3$ derive zero utility; that is, $q - p_i - t_{i3} = 0$, leading to $p_1^* = q - t_{13}$ and $p_2^* = q - t$. We can verify neither firm has incentive to deviate under $-3q + 7t < t_{13}$.

(b) $\frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}} < q \leq \frac{7t-t_{13}}{3}$: Because $\frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}} < q$, Firm 1 still has incentive to charge a price high enough such that the consumer with the highest misfit cost in Submarkets $\bar{1}3$ derive zero utility; that is, $q - p_1 - t_{13} = 0$, leading to $p_1^* = q - t_{13}$. On the other hand, when $q \leq \frac{7t-t_{13}}{3}$ such that some consumers in Submarket $\bar{2}3$ do not purchase, the profit function for Firm 2 is

$$\pi_2 = p_2 \left(\frac{1}{3} \frac{t - p_2 + p_1}{2t} + \frac{1}{3} \frac{q - p_2}{t} \right) \tag{72}$$

Substituting p_1^* into Equation (72), by the first-order condition for π_2, we have $p_2^* = \frac{1}{6}(3q + t - t_{13})$. We can verify neither firm has incentive to deviate.

(c) $q \leq \frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}}$: In this case, some consumers in Submarkets $\bar{1}3$ and $\bar{2}3$ do not purchase. The profit function for Firm 2 is in Equation (72). Similarly, the profit function for Firm 1 is

$$\pi_1 = p_1 \left(\frac{1}{3} \frac{t - p_1 + p_2}{2t} + \frac{1}{3} \frac{q - p_1}{t_{13}} \right) \tag{73}$$

Solving the first-order conditions for the two firms simultaneously,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{1}{3}\frac{t-2p_1+p_2}{2t} + \frac{1}{3}\frac{q_1-2p_1}{t_{13}} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{1}{3}\frac{t-2p_2+p_1}{2t} + \frac{1}{3}\frac{q_2-2p_2}{t} = 0 \end{cases}$$

we can derive $p_1^* = \frac{12qt+2qt_{13}+7tt_{13}}{24t+11t_{13}}$ and $p_2^* = \frac{10qt+4t^2+4qt_{13}+3tt_{13}}{24t+11t_{13}}$. We can verify neither firm has incentive to deviate.

Substituting p_i^* into Equations (68) and (4), respectively, we have the equilibrium demands and profits as specified in the lemma. □

As in the main model, Firm 1 and Firm 2 consider both the competitive segment and their respective monopoly segments to determine their product prices, and the optimal prices reflect their trade-offs in both segments (as in the third case in Equation (69)). However, when the products' value becomes high, the two firms would like to price their products as if they focus only on the monopoly segment, in which the consumers with the highest misfit cost in the monopoly segment derive zero utility. Because $t_{13} < t$, the consumers with the highest misfit cost in Submarket $\bar{13}$ have lower misfit cost than those in Submarket $\bar{23}$, and there exists case in which Firm 1 focuses only on the monopoly segment to price Product 1 while Firm 2 still considers both the competitive segment and its monopoly segment to optimally choose its price (as in the second case in Equation (69)). When the products' value is sufficiently higher, both firms price their products as if they consider only their respective monopoly segments (as in the first case in Equation (69)).

Similarly, we can derive the equilibrium with Firm 2 introducing the new product. As in the main model, if Firm 2 serves the market with both Products 2 and 3, all consumers' two preferred products are both available in the market. In Submarket $\bar{ij}$, the consumer at a distance of x_i from the preferred Product i derives a utility of u_i from buying product i and u_j from buying the other preferred product j, and

$$\begin{cases} u_i = q_i - x_i t_{ij} - p_i \\ u_j = q_j - (1-x_i)t_{ij} - p_j. \end{cases} \tag{74}$$

By letting $u_i = u_j$ in Equation (74), we can obtain the marginal consumer who is indifferent to purchasing Products i and j located at m_{ij}, the distance from Product i. Specifically, substituting

$t_{12} = t_{23} = t$ and $q_1 = q_2 = q_3 = q$ into Equation (74), we have

$$\begin{cases} m_{12} = \frac{t - p_1 + p_2}{2t} \\ m_{13} = \frac{t_{13} - p_1 + p_3}{2t_{13}} \\ m_{23} = \frac{t - p_2 + p_3}{2t} \end{cases} \tag{75}$$

The products' demands and the two firms' profits take the same form as in Equation (9) and (10). Following the same approach as in the main model, we can derive the equilibrium outcome as summarized in Lemma 11.

