Contents

A Fuzzy Segmentation Results for Extended Parameter Space

In the main text, we focus on the most relevant (v, w) parameter space, where consumer valuation exceeds a threshold ($v > 3$) and at least 10% of consumers choose to preserve privacy ($w > 0.1$) (recall from Section 3). We now extend our analysis to the full parameter space, considering all possible values of $w \in (0, 1)$ and any $v \geq 2$, a standard assumption ensuring that a monopoly firm has the incentive to serve the entire market (e.g., Chen and Iyer 2002; Moorthy and Tehrani 2023). As we demonstrate below, our main insights remain largely consistent.

Below, we first summarize the fuzzy-segmentation equilibrium results for the extended parameter space in Proposition A1 and then discuss their implications for our main findings. The complete derivation steps are provided in Online Appendix C.1.9.

Proposition A1 ***Fuzzy-Segment Equilibrium for the Extended Parameter Space.*** *The sellers' equilibrium strategies depend on the parameter space (v, w), which is partitioned into three regions: Region 0 where $0 < w < \breve{w}'$, Region 1 where $\breve{w}' \leq w < \breve{w}$, and Region 2 where $\breve{w} \leq w < 1$, with $\breve{w}' = \max\{\frac{50-16v}{30v-25}, 0\}$ and $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)}$. This partition is illustrated in Figure A1. Each region corresponds to a set of equilibrium segmentation and pricing strategies, $(z^{FS,*}, \mathbf{p}^{FS,*})$, as summarized in Table A1.*

Table A1: Full Equilibrium Under Fuzzy Segmentation, when $v \geq 2$, $0 < w < 1$

Region	Range	$z^{FS,*}$	$p_{1A}^{FS,*}$	$p_{1T}^{FS,*}$	$p_{2A}^{FS,*}$	$p_{2T}^{FS,*}$
0	$0 < w < \breve{w}'$	$\frac{(40v-75)w}{32-50w}$	$\frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z^* - 1$	$\frac{2}{3}v + \frac{2}{3}\frac{1-w}{w}z^*$	$\frac{3}{2} - z^*$	$\frac{5}{4} - \frac{3}{2}z^*$
1	$\breve{w}' \leq w < \breve{w}$	$\frac{w}{2+w}v$	$2\frac{1+w}{w}z^* - 1$	$\frac{2}{w}z^*$	$\frac{3}{2} - z^*$	$\frac{5}{4} - \frac{3}{2}z^*$
2	$\breve{w} \leq w < 1$	$\frac{5+15w}{14+18w}$				

Notes. $\breve{w}' = \max\{\frac{50-16v}{30v-25}, 0\}$, $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)}$.

<u>Equilibrium Market Outcome Discussion.</u> Note from the above that, in extending the parameter space, we introduce a new region (Region 0). Regions 1 and 2 remain unchanged from Proposition 2; therefore, we focus only on equilibrium market outcomes in Region 0. In Region 0, Third-party in Segment 1 serves not only customers from $1 - z^{FS,*}$ to 1, but also some privacy-preserving customers from $\tilde{x}_1$ to $1 - z^{FS,*}$, where $\tilde{x}_1 = 1 - \frac{1}{3}v + \frac{2}{3}\frac{1-w}{w}z$. Segment 2 is unaffected, with Platform serving customers in $[0, \frac{3}{8} - \frac{1}{4}z^{FS,*}]$ and Third-party $[\frac{3}{8} - \frac{1}{4}z^{FS,*}, 1 - z^{FS,*}]$.

The sellers' equilibrium profits in Region 0 are $\pi_A^{FS,*} = w(1-\tilde{x}_1)\left[\frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z^{FS,*} - 1\right] + \frac{1}{4}(1-w)(\frac{3}{2} - z^{FS,*})^2$ *and* $\pi_T^{FS,*} = \left(w(1 - z^{FS,*} - \tilde{x}_1) + z^{FS,*}\right) \cdot \left[\frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z^{FS,*}\right] + \frac{1}{2}(1-w)(\frac{5}{4} - \frac{3}{2}z^{FS,*})^2$ *for Platform and Third-party, respectively.*

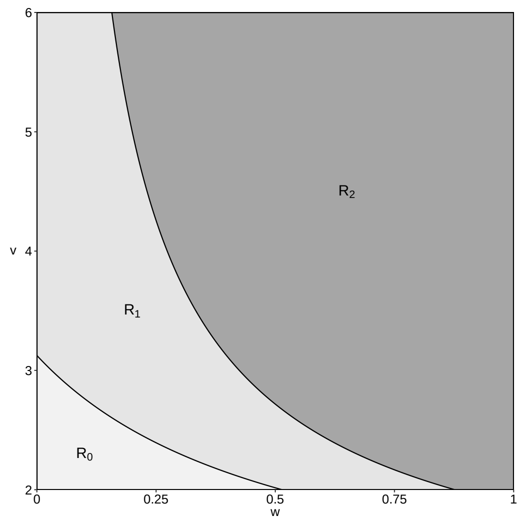
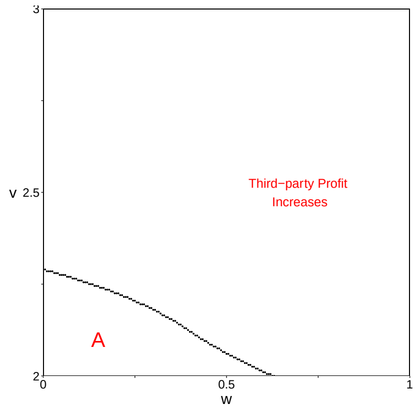


Figure A1: Partition of the Full Parameter Space.
Note: The two solid curves represent $\breve{w}'$ and $\breve{w}$, respectively, as defined in Proposition A1. The R_0 area represents Region 0, R_1 area Region 1 and R_2 area Region 2.

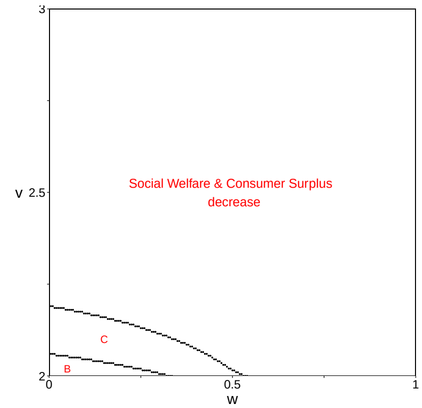
Main Insights Discussion. Below, we show that our main insights largely consistent in the expanded parameter space.

1. **Platform's Profit Increase (Theorem 1).** Consistent with Theorem 1, when any positive fraction of consumers remain privacy-preserving, adopting a fuzzy-segmentation strategy strictly increases Platform's total profit.[1]

[1]This can be reasoned as follows. When Platform solves its optimal segmentation problem, trivial segmentation rules such as $z = 0$ or $z = 1$ are always in the strategy space. The fact that they are not among optimal solutions shows that the profit associated with the optimal segmentation rule dominates these trivial segmentation rules.



(a) Parameter Space where Third-party Profit Decreases.



(b) Parameter Space where Social Welfare and Consumer Surplus Increases

Figure A2: Partition of the parameter space where Third-party profit decreases (left) and social welfare or consumer surplus increases (right), under optimal fuzzy segmentation.

Note: In these figures, we illustrate that in a narrow range of very low consumer valuation (when $2 \leq v < 3$), it is possible that Third-party experience a decrease in profit (Area A), consumer surplus increases (Area B + C), or social welfare increases (Area B), when Platform adopts the optimal fuzzy segmentation rule when w share of customers choose to preserve privacy, comparing to the full-data scenario. As illustrated, except the narrow areas where both w and v are small, our results in Theorems 1– 3 continue to hold.

2. **Third-Party Profit Effects (Theorem 1).** In most parameter settings, Third-party also benefits from the softened competition under fuzzy segmentation—consistent with Theorem 1. However, in a narrow range of very low consumer valuation, Third-party's gains in Segment 1 (where competition is softened) do not fully offset its losses in Segment 2 (where competition is intensified), resulting in a small decrease in its total profit, as illustrated in Figure A2a. This exception disappears once consumer valuation surpasses a moderate threshold.

3. **Consumer Surplus Implications (Theorem 2).** In most parameter settings, the presence of privacy-preserving consumers reduces the total consumer surplus compared to the full-data scenario, because of the elevated prices in Segment 1—consistent with Theorem 2. However, in a narrow range of very low consumer valuation, the consumer participation constraint limits how much Segment-1 prices could be raised, leading to less decrease in consumer surplus. Since consumers in Segment 2 always enjoy a higher surplus, the total consumer surplus may increase, as illustrated in Area B+C in Figure A2b. Notice that in Area C, despite a decreased social welfare compared to the full-data scenario, consumer surplus increases, because Third-party's profit decreases.

4. **Social Welfare Implications (Theorem 3).** In most parameter settings, social welfare decreases due to higher average transportation cost under fuzzy segmentation–consistent with Theorem 3. However, in a narrow range of very low consumer valuation, the price limitation in Segment 1 leads to a more balanced split of privacy-preserving customers between the two sellers, decreasing the transportation cost. Such decrease among privacy-preserving customers outweighs the increase in the transportation cost among data sharing customers, resulting in a small decrease in the total transportation cost. The relevant parameter space is Area B in Figure A2b. This exception disappears once consumer valuation surpasses a moderate threshold.

In summary, our study's key insights from Theorems 1–3 regarding how privacy-preserving consumers impact equilibrium market outcomes remain largely consistent in the extended parameter space.

B Model Extensions

B.1 Alternative Timing – When Third-party Serves as Stackelberg Leader

When Third-party serves as the price leader (sometimes denoted with the superscript "TL" to distinguish from the baseline analysis), the sequence of the game is modified as follows.

- Stage 1: When applicable, Platform will choose the segmentation rule z_0 or z.
- Stage 2: Platform and Third-party involves in a duopoly competition with segment-based differential pricing, where Third-party acts as a Stackelberg price leader.

To analyze the equilibrium outcomes, we employ the Subgame Perfect Nash equilibrium concept and use backward induction as our solution method.

B.1.1 Benchmarks

Because of the symmetry of the Hotelling model, there are no qualitative changes to the analysis of the no-segment and hard-segment benchmarks, except that all outcome variables (price, demand,

and profit) are flipped between two sellers. We summarize the results below and defer more detailed discussions to Online Appendices C.2.1 and C.2.2.

