Unlocking Profits in Generative AI: The Impact of AI Adaptive Learning on Freemium Strategy

I Proofs of Main and Supporting Results

A. Proof of Lemma 1

We solve the game through backward induction. Under the freemium strategy, both the premium version and the free version are available in the market.

(1) We first derive the demand functions $N_P^V(p)$, $N_F^V(p)$, and consumer surplus $cs^V(p)$ given p.

Consumers can choose among three options: purchasing the premium version, using the free version for free, or opting out. The boundary consumer who is indifferent between purchasing the premium version and using the free version, $\tilde{v}_{PF}$, can be derived by solving $\tilde{v}_{PF} + \alpha\Delta N - p = q\tilde{v}_{PF} + q\alpha\Delta N - c$, which yields $\tilde{v}_{PF} = \frac{p-c}{1-q} - \alpha\Delta N$, where $\Delta N = N_P + N_F$. Similarly, the boundary consumer who is indifferent between using the premium version and opting out is, $\tilde{v}_P = p - \alpha\Delta N$; the boundary consumer who is indifferent between using the free version and opting out is $\tilde{v}_F = \frac{c}{q} - \alpha\Delta N$. Consumers with $v > \tilde{v}_{PF}$ prefer to purchase the premium version over the free version, and vice versa; similarly, consumers with $v < \tilde{v}_P$ (or $\tilde{v}_F$) prefer to opt out of purchasing the premium version (or using the free version), and vice versa.

In a rational expectations equilibrium (REE), consumers have rational expectations about the total user base when making purchase decisions. Therefore, $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F\}$ and ΔN must be solved simultaneously. Because $\Delta N = N_P + N_F$ depends on $\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F$ and the support $v \in [0,1]$, we solve $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, \Delta N\}$ jointly. Comparing $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, 0\}$ results in two cases under different parameter conditions: (a) partially covered market, and (b) fully covered market.

(a) Partially covered market under the freemium strategy

Suppose $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ and the market is partially covered. Then $N_P = 1 - \tilde{v}_{PF}$, $N_F = \tilde{v}_{PF} - \tilde{v}_F$, and $cs^V = \int_{\tilde{v}_{PF}}^{1} (v + \alpha\Delta N - p)dv + \int_{\tilde{v}_F}^{\tilde{v}_{PF}} (qv + q\alpha\Delta N - c)dv$. Substituting $\tilde{v}_{PF}$ and $\tilde{v}_F$ into $N_P = 1 - \tilde{v}_{PF}$ and $N_F = \tilde{v}_{PF} - \tilde{v}_F$, then solving for N_P and N_F simultaneously gives

$$N_P^V(p) = \frac{1}{q}\left[\frac{q-c}{1-\alpha} - \frac{pq-c}{1-q}\right] \text{ and } N_F^V(p) = \frac{pq-c}{q(1-q)}, \tag{A.1}$$

$$cs^V(p) = \frac{1}{2q^2}\left[\frac{(q-c)^2}{(1-\alpha)^2} - \frac{(pq-c)^2}{1-q} - \frac{2(q-c)(pq-c)}{1-\alpha}\right]. \tag{A.2}$$

Hence $\Delta N = N_P^V + N_F^V = \frac{q-c}{q(1-\alpha)} < 1$. Substituting $\Delta N = \frac{q-c}{q(1-\alpha)}$ into $\tilde{v}_{PF}$, $\tilde{v}_P$, and $\tilde{v}_F$, we can have $\tilde{v}_{PF} = \frac{p-c}{1-q} - \alpha\frac{q-c}{q(1-\alpha)}$, $\tilde{v}_P = p - \alpha\frac{q-c}{q(1-\alpha)}$ and $\tilde{v}_F = \frac{c}{q} - \alpha\frac{q-c}{q(1-\alpha)}$. We then need to verify under what conditions $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ holds. Solving the inequalities after substituting the solutions of $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F\}$, we have the conditions: $\alpha q < c < q$, $\frac{c}{q} < p < \frac{[(1+c)q] - q^2 - c\alpha}{q(1-\alpha)}$. It is straightforward that (1) $p \geq \frac{[(1+c)q] - q^2 - c\alpha}{q(1-\alpha)}$ will result in non-positive profit for the firm; (2) $0 < p \leq \frac{c}{q}$ will result in non-positive free version demand, which will always be dominated by the paid-only strategy. Therefore, we only focus on the case where $\frac{c}{q} < p < \frac{[(1+c)q] - q^2 - c\alpha}{q(1-\alpha)}$ under the freemium strategy.

(b) Fully covered market under the freemium strategy

Suppose $(\tilde{v}_P <)\tilde{v}_F < 0 < \tilde{v}_{PF} < 1$, and the market is fully covered. Then $N_P = 1 - \tilde{v}_{PF}$, $N_F = \tilde{v}_{PF}$ and $cs^V = \int_{\tilde{v}_{PF}}^{1}(v + \alpha\Delta N - p)dv + \int_0^{\tilde{v}_{PF}}(qv + q\alpha\Delta N - c)dv$. Performing in a similar way to the partially covered market, we can have the demands in this case as follows.

$$\widehat{N}_P^V(p) = 1 + \alpha - \frac{p-c}{1-q}, \widehat{N}_F^V(p) = \frac{p-c}{1-q} - \alpha, \tag{A.3}$$

$$\widehat{cs}^V(p) = \frac{1+(p-c)^2-q}{2(1-q)} - p + (1+c-p)\alpha + \frac{1}{2}(1-q)\alpha^2, \tag{A.4}$$

where the superscript ˆ is used to distinguish the case of a fully covered market from the case of a partially covered market. Hence $\Delta\widehat{N} = \widehat{N}_P^V + \widehat{N}_F^V = 1$. Besides, $(\tilde{v}_P <)\tilde{v}_F < 0 < \tilde{v}_{PF} < 1$ holds if $0 < c < \alpha q$ and $c + (1-q)\alpha < p < c + (1-q)(1+\alpha)$. This is because that $p \geq c + (1-q)(1+\alpha)$ leads to non-positive profit and $p \leq c + (1-q)\alpha$ leads to non-positive free version demand, thus cannot be optimal.

(2) We next derive the firm's optimal choice of the price p^V.

(a) Partially covered market under the freemium strategy

Under the precondition of this case, i.e., $\frac{c}{q} < p < \frac{[(1+c)q]-q^2-c\alpha}{q(1-\alpha)}$, the firm's profit π^V, as a function of the premium price p, is

$$\pi^V(p) = pN_P - \frac{1}{2}s(N_P + N_F)^2. \tag{A.5}$$

Substituting (A.1) into (A.5) and solving for the first-order condition (FOC) gives the solution as

$$p^{V*} = \frac{(1+c)q-q^2-c\alpha}{2q(1-\alpha)}. \tag{A.6}$$

Besides, $\frac{c}{q} < p < \frac{[(1+c)q]-q^2-c\alpha}{q(1-\alpha)}$ requires $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$, where $\frac{q(1-q)}{2-q-\alpha} < q$ always holds. Substituting p^{V*} into (A.1) and (A.2) yields optimal demands and the optimal consumer surplus as follows

$$N_P^{V*} = \frac{1}{2q}\left[\frac{q-c}{1-\alpha} + \frac{c}{1-q}\right], N_F^{V*} = \frac{1}{2q}\left[\frac{q-c}{1-\alpha} - \frac{c}{1-q}\right], \tag{A.7}$$

$$cs^{V*} = \frac{1}{8q^2}\left[\frac{c^2}{1-q} + \frac{2c(q-c)}{1-\alpha} + \frac{(q-c)^2(1+3q)}{(1-\alpha)^2}\right]. \tag{A.8}$$

Finally, substituting p^{V*}, N_P^{V*} and N_F^{V*} into the firm's profit π^V, we have the firm's optimal profit as follows

$$\pi^{V*} = \frac{1}{4q^2}\left[\frac{c^2}{1-q} + \frac{2c(q-c)}{1-\alpha} + \frac{(q-c)^2(1-q-2s)}{(1-\alpha)^2}\right]. \tag{A.9}$$

It is worth noting that $\pi^{V*} \geq 0$ only if $0 < s \leq \min\left\{\frac{(\alpha c - cq + q^2 - q)^2}{2(1-q)(q-c)^2}, 1\right\}$.

In summary, the freemium strategy is viable when

$$q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha} \text{ and } 0 < s \leq \min\left\{\frac{(\alpha c - cq + q^2 - q)^2}{2(1-q)(q-c)^2}, 1\right\} \tag{A.10}$$

(b) Fully covered market under the freemium strategy

In a similar way to the partially covered market, for a fully covered market, we can have the firm's equilibrium price, resulting demands, profit, and consumer surplus as follows

$$\hat{p}^{V*} = \frac{1}{2}(1 + c - q + \alpha - q\alpha), \tag{A.11}$$

$$\widehat{N}_P^{V*} = 1 - \frac{1-c-q-\alpha+q\alpha}{2(1-q)}, \widehat{N}_F^{V*} = \frac{1-c-q-\alpha+q\alpha}{2(1-q)}, \tag{A.12}$$

$$\widehat{cs}^{V*} = \frac{1+(c-p)^2-2p-q+2pq}{2(1-q)} + (1 + c - p)\alpha + \frac{1}{2}(1 - q)\alpha^2, \tag{A.13}$$

$$\hat{\pi}^{V*} = \frac{(1+c-q+\alpha-q\alpha)^2}{4(1-q)} - \frac{s}{2}. \tag{A.14}$$

The conditions for positive demands and profit are:

$$0 < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } 0 < s \leq \min\left\{\frac{[c+(1-q)(1+\alpha)]^2}{2(1-q)}, 1\right\}. \tag{A.15}$$

B. Proof of Lemma 2

Under the paid-only strategy, there is only the premium version available in the market. Similar to the proof of Lemma 1, we first derive the demand function $N_P^S(p)$, and consumer surplus $cs^S(p)$ given p.

Consumers can only choose between purchasing the premium version and opting out. The boundary consumer who is indifferent between using the premium version and opting out is $\tilde{v}_P = p - \alpha\Delta N$. Consumers with $v > \tilde{v}_P$ will buy the premium version.

In a rational expectations equilibrium (REE), consumers have rational expectations about the total user base when making purchase decisions. Comparing the relative magnitude of $\{\tilde{v}_P, 1\}$ results in two cases under different parameter conditions: (a) partially covered market, and (b) fully covered market.

(a) Partially covered market under the paid-only strategy

Suppose $0 < \tilde{v}_P < 1$ and the market is partially covered. Then $N_P = 1 - \tilde{v}_P$. Solving $N_P = 1 - \tilde{v}_P$ and $\Delta N = N_P$ simultaneously gives

$$N_P^S(p) = \frac{1-p}{1-\alpha}, \tag{B.1}$$

$$cs^S(p) = \frac{(1-p)^2}{2(1-\alpha)^2}. \tag{B.2}$$

Now we verify under what conditions $0 < \tilde{v}_P < 1$ holds. Solving the inequalities after substituting the solutions of N_P^S into $\tilde{v}_P$, we arrive at the condition: $p > \alpha$.

For the case where $p > \alpha$, the firm's profit π^S, as a function of the premium price p, is

$$\pi^S(p) = pN_P - \frac{1}{2}sN_P{}^2. \tag{B.3}$$

Substituting (B.1) into (B.3), and solving the FOC gives the solution

$$p^{S*} = \frac{1+s-\alpha}{2+s-2\alpha}. \tag{B.4}$$

$p > \alpha$ requires $\max\{0,2\alpha-1\} < s < 1$. Substituting p^{S*} into (B.1) and (B.2) yields the optimal demand and the optimal consumer surplus as follows

$$N_P^{S*} = \frac{1}{2+s-2\alpha}, \tag{B.5}$$

$$cs^{S*} = \frac{1}{2(2+s-2\alpha)^2}. \tag{B.6}$$

Finally, substituting p^{S*} and N_P^{S*} into the firm's profit π^S, we have the firm's optimal profit as follows

$$\pi^{S*} = \frac{1}{2(2+s-2\alpha)}. \tag{B.7}$$

It is worth noting that $\pi^{S*} \geq 0$ always holds.

(b) Fully covered market under the paid-only strategy

Performing a similar method used in the partially covered market, we have the equilibria for the fully covered market under the paid-only strategy as follows:

$$\hat{N}_P^{S*} = 1, \tag{B.8}$$

$$\hat{p}^{S*} = \alpha, \tag{B.9}$$

$$\widehat{cs}^{S*} = \frac{1}{2}, \tag{B.10}$$

$$\hat{\pi}^{S*} = \alpha - \frac{1}{2}s. \tag{B.11}$$

Nevertheless, $\hat{\pi}^{S*} \geq 0$ requires $0 < s < 2\alpha$.

Taken together, note that if $\max\{0,2\alpha-1\} < s < 2\alpha$, the firm can choose either to set a high price to only attract high-valuation consumers and realize the profit outlined in (B.7), or set a low price to capture the whole market and realize the profit outlined in (B.11). Consequently, we need to compare these two profits to see which one is the global optimal profit. This is equivalent to comparing $\frac{1}{2(2+s-2\alpha)}$ and $\alpha - \frac{1}{2}s$. Actually, $\frac{1}{2(2+s-2\alpha)} < \left(\alpha - \frac{1}{2}s\right)$ always holds within $2\alpha - 1 < s < 2\alpha$. Therefore, in summary:

(i) If $0 < s < \max\{0,2\alpha-1\}$, the market is fully covered, and the optimal equilibria are outlined in (B.8) to (B.11);

(ii) If $\max\{2\alpha-1,0\} < s < 1$, the market is partially covered, and the optimal equilibria are outlined in (B.4) to (B.7).

C. Proof of Propositions 1 and 2

As Propositions 1 and 2 are related, we prove them together. To proceed, we consider four cases given the market coverage conditions under different strategies.

Case 1: Partially covered market under both the freemium strategy and the paid-only strategy

In this case, by Lemmas 1 and 2, the freemium strategy yields a partially covered market only if $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$ and $\max\{0,2\alpha-1\} < s \leq \min\left\{\frac{(\alpha c-cq+q^2-q)^2}{2(1-q)(q-c)^2},1\right\}$. Moreover, under the paid-only strategy, a partially covered market requires $\max\{2\alpha-1,0\} < s < 1$. Taking the intersection of these constraints, Case 1 is characterized by

$$q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha} \text{ and } \max\{2\alpha-1,0\} < s \leq \min\left\{\frac{(\alpha c-cq+q^2-q)^2}{2(1-q)(q-c)^2},1\right\}. \quad \text{(C.1)}$$

Therefore, under these two conditions, we next compare the firm's profits under the freemium strategy and the paid-only strategy. In the following, we first prove the conditions under which the firm prefers the freemium strategy. Then we discuss the necessary conditions for the optimality of the freemium strategy.