Lemma 11. *In the case with new-product entry, the equilibrium prices of the three products* (p_1^*, p_2^*, p_3^*) *are*

$$\begin{cases} \left(\frac{8tt_{13}}{3(t+t_{13})}, \frac{t\left(3t^2+17tt_{13}+10t_{13}^2\right)}{3(t+t_{13})(2t+t_{13})}, \frac{tt_{13}(17t+13t_{13})}{3(t+t_{13})(2t+t_{13})}\right) & \text{if } \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} < q \\ \left(\frac{4t_{13}[(4q-t)t+2(q+2t)t_{13}]}{3t^2+26tt_{13}+19t_{13}^2}, \frac{3(2q-t)t^2+(26q-15t)tt_{13}+2(8q-3t)t_{13}^2}{3t^2+26tt_{13}+19t_{13}^2}, \frac{t_{13}[(26q-11t)t+(22q-13t)t_{13}]}{3t^2+26tt_{13}+19t_{13}^2}\right) & \text{if } \frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} \\ \left(\frac{6(2q-t)t_{13}}{t+11t_{13}}, \frac{(2q-t)(t+5t_{13})}{t+11t_{13}}, \frac{6(2q-t)t_{13}}{t+11t_{13}}\right) & \text{otherwise,} \end{cases} \tag{76}$$

the equilibrium demands of the three products (d_1^*, d_2^*, d_3^*) *are*

$$\begin{cases} \left(\frac{4}{9}, \frac{7}{18} - \frac{2t}{9(t+t_{13})}, \frac{1}{6} + \frac{2t}{9(t+t_{13})}\right) & \text{if } \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} < q \\ \begin{aligned} &(\frac{2(t+t_{13})((4q-t)t+2(q+2t)t_{13})}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)}, \\ &\frac{12t^2(-q+t)+t(-10q+67t)t_{13}+(-2q+53t)t_{13}^2}{6t\left(3t^2+26tt_{13}+19t_{13}^2\right)}, \frac{2t^2(-2q+5t)+7t(-2q+11t)t_{13}+(-6q+45t)t_{13}^2}{6t\left(3t^2+26tt_{13}+19t_{13}^2\right)}) \end{aligned} & \text{if } \frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} \\ \left(\frac{2qt+t^2-2qt_{13}+23tt_{13}}{6t^2+66tt_{13}}, \frac{2(t(-q+t)+(q+5t)t_{13})}{3t(t+11t_{13})}, \frac{2qt+t^2-2qt_{13}+23tt_{13}}{6t^2+66tt_{13}}\right) & \text{otherwise,} \end{cases} \tag{77}$$

and the equilibrium profits for the two firms (π_1^*, π_2^*) *are*

$$\begin{cases} \left(\frac{32tt_{13}}{27(t+t_{13})}, \frac{1}{54}t\left(109 - \frac{64t}{t+t_{13}} - \frac{81t}{2t+t_{13}}\right)\right) & \text{if } \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} < q \\ \begin{aligned} &(\frac{8t_{13}(t+t_{13})((4q-t)t+2(q+2t)t_{13})^2}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2}, \\ &\frac{1}{6t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2}\left[-36t^4(-2q+t)(-q+t)\right. \\ &+t^3\left(-476q^2+1228qt-491t^2\right)t_{13}+t^2\left(-916q^2+4908qt-2213t^2\right)t_{13}^2 \\ &\left.+t\left(-676q^2+5652qt-2693t^2\right)t_{13}^3+\left(-164q^2+1928qt-903t^2\right)t_{13}^4\right]) \end{aligned} & \text{if } \frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} \\ \left(\frac{(2q-t)t_{13}(t(2q+t)+(-2q+23t)t_{13})}{t(t+11t_{13})^2}, \frac{(2q-t)\left[-2(q-t)t^2-(2q-23t)tt_{13}+(4q+119t)t_{13}^2\right]}{3t(t+11t_{13})^2}\right) & \text{otherwise.} \end{cases} \tag{78}$$

Proof. We distinguish three cases based on the value of q. (We can verify that the second-order conditions for profit maximization are satisfied.)