Lemma B1 ***Third-party as the Stackelberg leader.*** *With complete consumer data access,*

- *Under uniform pricing, Third-party sets the price at $\frac{3}{2}$, sells to customers located at $[\frac{5}{8}, 1]$ and realizes a profit of $\frac{9}{16}$. Platform sets the price at $\frac{5}{4}$, sells to customers located at $[0, \frac{5}{8}]$ and realizes a profit of $\frac{25}{32}$.*
- *Any non-trivial hard segmentation rule intensifies the price competition. Hence, Platform strictly prefers the uniform pricing scheme to setting hard segments for differential pricing.*

B.1.2 Fuzzy segmentation results

When analyzing the fuzzy segmentation model, a major modification concerns the response of the follower in the Stage-2 pricing game. Because the "L-shaped" fuzzy segmentation results in larger demand closer to the price leader (Third-party), the price follower (Platform) is more likely to lower its price to capture the additional demand to the right of $1 - z$, provided that this demand is sufficiently large. Anticipating this response, the price leader's ability to sustain a high price is limited. Nevertheless, there still exists optimal segmentation rules under which higher-than-uniform-pricing-benchmark equilibrium prices can be sustained in Segment-1, thereby resulting in increased total profit for Platform.

We state the full equilibrium when Third-party is the price leader in Proposition B2 and defer the full analysis to Online Appendix C.2.3.

Proposition B2 ***Third-party as the Stackelberg leader.*** *With a proportion* w *of privacy-preserving customers, the full equilibrium is characterized by $(z^{FS,TL}, \mathbf{p}^{FS,TL})$ as summarized in Table A2.*

Table A2: Equilibrium Under Fuzzy Segmentation, Third-party as Price Leader

Region	Range	$z^{FS,TL,*}$	$p_{1A}^{FS,TL,*}$	$p_{1T}^{FS,TL,*}$	$p_{2A}^{FS,TL,*}$	$p_{2T}^{FS,TL,*}$
1	$w < \breve{w}''$	ζ_a	$\frac{5}{4} + \frac{1-w}{w} z^*$	$\frac{3}{2} + 2\frac{1-w}{w} z^*$	$\frac{5}{4} - z^*$	$\frac{3}{2} - 2z^*$
2	$\breve{w}'' \leq w$	ζ_c				

Notes. The value of $\breve{w}''$ is $\frac{2(16\sqrt{13}+59)}{28\sqrt{13}+101} - \frac{2\sqrt{880\sqrt{13}+3173}}{28\sqrt{13}+101}$; $\zeta_a = \frac{-5+2\sqrt{13}}{12} \frac{w}{1-w}$ and $\zeta_c = \frac{4w^3+w-w(w+4)\sqrt{w}}{4(w^3-1)}$.

The corresponding equilibrium customer decisions are:

- *In Segment 1, Platform serves customers from* 0 *to* $\frac{5}{8}+\frac{1}{2}\frac{1-w}{w}z^{FS,TL,*}$*, all of which are privacy-preserving customers. Third-party serves customers from* $\frac{5}{8}+\frac{1}{2}\frac{1-w}{w}z^{FS,TL,*}$ *to* 1*, which consists of both are privacy-preserving customers and data-sharing customers.*

- *In Segment 2, Platform serves consumers from* 0 *to* $\frac{5}{8}-\frac{1}{2}z^{FS,TL,*}$*, all of which are data-sharing customers. Third-party serves customers from* $\frac{5}{8}-\frac{1}{2}z^{FS,TL,*}$ *to* $1-z^{FS,TL,*}$*, who are also all data-sharing consumers.*

The corresponding equilibrium profits are $\pi_A = \frac{1}{2}w\left(\frac{5}{4}+\frac{1-w}{w}z\right)^2+\frac{1}{2}(1-w)\left(\frac{5}{4}-z\right)^2$ *and* $\pi_T = \frac{1}{4}w\left(\frac{3}{2}+2\frac{1-w}{w}z\right)^2+\frac{1}{4}(1-w)\left(\frac{3}{2}-2z\right)^2$ *for Platform and Third-party, respectively.*

Figures A3 and A4 illustrate the optimal segmentation rule $z^{FS,TL,*}$ and the corresponding equilibrium Segment-1 prices with respect to w. Comparing to the equilibrium Segment-1 prices when Platform is the price leader (see those depict in Figure 4), the equilibrium Segment-1 prices when Third-party is the price leader is systematically lower across different values of w. This reflects an attenuation of the competition softening effect when the extra demand in the fuzzy segment is closer to the price leader.

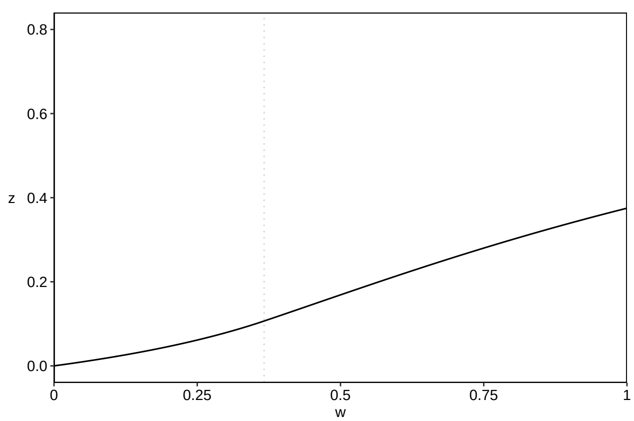
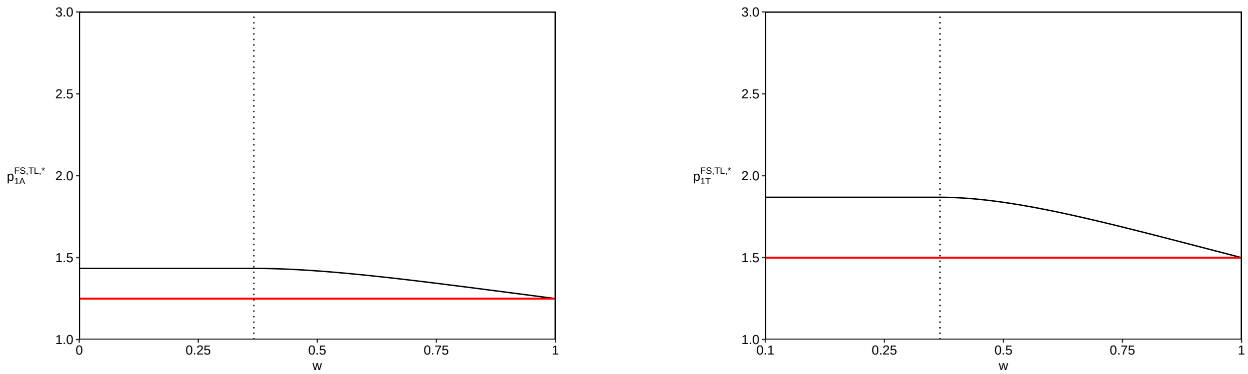


Figure A3: Equilibrium fuzzy segmentation rule, $z^{FS,TL,*}$, when Third-party serves as the price leader; $v \geq 3$

Note: The black solid line represents the optimal fuzzy-segmentation rule when the consumer valuation is at least 3. The vertical dotted line represents $\breve{w}''$, separating Region 1 and Region 2 as specified in Table A2.



(a) Platform Price

(b) Third-party Price

Figure A4: Segment-1 price for Platform and Third-party at fuzzy-segmentation equilibrium, when Third-party serves as the price leader; $v \geq 3$

Note: In this figure, we plot the equilibrium Segment-1 prices for Platform and Third-party, respectively, when Third-party serves as the price leader. The black curve represents each seller's Segment-1 price at fuzzy-segment equilibrium. The horizontal red line represents each seller's no-segment benchmark Segment-1 price. Observe that each seller's Segment-1 price is higher under fuzzy-segment than under no-segment. The vertical dotted line represents $\breve{w}''$, separating Region 1 and Region 2 as specified in Table A2.

B.1.3 Discussion on surplus and welfare

Implications for each seller's profit. It is straightforward to show that the equilibrium profit of each seller, as specified in Proposition B2, is strictly greater than their no-segment benchmark when $w \in (0, 1)$. Therefore, both seller's profit strictly increases when a portion w of consumers choose not to share data, even when Third-party serves as the price leader.

Each seller's profit under fuzzy segmentation comparing to the no-segment benchmark is illustrated in Figures A5a and A5b.

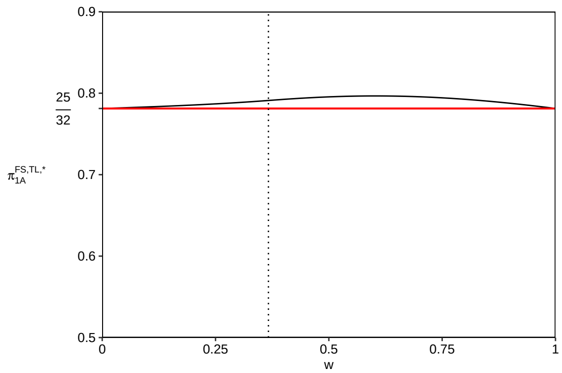
Implications to consumer surplus. As illustrated in Proposition B2, the indifferent customer lies at $\frac{5}{8} + \frac{1}{2}\frac{1-w}{w}z^{FS,TL,*}$ in Segment 1, moving towards the right from the no-segment benchmark, so the average traveling cost increases among privacy-preserving customers. Since privacy-preserving customers on average also pay higher prices, the total consumer surplus among privacy-preserving customers always decreases.

For data-sharing customers who prefer Third-party's product, after the adoption of fuzzy segmentation, they still purchase from Third-party, so the traveling cost does not change. However, they face higher prices as a result of the fuzzy segmentation. Therefore, the total consumer surplus among data-sharing Third-party preferring customers also decreases.