(1) Proof of the conditions under which the firm prefers the freemium strategy

Let $\Delta\pi_1 = \pi^{V*} - \pi^{S*}$. Substituting the profits in (A.9) and (B.7) into $\Delta\pi_1$ yields

$$\Delta\pi_1 = \pi^{V*} - \pi^{S*} = \frac{A_1s^2+B_1s+C_1}{4(1-q)q^2(1-\alpha)^2(2-2\alpha+s)}, \quad \text{(C.2)}$$

where $A_1 = -2(q-c)^2(1-q) < 0$;

$B_1 = 2cq(1-q)(4+q-5\alpha) - q^2(1-q)(3+q-4\alpha) + q^2c^2 + qc^2(4-6\alpha+\alpha(4+\alpha)-4)$;

$C_1 = 2(q-\alpha)(1-\alpha)(2cq+c^2q-q^2-2cq^2+q^3-c^2\alpha)$.

We can verify that in (C.2), the quadratic function of $A_1s^2+B_1s+C_1$ in s has two real solutions (i.e., ${B_1}^2-4A_1C_1>0$) if $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$. Let $A_1s^2+B_1s+C_1=0$, denoting the solutions as s_1 and s_2, where $s_1 = \frac{-B_1+\sqrt{{B_1}^2-4A_1C_1}}{2A_1}$ and $s_2 = \frac{-B_1-\sqrt{{B_1}^2-4A_1C_1}}{2A_1}$. Moreover, if $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$, we can also have $s_1 < s_2$. Therefore, if $s_1 < s < s_2$, then $A_1s^2+B_1s+C_1>0$, which leads to $\pi^{V*} > \pi^{S*}$. Besides, we can verify that s_2 is below the upper feasibility bound in (C.1) under the parameter restrictions in (C.1), which means that $s_1 < s < s_2$ can ensure the feasibility of the freemium strategy as the optimal freemium strategy. Hence, together with the precondition of this case in (C.1), in this case, $\pi^{V*} > \pi^{S*}$ if

$$q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha} \text{ and } \max\{s_1, 2\alpha-1,0\} < s \leq s_2. \quad \text{(C.3)}$$

Furthermore, we can demonstrate the following conditions:

(i) if $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$, $s_2 > 0$;

(ii) if $c_1 < c \leq \frac{q(1-q)}{2-q-\alpha}$, $s_1 > \max\{2\alpha-1,0\}$,

where $c_1 \triangleq \begin{cases} \frac{q\left(1-q-\sqrt{(1-q)(1-\alpha)}\right)}{\alpha-q}, & q < \alpha \leq \frac{1}{2} \\ \frac{q\left[(1-q)(2-q-3\alpha)-(1-q)(1-q-\alpha)\sqrt{2-2q}\right]}{2+q^2-2q(1-\alpha)-(4-\alpha)\alpha}, & \frac{1}{2} < \alpha \leq 1-q \end{cases}.$

Combining these conditions with the preconditions outlined in (C.3), we summarize conditions where $\pi^{V*} > \pi^{S*}$ in Proposition 2.

(2) Proof of the necessary and sufficient conditions for the optimality of the freemium strategy

Establishing the necessary and sufficient condition for the optimality of the freemium strategy reduces to identifying parameter regions where the feasibility set in (C.3) is non-empty. In particular, this requires the intersection of the following conditions: $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$, $s_1 \le s_2$, $2\alpha - 1 \le s_2$ and $0 \le s_2$.

First, $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$ is non-empty if $\alpha < 1-q$. Second, $s_1 < s_2$ if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$ as we have shown before. Third, $2\alpha - 1 \le s_2$ requires

$$\alpha < 1-q. \tag{C.4}$$

Finally, $0 \le s_2$ requires

$$0 < q < \frac{1}{2} \text{ and } q < \alpha < 1-q. \tag{C.5}$$

Taken together, we can have the necessary and sufficient conditions for the optimality of a freemium strategy in this case as

$$0 < q < \frac{1}{2}, q < \alpha < 1-q. \tag{C.6}$$

Case 2: Partially covered market under the freemium strategy and fully covered market under the paid-only strategy

Similarly, the preconditions for this case are

$$q\alpha < c \le \frac{q(1-q)}{2-q-\alpha} \text{ and } 0 < s < \max\{0, 2\alpha - 1\}. \tag{C.7}$$

Let $\Delta\pi_2 = \pi^{V*} - \hat{\pi}^{S*}$. Substituting the profits in (A.9) and (B.11) into $\Delta\pi_2$ yields

$$\Delta\pi_2 = \pi^{V*} - \hat{\pi}^{S*} = \frac{(H_1 s - H_2)}{4(1-q)q^2(1-\alpha)^2}, \tag{C.8}$$

where $H_1 = -2(1-q)(c-q\alpha)(c-2q+q\alpha)$;

$H_2 = -\{q^4 - 2c(1+c)q\alpha + c^2\alpha^2 - 2q^3[1+c-2(1-\alpha)^2\alpha] + q^2[1+c^2-4(1-\alpha)^2\alpha + 2c(1+\alpha)]\}$.

As we can show if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$, $H_1 > 0$, $\Delta\pi_2$ increases in s. Letting the numerator in (C.8) equals 0 and solving it for s gives $s_{E1} = \frac{H_2}{H_1}$. Then if $s_{E1} < s < 1$, $\Delta\pi_2 > 0$. Hence, together with the precondition of this case in (C.7), in the paid-only fully covered market, $\pi^{V*} > \hat{\pi}^{S*}$ if

$$q\alpha < c \le \frac{q(1-q)}{2-q-\alpha} \text{ and } \max\{0, s_{E1}\} < s \le \max\{0, 2\alpha - 1\}. \tag{C.9}$$

Besides, we can show that $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$ is non-empty if $\alpha < 1-q$. $\max\{0, s_{E1}\} < s \le 2\alpha - 1$ is non-empty if $\alpha > \frac{1}{2}$. Therefore, the necessary conditions for the optimality of the freemium strategy in this case is

$$\frac{1}{2} < \alpha < 1-q. \tag{C.10}$$

Case 3: Fully covered market under the freemium strategy and partially covered market under the paid-only strategy

In this case, the preconditions for this case are

$$0 < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } \max\{2\alpha - 1,0\} < s \le \min\left\{\frac{[c+(1-q)(1+\alpha)]^2}{2(1-q)}, 1\right\}. \quad \text{(C.11)}$$

Let $\Delta\pi_3 = \hat{\pi}^{V*} - \pi^{S*}$. Substituting the profits in (A.14) and (B.7) into $\Delta\pi_3$ yields

$$\Delta\pi_3 = \hat{\pi}^{V*} - \pi^{S*} = \frac{A_2 s^2 + B_2 s + C_2}{4(1-q)(2-2\alpha+s)}, \quad \text{(C.12)}$$

where $A_2 = -(1-q) < 0$;

$B_2 = c^2 + 2c(1-q)(1+\alpha) - (1-q)[3 + q(1+\alpha)^2 - \alpha(6+\alpha)]$;

$C_2 = 2c^2(1-\alpha) + 4c(1-q)(1-\alpha^2) + 2(1-q)\{\alpha - (1+\alpha)[q + (1-q)\alpha^2]\}$.

We can verify that in (C.12), the quadratic function of $A_2 s^2 + B_2 s + C_2$ in s has two real solutions (i.e., ${B_2}^2 - 4A_2C_2 > 0$) if $c > c_{E1} = 2\sqrt{\alpha - q\alpha} - (1-q)(1+\alpha)$. Let $A_2 s^2 + B_2 s + C_2 = 0$, denoting its solutions as s_{E2} and s_{E3}, where $s_{E2} = \frac{-B_2 + \sqrt{{B_2}^2 - 4A_2C_2}}{2A_2}$ and $s_{E3} = \frac{-B_2 - \sqrt{{B_2}^2 - 4A_2C_2}}{2A_2}$. Besides, $s_{E2} < \max\{2\alpha - 1,0\} < s_{E3} < \min\left\{\frac{[c+(1-q)(1+\alpha)]^2}{2(1-q)}, 1\right\}$. Therefore, if $\max\{2\alpha - 1,0\} < s < s_{E3}$, then $A_2 s^2 + B_2 s + C_2 > 0$, which leads to $\hat{\pi}^{V*} > \pi^{S*}$. Hence, together with the precondition of this case, in this case, $\hat{\pi}^{V*} > \pi^{S*}$ if

$$c_{E1} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } \max\{2\alpha - 1,0\} < s \le s_{E3}. \quad \text{(C.13)}$$

Similarly, $c_{E1} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\}$ is non-empty if $\alpha < 1 - q$. The second interval $\max\{2\alpha - 1,0\} < s \le s_{E3}$ is non-empty if $\alpha > 1/2$. Taken together, the necessary conditions for the optimality of the freemium strategy in this case is

$$\frac{1}{2} < \alpha < 1 - q. \quad \text{(C.14)}$$

Case 4: Fully covered market under both the freemium strategy and the paid-only strategy

In this case, the preconditions are

$$0 < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } 0 < s < \max\{2\alpha - 1,0\}. \quad \text{(C.15)}$$

Let $\Delta\pi_4 = \hat{\pi}^{V*} - \hat{\pi}^{S*}$. Substituting the profits in (A.14) and (B.11) into $\Delta\pi_4$ yields

$$\Delta\pi_4 = \hat{\pi}^{V*} - \hat{\pi}^{S*} = \frac{c^2 + 2(1-q)(1+\alpha)c + (1-q)\left(1 - q - 2\alpha - 2q\alpha + \alpha^2 - q\alpha^2\right)}{4(1-q)}, \quad \text{(C.16)}$$

We can verify that in (C.16), the numerator is a quadratic function in c and has two real solutions $c_{E1} = 2\sqrt{\alpha - q\alpha} - (1-q)(1+\alpha)$ and $c_{E2} = -2\sqrt{\alpha - q\alpha} - (1-q)(1+\alpha) < 0$, where $c_{E1} > c_{E2}$. It is straightforward that if $\max\{c_{E1}, 0\} < c < 1$, $\hat{\pi}^{V*} > \hat{\pi}^{S*}$. Thus, together with the precondition of this case, in this case, $\hat{\pi}^{V*} > \hat{\pi}^{S*}$ if

$$\max\{c_{E1},0\} < c \leq \min\{q\alpha, q\alpha-\alpha-q+1\} \text{ and } 0 < s < \max\{2\alpha-1,0\}. \quad \text{(C.17)}$$

$\max\{c_{E1},0\} < c < \min\{q\alpha, q\alpha-\alpha-q+1\}$ is non-empty if $\alpha < 1-q$. $0 < s < \max\{2\alpha-1,0\}$ is non-empty if $\alpha > \frac{1}{2}$. Hence, the necessary conditions for the optimality of a freemium strategy in this case is

$$\frac{1}{2} < \alpha < 1-q. \quad \text{(C.18)}$$

In summary, the firm's optimal freemium strategy in our base model (Case 1) is qualitatively consistent with the other three cases under different market coverage conditions.

D. Proof of Proposition 3

(a) Partially covered market under the freemium strategy

Given the precondition of this case outlined in (A.10), we next prove each point one by one.

(i) The first boundary condition in (A.10) for the feasibility of the freemium strategy can be rewritten as $q \in \left(q_{D1}, \min\left\{\frac{c}{\alpha}, q_{D2}\right\}\right]$, where $q_{D1} = \frac{1}{2}\left(1+c-\sqrt{1-6c+c^2+4c\alpha}\right)$, $q_{D1} > c$ and $q_{D2} = \frac{1}{2}\left(1+c+\sqrt{1-6c+c^2+4c\alpha}\right)$. Under this condition, $\frac{\partial p^{V*}(q)}{\partial q} = \frac{c\alpha-q^2}{2q^2(1-\alpha)}$ is positive if $q \in \left(q_{D1}, \min\left\{\frac{c}{\alpha}, \sqrt{c\alpha}\right\}\right]$. Therefore, for any s satisfying the boundary condition in (A.10), we can have that the premium version price increases with q if $q \in \left(q_{D1}, \min\left\{\frac{c}{\alpha}, \sqrt{c\alpha}\right\}\right]$ and decreases with q if $q \in \left(\max\{q_{D1}, \sqrt{c\alpha}\}, \min\left\{\frac{c}{\alpha}, q_{D2}\right\}\right]$.

(ii) Similarly, differentiate N_P^{V*} in (A.7) wrt to q yields $\frac{\partial N_P^{V*}(q)}{\partial q} = \frac{c(q^2+\alpha-2q\alpha)}{2(1-q)^2q^2(1-\alpha)} > 0$, indicating that the premium version demand always increases in q. Differentiate N_F^{V*} wrt to q yields $\frac{\partial N_F^{V*}(q)}{\partial q} = \frac{c(2-\alpha-q(4-q-2\alpha))}{2(1-q)^2q^2(1-\alpha)}$, which is positive under the boundary conditions in (A.10). Therefore, the free version demand always increases with q.

(iii) We next rewrite the feasibility condition (A.10) as a restriction on q. We can show that (A.10) is equivalent to $q \in (q_L, q_H)$, where $q_L = \frac{1+c}{2} - \frac{1}{2}\sqrt{1-6c+c^2+4c\alpha}$, $q_H = \min\left\{\frac{c}{\alpha}, \frac{1+c}{2} + \frac{1}{2}\sqrt{1-6c+c^2+4c\alpha}, q_{H1}\right\}$ and q_{H1} is the root of $-2c^2s + c^2\alpha^2 + (4cs+2c^2s-2c\alpha-2c^2\alpha)q + (1+2c+c^2-2s-4cs+2c\alpha)q^2 + (-2-2c+2s)q^3 + q^4 = 0$.

Next, differentiating $\pi^{V*}(q)$ in (A.9) wrt q gives $\frac{\partial \pi^{V*}(q)}{\partial q} = \frac{f(s)}{4q^3(1-q)^2(1-\alpha)^2}$, where $f(s) = 4c(c-q)(1-q)^2s - (q+cq-q^2-c\alpha)(q^2-cq^2-q^3-2c\alpha+3cq\alpha)$. For any $q \in (q_L, q_H)$, the denominator is positive while the numerator always increases with s. Solving $f(s)=0$ yields $s = \tilde{s}(q)$. Hence, $\text{sign}\left(\frac{\partial \pi^{V*}(q)}{\partial q}\right) = \text{sign}(f(s)) = -\text{sign}(s-\tilde{s}(q))$. In particular, If $s < \tilde{s}(q)$, $\frac{\partial \pi^{V*}(q)}{\partial q} > 0$; if $s > \tilde{s}(q)$, $\frac{\partial \pi^{V*}(q)}{\partial q} < 0$. Moreover, since $\frac{\partial \tilde{s}(q)}{\partial q} < 0$ always holds if $q < \alpha < 1-q$ and $\max\left\{q\alpha, \frac{(1-q)q^2}{q^2+2\alpha-3q\alpha}\right\} < c < \frac{q(1-q)}{2-q-\alpha}$, it follows that $s = \tilde{s}(q)$ is strictly decreasing in q. Hence $\tilde{s}(q)$ maps (q_L, q_H) bijectively onto the open interval $(\tilde{s}_{\min}, \tilde{s}_{\max})$ with $\tilde{s}_{\min} = \tilde{s}(q_H)$ and $\tilde{s}_{\max} = \tilde{s}(q_L)$. Because $\tilde{s}(q)$ is strictly decreasing, it admits a unique inverse on this range, denoted by $q_1 \in (q_L, q_H)$ such that: for any $q_L < q < q_1$, $s =$

$\tilde{s}(q) > \tilde{s}(q_1)$, indicating $-\text{sign}(s - \tilde{s}(q)) > 0$, ultimately sign $\left(\frac{\partial \pi^{V*}(q)}{\partial q}\right) > 0$; for any $q_1 < q < q_H$, $s = \tilde{s}(q) < \tilde{s}(q_1)$, indicating $-\text{sign}(s - \tilde{s}(q)) < 0$, ultimately sign $\left(\frac{\partial \pi^{V*}(q)}{\partial q}\right) < 0$. Therefore, when $s \in (\tilde{s}(q_H), \tilde{s}(q_L))$, $\pi^{V*}(q)$ is strictly increasing in q if $q_L < q < q_1$, and strictly decreasing in q if $q_1 < q < q_H$.