(I) $\frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} < q$: When all the marginal consumers derive positive utilities, the profit functions are well behaved. We can derive the first-order conditions of the firms' profit functions as

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{tp_3+(2t+p_2)t_{13}-2p_1(t+t_{13})}{6tt_{13}} = 0 \\ \frac{\partial \pi_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+2t}{6t} = 0 \\ \frac{\partial \pi_2}{\partial p_3} = \frac{tp_1+2(t+p_2)t_{13}-2p_3(t+t_{13})}{6tt_{13}} = 0 \end{cases} \tag{79}$$

Solving this system of equations, we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2} < q$ ensures marginal consumer m_{23} derives positive utility. We can verify that neither firm has profitable deviation.

(II) $\frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}$: In this case, marginal consumer m_{23} derives zero utility because $q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L(p_2, p_3, \lambda) = \pi_2 + \lambda(2q - t - p_2 - p_3)$. By solving the first-order conditions of π_1 and $L(p_2, p_3, \lambda)$,

$$\begin{cases} \frac{\partial \pi_1}{\partial p_1} = \frac{tp_3+(2t+p_2)t_{13}-2p_1(t+t_{13})}{6tt_{13}} = 0 \\ \frac{\partial L}{\partial p_2} = \frac{p_1-4p_2+2p_3+2t}{6t} - \lambda = 0 \\ \frac{\partial L}{\partial p_3} = \frac{tp_1+2(t+p_2)t_{13}-2p_3(t+t_{13})}{6tt_{13}} - \lambda = 0 \\ 2q - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. Condition $\frac{7t^2+29tt_{13}}{10t+14t_{13}} < q$ ensures marginal consumer m_{12} derives positive utility. We can verify that neither firm has profitable deviation.

(III) $q \leq \frac{7t^2+29tt_{13}}{10t+14t_{13}}$: In this case, both marginal consumers m_{23} and m_{12} derive zero utility because $q \leq \frac{7t^2+29tt_{13}}{10t+14t_{13}}$. When Firm 1 prices its product such that marginal consumer m_{12} derives zero utility, $2q - t - p_1 - p_2 = 0$. Using the Lagrange-multiplier method, we have $L_1(p_1, \lambda_1) = \pi_1 + \lambda_1(2q - t - p_1 - p_2)$. When Firm 2 prices its products such that marginal consumer m_{23} derives zero utility, $2q - t - p_2 - p_3 = 0$. Using the Lagrange-multiplier method, we have $L_2(p_2, p_3, \lambda_2) =$

$\pi_2 + \lambda_2(2q - t - p_2 - p_3)$. By solving the first-order conditions of $L_1(p_1, \lambda_1)$ and $L_2(p_2, p_3, \lambda_2)$,

$$\begin{cases} \frac{\partial L_1}{\partial p_1} = \frac{tp_3+(2t+p_2)t_{13}-2p_1(t+t_{13})}{6tt_{13}} - \lambda_1 = 0 \\ \frac{\partial L_2}{\partial p_2} = \frac{p_1-4p_2+2p_3+2t}{6t} - \lambda_2 = 0 \\ \frac{\partial L_2}{\partial p_3} = \frac{tp_1+2(t+p_2)t_{13}-2p_3(t+t_{13})}{6tt_{13}} - \lambda_2 = 0 \\ 2q - t - p_1 - p_2 = 0 \\ 2q - t - p_2 - p_3 = 0 \end{cases}$$

we can derive (p_1^*, p_2^*, p_3^*) as specified in the lemma. We can verify that neither firm has incentive to deviate.

Substituting p_i^* into Equations (9) and (10), we can derive the equilibrium demands and profits as specified in the lemma. □

In the first case in Equation (76), the valuation of the three products is high such that Firms 1 and 2 compete aggressively with each other in both Submarkets $\overline{12}$ and $\overline{13}$, which shapes the equilibrium prices. In the second case, due to the reduction in the product valuation, in equilibrium, Firm 2 chooses the optimal prices for its two products to just cover all the consumers in its exclusive Submarket $\overline{23}$ and extracts all the marginal consumer's surplus. Firm 1 reacts to its rival's prices by optimally choosing its price for Product 1 to compete against both Products 2 and 3. In the third case, due to the low product valuation, Firm 1 finds it optimal to set the highest price to serve the entire residual demand in Submarket $\overline{12}$ such that Submarket $\overline{12}$ is just fully covered, albeit it still competes aggressively with Product 3 in submarket $\overline{13}$.