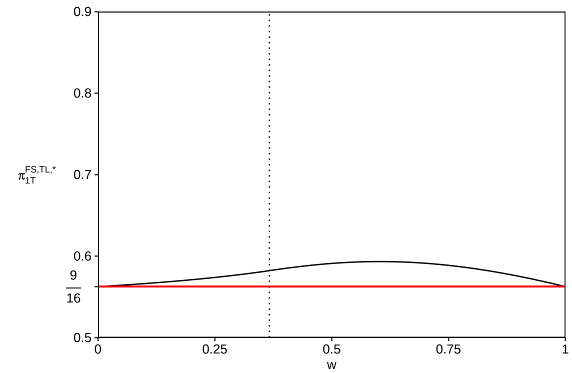
For data-sharing customers who prefer Platform's product, their average traveling cost decreases, as the indifferent customer lies at $\frac{5}{8} - \frac{1}{2}z^{FS,TL,*}$, moving toward the center from the no-segment benchmark. Furthermore, they on average face lower prices. Consequently, the total consumer surplus of data-sharing Platform-preferring customers always increases.

The total consumer surplus is illustrated in Figure A5c. We observe from the figure that when $w \in (0,1)$ (i.e., a non-zero fraction of customers are privacy-preserving)

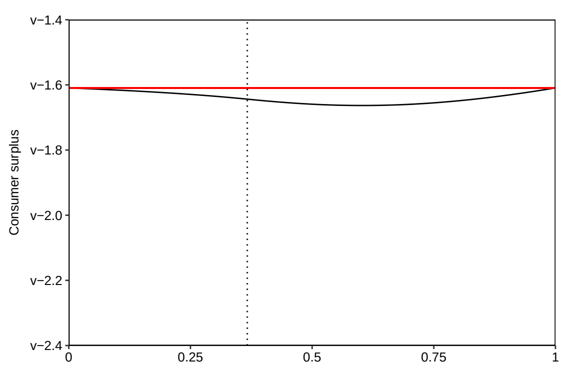
Implications to social welfare. Combining the consumer surplus and the firm profit, we are able to compute the social welfare under any portion of privacy-preserving customers. We illustrated it in Figure A5d.



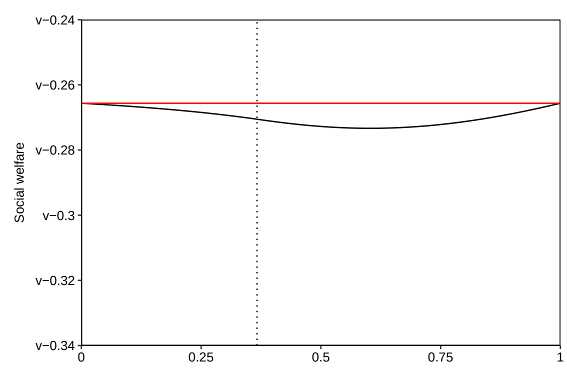
(a) Platform Profit



(b) Third-party Profit



(c) Consumer Surplus



(d) Social Welfare

Figure A5: Producer surplus (top two panels), consumer surplus (bottom left panel) and social welfare (bottom right panel), under optimal fuzzy segmentation, when Third-party serves as the price leader; $v \geq 3$

Note: In each sub-figure, the black curve represents the corresponding variable at fuzzy-segment equilibrium. The horizontal red line represents the no-segment benchmark of the same variable. Observe that each seller's profit is higher under fuzzy-segment than under no-segment across different size of w and both consumer surplus and social welfare decreases across different size of w. The vertical dotted line represents $\breve{w}''$, separating Region 1 and Region 2 as specified in Table A2.

B.2 Alternative Timing – When Both Sellers Move Simultaneously

Instead of a Stackelberg pricing sequence, another commonly used modeling setup in the price competition literature is that both sellers set prices simultaneously. We consider this alternative timing structure in the pricing stage (sometimes denoted with the superscript "S" to distinguish from the baseline analysis). The sequence of the game is modified as follows.

- Stage 1: When applicable, Platform will choose the segmentation rule z_0 or z.
- Stage 2: Platform and Third-party decide their respective price in each segment (when applicable) at the same time.

We adopt backward induction as our solution method. We first solve the pricing game under a given segmentation rule (when applicable), and then search for the optimal segmentation rule on behalf of Platform. When solving for the pricing game, we adopt the solution concept of the Nash equilibrium (NE).

B.2.1 Benchmarks

Solving the simultaneous pricing game in the Hotelling line framework is straightforward for the uniform-pricing and hard-segmentation benchmarks. We defer the technical details to Online Appendices C.3.1 and C.3.2. In short, the alternative timing structure does not alter the insight that hard-segmentation under full consumer data access intensify competition. We summarize these results in Lemma B2.

Lemma B2 ***Simultaneous pricing.*** *With complete consumer data access,*

- *Under uniform pricing, both Platform and Third-party set the price at* 1*, serve half of the market (with demand equal to* $\frac{1}{2}$*), and obtain a profit of* $\frac{1}{2}$*.*
- *Any non-trivial hard segmentation rule intensifies the price competition. Hence, Platform strictly prefers the uniform pricing scheme to setting hard segments for differential pricing.*

B.2.2 Fuzzy segmentation results

We summarize the full equilibrium when both sellers move simultaneously in the pricing game in Proposition B3. Figures A6 and A7 illustrate the optimal segmentation rule $z^{FS,S,*}$ and the corresponding equilibrium Segment-1 prices with respect to w. The mechanism that the equilibrium price in Segment 1 is higher than the uniform-pricing equilibrium price because of the asymmetric demand continues to hold. There always exists a fuzzy segmentation rule z under which the overall profit for Platform is enhanced beyond the uniform-pricing benchmark.

Proposition B3 ***Simultaneous pricing***. *With a proportion* w *of privacy-preserving customers, the pure-strategy Nash equilibrium is characterized by* $(z^{FS,S}, \mathbf{p}^{FS,S})$ *as summarized in Table A3.*

Table A3: Equilibrium Under Fuzzy Segmentation, Simultaneous Pricing

Region	Range	$z^{FS,S,*}$	$p_{1A}^{FS,S,*}$	$p_{1T}^{FS,S,*}$	$p_{2A}^{FS,S,*}$	$p_{2T}^{FS,S,*}$
1	$w < \breve{w}'''$	ζ_g	$1+\frac{2}{3}\frac{1-w}{w}z^*$	$1+\frac{4}{3}\frac{1-w}{w}z^*$	$1-\frac{2}{3}z^*$	$1-\frac{4}{3}z^*$
2	$\breve{w}''' \leq w$	ζ_f				

Notes. The value of $\breve{w}'''$ is $10-4\sqrt{6}$; $\zeta_f = \frac{9w^3-3w(w+2)\sqrt{w}}{9w^3-w^2-4w-4}$, $\zeta_g = \frac{3\sqrt{6}-3}{10}\frac{w}{1-w}$.

The corresponding equilibrium customer decisions are:

- *In Segment 1, Platform serves customers from* 0 *to* $\frac{1}{2}+\frac{1}{3}\frac{1-w}{w}z^{FS,S,*}$*, all of which are privacy-preserving customers. Third-party serves customers from* $\frac{1}{2}+\frac{1}{3}\frac{1-w}{w}z^{FS,S,*}$ *to* 1*, which consists of both privacy-preserving customers and data-sharing customers.*

- *In Segment 2, Platform serves consumers from* 0 *to* $\frac{1}{2}-\frac{1}{3}z^{FS,S,*}$*, all of which are data-sharing customers. Third-party serves customers from* $\frac{1}{2}-\frac{1}{3}z^{FS,S,*}$ *to* $1-z^{FS,S,*}$*, who are also all data-sharing consumers.*

The corresponding equilibrium profits are $\pi_A = \frac{1}{2}w\left(1+\frac{2}{3}\frac{1-w}{w}z\right)^2 + \frac{1}{2}(1-w)\left(1-\frac{2}{3}z\right)^2$ *and* $\pi_T = \frac{1}{2}w\left(1+\frac{4}{3}\frac{1-w}{w}z\right)^2 + \frac{1}{2}(1-w)\left(1-\frac{4}{3}z\right)^2$ *for Platform and Third-party, respectively.*

B.2.3 Discussion on surplus and welfare

Implications to each seller's profit. It is straightforward to show that the equilibrium profit of each seller, as specified in Proposition B3, is strictly greater than their no-segment benchmark

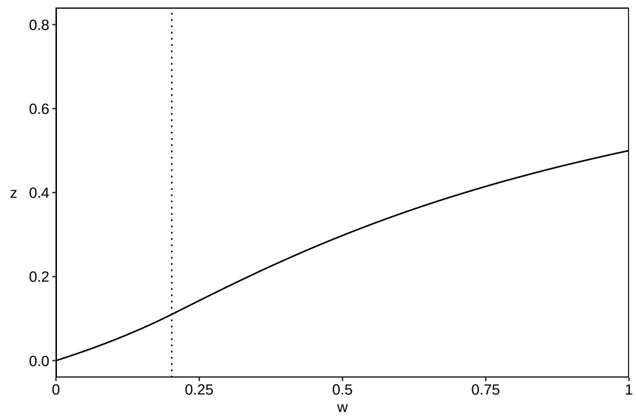


Figure A6: Equilibrium fuzzy segmentation rule, $z^{FS,S,*}$, when both sellers act simultaneously; $v \geq 3$

Note: The black solid line represents the optimal fuzzy-segmentation rule when consumer valuation is at least 3. The vertical dotted line represents $\breve{w}'''$, separating Region 1 and Region 2 as specified in Table A3.

when $w \in (0, 1)$. Therefore, both seller's profit strictly increases when a portion w of consumers choose not to share data, even when both sellers act simultaneously in the pricing subgame.

Each seller's profit under fuzzy segmentation comparing to the no-segment benchmark is illustrated in Figures A8a and A8b.