(b) Fully covered market under the freemium strategy

(i) Differentiating $\hat{p}^{V*}$ wrt q gives $\frac{\partial \hat{p}^{V*}}{\partial q} = -\frac{1}{2}(1+\alpha) < 0$. Therefore, as q increases, the optimal premium version price always decreases.

(ii) Differentiating $\hat{N}_P^{V*}$ wrt q gives $\frac{\partial \hat{N}_P^{V*}}{\partial q} = \frac{c}{2(1-q)^2} > 0$, and differentiating $\hat{N}_F^{V*}$ wrt q gives $\frac{\partial \hat{N}_F^{V*}}{\partial q} = -\frac{c}{2(1-q)^2} < 0$. Therefore, as q increases, the demand for the premium version increases while the demand for the free version decreases.

(iii) Differentiating $\hat{\pi}^{V*}$ wrt q gives $\frac{\partial \hat{\pi}^{V*}}{\partial q} = \frac{c^2-(1-q)^2(1+\alpha)^2}{4(1-q)^2} < 0$. Therefore, as q increases, the firm's profit always decreases.

The main difference in the impact of q between the partially covered market and the fully covered market is the learning dataset size during the fine-tuning stage. For the fully covered market, the total number of users is 1, indicating that the learning dataset size in the fine-tuning stage remains the same for any α and q. In this case, an increase in q only intensifies the competition between versions, without improving the AI capability of the premium version. Consequently, the AI adaptive learning benefit cannot enable the firm to raise the price and extract more premium user surplus, leading to a constant decrease in both the premium version price and the firm's profit as q increases. Nevertheless, a lower price can attract more users, thereby high-valuation free users are incentivized to upgrade to the premium version, leading to an increase in the premium version demand while a decrease in the free version demand.

E. Proof of Proposition 4

Case 1: Partially covered market under both the freemium strategy and the paid-only strategy

Let $\Delta cs_1 = cs^{V*} - cs^{S*}$. Substituting the consumer surplus in (A.8) and (B.6) into Δcs_1 yields

$$\Delta cs_1 = cs^{V*} - cs^{S*} = \frac{A_3 s^2 + B_3 s + C_3}{8q^2(1-q)(1-\alpha)^2(2+s-2\alpha)^2}, \tag{E.1}$$

where $A_3 = [\alpha^2 - 3q^2 + 2q(2-\alpha)]c^2 - 2q(1-q)(3q+\alpha)c + q^2 + 2q^3 - 3q^4 > 0$;

$B_3 = 4(1-\alpha)\{[\alpha^2 - 3q^2 + 2q(2-\alpha)]c^2 - 2q(1-q)(3q+\alpha)c + q^2 + 2q^3 - 3q^4\}$;

$C_3 = 4(1-\alpha)^2\{[\alpha^2 - 3q^2 + 2q(2-\alpha)]c^2 - 2q(1-q)(3q+\alpha)c + 3(1-q)q^3\}$.

Similarly, in (E.1), we can verify that $A_3 s^2 + B_3 s + C_3$ is a quadratic function in s which has two real solutions (i.e., ${B_3}^2 - 4A_3 C_3 > 0$) if $q\alpha < c \leq \frac{q(1-q)}{2-q-\alpha}$. Let $A_3 s^2 + B_3 s + C_3 = 0$, denoting the solutions as

s_{F1} and s_3, where $s_{F1} = \frac{-B_3 - \sqrt{{B_3}^2 - 4A_3C_3}}{2A_3}$ and $s_3 = \frac{-B_3 + \sqrt{{B_3}^2 - 4A_3C_3}}{2A_3}$. Moreover, if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$, we can have $s_{F1} < \max\{0{,}2\alpha - 1\} < s_3 < \min\left\{\frac{\left(\alpha c - cq + q^2 - q\right)^2}{2(1-q)(q-c)^2}, 1\right\}$. Therefore, if $s_3 < s < \min\left\{\frac{\left(\alpha c - cq + q^2 - q\right)^2}{2(1-q)(q-c)^2}, 1\right\}$, $A_3s^2 + B_3s + C_3 > 0$, which leads to $cs^{V*} > cs^{S*}$. Hence, together with the precondition of this case in (C.1), in the partially covered market, $cs^{V*} > cs^{S*}$ if

$$q\alpha < c \le \frac{q(1-q)}{2-q-\alpha} \text{ and } s_3 < s < \min\left\{\frac{\left(\alpha c - cq + q^2 - q\right)^2}{2(1-q)(q-c)^2}, 1\right\}. \tag{E.2}$$

Case 2: Partially covered market under the freemium strategy and fully covered market under the paid-only strategy

Let $\Delta cs_2 = cs^{V*} - \widehat{cs}^{S*}$. Substituting the consumer surplus in (A.8) and (B.10) into Δcs_2 yields

$$\Delta cs_2 = cs^{V*} - \widehat{cs}^{S*} = \frac{D_2c^2 + E_2c + F_2}{8(1-q)q^2(1-\alpha)^2}, \tag{E.3}$$

where $D_2 = 4q - 3q^2 - 2q\alpha + \alpha^2 > 0$, $E_2 = -2q(1-q)(3q+\alpha)$, $F_2 = -(1-q)q^2(3 - 3q - 8\alpha + 4\alpha^2)$. It is straightforward that the numerator of (E.3) is a quadratic function in c, and has two real solutions (i.e., ${E_2}^2 - 4D_2F_2 > 0$) if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$. Let $D_2c^2 + E_2c + F_2 = 0$, denoting the solutions as c_{F1} and c_{F2}, where $c_{F1} = \frac{-E_2 - \sqrt{{E_2}^2 - 4D_2F_2}}{2D_2}$ and $c_{F2} = \frac{-E_2 + \sqrt{{E_2}^2 - 4D_2F_2}}{2D_2}$. Moreover, if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$, we can have $c_{F1} < q\alpha < \frac{q(1-q)}{2-q-\alpha} < c_{F2}$. Therefore, given the condition in (C.7), for all $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$ and $0 < s < \max\{0{,}2\alpha - 1\}$, $cs^{V*} < \widehat{cs}^{S*}$.

Case 3: Fully covered market under the freemium strategy and partially covered market under the paid-only strategy

Let $\Delta cs_3 = \widehat{cs}^{V*} - cs^{S*}$. Substituting the consumer surplus in (A.13) and (B.6) into Δcs_3 yields

$$\Delta cs_3 = \widehat{cs}^{V*} - cs^{S*} = \frac{A_4s^2 + B_4s + C_4}{8(1-q)(2+s-2\alpha)^2}, \tag{E.4}$$

where $A_4 = c^2 - 2c(1-q)(3-\alpha) + (1-q)(1 + 3q + 2\alpha + 6q\alpha + \alpha^2 - q\alpha^2) > 0$;

$B_4 = 4(1-\alpha)[c^2 - 2c(1-q)(3-\alpha) + (1-q)(1 + 3q + 2\alpha + 6q\alpha + \alpha^2 - q\alpha^2)]$;

$C_4 = 4\{[c^2 - 2c(1-q)(3-\alpha)](1-\alpha)^2 + (1-q)(3q - 2\alpha^2 - 10q\alpha^2 + 8q\alpha^3 + \alpha^4 - q\alpha^4)\}$.

Similarly, in (E.4), we can verify that $A_4s^2 + B_4s + C_4$ is a quadratic function in s which has two real solutions (i.e., ${B_4}^2 - 4A_4C_4 > 0$) if $q\alpha < c \le \frac{q(1-q)}{2-q-\alpha}$. Let $A_4s^2 + B_4s + C_4 = 0$, denoting the solutions as s_{F2} and s_{F3}, where $s_{F2} = \frac{-B_4 - \sqrt{{B_4}^2 - 4A_4C_4}}{2A_4} < 0$ and $s_{F3} = \frac{-B_4 + \sqrt{{B_4}^2 - 4A_4C_4}}{2A_4}$. Therefore, given the preconditions of this case outlined in (C.11), if $s_{F3} < s < \min\left\{\frac{[c + (1-q)(1+\alpha)]^2}{2(1-q)}, 1\right\}$, then $A_4s^2 + B_4s + C_4 > 0$, which leads to $\widehat{cs}^{V*} > cs^{S*}$. Moreover, if $c_{F3} = (1-q)(3-\alpha) - 2\sqrt{(1-q)(3 - 3q - 2\alpha)} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\}$, we can have $s_{F3} > \max\{2\alpha - 1{,}0\}$. Hence, together with the precondition in (C.11), $\widehat{cs}^{V*} < cs^{S*}$ if

$$0 < c < c_{F3} \text{ and } \max\{2\alpha - 1{,}0\} < s \leq \min\left\{\frac{[c+(1-q)(1+\alpha)]^2}{2(1-q)}, 1\right\}, \text{ or} \tag{E.5}$$

$$c_{F3} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } \max\{0{,}2\alpha - 1\} < s < s_{F3}. \tag{E.6}$$

Case 4: Fully covered market under both the freemium strategy and the paid-only strategy

Let $\Delta cs_4 = \widehat{cs}^{V*} - \widehat{cs}^{S*}$. Substituting the consumer surplus in (A.13) and (B.10) into Δcs_4 yields

$$cs_4 = \widehat{cs}^{V*} - \widehat{cs}^{S*} = \frac{c^2 - 2(1-q)(3-\alpha)c + (1-q)(3+3q+2\alpha+6q\alpha+\alpha^2-q\alpha^2)}{8(1-q)}. \tag{E.7}$$

The numerator of (E.7) is a quadratic function in c and has two real solutions $c_{F3} = (1-q)(3-\alpha) - 2\sqrt{(1-q)(3-3q-2\alpha)}$ and $c_{F4} = (1-q)(3-\alpha) + 2\sqrt{(1-q)(3-3q-2\alpha)}$. Moreover, $c_{F4} > \min\{q\alpha, q\alpha - \alpha - q + 1\}$. Therefore, if $c_{F3} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\}$, $\widehat{cs}^{V*} < \widehat{cs}^{S*}$. Together with the precondition of this case in (C.15), $\widehat{cs}^{V*} < \widehat{cs}^{S*}$ if

$$c_{F3} < c < \min\{q\alpha, q\alpha - \alpha - q + 1\} \text{ and } 0 < s < \max\{2\alpha - 1{,}0\}. \tag{E.8}$$

In summary, similarly, in terms of the consumer surplus, the latter three cases are also qualitatively consistent with our base model (i.e., Case 1).

F. Proof of Proposition 5

In the two-stage game, we first derive the equilibrium price, demands, and profits, then proceed to analyze the firm's optimal freemium strategy.

Accordingly, in this two-stage game, the utility for a consumer choosing the premium version is given by

$$U_P^V(v) = \underbrace{v \cdot \theta^B}_{\substack{\textbf{Current utility:}\\ \text{Perceived utility}\\ \text{at pre-training stage}}} + \underbrace{(v \cdot \theta^B + \alpha \cdot \theta^B \cdot \Delta N)}_{\substack{\textbf{Fureture utility:}\\ \text{Perceived Utility}\\ \text{at fine-tuning stage}}} - \underbrace{p}_{\substack{\text{Price of the}\\ \text{Premium}\\ \text{Version}}} = 2v + \alpha(N_P + N_F) - p,$$

while the utility for a consumer choosing the free version is given by

$$U_P^V(v) = \underbrace{v \cdot \theta^B}_{\substack{\textbf{Current utility:}\\ \text{Perceived utility}\\ \text{at pre-training stage}}} + \underbrace{(v \cdot \theta^B + \alpha \cdot \theta^B \cdot \Delta N)}_{\substack{\textbf{Fureture utility:}\\ \text{Perceived Utility}\\ \text{at fine-tuning stage}}} - \underbrace{c}_{\substack{\text{Outside}\\ \text{options}}} = 2qv + q\alpha(N_P + N_F) - c.$$

(1) Equilibrium under the freemium strategy

Performing a similar method in Lemma 1, we can have the demand functions as follows:

$$N_P^V = \frac{4(1-q)q + c(2q-\alpha) - pq(2-\alpha)}{2q(1-q)(2-\alpha)} \text{ and } N_F^V = \frac{pq-c}{2q(1-q)}, \tag{F.1}$$

$$cs^V = \frac{1}{4q^2}\left[\frac{q(c-pq)^2}{(1-q)^2} - \frac{4(c-2q)^2}{(2-\alpha)^2} - \frac{4(c-2q)(c-pq)}{2-\alpha}\right]. \tag{F.2}$$

The following constraints are necessary conditions for a partially covered market,

$$\alpha q < c < q \text{ and } \frac{c}{q} < p < \frac{c(2q-\alpha)-4q(1-q)}{q(2-\alpha)}. \tag{F.3}$$

Given these two conditions in (F.3), we next derive the firm's optimal premium price p. The firm's profit function is the same as $\pi^V(p)$ in (A.5). Performing a similar method, we can have the optimal price p^{V*} as

$$p^{V*} = \frac{4(1-q)q+c(2q-\alpha)}{2q(2-\alpha)}. \tag{F.4}$$

Besides, conditions in (F.3) require $q\alpha < c \le \min\left\{\frac{4(1-q)q}{4-2q-\alpha}, q\right\}$. Substituting p^{V*} into (F.1) and (A.5) yields the optimal demands and the firm's profit π^{V_c} as follows

$$N_P^{V*} = \frac{4(1-q)q+c(2q-\alpha)}{4q(1-q)(2-\alpha)}, \tag{F.5}$$

$$N_F^{V*} = \frac{4(1-q)q+c[\alpha-2(2-q)]}{4q(1-q)(2-\alpha)}, \tag{F.6}$$

$$\pi^{V*} = \frac{(4q+2cq-4q^2-c\alpha)^2}{8q^2(1-q)(2-\alpha)^2} - \frac{(c-2q)^2 s}{2q^2(2-\alpha)^2}. \tag{F.7}$$

Note that $\pi^{V*} > 0$ requires $0 < s < \min\left\{\frac{(4q+2cq-4q^2-c\alpha)^2}{4(c-2q)^2(1-q)}, 1\right\}$.