By comparing the equilibrium profits with and without the introduction of Product 3, we can derive the conditions under which the focal firm would prefer to share its technology in equilibrium. Under this model variant, because Product 3 offers value comparable to Product 1, we can show that Firm 2 always has incentive to adopt the technology if shared. Therefore, the conditions for the focal firm to share its technology are the same as the conditions for the new-product introduction in equilibrium. We summarize the conditions in the following proposition.

Proposition 14. *In equilibrium, the new product would be introduced to the market if and only if* *where* $\underline{\underline{q}}(\theta) \leq q \leq \bar{\bar{q}}(\theta)$, *where*

$$\bar{\bar{q}}(\theta) = \begin{cases} \frac{82t^2t_{13}+27tt_{13}^2+9t_{13}^3}{18t^2+27tt_{13}+9t_{13}^2} & \text{if } \ddddot{t}_{13} < t_{13} \\ \begin{array}{l} \frac{1}{128}\left[\frac{9t^2}{2t_{13}}+361t_{13}+\frac{1}{2}t\left(-702-\frac{32t}{t+t_{13}}+\frac{1881t}{2t+t_{13}}\right)\right]- \\ \frac{1}{128}\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}(t+t_{13})(2t+t_{13})} \end{array} & \text{if } \ddot{t}_{13} < t_{13} \leq \ddddot{t}_{13} \\ \begin{cases} \begin{array}{l} \frac{1}{128}\left[\frac{9t^2}{2t_{13}}+361t_{13}+\frac{1}{2}t\left(-702-\frac{32t}{t+t_{13}}+\frac{1881t}{2t+t_{13}}\right)\right]- \\ \frac{1}{128}\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}(t+t_{13})(2t+t_{13})} \end{array} & \text{if } \tilde{q} < q \\ \begin{array}{l} \frac{171t^5+3804t^4t_{13}+14338t^3t_{13}^2+20868t^2t_{13}^3+13227tt_{13}^4+2888t_{13}^5}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \\ -\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{\left(3249t^6+85032t^5t_{13}+127738t^4t_{13}^2-47308t^3t_{13}^3-143175t^2t_{13}^4-60772tt_{13}^5-6236t_{13}^6\right)}}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \end{array} & \text{otherwise,} \end{cases} & \text{if } \dot{t}_{13} < t_{13} \leq \ddot{t}_{13} \\ \begin{array}{l} \frac{1}{128}\left[\frac{9t^2}{2t_{13}}+361t_{13}+\frac{1}{2}t\left(-702-\frac{32t}{t+t_{13}}+\frac{1881t}{2t+t_{13}}\right)\right]- \\ \frac{1}{128}\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}(t+t_{13})(2t+t_{13})} \end{array} & \text{otherwise,} \end{cases}$$

and

$$\underline{\underline{q}}(\theta) = \begin{cases} \begin{array}{l} \frac{1}{2\left(72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}(54t+11t_{13})\right)\right)\right)\right)} \\ \left[tt_{13}\left(-84t^4-1904t^3t_{13}+30069t^2t_{13}^2+31086tt_{13}^3+7865t_{13}^4\right)-\right. \\ \left.6tt_{13}\left(t+11t_{13}\right)\left(24t+11t_{13}\right)\sqrt{-12t^4-286t^3t_{13}+24t^2t_{13}^2+242tt_{13}^3+81t_{13}^4}\right] \end{array} & \text{if } \dddot{t}_{13} < t_{13} \\ \begin{array}{l} -6\frac{t(t-t_{13})t_{13}(-2t+t_{13})(24t+11t_{13})\left(3t^2+26tt_{13}+19t_{13}^2\right)}{324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6}\sqrt{\frac{t+t_{13}}{2t+t_{13}}} \\ +\frac{tt_{13}\left(-378t^5-15831t^4t_{13}-15072t^3t_{13}^2+13246t^2t_{13}^3+17522tt_{13}^4+5217t_{13}^5\right)}{2\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)} \end{array} & \text{if } \ddot{t}_{13} < t_{13} \leq \dddot{t}_{13} \\ \begin{cases} \begin{array}{l} \frac{171t^5+3804t^4t_{13}+14338t^3t_{13}^2+20868t^2t_{13}^3+13227tt_{13}^4+2888t_{13}^5}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \\ +\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{\left(3249t^6+85032t^5t_{13}+127738t^4t_{13}^2-47308t^3t_{13}^3-143175t^2t_{13}^4-60772tt_{13}^5-6236t_{13}^6\right)}}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \end{array} & \text{if } \tilde{q} < q \\ \begin{array}{l} -6\frac{t(t-t_{13})t_{13}(-2t+t_{13})(24t+11t_{13})\left(3t^2+26tt_{13}+19t_{13}^2\right)}{324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6}\sqrt{\frac{t+t_{13}}{2t+t_{13}}} \\ +\frac{tt_{13}\left(-378t^5-15831t^4t_{13}-15072t^3t_{13}^2+13246t^2t_{13}^3+17522tt_{13}^4+5217t_{13}^5\right)}{2\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)} \end{array} & \text{otherwise,} \end{cases} & \text{if } \dot{t}_{13} < t_{13} \leq \ddot{t}_{13} \\ \begin{array}{l} \frac{171t^5+3804t^4t_{13}+14338t^3t_{13}^2+20868t^2t_{13}^3+13227tt_{13}^4+2888t_{13}^5}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \\ +\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{\left(3249t^6+85032t^5t_{13}+127738t^4t_{13}^2-47308t^3t_{13}^3-143175t^2t_{13}^4-60772tt_{13}^5-6236t_{13}^6\right)}}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \end{array} & \text{otherwise.} \end{cases}$$