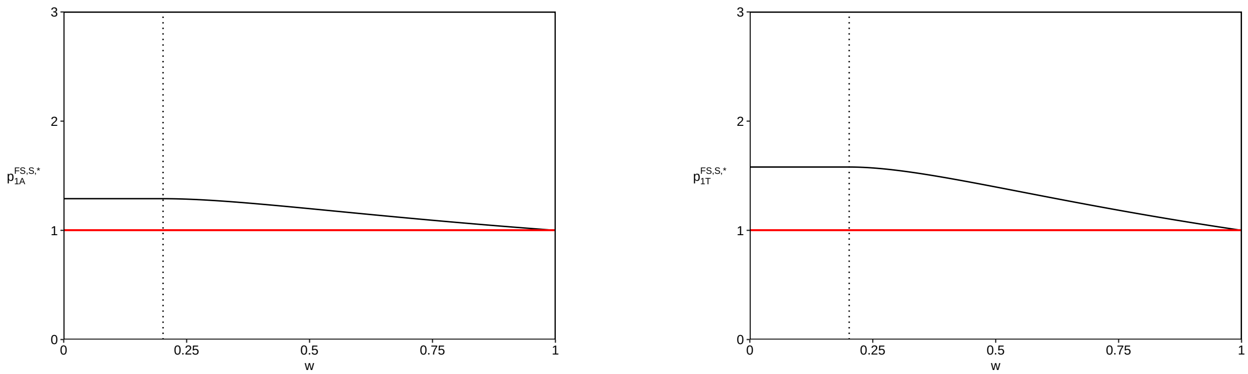
Implications to consumer surplus. As illustrated in Proposition B3, the indifferent customer lies at $\frac{1}{2} + \frac{1}{3}\frac{1-w}{w} z^{FS,S,*}$ in Segment 1, indicating that the average traveling cost increases for both Segment-1 customers and Segment-2 customers. Since privacy-preserving customers also pay higher prices, the total consumer surplus among privacy-preserving customers always decreases.

For data-sharing customers who prefer Third-party's product, after the adoption of fuzzy segmentation, they still purchase from Third-party, so the traveling cost does not change. However, they face higher prices as a result of the fuzzy segmentation. Therefore, the total consumer surplus among data-sharing Third-party preferring customers also decreases.

For data-sharing customers who prefer Platform, their average traveling cost increases, as the indifferent customer lies at $\frac{1}{2} - \frac{1}{3} z^{FS,S,*}$ in Segment 2, deviating from $\frac{1}{2}$. However, they face lower prices. In sum, their average consumer surplus increases.

The total consumer surplus is illustrated in Figure A8c.

Implications to social welfare. Combining the consumer surplus and the firm profit, we are able to compute the social welfare under any portion of privacy-preserving customers. We illustrate it in Figure A8d.



(a) Platform Price

(b) Third-party Price

Figure A7: Segment-1 price for Platform and Third-party at fuzzy-segmentation equilibrium, when both sellers act simultaneously; $v \geq 3$

Note: In this figure, we plot the equilibrium Segment-1 prices for Platform and Third-party, respectively, when both sellers act simultaneously in the pricing game. The black curve represents each seller's Segment-1 price at fuzzy-segment equilibrium. The horizontal red line represents each seller's no-segment benchmark Segment-1 price. Observe that each seller's Segment-1 price is higher under fuzzy-segment than under no-segment. The vertical dotted line represents $\breve{w}'''$, separating Region 1 and Region 2 as specified in Table A3.

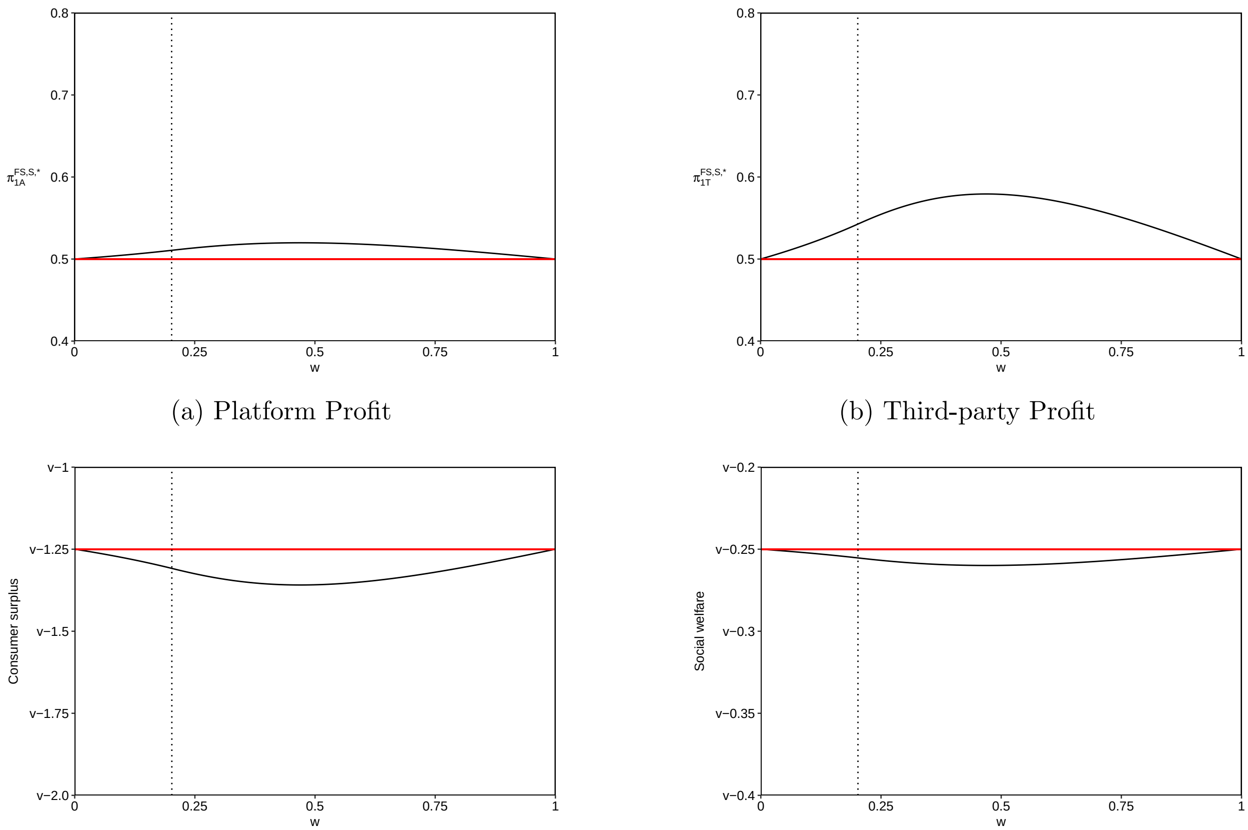
B.3 Platform Receives Revenue-sharing from Third-party

We consider a model extension by incorporating a percentage-based revenue-sharing scheme, where the third-party seller pays a fraction s of its revenue to the platform. Its analysis closely parallels the derivations in our main model, with one important difference: the platform can now earn revenue even when it does not directly sell a product. This introduces an additional consideration: whether the platform chooses to enter the market as a direct seller or instead exits and earns revenue purely through commissions.

Specifically, when the revenue-sharing rate is sufficiently high ($s > \bar{s}$, where $\bar{s}$ depends on the consumer valuation, v), the platform's incentives change qualitatively: it may optimally exit direct selling to consumers. Because our primary interest lies in platform–third-party competition, we therefore restrict attention to parameter regions and segmentation rules under which both sellers remain active in both segments.

While this restriction is conceptually intuitive, explicitly accounting for the platform's potential exit adds substantial analytical complexity. We tackle this challenge in a two-step routine:

- In the no-segment benchmark case, we solve the platform's entry decision analytically—characterizing the parameter space in which Platform will choose to enter.



(a) Platform Profit

(b) Third-party Profit

(c) Consumer Surplus

(d) Social Welfare

Figure A8: Consumer surplus (left) and social welfare (right), under optimal fuzzy segmentation, when both sellers act simultaneously; $v \geq 3$

Note: In each sub-figure, the black curve represents the corresponding variable at fuzzy-segment equilibrium. The horizontal red line represents the no-segment benchmark of the same variable. Observe that each seller's profit is higher under fuzzy-segment than under no-segment across different size of w and both consumer surplus and social welfare decreases across different size of w. The vertical dotted line represents $\breve{w}'''$, separating Region 1 and Region 2 as specified in Table A3.

- In the hard-segment (HS) and fuzzy-segment (FS) cases, because multiple model primitives are involved—(v, s, z_0) in the HS case and (v, s, w, z) in the FS case—although the equilibrium is theoretically well defined, deriving a complete analytical characterization is very involved and adds limited additional insights. Therefore, we provide the analytical expression of Third-party's best response price conditional on the model primitives and Platform's segmentation and pricing decisions, and use a numerical routine to solve for Platform's optimal price under any segmentation rule and, ultimately, its optimal segmentation decision. In this numerical analysis, the platform's entry decision is evaluated explicitly to ensure that we focus on the strategy space in which the platform remains an active seller.

In what follows, we present the key analytical results, the pseudo code for the numerical routine and discuss the robustness of our main results under this model extension.

Firstly, we provide the analytical results for the uniform-pricing equilibrium when Third-party pays a share s of its total revenue as a commission to Platform in Lemma B3. We defer the proof of this lemma to Online Appendix C.4.1. This equilibrium serves as the benchmark against which we later compare hard segmentation and fuzzy segmentation under revenue sharing. Two observations are worth noting. When $s \leq \bar{s}$, both sellers' equilibrium prices are higher than the no-revenue-sharing counterparts, $\frac{3}{2}$ and $\frac{5}{4}$, respectively, and both are increasing in s. Furthermore, when $s > \bar{s}$, Platform has a financial incentive to stay out of the market—not the focus of this study.

Lemma B3 ***No-Segment / Uniform-Pricing Equilibrium with Revenue-Sharing***. *The equilibrium under uniform pricing when Third-party pays a share s of its total revenue as a commission to Platform is characterized by (p_A, p_T) as summarized in Table A4. The equilibrium profits are summarized in Table A5.*

Table A4: Full Equilibrium Under Uniform Pricing (When Third-Party Pays a Commission)

Segment	Platform Price	Third-Party Price	Range of v	Range of s
n.a.	$p_A^\bullet = \frac{3+s}{2-s}$	$p_T^\bullet = \frac{5}{4-2s}$	$v \geq 3$	$0 < s \leq \bar{s}$
	$p_A^\bullet = v$	$p_T^\bullet = v - 1$	$v \geq 3$	$\bar{s} < s \leq 1$

where $\bar{s} = \frac{1}{4}\left(\frac{4v-6}{v-1} - \sqrt{2}\sqrt{\frac{8v^2-33v+27}{(v-1)^2}}\right)$.