Taken together, the freemium strategy in a multi-stage game is feasible only if the following conditions hold simultaneously:

$$\alpha q < c < \min\left\{\frac{4(1-q)q}{4-2q-\alpha}, q\right\},\ 0 < s < \min\left\{\frac{(4q+2cq-4q^2-c\alpha)^2}{4(c-2q)^2(1-q)}, 1\right\} \tag{F.8}$$

(2) Equilibrium under the paid-only strategy

Similarly, under the paid-only strategy, the demand for the premium version is given as

$$N_P^S = \frac{2-p}{2-\alpha}, \tag{F.9}$$

The condition for a partially covered market is $p < 2$. The firm's profit $\pi^S(p)$ is the same as $\pi^S(p)$ in (B.3). Solving the FOC gives the solution

$$p^{S*} = \frac{2(2+s-\alpha)}{4+s-2\alpha}. \tag{F.10}$$

$p < 2$ requires $0 < s < 1$ which always holds. Substituting p^{S*} into (F.9) and (B.3), we can have the optimal demands and the firm's optimal profit as follows

$$N_P^{S*} = \frac{2}{4+s-2\alpha}, \tag{F.11}$$

$$\pi^{S*} = \frac{2}{4+s-2\alpha}, \tag{F.12}$$

It is worth noting that $\pi^{S*} \ge 0$ always holds.

(3) Proof of the conditions under which the firm prefers the freemium strategy

Let $\Delta\pi_5 = \pi^{V*} - \pi^{S*}$, where π^{V*} is given in (F.7) and π^{S*} is given in (F.12). We can have

$$\Delta\pi_5 = \pi^{V*} - \pi^{S*} = \frac{A_5 s^2 + B_5 s + C_5}{8q^2(1-q)(4+s-2\alpha)(2-\alpha)^2}, \tag{F.13}$$

where $A_5 = -4(c-2q)^2(1-q) < 0$;

$B_5 = 8cq(1-q)(8+2q-5\alpha) - 16q^2(1-q)(3+q-2\alpha) + c^2[4q(4+q) - 8(2-\alpha) - 12q\alpha + \alpha^2]$;

$C_5 = -2(2q-\alpha)(2-\alpha)[8q^2(1-q) - 8cq(1-q) - c^2(2q-\alpha)]$.

We can verify that in (F.13), the quadratic function of $A_5s^2 + B_5s + C_5$ in s has two real solutions (i.e., ${B_5}^2 - 4A_5C_5 > 0$) with the constraints in (F.8). Let $A_5s^2 + B_5s + C_5 = 0$, denoting the solutions as s_4 and s_{41}, where $s_4 = \frac{-B_5 + \sqrt{{B_5}^2 - 4A_5C_5}}{2A_5}$ and $s_{41} = \frac{-B_5 - \sqrt{{B_5}^2 - 4A_5C_5}}{2A_5}$. Moreover, if $q\alpha < c \le \min\left\{\frac{4(1-q)q}{4-2q-\alpha}, q\right\}$, we can also have $s_4 > s_{41}$. Therefore, if $s_{41} < s < s_4$, then $A_5s^2 + B_5s + C_5 > 0$, which leads to $\pi^{V*} > \pi^{S*}$. However, we can show that with the constraints in (F.8), $s_{41} < 0$ while $0 < s_4 < \min\left\{\frac{(4q+2cq-4q^2-c\alpha)^2}{4(c-2q)^2(1-q)}, 1\right\}$ if $0 < q < \frac{1}{2}$, $2q < \alpha < 1$ and $q\alpha < c < q$. Hence, together with the precondition of this case in (F.8), in this case, $\pi^{V*} > \pi^{S*}$ if

$$0 < q < \frac{1}{2},\ 2q < \alpha < 1 \text{ and } q\alpha < c < q. \tag{F.14}$$

G. Numerical Illustration for Observation 1

We resort to numerical analysis to characterize the firm's optimal q^*. Focusing on the parameter region in which the firm prefers the freemium strategy (as characterized in Propositions 1 and 2), we plot the firm's profit as a function of q over its feasible interval and identify the corresponding maximizer.

Figure A1. The Firm's Profit with Respect to q under a Lower α

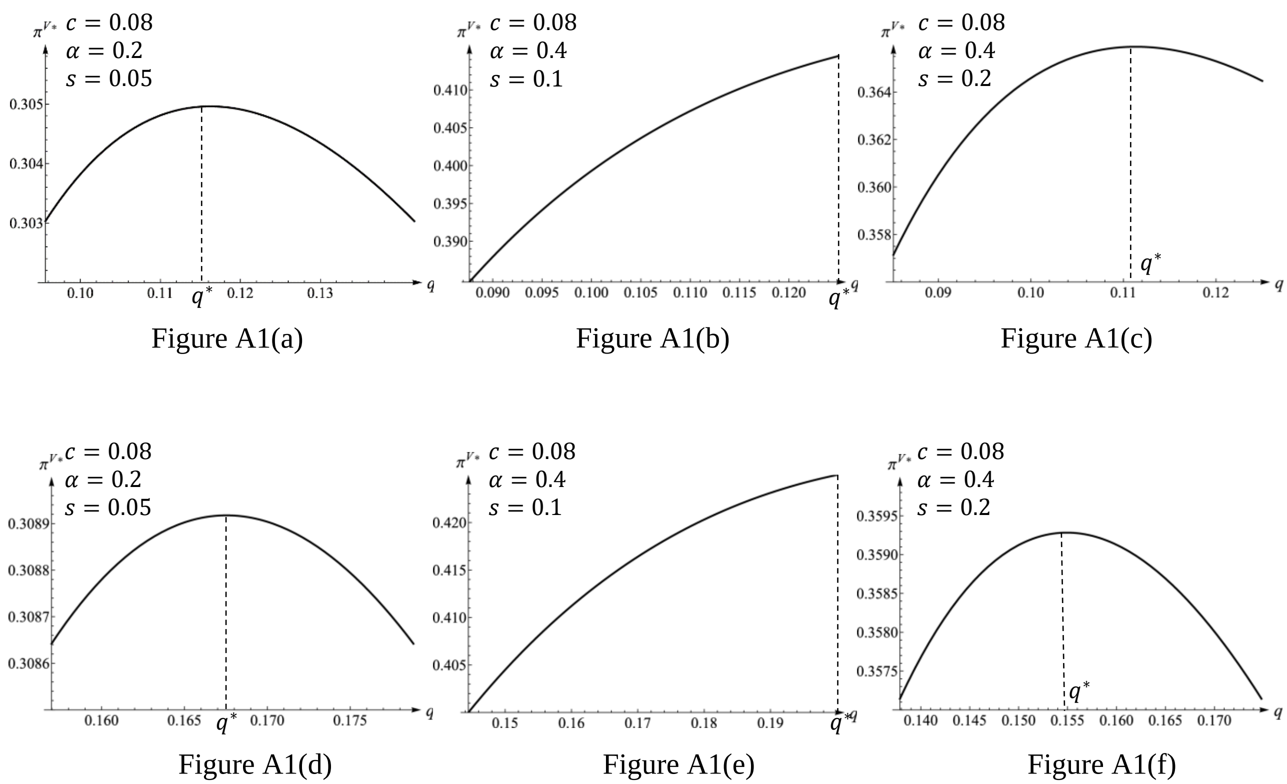


Figure A1(a) Figure A1(b) Figure A1(c)

Figure A1(d) Figure A1(e) Figure A1(f)

When α is relatively low (see Figures A1(a) and A1(d), under different values of c), the firm's profit increases and then decreases in q over the feasible interval. The profit function is therefore single-peaked, and the optimal q^* is an interior solution.

When α is relatively high (see Figures A1(b) and A1(c), holding c fixed and varying s), the shape of the firm's profit depends on the service-cost coefficient s. For low s (Figure A1(b)), the firm's profit is increasing in q, so the maximizer occurs at the upper boundary of the feasible set. For high s (Figure A1(c)), the firm's profit again exhibits an inverted-U shape, implying an interior optimal q^*. The same qualitative pattern holds when c is larger, as illustrated in Figures A1(e) and A1(f).

H. Equilibrium Characterization and Numerical Illustration for Observation 2

In the extension, we use superscripts "V_c" and "S_c" to denote the concave-learning case in which the adaptive-learning increment satisfies $\theta^I = \alpha \cdot \theta^B \cdot \sqrt{\Delta N}$. We first characterize the equilibrium outcomes under the freemium strategy and the paid-only strategy, and then use numerical analysis to illustrate Observation 2.

(1) Under the freemium strategy

Similar to Lemma 1, when the firm adopts the freemium strategy, a consumer using the premium version derives the utility of

$$U_P^{V_c}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \sqrt{\Delta N} - p = \underbrace{v}_{\substack{\textit{Perceived} \\ \textit{base model} \\ \textit{capability}}} + \underbrace{\alpha\sqrt{N_P + N_F}}_{\substack{\textit{Perceived AI} \\ \textit{adaptive} \\ \textit{learning} \\ \textit{increment}}} - \underbrace{p}_{\substack{\textit{Price of the} \\ \textit{Premium} \\ \textit{Version}}}.$$

A consumer using the free version gets the utility of

$$U_F^{V_c}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \sqrt{\Delta N} - c = \underbrace{qv}_{\substack{\textit{Perceived} \\ \textit{base model} \\ \textit{capability}}} + \underbrace{q\alpha\sqrt{N_P + N_F}}_{\substack{\textit{Perceived AI} \\ \textit{adapative} \\ \textit{learning} \\ \textit{increment}}} - \underbrace{c}_{\substack{\textit{Outside} \\ \textit{options}}}.$$

The boundary consumer $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F\}$ are defined as in Lemma 1. The key difference is that the adaptive-learning term now depends on $\sqrt{\Delta N}$, where $\Delta N = \sqrt{N_P + N_F}$. Solving $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, \Delta N\}$ jointly yields multiple market-coverage regimes. To demonstrate the robustness of our baseline insights, we focus on the partially covered case.

Suppose $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ and the market is partially covered. Then $N_P = 1 - \tilde{v}_{PF}$, $N_F = \tilde{v}_{PF} - \tilde{v}_F$, and $cs^V = \int_{\tilde{v}_{PF}}^{1} (v + \alpha\Delta N - p)dv + \int_{\tilde{v}_F}^{\tilde{v}_{PF}} (qv + q\alpha\Delta N - c)dv$. Substituting $\tilde{v}_{PF}$ and $\tilde{v}_F$ into $N_P = 1 - \tilde{v}_{PF}$ and $N_F = \tilde{v}_{PF} - \tilde{v}_F$ and solving for N_P and N_F simultaneously, we can have

$$N_P^{V_c} = \frac{pq-c}{(1-q)q} + \frac{(q\alpha+\sqrt{q(q\alpha^2+4q-4c)})^2}{4q^2} \text{ and } N_F^{V_c} = \frac{pq-c}{q(1-q)}, \tag{H.1}$$

Substituting $N_P = N_P^{V_c}$ and $N_F = N_F^{V_c}$ in (H.1) into $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, \Delta N\}$, we have the conditions for $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ as follows:

$$\alpha q < c < q \text{ and } \frac{c}{q} < p < \frac{q(2+2c-2q+\alpha^2-q\alpha^2)+\alpha(1-q)\sqrt{q(q\alpha^2+4q-4c)})}{2q}. \tag{H.2}$$

Given these two conditions in (H.2), we next derive the firm's optimal choice of the premium price p. Note that the firm's profit function under freemium in this extension takes the same form as (A.5). Therefore, substituting $N_P^{V_c}$ and $N_F^{V_c}$ from (H.1) into (A.5), and solving for the FOC gives the solution as

$$p^{V_c*} = \frac{q(2+2c-2q+\alpha^2-q\alpha^2)+\alpha(1-q)\sqrt{q(q\alpha^2+4q-4c)})}{4q}. \tag{H.3}$$

The conditions in (H.2) require $q\alpha < c \le \frac{(1-q)q(4-2q+\alpha^2+\alpha\sqrt{8-4q+\alpha^2})}{2(2-q)^2}$, where $\frac{(1-q)q(4-2q+\alpha^2+\alpha\sqrt{8-4q+\alpha^2})}{2(2-q)^2} < q$ always holds. Substituting p^{V_c*} into (H.1) and (A.5), we can have the optimal demands and the firm's profit π^{V_c} as follows

$$N_P^{V_c*} = \frac{1}{4}\left[2 + \frac{2c}{1-q} + \alpha^2 + \frac{\alpha\sqrt{q(q\alpha^2+4q-4c)}}{q}\right], \tag{H.4}$$

$$N_F^{V_c*} = \frac{1}{4}\left[2 + \alpha^2 - \frac{2c}{1-q} - \frac{4c}{q} + \frac{\alpha\sqrt{q(q\alpha^2+4q-4c)}}{q}\right], \tag{H.5}$$

$$\pi^{V_c*} = \frac{H_3+H_4s+H_5\sqrt{q(q\alpha^2+4q-4c)}}{16(1-q)q^3}. \tag{H.6}$$

Note that $\pi^{V_c*} > 0$ requires $0 < s < \min\left\{\frac{H_6+2q^2\alpha(1-q)[2c+(1-q)(2+\alpha^2)]\sqrt{q(q\alpha^2+4q-4c)}}{H_7-4q\alpha(1-q)(2c-2q-q\alpha^2)\sqrt{q(q\alpha^2+4q-4c)}}, 1\right\}$. Moreover,

$H_3 = 2q^2\{2(1+c-q)^2 q - 2(1-q)[c-2cq-2(1-q)q]\alpha^2 + (1-q)^2 q\alpha^4\}$;

$H_4 = 4sq(1-q)(2c^2 - 4cq + 2q^2 - 4cq\alpha^2 + 4q^2\alpha^2 + q^2\alpha^4)$;

$H_5 = 2q\alpha(1-q)\{2cq + 4cs + q(2+\alpha^2)[(1-q)-2s]\}$;

$H_6 = 2q^2\{q[2c^2 + (1-q)^2(2+4\alpha^2+\alpha^4)] + 2c(1-q)(2q - \alpha^2 + 2q\alpha^2)\}$;

$H_7 = 4q(1-q)(2c^2 - 4cq + 2q^2 - 4cq\alpha^2 + 4q^2\alpha^2 + q^2\alpha^4)$.

Taken together, the freemium strategy is feasible only if the following conditions hold simultaneously:

$$q\alpha < c \le \frac{(1-q)q(4-2q+\alpha^2+\alpha\sqrt{8-4q+\alpha^2})}{2(2-q)^2},\ 0 < s < \min\left\{\frac{H_6+2q^2\alpha(1-q)[2c+(1-q)(2+\alpha^2)]\sqrt{q(q\alpha^2+4q-4c)}}{H_7-4q\alpha(1-q)(2c-2q-q\alpha^2)\sqrt{q(q\alpha^2+4q-4c)}}, 1\right\} \quad \text{(H.7)}$$

(2) Under the paid-only strategy

Similarly, when the firm adopts the paid-only strategy, a consumer using the premium version derives the utility of

$$U_P^{S_c}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \sqrt{\Delta N} - p = v + \alpha\sqrt{N_P} - p.$$

The boundary consumer $\tilde{v}_P$ is defined as in Lemma 2. The key difference is that the adaptive-learning term depends on $\sqrt{\Delta N} = \sqrt{N_P}$. Solving $\{\tilde{v}_P, \Delta N\}$ jointly yields multiple market-coverage regimes; we focus on the partially covered case..