where $\tilde{q} = \frac{171t^5+3804t^4\ddot{t}_{13}+14338t^3\ddot{t}_{13}{}^2+20868t^2\ddot{t}_{13}{}^3+13227t\ddot{t}_{13}{}^4+2888\ddot{t}_{13}{}^5}{6\left(9t^4+668t^3\ddot{t}_{13}+1814t^2\ddot{t}_{13}{}^2+1628t\ddot{t}_{13}{}^3+489\ddot{t}_{13}{}^4\right)}$, $\ddddot{t}_{13} = \frac{8+(2366+54\sqrt{2987})^{1/3}-(-2366+54\sqrt{2987})^{1/3}}{18}t$, $\dddot{t}_{13}$ *is the highest root of*

$$-98t^5-1057t^4x-1156t^3x^2+1066t^2x^3+1322tx^4+315x^5=0$$

$\ddot{t}_{13}$ *is the highest root of*

$$-3249t^6-85032t^5x-127738t^4x^2+47308t^3x^3+143175t^2x^4+60772tx^5+6236x^6=0$$

and $t_{13}^{\cdot}$ *is the fifth highest root of*

$$-4194t^7-177537t^6x-343180t^5x^2-28946t^4x^3+335130t^3x^4+265999t^2x^5+73436tx^6+6300x^7=0.$$

Proof. Because the conditions for the focal firm to share its technology are the same as the conditions for new-product introduction in equilibrium, we compare Firm 1's equilibrium profits before and after the introduction of Product 3 for different cases, based on Lemmas 10 and 11.

The case with $\frac{7t-t_{13}}{3}<q$**:** The first case of Lemma 10 overlaps with the first and the second cases of Lemma 11.

(I) When $\frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}<q$, we have

$$\begin{aligned}\hat{\pi}_1^*-\pi_1^* &= \tfrac{32tt_{13}}{27(t+t_{13})}-\tfrac{(q-t_{13})(2t+t_{13})}{6t}\\ &= -\tfrac{(2t+t_{13})}{6t}\left[q-\tfrac{82t^2t_{13}+27tt_{13}^2+9t_{13}^3}{18t^2+27tt_{13}+9t_{13}^2}\right]\end{aligned}$$

Therefore, $\hat{\pi}_1^*\geq\pi_1^*$ if and only if $q\leq\frac{82t^2t_{13}+27tt_{13}^2+9t_{13}^3}{18t^2+27tt_{13}+9t_{13}^2}=\bar{\bar{q}}_a(\theta)$.

(II) When $\frac{7t^2+29tt_{13}}{10t+14t_{13}}<q\leq\frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}$, we have

$$\begin{aligned}\hat{\pi}_1^*-\pi_1^* &= \tfrac{8t_{13}(t+t_{13})((4q-t)t+2(q+2t)t_{13})^2}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2}-\tfrac{(q-t_{13})(2t+t_{13})}{6t}\\ &= \tfrac{32t_{13}(t+t_{13})(2t+t_{13})^2}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2}\\ &\times\left[q-\tfrac{1}{128}\left(\tfrac{9t^2}{2t_{13}}+361t_{13}-\tfrac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}(t+t_{13})(2t+t_{13})}-\tfrac{1}{2}t\left(702+\tfrac{32t}{t+t_{13}}-\tfrac{1881t}{2t+t_{13}}\right)\right)\right]\\ &\times\left[q-\tfrac{1}{128}\left(\tfrac{9t^2}{2t_{13}}+361t_{13}+\tfrac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}(t+t_{13})(2t+t_{13})}-\tfrac{1}{2}t\left(702+\tfrac{32t}{t+t_{13}}-\tfrac{1881t}{2t+t_{13}}\right)\right)\right]\end{aligned}$$