Table A5: Equilibrium Profits Under Uniform Pricing (When Third-Party Pays a Commission)

Segment	Platform Profit	Third-Party Profit	Range of v	Range of s
n.a.	$\pi_A^\bullet = \frac{9+8s}{16-8s}$	$\pi_T^\bullet = \frac{25(1-s)}{8(2-s)^2}$	$v \geq 3$	$0 < s \leq \bar{s}$
	$\pi_A^\bullet = (v-1)s$	$\pi_T^\bullet = (v-1)(1-s)$	$v \geq 3$	$\bar{s} < s \leq 1$

where $\bar{s} = \frac{1}{4}\left(\frac{4v-6}{v-1} - \sqrt{2}\sqrt{\frac{8v^2-33v+27}{(v-1)^2}}\right)$.

Next, we examine whether hard segmentation continues to intensify competition relative to uniform pricing under this model extension with revenue sharing. As discussed earlier, the analytical characterization of hard segmentation is very involved; accordingly, we proceed as follows. First, we analytically derive the third-party seller's best-response pricing for any given hard-segmentation rule z_0 and Platform prices p_{iA} and present the results in Lemma B4. Next, we use Algorithm 1 to (i) numerically solve for the platform's optimal segment-specific prices $p_{1A}^{rs,HS,\bullet}$ and $p_{2A}^{rs,HS,\bullet}$ conditional on a hard-segmentation rule z_0, and (ii) compare the corresponding market outcomes across z_0 to numerically determine the optimal segmentation strategy.

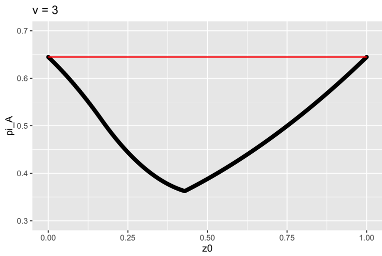
Lemma B4 ***Hard-Segmentation Third-party Best-Response Prices with Revenue-Sharing****.* *Third-party's best response price, conditional on Platform's price and a hard-segmentation rule, when it pays a share* s *of its total revenue as a commission to Platform and the consumer valuation is at least* 3*, is characterized by* $(p_{1T}^{rs,HS,\circ}, p_{2T}^{rs,HS,\circ})$ *as summarized in Equations B1 and B2.*

$$p_{1T}^{rs,HS,\circ}(p_{1A}) = \begin{cases} 0 & \text{if} \quad 0 \leq p_{1A} \leq 1 - 2z_0 \\ \frac{1}{2}(p_{1A} + 2z_0 - 1) & \text{if} \quad 1 - 2z_0 \leq p_{1A} \leq 2z_0 + 1 \\ p_{1A} - 1 & \text{if} \quad 2z_0 + 1 \leq p_{1A} \leq v \end{cases} \tag{B1}$$

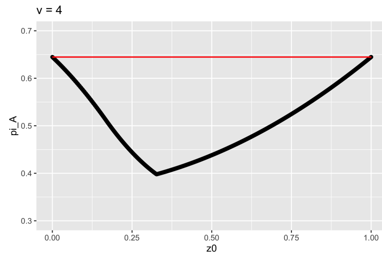
$$p_{2T}^{rs,HS,\circ}(p_{2A}) = \begin{cases} \frac{1}{2}(p_{2A} + 1) & \text{if} \quad 0 \leq p_{2A} \leq 3 - 4z_0 \\ p_{2A} + 2z_0 - 1 & \text{if} \quad 3 - 4z_0 \leq p_{2A} \leq v - z_0 \\ v + z_0 - 1 & \text{if} \quad v - z_0 \leq p_{2A} \leq v \end{cases}. \tag{B2}$$

With the analytical results in Lemma B4, we proceed to a numerical evaluation of the full equilibrium and the resulting market outcomes. Our numerical analysis is conducted over a systematic grid of parameter values, with the model primitives $s \in \{0.05, 0.1, 0.15, 0.2, 0.25\}$ and

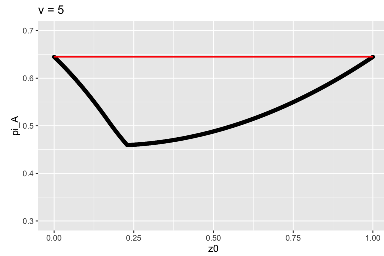
$v \in \{3, 4, 5, 6, 7, 8\}$. Across all parameter configurations examined, we find that any non-trivial hard segmentation intensifies price competition in both segments. Consequently, hard segmentation is strictly dominated by uniform pricing under the revenue-sharing scheme, consistent with our main findings in the baseline model. The numerical results are illustrated in Figure A9.



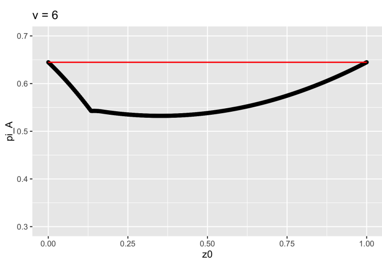
(a) $v = 3$



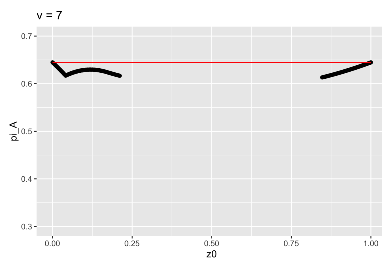
(b) $v = 4$



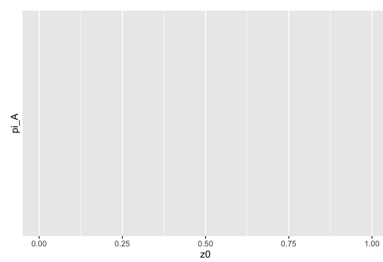
(c) $v = 5$



(d) $v = 6$



(e) $v = 7$

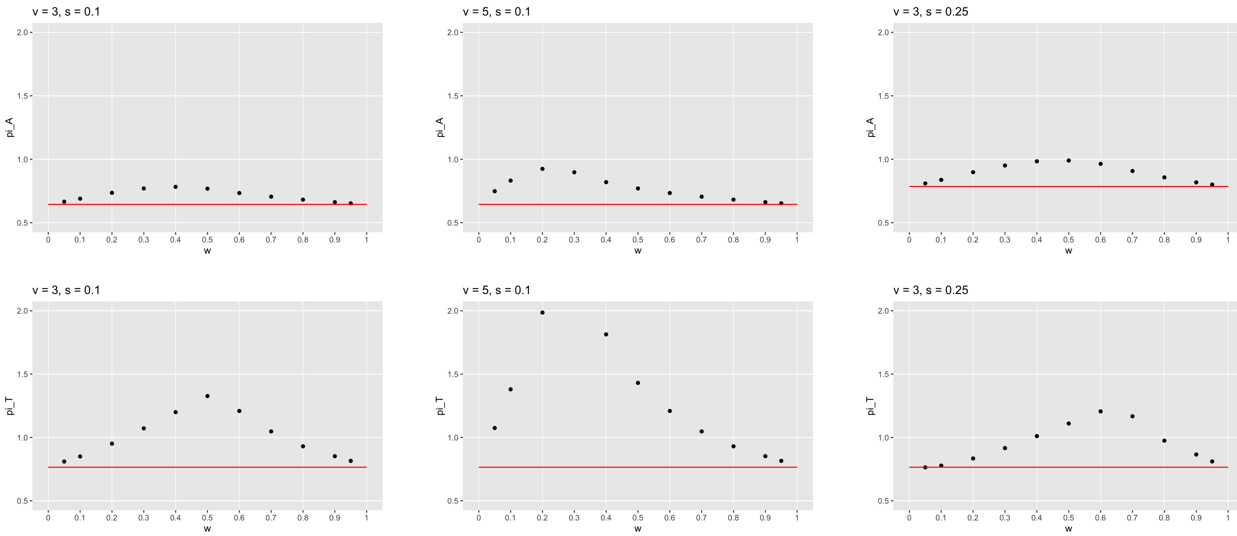


(f) $v = 8$

Figure A9: Platform's profit under different hard segmentation rule, $s = 0.1$.
Note: In each subplot, the black solid line represents Platform's total profit under each hard segmentation rule z_0 under the corresponding customer valuation v; the red solid line represents Platform's no-segment benchmark profit. When $v = 7$, the black solid line is disconnected, because certain hard segmentation rules resulting in Platform existing in both at least one of the segment and are hence excluded from the discussion. When $v = 8$, $\bar{s} = 0.091 < 0.1$, so Platform will exist the market even in the no-segment scenario. In all non-trivial cases, Platform's total profit under hard segmentation is lower than that under uniform pricing.

Lastly, we perform a similar two-step numerical procedure to characterize the market outcome with the presence of privacy-preserving customers, with an option for Platform to form a fuzzy segmentation. Across the same set of model primitives considered in the hard-segmentation analysis, across different fractions of privacy-preserving customers $w \in \{0.05\} \cup \{0.1, 0.2, ..., 0.9\} \cup \{0.95\}$, our numerical results show that there always exists a fuzzy segmentation that strictly increases the platform's profit relative to uniform pricing under the revenue-sharing scheme, consistent with our main findings in the baseline model. Third-party also benefits from fuzzy segmentation for the majority of the parameter space. We illustrate the changes in each seller's profit from their respective no-segment benchmark profit in Figure A10.[2]

[2]As discussed in Online Appendix A, Third-party's profit might slightly fall below its no-segment benchmark as



(a) $v = 3, s = 0.1$ (b) $v = 5, s = 0.1$ (c) $v = 3, s = 0.25$

Figure A10: Platform's (top row) and Third-party's (bottom row) profits under different model primitives.
Note: In each subplot, the black dots represent the total profit of the corresponding seller under different size of privacy-preserving customers with optimal fuzzy segmentation, under the corresponding customer valuation v and revenue-sharing share s; the red solid line represents each seller's no-segment benchmark profit. While Platform's total profit always increases from its no-segment benchmark profit, we note that Third-party may experience a minor drop in total profit as illustrated in Online Appendix A. Specifically, when $v = 3$ and $s = 0.25$, for a small share of privacy-preserving customers ($w = 0.05$), Third-party's total profit slightly decreases from its no-segment benchmark profit.