Suppose $0 < \tilde{v}_P < 1$ and the market is partially covered. Then $N_P = 1 - \tilde{v}_P$. Solving $N_P = 1 - \tilde{v}_P$ and $\sqrt{\Delta N} = \sqrt{N_P}$ simultaneously gives

$$N_P^{S_c} = \frac{1}{4}(\alpha + \sqrt{4 - 4p + \alpha^2})^2, \quad \text{(H.8)}$$

Substituting $N_P = N_P^{S_c}$ into $\{\tilde{v}_P, \Delta N\}$, we can have the condition for $0 < \tilde{v}_P < 1$ as

$$p > \alpha. \quad \text{(H.9)}$$

The firm's profit π^{S_c} is the same as π^S in (B.3). Solving the FOC gives the solution

$$p^{S_c*} = \frac{16+24s+8s^2+3\alpha^2+6s\alpha^2+(\alpha+2s\alpha)\sqrt{32+16s+9\alpha^2}}{8(2+s)^2}. \quad \text{(H.10)}$$

$p > \alpha$ requires $\max\{0, 3\alpha/2 - 1\} < s < 1$. Substituting p^{S_c*} into (H.8) and (B.3) yields optimal demands and the firm's optimal profit as follows

$$N_P^{S_c*} = \frac{1}{16}\left[2\alpha + \frac{\sqrt{2(16+8s+5\alpha^2+2s\alpha^2+2s^2\alpha^2)-2(1+2s)\alpha\sqrt{32+16s+9\alpha^2}}}{2+s}\right]^2, \quad \text{(H.11)}$$

$$\pi^{S_c*} = \frac{\left[2(2+s)\alpha+\sqrt{2}H_8\right]^2\left[16+8s+3\alpha^2-2s\alpha^2-4s^2\alpha^2-2\sqrt{2}s\alpha H_8+(\alpha+2s\alpha)\sqrt{32+16s+9\alpha^2}\right]}{256(2+s)^3}, \quad \text{(H.12)}$$

where $H_8 = \sqrt{16 + 8s + 5\alpha^2 + 2s\alpha^2 + 2s^2\alpha^2 - (\alpha + 2s\alpha)\sqrt{32 + 16s + 9\alpha^2}}$. It is worth noting that $\pi^{S_c*} \geq 0$ always holds.

(3) Numerical evidence for the firm's preferred strategy

The closed-form profit expressions in (H.6) and (H.12) are algebraically involved due to the concave learning function. We therefore resort to numerical analysis to compare π^{V_c*} and π^{S_c*}. Varying q and α, we illustrate the parameter region in which the firm prefers freemium when the marginal capability gain decreases with the learning dataset size.

Figure A2. The Firm's Optimal Freemium Strategy under Concave $\boldsymbol{\theta^I}$ Scenario when $\boldsymbol{\alpha = 0.65}$

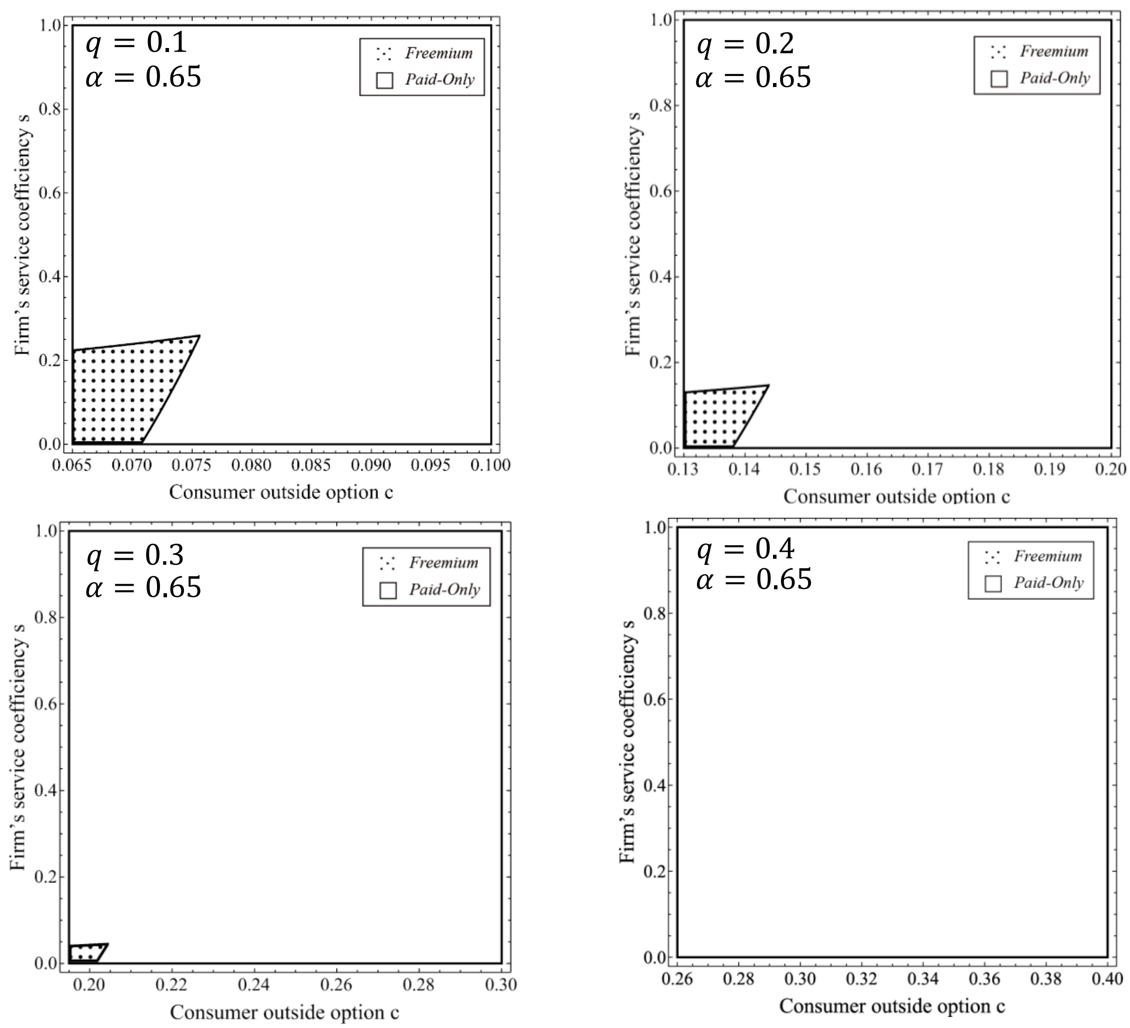
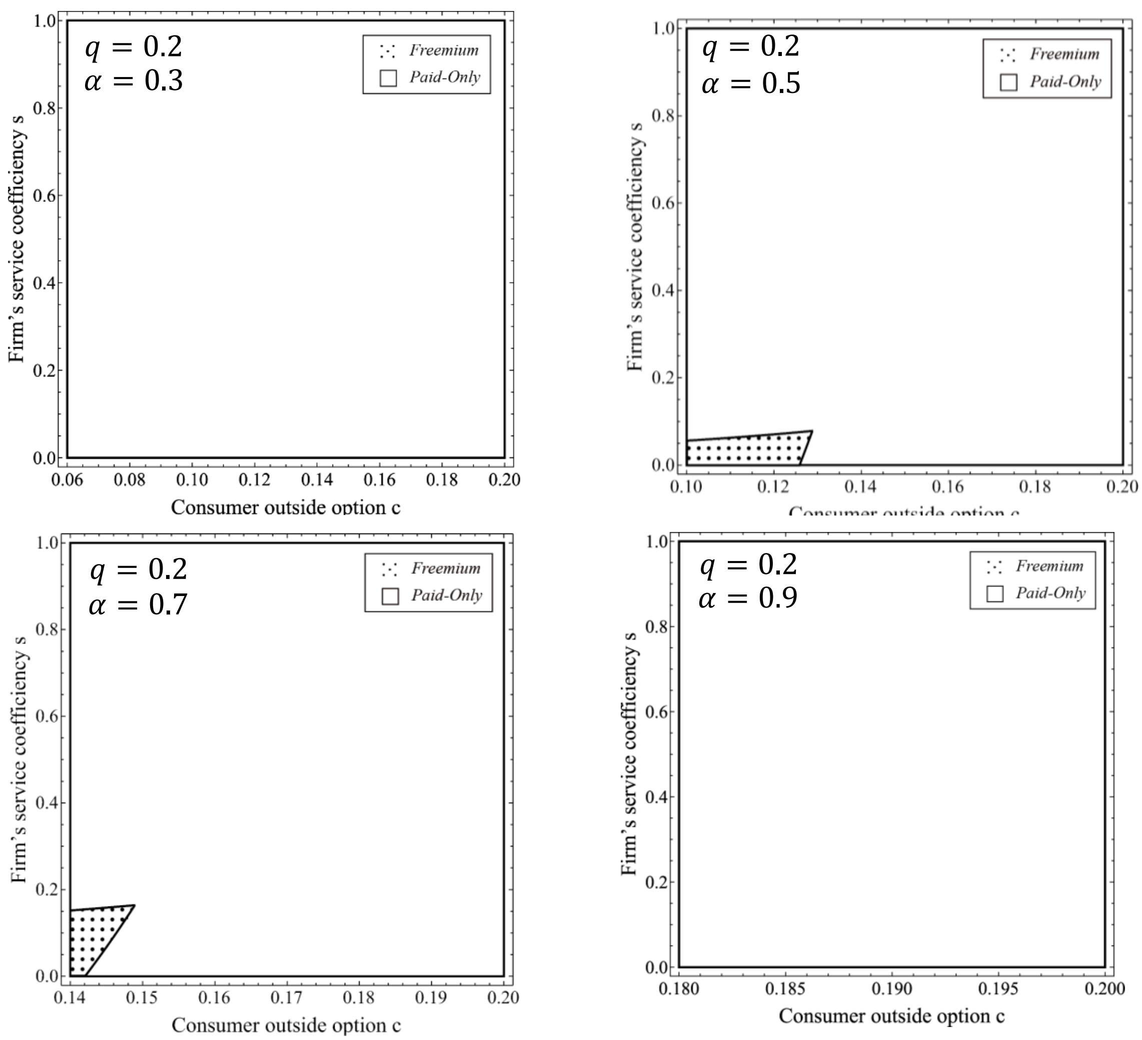


Figure A2 shows that freemium strategy yields a higher profit than the paid-only strategy only when both the service-cost coefficient s and the consumer outside option c are relatively low. Comparing the four panels also reveals that, as q increases, the region in which the freemium strategy is optimal shrinks. Moreover, relative to the baseline model, the concave-learning specification substantially narrows the range of q for which the freemium strategy can be optimal; for sufficiently large q (e.g., $q = 0.4$ in the bottom-right panel), the region where the freemium strategy dominates may disappear. In other words, for the freemium strategy to be optimal, the value of q should fall within a narrower range than that of the base model.

Recall that in the baseline model, when $0 < q < 1/2$ and $q < \alpha < 1 - q$, there always exists a non-empty region in which the firm benefits from the freemium strategy. Under the concave-learning specification, however, Figure A3 shows that even when $0 < q < 1/2$, the freemium strategy region can vanish for some parameter values (e.g., when α is sufficiently large relative to q). Put differently, compared with the baseline model, the concave-learning case requires a tighter admissible range of α for the freemium strategy to be optimal. In particular, the lower bound on α becomes higher (relative to q), whereas the upper bound ($1 - q$) remains unchanged.

Figure A3. The Firm's Optimal Freemium Strategy under Concave θ^I Scenario when $q = 0.2$



Taken together, under the concave-learning specification, the optimality of the freemium strategy requires both α and q to lie in a more restricted range than in the baseline model.

I. Proof of Proposition 6

When consumers have heterogeneous preferences for both θ^B and θ^I, we first derive the equilibrium prices, demands, and profits, and then analyze the firm's optimal freemium strategy. Specifically, the subscript P/F denotes the premium/free version, and the subscript H/L denotes the consumer type with a high/low willingness to pay for the adaptive learning increment θ^I. Thus, $N_{PH}(N_{FH})$ and $N_{PL}(N_{FL})$ represent the demands for the premium (free) version from high-type and low-type consumers, respectively.

(1) Equilibrium under the freemium strategy

We assume that consumers have heterogeneous preferences for the pre-trained capability, $v \sim U[0,1]$, and belong to one of two types in their willingness to pay for the adaptive-learning increment θ^I: a high type t_H or a low type t_L, where $0 < t_L < t_H < 1$. The probability of being a high type is λ, and the probability of being a low type is $1-\lambda$. Accordingly, a consumer choosing the premium version has the net utility

$$U_P^V(v,t_i) = v\cdot\theta^B + t_i\theta^I - p = \underbrace{v}_{\textit{Perceived base model capability}} + \underbrace{t_i\alpha(N_P+N_F)}_{\textit{Perceived AI adaptive learning increment}} - \underbrace{p}_{\textit{Price of the Premium Version}},$$

A consumer using the free version gets the utility of

$$U_F^V(v,t_i) = v\cdot\theta^B + t_i\theta^I - c = \underbrace{qv}_{\textit{Perceived base model capability}} + \underbrace{t_i q\alpha(N_P+N_F)}_{\textit{Perceived AI adaptive learning increment}} - \underbrace{c}_{\textit{Outside option}}.$$

For each type $i \in \{H, L\}$, define the boundary consumers by the standard indifference conditions: $\tilde{v}_{PFi}$ is indifferent between the premium and free versions ($U_P^V(v,t_i) = U_F^V(v,t_i)$), $\tilde{v}_{Pi}$ is indifferent between purchasing the premium version and opting out ($U_P^V(v,t_i) = 0$), and $\tilde{v}_{Fi}$ is indifferent between using the free version and opting out ($U_F^V(v,t_i) = 0$). Solving these conditions yields

$$\tilde{v}_{PFi} = \frac{p-c-\Delta N t_i\alpha + \Delta N q t_i\alpha}{1-q},\ \tilde{v}_{Pi} = p - \Delta N t_i\alpha,\ \tilde{v}_{Fi} = \frac{c-\Delta N q t_i\alpha}{q}, \tag{I.1}$$

Here $\Delta N = N_{PH} + N_{PL} + N_{FH} + N_{FL}$. Solving $\{\tilde{v}_{PFi}, \tilde{v}_{Pi}, \tilde{v}_{Fi}, \Delta N\}$ jointly can generate different market-coverage regimes. Since this extension is intended to test robustness, we focus on the partially covered regime.