We can verify that the term in the second bracket is negative. Therefore, $\hat{\pi}_1^*\geq\pi_1^*$ if and only if

$$q\leq\frac{1}{128}\left(\frac{9t^2}{2t_{13}}+361t_{13}-\frac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{(t-t_{13})\left(9t^3+293t^2t_{13}+187tt_{13}^2-105t_{13}^3\right)}}{t_{13}\left(t+t_{13}\right)\left(2t+t_{13}\right)}-\frac{1}{2}t\left(702+\frac{32t}{t+t_{13}}-\frac{1881t}{2t+t_{13}}\right)\right)=\bar{\bar{q}}_b(\theta).$$

The case with $\frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}}<q\leq\frac{7t-t_{13}}{3}$**:** The second case of Lemma 10 overlaps with the first, second, and the third cases of Lemma 11.

(I) When $\frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}<q$, we can verify that $\hat{\pi}_1^*<\pi_1^*$.

(II) When $\frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}$, we have

$$\begin{aligned}
\hat{\pi}_1^* - \pi_1^* \quad &= \tfrac{8t_{13}(t+t_{13})((4q-t)t+2(q+2t)t_{13})^2}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2} - \tfrac{(q-t_{13})(-3q+19t+5t_{13})}{36t} \\
&= \tfrac{9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4}{12t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2} \\
&\times \left[q - \left(\frac{171t^5 + 3804t^4t_{13} + 14338t^3t_{13}^2 + 20868t^2t_{13}^3 + 13227tt_{13}^4 + 2888t_{13}^5}{6\left(9t^4 + 668t^3t_{13} + 1814t^2t_{13}^2 + 1628tt_{13}^3 + 489t_{13}^4\right)}\right.\right. \\
&\quad - \left(3t^2 + 26tt_{13} + 19t_{13}^2\right) \times \\
&\quad \left.\left.\frac{\sqrt{\left(3249t^6 + 85032t^5t_{13} + 127738t^4t_{13}^2 - 47308t^3t_{13}^3 - 143175t^2t_{13}^4 - 60772tt_{13}^5 - 6236t_{13}^6\right)}}{6\left(9t^4 + 668t^3t_{13} + 1814t^2t_{13}^2 + 1628tt_{13}^3 + 489t_{13}^4\right)}\right)\right] \\
&\times \left[q - \left(\frac{171t^5 + 3804t^4t_{13} + 14338t^3t_{13}^2 + 20868t^2t_{13}^3 + 13227tt_{13}^4 + 2888t_{13}^5}{6\left(9t^4 + 668t^3t_{13} + 1814t^2t_{13}^2 + 1628tt_{13}^3 + 489t_{13}^4\right)}\right.\right. \\
&\quad + \left(3t^2 + 26tt_{13} + 19t_{13}^2\right) \times \\
&\quad \left.\left.\frac{\sqrt{\left(3249t^6 + 85032t^5t_{13} + 127738t^4t_{13}^2 - 47308t^3t_{13}^3 - 143175t^2t_{13}^4 - 60772tt_{13}^5 - 6236t_{13}^6\right)}}{6\left(9t^4 + 668t^3t_{13} + 1814t^2t_{13}^2 + 1628tt_{13}^3 + 489t_{13}^4\right)}\right)\right]
\end{aligned}$$

We can verify that $\hat{\pi}_1^* \geq \pi_1^*$ if and only if

$$\begin{aligned}
q < &\tfrac{171t^5+3804t^4t_{13}+14338t^3t_{13}^2+20868t^2t_{13}^3+13227tt_{13}^4+2888t_{13}^5}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \\
&- \tfrac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{\left(3249t^6+85032t^5t_{13}+127738t^4t_{13}^2-47308t^3t_{13}^3-143175t^2t_{13}^4-60772tt_{13}^5-6236t_{13}^6\right)}}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} = \bar{\bar{q}}_c(\theta)
\end{aligned}$$