Across all parameter configurations examined, total consumer surplus decreases under fuzzy segmentation in all but one case[3], and economic social welfare decreases in all but three cases[4]. All exceptional cases corresponds to $w \geq 0.9$, in which the competition-softening effect of the fuzzy segmentation is limited, while the product matching efficiency increases in the fuzzy segment (Segment 1). We illustrate the changes in each seller's profit from their respective no-segment benchmark profit in Figure A11.

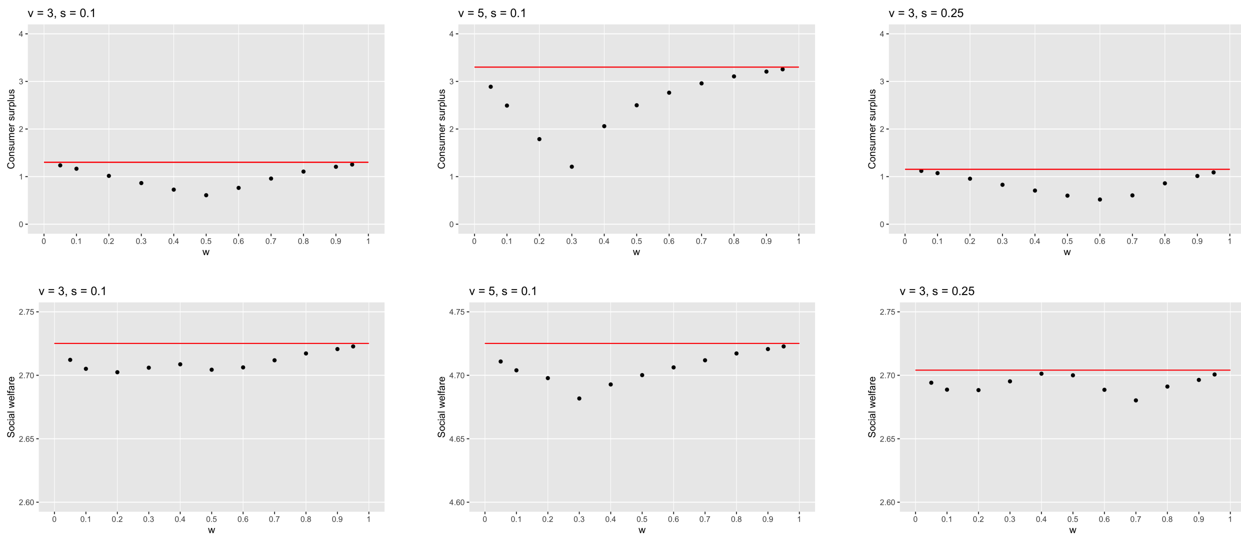
Last but not least, consistent with our main insights, the decline in consumer surplus is unevenly distributed across consumers: (1) the privacy-preserving customers incur a lower consumer surplus, (2) the data-sharing Third-party-preferring customers suffer from a negative spillover effect and

Platform chooses its optimal fuzzy segmentation rule under a more constrained setting. Specifically, in the revenue sharing model extension, these exceptions occur when $s = 0.25, v = 3, w = 0.05$ and when $s = 0.25, v = 4, w = 0.95$. We note that when the revenue-sharing rate s is high and consumers' willingness to pay is low, there is a larger set of segmentation rules that would lead Platform to exit the market in at least one of the segments. Due to our restrictions to segmentation rules under which Platform would stay in the market, the strategy space is limited and Third-party's profit could be compromised.

[3] When $s = 0.25, v = 4, w = 0.95$.

[4] When $s = 0.25, v = 4, w = 0.95$; $s = 0.1, v = 7, w = 0.90$; $s = 0.1, v = 7, w = 0.95$.

incur a lower consumer surplus, and (3) the data-sharing Platform-preferring customers enjoy a higher consumer surplus.



(a) $v = 3, s = 0.1$ (b) $v = 5, s = 0.1$ (c) $v = 3, s = 0.25$

Figure A11: Consumer surplus (top row) and social welfare (bottom row) under different model primitives.

Note: In each subplot, the black dots represent the consumer surplus (social welfare) under different size of privacy-preserving customers with optimal fuzzy segmentation, under the corresponding customer valuation v and revenue-sharing share s; the red solid line represents the corresponding no-segment benchmark consumer surplus (social welfare). Both consumer surplus and social welfare decrease for majority of the parameter space.

Algorithm 1: Grid Search for Optimal $(z^{rs,HS,*}, \mathbf{p}^{rs,HS,*})$ Given (s, v).

Input : Model primitives (s, v); grid step sizes for z and p_A.

Construct grids $\mathcal{Z} \subset [0, 1]$ and $\mathcal{P}_{\mathcal{A}} \subset [0, v]$

Output: Set of $(z^{rs,HS,*}, \{p_{ij}^{rs,HS,*}\})$, with $i = 1, 2$ and $j = A, T$.

Construct grids:

Choose grids $\mathcal{Z} \subset [0, 1]$ and $\mathcal{P}_{\mathcal{A}} \subset [0, v]$

for $z \in \mathcal{Z}$ **do**

`// Initialize a grid over potential segmentation rules`

foreach $i \in \{1, 2\}$ **do**

`// Initialize a grid over potential Platform prices in Segment` i

foreach $p_{iA} \in \mathcal{P}_A$ **do**

Compute Third-party's best response price, p_{iT}, according to Lemma B4

Compute the location of the indifferent customers, $\tilde{x}_i, \bar{x}_{iA}, \bar{x}_{iT}$

Compute the demand for each seller, q_{iA} and q_{iT}

Compute the profit for each seller, $\pi_{iA} \leftarrow p_{iA} \cdot q_{iA} + s \cdot p_{iT} \cdot q_{iT}$,

$\pi_{iT} \leftarrow (1 - s) \cdot p_{iT} \cdot q_{iT}$

end

`// Check constraint` $p_{iA} < v$

Define feasible price grid:

$$\mathcal{P}_{iA} \leftarrow \{p_{iA}^{feas} \in \mathcal{P}_A : p_{iA} < v\}$$

`// Determine the optimal segment-specific Platform price,` p_{iA}

$p_{iA}^{rs,HS,\bullet}(z) \leftarrow argmax_{p_{iA} \in \mathcal{P}_{1A}^{feas}} \pi_{iA}(p_{iA}|z)$

$\pi_{iA}^{rs,HS,\bullet}(z) \leftarrow \pi_{iA}(p_{iA}^{rs,HS,\bullet}(z)|z)$

end

end

$z^* \leftarrow argmax_z \pi_{1A}^{rs,FS,\bullet}(z) + \pi_{2A}^{rs,HS,\bullet}(z)$

return $(z^{rs,HS,*}, \{p_{ij}^{rs,HS,*}\})$, *with* $i = 1, 2$ *and* $j = A, T$

B.4 Platform as a Pure Marketplace

When the platform does not sell its own product but functions purely as an intermediary, it hosts competing sellers A and T, and receives a commission rate k on each seller's revenue. The platform's profit is therefore $\pi_{plat}^{seg}(z) = k \cdot \left[\pi_A^{seg} + \pi_T^{seg}\right]$, with $seg \in \{NS, HS, FS\}$. We continue to analyze the market outcome under full consumer information (i.e., uniform pricing and hard segmentation), and when some consumer choose to preserve privacy (i.e., creating the possibility for fuzzy segmentation).

Under no-segment benchmark case, the platform's profit is $\frac{43}{32}k$. Because any non-trivial hard-segmentation rule strictly reduces each seller's profit, it also lowers the platform's commission revenue. Consequently, even when acting purely as a marketplace intermediary, the platform has no incentive to impose hard segmentation under full consumer information.

When there exist privacy-preserving consumers not sharing data, we next examine whether the platform has the incentive to adopt a fuzzy segmentation as illustrated in Figure 1c. In this marketplace setting, the two sellers are symmetric third-party sellers, so there is no natural Stackelberg ordering. We therefore consider all three possible pricing timing structures: (i) Seller A as price leader,[5] (ii) Seller T as price leader,[6] and (iii) simultaneous pricing.[7] Across all timing structures and all parameter configurations examined,[9] our analysis yields the following robust findings:

[5]When Seller A is the price leader, we adopt the analytical solution of Seller T's best response in Lemma C6 and use a grid search similar to Algorithm 2 to first find $p_{1A}^{FS,\bullet}(z)$ and $p_{2A}^{FS,\bullet}(z)$, and then find the optimal fuzzy segmentation rule z that maximizes the platform's profit $\pi_{plat}^{FS}(z)$. This yields the optimal fuzzy segmentation rule z in Figure A12a. The resulting optimal z is higher than the one when the platform also competes in selling the product as Seller A.

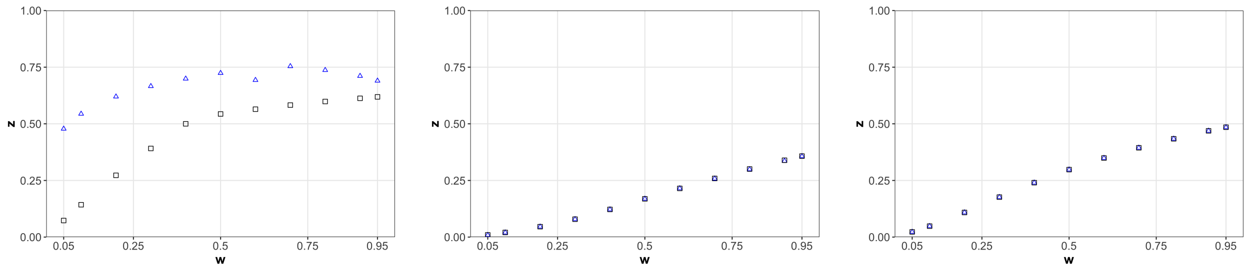
[6]When Seller T is the price leader, we adopt the analytical solutions of the pricing game in Tables A11 and A12, and use a grid search to find the optimal fuzzy segmentation rule z that maximizes the platform's profit. This yields the optimal fuzzy segmentation rule z in Figure A12b. The resulting optimal z is identical to the one when the platform also competes in selling the product as Seller A.