Suppose $0 < \tilde{v}_{Fi} (< \tilde{v}_{Pi}) < \tilde{v}_{PFi} < 1$, indicating partial market coverage. Then, demands for premium and free versions become $N_{PL} = (1-\lambda)(1-\tilde{v}_{PFL})$, $N_{PH} = \lambda(1-\tilde{v}_{PFH})$, $N_{FH} = \lambda(\tilde{v}_{PFH} - \tilde{v}_{FH})$, and $N_{FL} = (1-\lambda)(\tilde{v}_{PFL} - \tilde{v}_{FL})$. Solving for N_{Pi} and N_{Fi}, yields the equilibrium demands reported in (I.2).

$$N_{PH}^{V} = \lambda \frac{q\{1-p+\mathrm{th}\alpha+c[1+\alpha(t_H-t_L)(1-\lambda)]-\alpha(1-p)[t_H\lambda+t_L(1-\lambda)]\}-c\mathrm{th}\alpha-q^2[1+\alpha(t_H-t_L)(1-\lambda)]}{q(1-q)[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]},\ N_{FH}^{V} = \frac{(pq-c)\lambda}{(1-q)q},$$

$$N_{PL}^{V} = \frac{(1-\lambda)[q+cq-pq-q^2-ct_L\alpha+pqt_L\alpha-q(1+c-p-q)(t_H-t_L)\alpha\lambda]}{q(1-q)[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]},\ N_{FL}^{V} = \frac{(pq-c)(1-\lambda)}{(1-q)q}, \tag{I.2}$$

Substituting N_{Pi}^{V} and N_{Fi}^{V} in (I.2) into $\{\tilde{v}_{PFi}, \tilde{v}_{Pi}, \tilde{v}_{Fi}, \Delta N\}$, we identify conditions ensuring the scenario $0 < \tilde{v}_{FH} (< \tilde{v}_{PH}) < \tilde{v}_{PFH} < 1$ as follows:

$$\frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < q \text{ and } \frac{c}{q} < p < \frac{-ct_H\alpha+q(1+c)[1+\alpha(t_H-t_L)(1-\lambda)]-q^2[1+\alpha(t_H-t_L)(1-\lambda)]}{q[1-t_H\alpha\lambda-t_L(1-\alpha)\lambda]}, \tag{I.3}$$

and conditions ensuring the scenario $0 < \tilde{v}_{FL} (< \tilde{v}_{PL}) < \tilde{v}_{PFL} < 1$ as follows:

$$\frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < q \text{ and } \frac{c}{q} < p < \frac{q+cq-q^2-ct_L\alpha-q\alpha\lambda(1+c-q)(t_H-t_L)}{q[1-t_H\alpha\lambda-t_L(1-\alpha)\lambda]}. \tag{I.4}$$

Combining the conditions given in (I.3) and (I.4), the necessary conditions to sustain the demands specified in (I.2) become

$$\frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < q \text{ and} \frac{c}{q} < p < \frac{q+cq-q^2-ct_L\alpha-q\alpha\lambda(1+c-q)(t_H-t_L)}{q[1-t_H\alpha\lambda-t_L(1-\alpha)\lambda]}, \tag{I.5}$$

Under (I.5), the market is partially covered and all equilibrium demands in (I.2) are strictly positive.

Given these two conditions in (I.5), we next derive the firm's optimal choice of the premium price p. Note that the firm's profit π^V in this extension is the same as π^V in (A.5). Therefore, substituting N_{Pi}^{V} and N_{Fi}^{V} in (I.2) into (A.5), and solving for the FOC gives the solution as

$$p^{V*} = \frac{q(1+c-q)-c\alpha(t_L+t_H\lambda-t_L\lambda)}{2q[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]}. \tag{I.6}$$

The conditions in (I.5) require $0 < \alpha < \min\left\{\frac{1-q}{t_H}, 1\right\}$ and $\frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < \frac{(1-q)q}{2-q-\alpha(t_L+t_H\lambda-t_L\lambda)}$. Substituting p^{V*} into (I.2) and (A.5), we can have the optimal demands and the firm's profit π^V as follows

$$N_{PH}^{V*} = \frac{\lambda\{[q(1+c)-q^2][1-2\alpha(t_H-t_L)(1-\lambda)]+c\alpha[t_L-t_H(2-\lambda)-t_L\lambda]\}}{2q(1-q)[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]}, \tag{I.7}$$

$$N_{FH}^{V*} = \frac{\lambda\{q(1-q)-c[2-q-\alpha(t_L+t_H\lambda-t_L\lambda)]\}}{2q(1-q)[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]}, \tag{I.8}$$

$$N_{PL}^{V*} = \frac{(1-\lambda)\{q(1-q)-c[2-q-\alpha(t_L+t_H\lambda-t_L\lambda)]\}}{2q(1-q)[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]}, \tag{I.9}$$

$$N_{FL}^{V*} = \frac{(1-\lambda)\{q(1+c)-q^2-c\alpha t_L+\alpha\lambda(t_H-t_L)[c-2cq-2q(1-q)]\}}{2q(1-q)[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]}, \tag{I.10}$$

$$\pi^{V*} = \frac{[q(1+c-q)-c\alpha(t_L+t_H\lambda-t_L\lambda)]^2}{4q^2(1-q)[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]^2} - \frac{(q-c)^2 s}{2q^2[1-t_H\alpha\lambda-t_L\alpha(1-\lambda)]^2}. \tag{I.11}$$

Note that $\pi^{V*} > 0$ requires $0 < s < \min\left\{\frac{\{q(1+c)-q^2-c\alpha[t_H\lambda+t_L(1-\lambda)]\}^2}{2(q-c)^2(1-q)}, 1\right\}$. Taken together, the freemium strategy is feasible only if the following conditions hold simultaneously:

$$0 < \alpha < \min\left\{\frac{1-q}{t_H}, 1\right\}, \frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < \frac{(1-q)q}{2-q-\alpha(t_L+t_H\lambda-t_L\lambda)},$$

$$0 < s < \min\left\{\frac{\{q(1+c)-q^2-c\alpha[t_H\lambda+t_L(1-\lambda)]\}^2}{2(q-c)^2(1-q)}, 1\right\}. \tag{I.12}$$

(2) Equilibrium under the paid-only strategy

Similarly, under the paid-only strategy, a consumer choosing the premium version has a net utility of

$$U_P^S(v,t_i) = v \cdot \theta^B + t_i\theta^I - p = v + t_i\alpha N_P - p.$$

Solving $U_P^S(v,t_i) = 0$ yields the type-specific cutoff

$$\tilde{v}_{Pi} = p - \Delta N t_i \alpha, \tag{I.13}$$

where $\Delta N = N_{Pi}$ under the paid-only strategy. Focusing on the partially covered regime ($0 < \tilde{v}_{Pi} < 1$ for both types), we have $N_{PL} = (1-\lambda)(1-\tilde{v}_{PL})$, and $N_{PH} = \lambda(1-\tilde{v}_{PH})$. Solving these equations jointly yields the equilibrium outcomes reported in (I.15)–(I.18).

$$N_{PH}^S = \lambda\frac{(1-p)[1+\alpha(t_H-t_L)(1-\lambda)]}{1-t_H\alpha\lambda-t_L(1-\lambda)\alpha},$$
$$N_{PL}^S = \frac{(1-\lambda)(1-p)(1-t_H\alpha\lambda+t_L\alpha\lambda)}{1-t_H\alpha\lambda-t_L(1-\lambda)\alpha}. \tag{I.14}$$

Substituting N_{Pi}^S in (I.14) into $\{\tilde{v}_{Pi}, \Delta N\}$, one can readily prove that $0 < \tilde{v}_{Pi} < 1$ always holds and thus the market is partially covered. The firm's profit π^S is the same as π^S in (B.3). Solving the FOC gives the solution

$$p^{S*} = \frac{s+1-t_H\alpha\lambda-t_L(1-\lambda)\alpha}{s+2[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]}. \tag{I.15}$$

Substituting p^{S*} into (I.14) and (B.3) yields optimal demands and the firm's optimal profit as follows

$$N_{PH}^{S*} = \frac{\lambda[1+\alpha(t_H-t_L)(1-\lambda)]}{s+2[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]}, \tag{I.16}$$

$$N_{PL}^{S*} = \frac{(1-\lambda)(1-t_H\alpha\lambda+t_L\alpha\lambda}{s+2[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]}, \tag{I.17}$$

$$\pi^{S*} = \frac{1}{2\{s+2[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]\}}, \tag{I.18}$$

It is worth noting that $\pi^{S*} \geq 0$ always holds.

(3) Proof of the conditions under which the firm prefers the freemium strategy

We focus on the parameter region in which the partially covered equilibria characterized above apply. Under (I.12), the freemium strategy is feasible; the paid-only equilibrium is given in (I.15)–(I.18). We next compare the resulting optimal profits.

Let $\Delta\pi_6 = \pi^{V*} - \pi^{S*}$, where π^{V*} is given in (I.11) and π^{S*} is given in (I.18). We can have

$$\Delta\pi_6 = \pi^{V*} - \pi^{S*} = \frac{A_6 s^2 + B_6 s + C_6}{4q^2(1-q)[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]^2\{s+2[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]\}}, \tag{I.19}$$

where $A_6 = -2(c-q)^2(1-q) < 0$;

$B_6 = 2cq(1-q)\{q-1+5[1-t_H\alpha\lambda - t_L(1-\lambda)\alpha]\} - q^2(1-q)\{q-1+4[1-t_H\alpha\lambda - t_L(1-\lambda)\alpha]\} - c^2\{4-q^2-q\{6[1-t_H\alpha\lambda - t_L(1-\lambda)\alpha]-2\} - \alpha[t_H\lambda + t_L(1-\lambda)]\{4+[t_H\lambda+t_L(1-\lambda)]\}\}$;

$C_6 = 2[1-t_H\alpha\lambda - t_L(1-\lambda)\alpha][q-t_H\alpha\lambda - t_L(1-\lambda)\alpha]\{2c(1-q)q-(1-q)q^2+c^2[q-t_H\alpha\lambda - t_L(1-\lambda)\alpha]\}$.

It can be shown that the denominator of $\Delta\pi_6$ is positive. Additionally, we verify that under the constraints specified in (I.12), the quadratic function $A_6s^2 + B_6s + C_6$ in equation (I.19) has two real solutions, as indicated by the discriminant ${B_5}^2 - 4A_5C_5 > 0$. Let $A_6s^2 + B_6s + C_6 = 0$, denoting the solutions as $s_5 = \frac{-B_6 + \sqrt{{B_6}^2 - 4A_6C_6}}{2A_6}$ and $s_6 = \frac{-B_6 - \sqrt{{B_6}^2 - 4A_6C_6}}{2A_6}$, where it holds that $s_5 < s_6 < \min\left\{\frac{\{q(1+c)-q^2-c\alpha[t_H\lambda+t_L(1-\lambda)]\}^2}{2(q-c)^2(1-q)}, 1\right\}$. Therefore, if $s_5 < s < s_6$, then we have $A_6s^2 + B_6s + C_6 > 0$, which leads to $\pi^{V*} > \pi^{S*}$. However, we must also verify the positivity of s_5 and s_6. Indeed, we can demonstrate the following conditions

(i) if $\frac{q}{(t_L+t_H\lambda-t_L\lambda)} < \alpha < \min\left\{\frac{1-q}{t_H}, 1\right\}$ and $\frac{qt_H\alpha}{1-\alpha(t_H-t_L)(1-\lambda)} < c < \frac{(1-q)q}{2-q-\alpha(t_L+t_H\lambda-t_L\lambda)}$, $s_6 > 0$;

(ii) if $\frac{q}{(t_L+t_H\lambda-t_L\lambda)} < \alpha < \min\left\{\frac{1-q}{t_H}, 1\right\}$ and $\max\left\{\frac{q(\sqrt{(1-q)[1-t_H\alpha\lambda-t_L(1-\lambda)\alpha]}+q-1)}{q-t_H\alpha\lambda-t_L(1-\lambda)\alpha}, 0\right\} < c \leq \frac{(1-q)q}{2-q-\alpha(t_L+t_H\lambda-t_L\lambda)}$, $s_5 > 0$.

Combining these conditions with the preconditions outlined in (I.12), we summarize the necessary conditions ensuring $\pi^{V*} > \pi^{S*}$ in Proposition 6.

J. Proofs of the case without adaptive learning

To establish that adaptive learning is the primary mechanism driving the adoption of the freemium strategy, we construct a benchmark in which adaptive learning is excluded. We first derive the firm's equilibrium outcomes under the paid-only strategy and the freemium strategy, respectively, and then compare the resulting profits. In this section, we use superscripts "V_B" and "S_B" to denote the no adaptive learning benchmark.

(1) Equilibrium under the paid-only strategy

Under the paid-only strategy, a consumer using the premium version derives the utility of $U_P^{S_B}(v) = v \cdot \theta^B - p = v - p$. Consumers with $U_P^{S_B}(v) > 0$ purchase the premium version; otherwise they opt out. Therefore, the demand for the premium version is given as

$$N_P^{S_B} = 1 - p. \tag{J.1}$$

Note that $N_P^{S_B} > 0$ holds as long as $0 < p < 1$. The firm's profit $\pi^{S_B}(p)$ is the same as $\pi^S(p)$ in (B.3). Solving the FOC gives the solution

$$p^{S_B*} = \frac{1+s}{2+s}. \tag{J.2}$$

$0 < p < 1$ requires $0 < s < 1$ which always holds. Substituting p^{S_B*} into (J.1) and (B.3), we can have the optimal demands and the firm's optimal profit as follows

$$N_P^{S_B*} = \frac{1}{2+s}, \tag{J.3}$$

$$\pi^{S_B*} = \frac{1}{2(2+s)}. \tag{J.4}$$

It is worth noting that $\pi^{S_B*} \geq 0$ always holds.

(2) Equilibrium under the freemium strategy

Under the freemium strategy, a consumer using the premium version obtains $U_P^{V_B}(v) = v \cdot \theta^B - p = v - p$, while a consumer using the free version obtains $U_F^{V_B}(v) = v \cdot \theta^B - c = qv - c$. Consumers choose the option that yields the highest nonnegative utility. Therefore, the demand for the premium version is given by

$$N_P^{V_B} = \frac{1+c-q-p}{1-q}. \tag{J.5}$$

The demand for the free version is given as

$$N_F^{V_B} = \frac{pq-c}{q(1-q)}. \tag{J.6}$$

$N_P^{V_B} > 0$ and $N_F^{V_B} > 0$ hold under the condition of $\frac{c}{q} < p < 1 + c - q$. The firm's profit $\pi^{V_B}(p)$ is the same as $\pi^V(p)$ in (A.5).

Substituting (J.5) into (A.5) and solving for the first-order condition (FOC) gives the solution as

$$p^{V_B*} = \frac{1}{2}(1 + c - q). \tag{J.7}$$

Besides, $\frac{c}{q} < p < 1 + c - q$ requires $0 < c < \frac{(1-q)q}{2-q}$, where $\frac{(1-q)q}{2-q} < q$ always holds. Substituting p^{V_B*} into (J.5) and (A.5) yields optimal demands as follows

$$N_P^{V_B*} = \frac{2c-q-cq-q^2}{2q(1-q)},\ N_F^{V_B*} = \frac{1+c-q}{2(1-q)}, \tag{J.8}$$

Finally, substituting p^{V*}, N_P^{V*} and N_F^{V*} into the firm's profit π^V, we have the firm's optimal profit as follows

$$\pi^{V_B*} = \frac{(1+c-q)^2}{4(1-q)} - \frac{(q-c)^2 s}{2q^2}. \tag{J.9}$$

It is worth noting that $\pi^{V_B*} \geq 0$ only if $0 < s \leq \min\left\{\frac{(1+c-q)^2 q^2}{2(c-q)^2(1-q)}, 1\right\}$.