or

$$\begin{aligned}
q > &\tfrac{171t^5+3804t^4t_{13}+14338t^3t_{13}^2+20868t^2t_{13}^3+13227tt_{13}^4+2888t_{13}^5}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} \\
&+ \tfrac{\left(3t^2+26tt_{13}+19t_{13}^2\right)\sqrt{\left(3249t^6+85032t^5t_{13}+127738t^4t_{13}^2-47308t^3t_{13}^3-143175t^2t_{13}^4-60772tt_{13}^5-6236t_{13}^6\right)}}{6\left(9t^4+668t^3t_{13}+1814t^2t_{13}^2+1628tt_{13}^3+489t_{13}^4\right)} = \underline{\underline{q_a}}(\theta)
\end{aligned}$$

(III) When $q \leq \frac{7t^2+29tt_{13}}{10t+14t_{13}}$, we can verify that $\hat{\pi}_1^* < \pi_1^*$.

The case with $q \leq \frac{31tt_{13}+11t_{13}^2}{12t+9t_{13}}$: The third case of Lemma 10 overlaps with the second and the third cases of Lemma 11.

(I) When $\frac{7t^2+29tt_{13}}{10t+14t_{13}} < q \leq \frac{9t^3+43t^2t_{13}+26tt_{13}^2}{12t^2+18tt_{13}+6t_{13}^2}$, we have

$$\begin{aligned}
\hat{\pi}_1^* - \pi_1^* &= \tfrac{8t_{13}(t+t_{13})((4q-t)t+2(q+2t)t_{13})^2}{3t\left(3t^2+26tt_{13}+19t_{13}^2\right)^2} - \tfrac{(2t+t_{13})[12qt+(2q+7t)t_{13}]^2}{6tt_{13}(24t+11t_{13})^2} \\
&= -\tfrac{2(2t+t_{13})\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)}{3tt_{13}(24t+11t_{13})^2\left(3t^2+26tt_{13}+19t_{13}^2\right)^2} \\
&\quad \times \left[q - \left(\frac{tt_{13}\left(-378t^5-15831t^4t_{13}-15072t^3t_{13}^2+13246t^2t_{13}^3+17522tt_{13}^4+5217t_{13}^5\right)}{2\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)}\right.\right. \\
&\quad \left.\left. - 6\frac{t\left(t-t_{13}\right)t_{13}\left(-2t+t_{13}\right)\left(24t+11t_{13}\right)\left(3t^2+26tt_{13}+19t_{13}^2\right)}{324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6}\sqrt{\frac{t+t_{13}}{2t+t_{13}}}\right)\right] \\
&\quad \times \left[q - \left(\frac{tt_{13}\left(-378t^5-15831t^4t_{13}-15072t^3t_{13}^2+13246t^2t_{13}^3+17522tt_{13}^4+5217t_{13}^5\right)}{2\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)}\right.\right. \\
&\quad \left.\left. + 6\frac{t\left(t-t_{13}\right)t_{13}\left(-2t+t_{13}\right)\left(24t+11t_{13}\right)\left(3t^2+26tt_{13}+19t_{13}^2\right)}{324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6}\sqrt{\frac{t+t_{13}}{2t+t_{13}}}\right)\right]
\end{aligned}$$

Because $-2t+t_{13} < 0$, and we can verify that $324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6 < 0$ and the term in the second square bracket is positive, $\hat{\pi}_1^* \geq \pi_1^*$ if and only if

$$\begin{aligned}
q > & \frac{tt_{13}\left(-378t^5-15831t^4t_{13}-15072t^3t_{13}^2+13246t^2t_{13}^3+17522tt_{13}^4+5217t_{13}^5\right)}{2\left(324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6\right)} \\
& -6\frac{t\left(t-t_{13}\right)t_{13}\left(-2t+t_{13}\right)\left(24t+11t_{13}\right)\left(3t^2+26tt_{13}+19t_{13}^2\right)}{324t^6+5724t^5t_{13}+11889t^4t_{13}^2+660t^3t_{13}^3-12790t^2t_{13}^4-8936tt_{13}^5-1575t_{13}^6}\sqrt{\frac{t+t_{13}}{2t+t_{13}}} = \underline{\underline{\underline{q_b}}}(\theta)
\end{aligned}$$