[7]When both sellers act simultaneously in the pricing game, we adopt the analytical solutions of the pricing game in Eq. (C22) and Eq. (C24), and the corresponding profits in each segment in Eq. (C23) and Eq. (C25) to solve the profit maximizing problem analytically. The optimization problem is

$$\begin{aligned} \max_{z} \quad & \frac{1}{2} \cdot w \left(1 + \frac{2}{3}\,\frac{1-w}{w} z\right)^2 + \frac{1}{2} \cdot w \left(1 + \frac{4}{3}\,\frac{1-w}{w} z\right)^2 + \frac{1}{2} \cdot (1-w) \left(1 - \frac{2}{3}\, z\right)^2 + \frac{1}{2} \cdot (1-w) \left(1 - \frac{4}{3}\, z\right)^2, \\ \text{s.t.} \quad & 0 \le z \le \bar{z} \end{aligned}$$

where $\bar{z} = \max\{\zeta_f, \zeta_g\}$.[8] The F.O.C. of the objective function is equals to $\frac{20}{9} \cdot \frac{1-w}{w} z$ and is always positive when $0 < w < 1$. Therefore, the optimal solution is still the largest z in the feasible region. This yields the optimal fuzzy segmentation rule z in Figure A12c. The resulting optimal z is identical to the one when the platform also competes in selling the product as Seller A.

[9]We consider the fraction of privacy-preserving customers $w \in \{0.05, 0.1, 0.2, ..., 0.9, 0.95\}$. Furthermore, when Seller A acts as a price leader, consumer valuation actively impacts the equilibrium outcome; hence we consider consumer valuation $v \in \{3, 4, 5, 6, 7, 8\}$.



(a) Seller A as Price Leader (b) Seller T as Price Leader (c) Simultaneous

Figure A12: Illustration of Optimal Segmentation Rule Under Three Timing Structures (Square representing the setup where the platform competes with the third-party seller; triangle representing the setup where the platform serves only as a marketplace.)
Note: w represents the fraction of privacy-preserving customers. z denotes the optimal size of data-sharing customer the platform chooses to mix with privacy-preserving customers. The black mark represents the main case, when the platform also acts as Seller A and competes with the third-party seller (Seller T); the blue mark represents the case when the platform serves only as a marketplace.

- It continues to hold that there always exits a fuzzy segmentation rule that strictly increases the platform's total profit compared to uniform pricing. The optimal segmentation rules when the platform serves only as a marketplace under each timing structure are illustrated in Figure A12.
- The total consumer surplus decreases. The total social welfare decreases.
- The decreases in consumer welfare is not evenly borne by all customers. Regardless of the timing structure, the data-sharing customers who prefer Seller T faces a negative spillover effect as they incur the same transportation cost and a higher price.[10]

[10] The consumer surplus change among privacy-preserving customers and data-sharing customers who prefer Seller A depends on the tradeoff between the change in the transportation cost and the change in the prices.

C Mathematical Appendix (Proofs of All Results)

C.1 Proofs of Results in the Main Text

C.1.1 Proof of Lemma 1

We solve the game by backward induction with Platform as a price leader. For a given Platform price p_A, Third-party chooses its price p_T to maximize its profit, $\pi_T^{NS} = D_T^{NS}\ p_T$. We use the location of the indifferent customer, $\tilde{x} = \frac{1}{2}(p_T - p_A) + \frac{1}{2}$, to help illustrate the demand, where $\tilde{x}$ solves $v - p_A - x = v - p_T - (1-x)$. Third-party's demand function is as follows.

$$D_T^{NS} = \begin{cases} 1 & \text{if} \quad p_T \leq p_A - 1 \\ 1 - \tilde{x} = \frac{1}{2} + \frac{1}{2}(p_A - p_T) & \text{if} \quad p_A - 1 \leq p_T \leq p_A + 1 \\ 0 & \text{if} \quad p_T \geq p_A + 1 \text{ (trivially dominated)} \end{cases}.$$

Note that Third-party's profit function is continuous and piecewise concave function. We first obtain the optimal prices within each of the following two intervals $E_1 \equiv (0, p_A - 1]$ and $E_2 \equiv [p_A - 1, p_A + 1]$; among the two, the price that yields the highest profit is Third-party's optimal price. Let $p_T^{E_i}$ denote Third-party's optimal price within the interval E_i.

- Suppose that $p_T \in E_1$. The demand is constant, so the optimal price is $p_T^{E_1} = p_A - 1$.
- Suppose that $p_T \in E_2$. If $p_A \leq 3$, one can show that the optimal price is $p_T^{E_2} = \frac{1}{2}p_A + \frac{1}{2}$; if $p_A > 3$, one can show that the optimal price is $p_T^{E_2} = p_A - 1$.

Comparing the profits corresponding to prices $p_T^{E_1}$ and $p_T^{E_2}$, one can show that Third-party's optimal price is as follows.

$$p_T^{NS,\circ}(p_A) = \begin{cases} \frac{1}{2}p_A + \frac{1}{2} & \text{if} \quad p_A \leq 3 \\ p_A - 1 & \text{if} \quad p_A > 3 \end{cases}.$$

Using $p_T^{NS,\circ}$, we then solve for Platform's optimal price to maximize profit, i.e., $\pi_A^{NS} = D_A^{NS}\ p_A$. Platform's demand function is as follows.

$$D_A^{NS} = \begin{cases} -\frac{1}{4}p_A + \frac{3}{4} & \text{if} \quad p_A \leq 3 \\ 0 & \text{if} \quad p_A > 3 \end{cases}$$

The case where $p_A > 3$ is trivially dominated, as it results in zero profit. Hence, we focus on the case where $p_A \leq 3$. One can show that the optimal price is $p_A^{NS,*} = \frac{3}{2}$. The corresponding $p_T^{NS,*} = \frac{5}{4}$. The indifferent customer lies at $\frac{3}{8}$ and the equilibrium profit for Platform and Third-party is $\frac{9}{16}$ and $\frac{25}{32}$, respectively. $\square$

C.1.2 Proof of Lemma 2

We solve the pricing subgames by backward induction. For a given segmentation rule z_0, both sellers engage in a sequential pricing game in each segment, with Platform as a price leader. Since the pricing subgame in each section is independent from each other, we solve the subgames in Segment 1 and Segment 2 separately.

Segment 1. For a given hard segmentation rule z_0 and Platform price p_{1A}, Third-party chooses its price p_{1T} to maximize its profit in Segment 1, $\pi_{1T} = D_{1T}\ p_{1T}$. Third-party's demand function in Segment 1 is as follows.

$$D_{1T} = \begin{cases} z_0 & \text{if} \quad 0 \leq p_{1T} \leq p_{1A} - 1 \\ z_0 - \frac{1}{2} - \frac{1}{2}(p_{1T} - p_{1A}) & \text{if} \quad p_{1A} - 1 \leq p_{1T} \leq p_{1A} + 2z_0 - 1 \\ 0 & \text{if} \quad p_{1A} + 2z_0 - 1 \leq p_{1T} \leq v \text{ (trivially dominated)} \end{cases}$$

Note that Third-party's profit function is continuous and piecewise concave on each interval. We first obtain the optimal prices within each of the following three intervals $I_1 \equiv (0, p_{1A} - 1]$, and $I_2 \equiv [p_{1A} - 1, p_{1A} + 2z_0]$; between the two, the price that yields the highest profit is Third-party's optimal price. Let $p_{1T}^{I_i}$ denote Third-party's optimal price within the interval I_i.

- Suppose that $p_{1T} \in I_1$. One can show that the optimal price is $p_{1T}^{I_1} = p_{1A} - 1$.
- Suppose that $p_{1T} \in I_2$. If $p_{1A} < 1 - 2z_0$, the optimal price is $p_{1T}^{I_2} = 0$ (since $p_{1A} + 2z_0 - 1 < 0$ and the price cannot be negative); if $1 - 2z_0 < p_{1A} < 2z_0 + 1$, the optimal price is $p_{1T}^{I_2} = \frac{1}{2}p_{1A} + z_0 - \frac{1}{2}$; if $p_{1A} \geq 2z_0 + 1$, the optimal price is $p_{1T}^{I_2} = p_{1A} - 1$.

Comparing the profits corresponding to prices $p_{1T}^{I_1}$ and $p_{1T}^{I_2}$, one can show that Third-party's optimal Segment-1 price is as follows.

$$p_{1T}^{HS,\circ} = \begin{cases} 0 & \text{if} \quad 0 \leq p_{1A} \leq 1 - 2z_0 \\ \frac{1}{2}p_{1A} + z_0 - \frac{1}{2} & \text{if} \quad 1 - 2z_0 \leq p_{1A} \leq 2z_0 + 1 \\ p_{1A} - 1 & \text{if} \quad p_{1A} \geq 2z_0 + 1 \end{cases}$$

Using $p_{1T}^{HS,\circ}$, we can derive Platform's profit-maximizing price in the Segment-1 subgame, where $\pi_{1A} = D_{1A}\ p_{1A}$. To find the optimal p_{1A}, we first find the locally optimal p_{1A} within each of the intervals $K_1 \equiv (0, 1 - 2z_0]$, $K_2 \equiv [1 - 2z_0, 1 + 2z_0]$, and $K_3 \equiv [1 + 2z_0, \infty]$. Then, comparing the profits under each local optimum, we can find the globally optimal price, p_{1A}.