In summary, the freemium strategy is viable when

$$0 < c < \frac{(1-q)q}{2-q} \text{ and } 0 < s \leq \min\left\{\frac{(1+c-q)^2 q^2}{2(c-q)^2(1-q)}, 1\right\} \tag{J.10}$$

(3) Conditions under which the firm prefers the freemium strategy

Let $\Delta\pi_7 = \pi^{V_B*} - \pi^{S_B*}$, where π^{V_B*} is given in (J.9) and π^{S_B*} is given in (J.4). We can have

$$\Delta\pi_7 = \pi^{V_B*} - \pi^{S_B*} = \frac{A_7 s^2 + B_7 s + C_7}{4q^2(1-q)(2+s)}, \tag{J.11}$$

where $A_7 = -2(c-q)^2(1-q) < 0$; $B_7 = -4c^2 + 8cq + 4c^2q - 3q^2 - 6cq^2 + c^2q^2 + 2q^3 - 2cq^3 + q^4$; $C_7 = 2q^2(2c + c^2 - q - 2cq + q^2)$.

It can be shown that the denominator of $\Delta\pi_7$ is positive. Moreover, under the feasibility conditions in this benchmark (J.10), the quadratic function $A_7s^2 + B_7s + C_7$ in equation (J.11) admits two real roots, as the discriminant satisfies ${B_7}^2 - 4A_7C_7 > 0$. Letting $A_7s^2 + B_7s + C_7 = 0$, the roots are given by $s_{J1} = \frac{-B_7+\sqrt{{B_7}^2-4A_7C_7}}{2A_7}$ and $s_{J2} = \frac{-B_7-\sqrt{{B_7}^2-4A_7C_7}}{2A_7}$, with $s_7 < s_8 < 0$ whenever $0 < c < \frac{(1-q)q}{2-q}$. Consequently, for all $0 < c < \frac{(1-q)q}{2-q}$ and $0 < s \leq \min\left\{\frac{(1+c-q)^2q^2}{2(c-q)^2(1-q)}, 1\right\}$, we obtain $\Delta\pi_7 = \pi^{V_{B^*}} - \pi^{S_{B^*}} < 0$, which implies that the firm strictly prefers not to adopt a freemium strategy in the absence of adaptive learning.

K. Proof of Proposition 7

In the extension, we use superscripts "V_o" and "S_o" to represent the case where outside options for both versions are included. In the following, we first derive the equilibrium solutions under the freemium strategy and the paid-only strategy.

(1) Under the freemium strategy

Similar to Lemma 1, when the firm adopts the freemium strategy, a consumer using the premium version derives the utility of

$$U_P^{V_o}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \Delta N - \underbrace{p}_{\substack{\textit{Price of}\\ \textit{the}\\ \textit{Premium}\\ \textit{Version}}} - kc = \underbrace{v}_{\substack{\textit{Perceived}\\ \textit{base model}\\ \textit{capability}}} + \underbrace{\alpha(N_P + N_F)}_{\substack{\textit{Perceived AI}\\ \textit{adpative}\\ \textit{learning}\\ \textit{increment}}} - p - \underbrace{kc}_{\substack{\textit{Outside}\\ \textit{options}}}.$$

A consumer using the free version gets the utility of

$$U_F^{V_o}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \Delta N - c = \underbrace{qv}_{\substack{\textit{Perceived}\\ \textit{base model}\\ \textit{capability}}} + \underbrace{q\alpha(N_P + N_F)}_{\substack{\textit{Perceived AI}\\ \textit{adaptive}\\ \textit{learning}\\ \textit{increment}}} - \underbrace{c}_{\substack{\textit{Outside}\\ \textit{options}}}.$$

Solving the equations $U_P^{V_o}(v) = U_F^{V_o}(v)$, $U_P^{V_o}(v) = 0$ and $U_F^{V_o}(v) = 0$, we identify the boundary consumers as follows

$$\tilde{v}_{PF} = \frac{p-c+ck}{(1-q)} - \alpha\Delta N,\ \tilde{v}_P = p + ck - \alpha\Delta N,\ \tilde{v}_F = \frac{c}{q} - \alpha\Delta N, \tag{K.1}$$

where $\Delta N = (N_P + N_F)$. Solving $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, \Delta N\}$ simultaneously results in two different market coverage cases. Since this extension is intended to verify the robustness of our base model, we concentrate solely on the partial market coverage scenario.

Suppose $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ and the market is partially covered. Then $N_P = 1 - \tilde{v}_{PF}$, $N_F = \tilde{v}_{PF} - \tilde{v}_F$. Substituting $\tilde{v}_{PF}$ and $\tilde{v}_F$ into $N_P = 1 - \tilde{v}_{PF}$ and $N_F = \tilde{v}_{PF} - \tilde{v}_F$ and solving for N_P and N_F simultaneously, we can have

$$N_P^{V_O} = \frac{pq-c(1-kq)}{q(1-q)} + \frac{q-c}{1-\alpha}, \tag{K.2}$$

$$N_F^{V_O} = \frac{(ck+p)q-c}{q(1-q)}. \tag{K.3}$$

Substituting $N_P = N_P^{V_O}$ and $N_F = N_F^{V_O}$ in (K.2) and (K.3) into $\{\tilde{v}_{PF}, \tilde{v}_P, \tilde{v}_F, \Delta N\}$, we have the conditions for $0 < \tilde{v}_F (< \tilde{v}_P) < \tilde{v}_{PF} < 1$ as follows:

$$q\alpha < c < q \text{ and } \max\left\{\frac{c(1-kq)}{q}, 0\right\} < p < \frac{q+cq-ckq-q^2-c\alpha+ckq\alpha}{q(1-\alpha)}. \tag{K.4}$$

Under conditions in (K.4), the market is partially covered, and equilibrium demands $N_P^{V_O}$ and $N_F^{V_O}$ remain positive.

Note that the firm's profit π^{V_O} in this extension is the same as π^V in (A.5). Given conditions in (K.4), substituting $N_P^{V_O}$ and $N_F^{V_O}$ in (K.2) and (K.3) into (A.5), and solving for the FOC gives the solution as

$$p^{V_O*} = \frac{q+cq-ckq-q^2-c\alpha+ckq\alpha}{2q(1-\alpha)}. \tag{K.5}$$

The conditions in (K.4) require

$$q\alpha < c < c_2, \tag{K.6}$$

where $c_2 = \begin{cases} c_{O1}, & \text{if } 0 < k < 1, 0 < q < 1 \; and \; 0 < \alpha < \frac{1-q}{1-kq}, or \; k > 1, 0 < q < \frac{1}{k} \; and \; 0 < \alpha < 1 \\ c_{O2}, & \text{if } k > 1, \frac{1}{k} < q < \sqrt{\frac{1}{k}} \; and \; 0 < \alpha < q, or \; k > 1, \sqrt{\frac{1}{k}} < q < 1 and \; 0 < \alpha < 1 \\ \frac{1}{k}, & \text{if } k > 1, \frac{1}{k} < q < \sqrt{\frac{1}{k}} \; and \; q < \alpha < \frac{1}{kq} \end{cases}$,

$c_{O1} = \frac{q(1-q)}{2-q-kq-\alpha+kq\alpha}$, $c_{O2} = \frac{q(1-q)}{-q+kq+\alpha-kq\alpha}$.

Substituting p^{V_O*}into (K.2), (K.3) and (A.5), we have the optimal demands and the firm's profit π^{V_O} as follows

$$N_P^{V_O^*} = \frac{q+cq-ckq-q^2-c\alpha+ckq\alpha}{2q(1-q)(1-\alpha)}, \tag{K.7}$$

$$N_F^{V_O^*} = \frac{-2c+q+cq+ckq-q^2+c\alpha-ckq\alpha}{2q(1-q)(1-\alpha)}, \tag{K.8}$$

$$\pi^{V_O*} = \frac{\left(q+cq-ckq-q^2-c\alpha+ckq\alpha\right)^2}{4q^2(1-q)(1-\alpha)^2} - \frac{(q-c)^2 s}{2q^2(1-\alpha)^2}, \tag{K.9}$$

Note that $\pi^{V_O*} > 0$ requires $0 < s < \min\{s_7, 1\}$, where $s_7 = \frac{\left(q+cq-ckq-q^2-c\alpha+ckq\alpha\right)^2}{2q^2(1-q)(q-c)^2}$.

Taken together, the freemium strategy is feasible only if the following conditions hold simultaneously:

$$q\alpha < c < c_2, 0 < s < \min\{s_7, 1\}. \tag{K.10}$$

(2) Under the paid-only strategy

Similarly, when the firm adopts the paid-only strategy, a consumer using the premium version derives the utility of

$$U_P^{S_O}(v) = v \cdot \theta^B + \alpha \cdot \theta^B \cdot \Delta N - p - kc = v + \alpha N_P - p - kc.$$

The details are similar to K(1), so we omit them and present the optimal demands and the firm's profit π^{S_O} as follows

$$p^{S_O*} = \frac{(1-ck)(1+s-\alpha)}{2+s-2\alpha}, \tag{K.11}$$

$$N_P^{S_O^*} = \frac{1-ck}{2+s-2\alpha}, \tag{K.12}$$

$$\pi^{S_O*} = \frac{(1-ck)^2}{2(2+s-2\alpha)}, \tag{K.13}$$

The single version strategy is feasible only if the following conditions hold simultaneously:

$$0 < c < \min\left\{q, \frac{1}{k}\right\} \text{ and } s > \max\{2\alpha - 1 - ck, 0\}. \tag{K.14}$$

(3) Proof of the conditions under which the firm prefers the freemium strategy

Let $\Delta\pi_8 = \pi^{V_O*} - \pi^{S_O*}$. Substituting the profits in (K.9) and (K.13) into $\Delta\pi_8$ yields

$$\Delta\pi_8 = \pi^{V_O*} - \pi^{S_O*} = \frac{A_8 s^2 + B_8 s + C_8}{4q^2(1-q)(1-\alpha)^2(2-2\alpha+s)}, \tag{K.15}$$

where $A_8 = -2(q-c)^2(1-q) < 0$;

$B_8 = 2cq(1-q)[4+q-kq(1-\alpha)-5\alpha] - q^2(1-q)(3+q-4\alpha) + c^2\{[q-kq(1-\alpha)]^2 + \alpha(4+\alpha) + q[4+2\alpha(k-k\alpha-3)] - 4\}$;

$C_8 = 2(q-\alpha)(1-\alpha)\{[\alpha - q(1-k(2-kq)(1-\alpha)] - q(1-q)(2c-q)\}$.

The quadratic function of $A_8 s^2 + B_8 s + C_8$ in s has two real solutions (i.e., ${B_8}^2 - 4A_8C_8 > 0$) if $0 < c < c_2$. Let $A_8 s^2 + B_8 s + C_8 = 0$, denoting the solutions as s_7 and s_8, where $s_7 = \frac{-B_8 + \sqrt{{B_8}^2 - 4A_8C_8}}{2A_8}$ and $s_8 = \frac{-B_8 - \sqrt{{B_8}^2 - 4A_8C_8}}{2A_8}$. Moreover, if $q\alpha < c < c_2$, then $s_7 < s_8$ holds, indicating $A_8 s^2 + B_8 s + C_8 > 0$ within if $s_7 < s < s_8$, which leads to $\pi^{V_O*} > \pi^{S_O*}$. Besides, under (K.14), we can show that $s_8 < \min\{s_7, 1\}$ always holds, which means that $s_7 < s < s_8$ can ensure the feasibility of the freemium strategy as the optimal freemium strategy. Hence, together with the precondition of this case in (K.10) and (K.14), $\pi^{V_O*} > \pi^{S_O*}$ if

$$q\alpha < c < c_2 \text{ and } \max\{s_7, 2\alpha - 1 - ck, 0\} < s \le s_8. \tag{K.16}$$

Furthermore, we examine the conditions under which $s_7 > \max\{2\alpha - 1 - ck, 0\}$. It is straightforward to see that $\frac{\partial(2\alpha-1-ck)}{\partial c} < 0$. In addition, $\frac{\partial s_7(c)}{\partial c} < 0$ if $0 < k < \frac{1}{q}$ and $q\alpha < c < c_2$, $\frac{\partial s_7(c)}{\partial c} > 0$ if $k > \frac{1}{q}$ and $q\alpha < c < c_2$. Given this monotonicity of $s_7(c)$ wrt c, we next analyze the conditions under which $s_7 > \max\{2\alpha - 1 - ck, 0\}$ in two cases.

Case 1: $\boldsymbol{0 < k < \frac{1}{q}}$ and $\boldsymbol{q\alpha < c < c_2}$

First, we consider when $\max\{2\alpha - 1 - ck, 0\} = 0$, which is equivalent to $\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\} < c < c_2$. Since $s_7(c)$ is monotonically increasing in c, it achieves its minimum at c_2 and its maximum at $\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\}$. It can be shown that $s_7(c)|_{c=\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\}} < 0$ and $s_7(c)|_{c=c_2} > 0$ can be true if

$$0 < k < 1,\ 0 < q < \frac{1-\sqrt{1-k}}{k} \text{ and } q < \alpha \le \alpha_{O1}, \text{ or } k > 1,\ 0 < q < \frac{1}{k} \text{ and } q < \alpha \le \alpha_{O1}, \quad \text{(K.17)}$$

where α_{O1} is the root of $4(1-kq)^2\alpha^3 - 4(1+q-4kq+2k^2q^2)\alpha^2 + (1+4q-14kq+4kq^2+5k^2q^2)\alpha - q(1-4k+2kq+k^2q^2) = 0$ when $0 < k < 1$ and $0 < q < \frac{1-\sqrt{1-k}}{k}$ or $k > 1$ and $0 < q < \frac{1}{k}$.

Under these conditions, the equation $s_7(c) = 0$ has exactly one root, denoted c_{O3} within the region of $\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\} < c < c_2$. Therefore, for any $c_{O3} < c < c_2$, we obtain $s_7(c) > 0 \ge \max\{2\alpha-1-ck, 0\}$ with conditions in (K.17).

In addition, $s_7(c)|_{c=\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\}} > 0$ and $s_7(c)|_{c=c_2} > 0$ can be true if

$$0 < k < 1,\ 0 < q < \frac{1-\sqrt{1-k}}{k} \text{ and } \alpha_{O1} < \alpha < \frac{\sqrt{9-12q-14kq+4q^2+12kq^2+9k^2q^2-8k^2q^3}-5+2q+3kq}{4(1-kq)},$$

$$\text{or } k > 1,\ 0 < q < \frac{1}{k} \text{ and } \alpha_{O1} < \alpha < \frac{\sqrt{9-12q-14kq+4q^2+12kq^2+9k^2q^2-8k^2q^3}-5+2q+3kq}{4(1-kq)}. \quad \text{(K.18)}$$

Under these conditions, the equation $s_7(c) = 0$ has no root within the region of $\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\} < c < c_2$. Therefore, for any $\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\} < c < c_2$, we obtain $s_7(c) > 0 \ge \max\{2\alpha-1-ck, 0\}$ with conditions in (K.18).