(II) When $q \leq \frac{7t^2+29tt_{13}}{10t+14t_{13}}$, we have

$$\begin{aligned}
\hat{\pi}_1^* - \pi_1^* &= \tfrac{(2q-t)t_{13}(t(2q+t)+(-2q+23t)t_{13})}{t(t+11t_{13})^2} - \tfrac{(2t+t_{13})[12qt+(2q+7t)t_{13}]^2}{6tt_{13}(24t+11t_{13})^2} \\
&= -\tfrac{2\left(72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}(54t+11t_{13})\right)\right)\right)\right)}{3tt_{13}(t+11t_{13})^2(24t+11t_{13})^2} \\
&\quad \times \left[q - \left(\frac{tt_{13}\left(-84t^4-1904t^3t_{13}+30069t^2t_{13}^2+31086tt_{13}^3+7865t_{13}^4\right)}{2\left(72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}\left(54t+11t_{13}\right)\right)\right)\right)\right)}\right.\right. \\
&\quad \left.\left. - \frac{3tt_{13}\left(t+11t_{13}\right)\left(24t+11t_{13}\right)\sqrt{-12t^4-286t^3t_{13}+24t^2t_{13}^2+242tt_{13}^3+81t_{13}^4}}{72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}\left(54t+11t_{13}\right)\right)\right)\right)}\right)\right] \\
&\quad \times \left[q - \left(\frac{tt_{13}\left(-84t^4-1904t^3t_{13}+30069t^2t_{13}^2+31086tt_{13}^3+7865t_{13}^4\right)}{2\left(72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}\left(54t+11t_{13}\right)\right)\right)\right)\right)}\right.\right. \\
&\quad \left.\left. + \frac{3tt_{13}\left(t+11t_{13}\right)\left(24t+11t_{13}\right)\sqrt{-12t^4-286t^3t_{13}+24t^2t_{13}^2+242tt_{13}^3+81t_{13}^4}}{72t^5+t_{13}\left(1644t^4+t_{13}\left(6590t^3+t_{13}\left(7857t^2+77t_{13}\left(54t+11t_{13}\right)\right)\right)\right)}\right)\right]
\end{aligned}$$

We can verify that the term in the second square bracket is negative. Therefore, $\hat{\pi}_1^* \geq \pi_1^*$ if and only if

$$q > \frac{tt_{13}\left(-84t^4 - 1904t^3t_{13} + 30069t^2t_{13}^2 + 31086tt_{13}^3 + 7865t_{13}^4\right)}{2\left(72t^5 + t_{13}\left(1644t^4 + t_{13}\left(6590t^3 + t_{13}\left(7857t^2 + 77t_{13}\left(54t + 11t_{13}\right)\right)\right)\right)\right)} - \frac{3tt_{13}\left(t + 11t_{13}\right)\left(24t + 11t_{13}\right)\sqrt{-12t^4 - 286t^3t_{13} + 24t^2t_{13}^2 + 242tt_{13}^3 + 81t_{13}^4}}{72t^5 + t_{13}\left(1644t^4 + t_{13}\left(6590t^3 + t_{13}\left(7857t^2 + 77t_{13}\left(54t + 11t_{13}\right)\right)\right)\right)} = \underline{\underline{q_c}}(\theta)$$

We can verify that $\bar{\bar{q}}_a(\theta)$ and $\bar{\bar{q}}_b(\theta)$ intersect at $\ddddot{t}_{13}$, $\underline{\underline{q_b}}(\theta)$ and $\underline{\underline{q_c}}(\theta)$ intersect at $\dddot{t}_{13}$, $\underline{\underline{q_a}}(\theta)$ and $\bar{\bar{q}}_c(\theta)$ intersect at $\left(\ddot{t}_{13},\tilde{q}\right)$, and $\underline{\underline{q_b}}(\theta)$ and $\bar{\bar{q}}_c(\theta)$ intersect at $\dot{t}_{13}$, with $\ddddot{t}_{13} > \dddot{t}_{13} > \ddot{t}_{13} > \dot{t}_{13}$. Altogether, we have the results as in Proposition 14. $\square$

Evidently, Proposition 14 reveals the same insights as those delivered by our baseline model—the new product can be introduced in equilibrium only when q is neither too high nor too low (i.e., $\underline{\underline{q}}(\theta) \leq q \leq \bar{\bar{q}}(\theta)$ in this case). This ensures that the entry of the new product exerts cannibalization externality on the focal firm without imposing excessive competition pressure on it.

In summary, even if the differentiation between Products 1 and 3 is lower than that between Products 1 and 2 or Products 2 and 3, our main insights remain unchanged.