Second, we consider when $\max\{2\alpha-1-ck, 0\} = 2\alpha-1-ck$, which is equivalent to $q\alpha < c < \min\left\{c_2, \frac{2\alpha-1}{k}\right\}$. Under this condition, $[s_7(c)-(2\alpha-1-ck)]|_{c=q\alpha} > 0$ cannot hold. However, $[s_7(c)-(2\alpha-1-ck)]|_{c=\min\left\{c_2, \frac{2\alpha-1}{k}\right\}} > 0$ is true if the following hold:

$$0 < k < 1,\ 0 < q < \frac{1-\sqrt{1-k}}{k} \text{ and } \alpha_{O1} < \alpha < \frac{-1+q}{-1+kq}, \text{ or } k > 1,\ 0 < q < \frac{1}{k} \text{ and } \alpha_{O1} < \alpha < 1. \quad \text{(K.19)}$$

Under these conditions, the equation $s_7(c) = 2\alpha-1-ck$ has exactly one root, denoted c_{O4} within the region of $q\alpha < c < \min\left\{c_2, \frac{2\alpha-1}{k}\right\}$. Therefore, for any $c_{O4} < c < \min\left\{c_2, \frac{2\alpha-1}{k}\right\}$, we obtain $s_7(c) > 0 \ge \max\{2\alpha-1-ck, 0\}$ with conditions in (K.19).

In addition, $s_7(c)|_{c=\max\left\{q\alpha, \frac{2\alpha-1}{k}\right\}} > 0$ and $s_7(c)|_{c=c_2} > 0$ can be true if

$$0 < k < 1,\ 0 < q < \frac{1-\sqrt{1-k}}{k} \text{ and } \alpha_{O1} < \alpha < \frac{-1+q}{-1+kq}, \text{ or } k > 1,\ 0 < q < \frac{1}{k} \text{ and } \alpha_{O1} < \alpha < 1. \quad \text{(K.20)}$$

Case 2: $k > \frac{1}{q}$ and $q\alpha < c < c_2$

In this case, $2\alpha-1-ck < 0$ always holds, leading to $\max\{2\alpha-1-ck, 0\} = 0$. Since $s_7(c)$ is monotonically decreasing in c, it achieves its maximum at $q\alpha$. However, $s_7(c)|_{c=q\alpha} < 0$ always holds, indicating $s_7(c) < \max\{2\alpha-1-ck, 0\}$ within all feasible regions.

In conclusion, summarizing conditions in (K.18) to (K.20), we can have $s_7(c) > \max\{2\alpha-1-ck, 0\}$ if the following hold:

(i) When $0 < k < 1$, $0 < q < \frac{1-\sqrt{1-k}}{k}$, $q < \alpha \le \frac{1-q}{1-kq}$ and $\max\{c_{O3}, c_{O4}\} < c < c_2$;

(ii) When $k > 1$, $0 < q < \frac{1}{k}$, $q < \alpha \le 1$ and $\max\{c_{O3}, c_{O4}\} < c < c_2$.

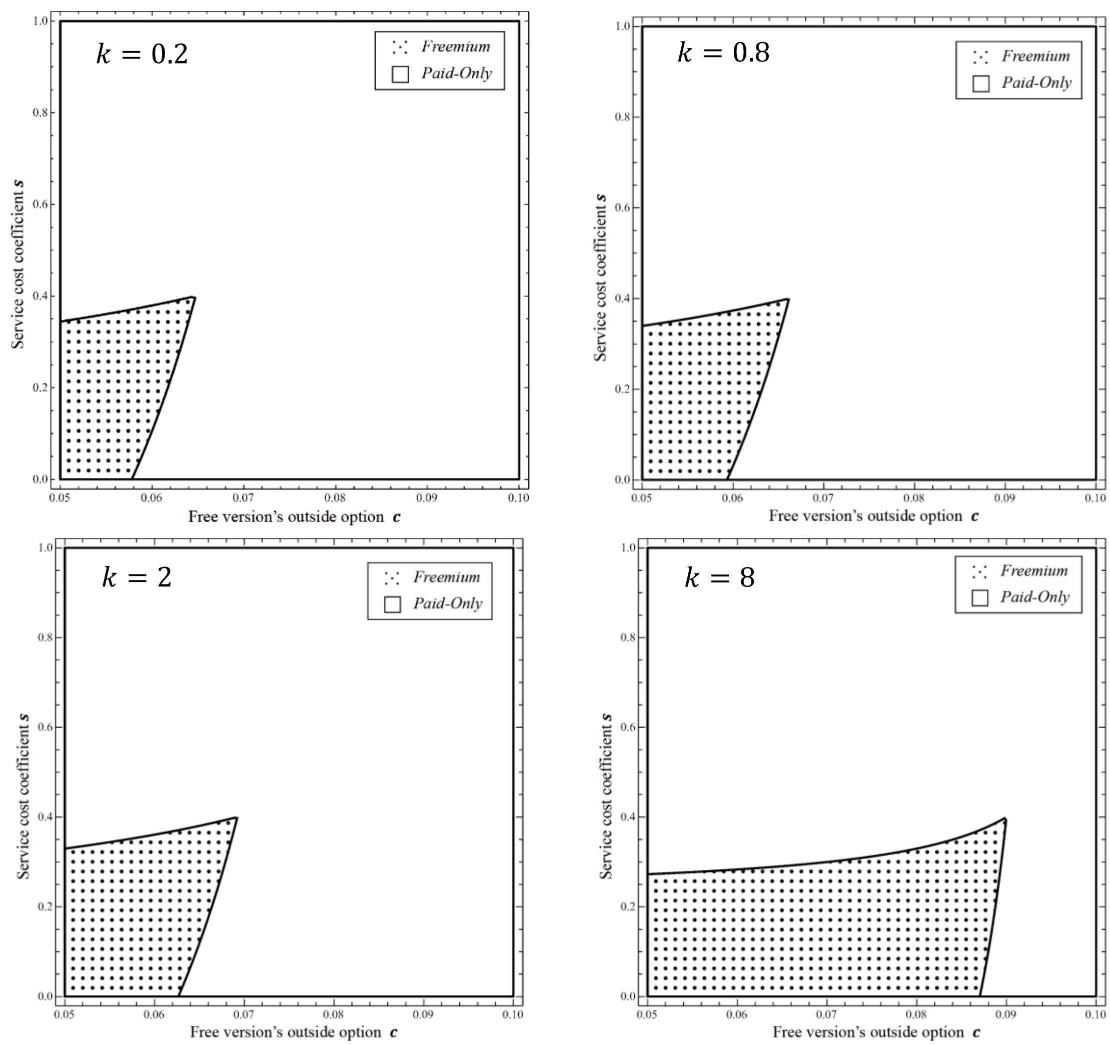
Finally, using a similar way, we can verify that if $q < \alpha < 1$, then any $q\alpha < c < c_2$, $s_8 > \max\{2\alpha - 1 - ck, 0, s_7\}$.

Combining conditions for $s_7(c) > \max\{2\alpha - 1 - ck, 0\}$ and $s_8 > \max\{2\alpha - 1 - ck, 0, s_7\}$ and denoting $c_3 = \max\{c_{O3}, c_{O4}\}$, we can summarize conditions where $\pi^{Vo*} > \pi^{So*}$ in Proposition 8.

(4) Impact of k on the firm's optimal freemium strategy

Next, we resort to numerical analysis to explore how k impacts the firm's optimal freemium strategy. In the following, given q and α, we illustrate the firm's feasible region for an optimal freemium strategy under different value of k.

Figure A4. The Firm's Optimal Freemium Strategy considering k ($q = 0.1$, $\alpha = 0.5$)



Comparing the four panels in Figure A4 shows that the region favoring the freemium strategy expands as k increases. Intuitively, a higher premium version outside option (a larger kc) weakens the attractiveness of the premium version, making the firm rely more on the free version to grow the user base and generate learning-driven gains.

L. Equilibrium Characterization and Numerical Illustration for Observation 3

In this extension, we use superscripts "V_S" and "S_S" to denote the freemium and paid-only strategies when service costs differ across versions. Let s be the baseline service-cost coefficient, and let $t \in (0,1)$ capture the free version's service-cost advantage. Specifically, the free version has a service-cost coefficient $s_F = ts$, while the premium version has $s_P = s$ (so a smaller t implies a larger cost advantage for the free version). Since this heterogeneity affects only the firm's cost side (not consumer utility), the demand functions remain the same as in the base model. We focus on the partially covered regime, using (A.1) for the freemium strategy and (B.1) for the paid-only strategy. Because t only affects the firm's decision under the freemium strategy, we derive the equilibrium outcomes under the freemium strategy below; the paid-only equilibrium follows directly from Lemma 2.

(1) Equilibrium under the freemium strategy

Given the precondition for demand functions in (A.1), i.e., $\frac{c}{q} < p < \frac{[(1+c)q]-q^2-c\alpha}{q(1-\alpha)}$, the firm's profit π_S^V, as a function of the premium price p, is

$$\pi^{V_S}(p) = pN_P - \frac{1}{2}sN_P^2 - \frac{1}{2}tsN_F^2. \tag{L.1}$$

Substituting (A.1) into (L.1) and solving for the FOC yields

$$p^{V_S*} = \frac{(1-q)(q+cq-q^2-c\alpha)+s(ck+q+cq-q^2-c\alpha-ck\alpha)}{q(2-2q+s+ks)(1-\alpha)}. \tag{L.2}$$

Feasibility of $\frac{c}{q} < p < \frac{[(1+c)q]-q^2-c\alpha}{q(1-\alpha)}$ further requires $s > \max\left\{0, \frac{q-q^2-c(2-q-\alpha)}{c-q}\right\}$. Substituting p^{V_S*} into (A.1) and (L.1) gives the optimal demands:

$$N_P^{V_S*} = \frac{q(1+c+ks)-q^2-c(ks+\alpha)}{q(2-2q+s+ks)(1-\alpha)},\ N_F^{V_S*} = \frac{q(1-q+s)-c(2-q+s-\alpha)}{q(2-2q+s+ks)(1-\alpha)}, \tag{L.3}$$

Finally, substituting p^{V_S*}, $N_P^{V_S*}$ and $N_F^{V_S*}$ into the firm's profit π^{V_S}, the firm's optimal profit is

$$\pi^{V_S*} = \frac{\left(q+cq-q^2-c\alpha\right)^2+2cks(c-q)(1-\alpha)-k(c-q)^2s^2}{2q^2(2-2q+s+ks)(1-\alpha)^2}. \tag{L.4}$$

$\pi^{V_S*} \geq 0$ only if $0 < s \leq \min\left\{\frac{\sqrt{\{(1-q)^2q^2+[(\alpha-q)^2+k(1-\alpha)^2]c^2-2(1-q)(\alpha-q)cq\}k}+(1-\alpha)ck}{k(q-c)}, 1\right\}$.

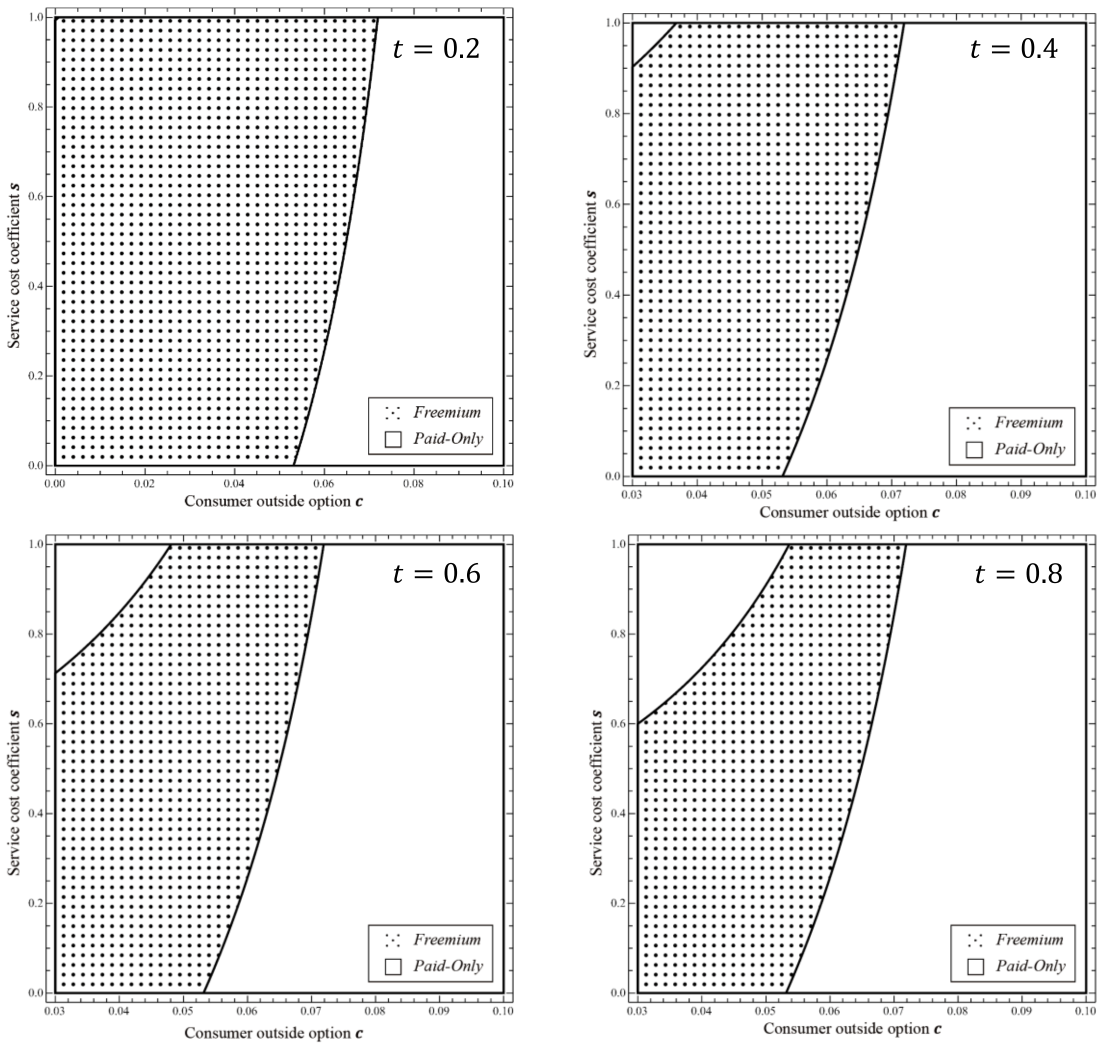
In summary, the freemium strategy is viable when

$$\max\left\{0, \frac{q-q^2-c(2-q-\alpha)}{c-q}\right\} < s \leq \min\left\{\frac{\sqrt{\{(1-q)^2q^2+[(\alpha-q)^2+k(1-\alpha)^2]c^2-2(1-q)(\alpha-q)cq\}k}+(1-\alpha)ck}{k(q-c)}, 1\right\}. \tag{L.5}$$

(2) Numerical evidence for the firm's preferred strategy

Next, we resort to numerical analysis to examine how t affects the firm's preferred strategy. Holding q and α fixed, we illustrate the parameter region in which the freemium strategy is optimal for different values of t.

Figure A5. The Firm's Optimal Freemium Strategy considering t ($q = 0.1$, $\alpha = 0.5$)



Comparing the four panels in Figure A5 shows that when t is low (i.e., the free version is substantially cheaper to serve), the firm prefers the freemium strategy for a wide range of s, provided that the outside option c is sufficiently low. As t increases, the region in which the freemium strategy is optimal shrinks: offering a free version becomes more costly, so the freemium strategy is preferred only when both s and c are relatively small.