Observe that when $p_{1A} \in K_1$, Third-party's best response price reaches the zero-lower-bound, meaning that Platform can capture the entire demand in Segment 1 by charging $p_{1A} = 1 - 2z_0$, coinciding with the left boundary of K_2. Furthermore, when $p_{1A} \in K_3$, Third-party can capture the entire demand in Segment 1 by charging $p_{1T}^{HS,\circ} = p_{1A} - 1$, so this case is trivial and weakly dominated by the right boundary of K_2 for Platform. As a result, the locally optimal p_{1A} in K_2 is the global optimal. Platform's profit function when p_{1A} in K_2 is $\pi_{1A} = (\frac{1}{2}p_{1A} + z_0 - \frac{1}{2}) \cdot p_{1A}$. Solving for this optimal p_{1A} in K_2 requires a straightforward constrained optimization analysis. One can show that

$$p_{1A}^{HS,\bullet} = \begin{cases} 1 - 2z_0 & \text{if} \quad 0 \leq z_0 < \frac{1}{6} \\ z_0 + \frac{1}{2} = \frac{3}{2} - (1 - z_0) & \text{if} \quad \frac{1}{6} \leq z_0 \leq 1 \end{cases}.$$

Plugging $p_{1A}^{HS,\bullet}$ into the expression for $p_{1T}^{HS,\circ}$, we find that

$$p_{1T}^{HS,\bullet} = \begin{cases} 0 & \text{if} \quad 0 \leq z_0 < \frac{1}{6} \\ \frac{3}{2}z_0 - \frac{1}{4} = \frac{5}{4} - \frac{3}{2}(1 - z_0) & \text{if} \quad \frac{1}{6} \leq z_0 \leq 1 \end{cases}.$$

Segment 2. Similarly, for a given segmentation rule z_0 and Platform price p_{2A}, Third-party chooses its price p_{2T} to maximize its profit in Segment 2, $\pi_{2T} = D_{2T}\ p_{2T}$. Third-party's demand function in Segment 2 is as follows.

$$D_{2T} = \begin{cases} 1 - z_0 & \text{if} \quad 0 \leq p_{2T} \leq p_{2A} + 2z_0 - 1 \\ \frac{1}{2} + \frac{1}{2}(p_{2A} - p_{2T}) & \text{if} \quad p_{2A} + 2z_0 - 1 \leq p_{2T} \leq p_{2A} + 1 \\ 0 & \text{if} \quad p_{2T} \geq p_{2A} + 1 \text{ (trivially dominated)} \end{cases}$$

Third-party's profit function is continuous and piecewise concave on each interval. To find the optimal p_{2T}, we first find the locally optimal p_{2T} within each of the intervals $M_1 \equiv (0, p_{2A}+2z_0-1]$ and $M_2 \equiv [p_{2A} + 2z_0 - 1, p_{2A} + 1]$. Then, comparing the profits under each local optimum, we can find the globally optimal price, p_{2T}.

- Suppose that $p_{2T} \in M_1$. One can show that the optimal price is $p_{2T}^{I_1} = p_{2A} + 2z_0 - 1$.
- Suppose that $p_{2T} \in M_2$. If $p_{2A} \leq 3-4z_0$, the optimal price is $p_{2T}^{I_2} = \frac{1}{2}p_{2A}+\frac{1}{2}$; if $p_{2A} > 3-4z_0$, the optimal price is $p_{2T}^{I_2} = p_{2A} + 2z_0 - 1$.

Comparing the profits corresponding to prices $p_{2T}^{I_1}$ and $p_{2T}^{I_2}$, one can show that Third-party's optimal Segment-2 price is as follows.

$$p_{2T}^{HS,\circ} = \begin{cases} \frac{1}{2}p_{2A} + \frac{1}{2} & \text{if} \quad p_{2A} \leq 3 - 4z_0 \\ p_{2A} + 2z_0 - 1 & \text{if} \quad p_{2A} > 3 - 4z_0 \end{cases}.$$

Using $p_{2T}^{HS,\circ}$, we can derive Platform's profit-maximizing price in the Segment-2 subgame, where $\pi_{2A} = D_{2A}\, p_{2A}$. Observe that when $p_{2A} > 3 - 4z_0$, Third-party's best response results in $\tilde{x}_2 = z_0$ (recall that $\tilde{x}_\cdot = \frac{1}{2}(p_{\cdot T} - p_{\cdot A}) + \frac{1}{2}$), yielding zero demand for Platform. Therefore, Platform has no incentive to choose p_{2A} above $3 - 4z_0$. Specifically,

- When $z_0 \leq \frac{3}{4} \Leftrightarrow 3 - 4z_0 \geq 0$, Platform will choose $p_{2A} \in [0, 3 - 4z_0]$, with $p_{2T}^{HS,\circ} = \frac{1}{2}p_{2A} + \frac{1}{2}$. Platform's profit maximization problem is $\max_{p_{2A}}(-\frac{1}{4}p_{2A}+\frac{3}{4}-z_0)\cdot p_{2A}$. Solving the first-order condition, we get $p_{2A}^{HS,\bullet} = \frac{3}{2} - 2z_0$. The corresponding $p_{2T}^{HS,\bullet} = \frac{5}{4} - z_0$.
- When $z_0 > \frac{3}{4} \Leftrightarrow 3 - 4z_0 < 0$, Platform cannot achieve any positive demand in this segment, even though it lowers its price to 0. As a result, $p_{2A}^{HS,\bullet} = 0$. The corresponding $p_{2T}^{HS,\bullet} = 2z_0-1$.

This finishes the proof of Lemma 2. □

C.1.3 Proof of Proposition 1

Table A6 summarizes each seller's segment-level (Segments 1 and 2) and total profits in the hard-segment pricing equilibrium, under any given segmentation rule z_0. The results follow directly from the equilibrium prices and the corresponding demands as specified in Lemma 2.

Table A6: Sellers' Profits in Hard-Segment Pricing Equilibrium Under z_0

	Segment 1 Profits	Segment 2 Profits	Total Profits
$0 \le z_0 \le \frac{1}{6}$	$\pi_{1A} = (1-2z_0)z_0$ $\pi_{1T} = 0$	$\pi_{2A} = \frac{1}{4}(2z_0 - \frac{3}{2})^2$ $\pi_{2T} = \frac{1}{8}(2z_0 - \frac{5}{2})^2$	$\pi_A = \pi_{1A} + \pi_{2A}$ $\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{1}{6} \le z_0 \le \frac{3}{4}$	$\pi_{1A} = \frac{1}{4}(z_0 + \frac{1}{2})^2$ $\pi_{1T} = \frac{1}{8}(3z_0 - \frac{1}{2})^2$	$\pi_{2A} = \frac{1}{4}(2z_0 - \frac{3}{2})^2$ $\pi_{2T} = \frac{1}{8}(2z_0 - \frac{5}{2})^2$	$\pi_A = \pi_{1A} + \pi_{2A}$ $\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{3}{4} \le z_0 \le 1$	$\pi_{1A} = \frac{1}{4}(z_0 + \frac{1}{2})^2$ $\pi_{1T} = \frac{1}{8}(3z_0 - \frac{1}{2})^2$	$\pi_{2A} = 0$ $\pi_{2T} = (2z_0 - 1)(1 - z_0)$	$\pi_A = \pi_{1A} + \pi_{2A}$ $\pi_T = \pi_{1T} + \pi_{2T}$

To analyze how z_0 affects each seller's total profit and find the optimal z_0 that maximizes Platform's total profit, we consider the following three cases.

- When $0 \le z_0 \le \frac{1}{6}$, $\frac{d\pi_A}{dz_0} = -\frac{1}{2} - 2z_0 \le 0$ and $\frac{d\pi_T}{dz_0} = z_0 - \frac{5}{4} < 0$. Therefore, in this region, the total profit of each sellers decreases in z_0.
- When $\frac{1}{6} \le z_0 \le \frac{3}{4}$, $\frac{d\pi_A}{dz_0} = \frac{5}{2}z_0 - \frac{5}{4}$ and $\frac{d\pi_T}{dz_0} = \frac{13}{4}z_0 - \frac{13}{8}$. Therefore, in this region, the total profit of each seller decreases in z_0 when $z_0 < \frac{1}{2}$ and increases in z_0 when $z_0 > \frac{1}{2}$.
- When $\frac{3}{4} \le z_0 \le 1$, $\frac{d\pi_A}{dz_0} = \frac{1}{2}z_0 + \frac{1}{4} \ge 0$ and $\frac{d\pi_T}{dz_0} = -\frac{13}{4}z_0 + \frac{31}{8} \ge 0$. Therefore, in this region, the total profit of each seller increases in z_0.

Since each seller's profit function is continuous, we conclude that each seller's profit is maximized at $z_0 = 0$ or $z_0 = 1$, corresponding to the no-segment case. This suggests that Platform does not have incentive to impose a hard segmentation. □

C.1.4 Proof of Lemma 3

Similar to the Proof of Lemma 2, we solve the pricing subgames for fuzzy segmentation under a given segmentation rule z by backward induction. Lemma 3 focuses on the case without the consumer participation (CP) constraint. Furthermore, we solve the subgames in Segment 1 and Segment 2 separately.

Segment 1. For a given fuzzy segmentation rule z and Platform price p_{1A}, Third-party chooses its price p_{1T} to maximizes its profit in Segment 1, $\hat{\pi}_{1T} = \hat{D}_{1T}\, p_{1T}$. Third-party's demand function in Segment 1 (without CP constraint) is as follows.

$$\hat{D}_{1T} = \begin{cases} w - wz + z & \text{if} \quad p_{1T} \leq p_{1A} - 1 \\ (\frac{1}{2} + \frac{p_{1A}-p_{1T}}{2} - z)w + z & \text{if} \quad p_{1A} - 1 < p_{1T} < p_{1A} + 1 - 2z \\ \frac{p_{1A}-p_{1T}}{2} + \frac{1}{2} & \text{if} \quad p_{1A} + 1 - 2z \leq p_{1T} \leq p_{1A} + 1 \\ 0 & \text{if} \quad p_{1T} > p_{1A} + 1 \text{ (trivially dominated)} \end{cases}$$

Note that Third-party's profit function is continuous and piecewise concave on each interval. We first obtain the optimal prices within each of the following three intervals $G_1 \equiv (0, p_{1A} - 1]$, $G_2 \equiv [p_{1A} - 1, p_{1A} + 1 - 2z]$, and $G_3 \equiv [p_{1A} + 1 - 2z, p_{1A} + 1]$; among the three, the price that yields the highest profit is Third-party's optimal Segment-1 price. Let $p_{1T}^{G_i}$ denote Third-party's optimal price within the interval G_i.

- Suppose $p_{1T} \in G_1$. One can show that the optimal price is $p_{1T}^{G_1} = p_{1A} - 1$.
- Suppose $p_{1T} \in G_2$. If $p_{1A} < 2\frac{1+w}{w}z - 1$, the optimal price is $p_{1T}^{G_2} = p_{1A} + 1 - 2z$; if $2\frac{1+w}{w}z - 1 \leq p_{1A} \leq 3 + 2\frac{1-w}{w}z$, the optimal price is $p_{1T}^{G_2} = \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z$; if $p_{1A} > 3 + 2\frac{1-w}{w}z$, the optimal price is $p_{1T}^{G_2} = p_{1A} - 1$.
- Suppose $p_{1T} \in G_3$. If $p_{1A} \leq 4z - 1$, the optimal price is $p_{1T}^{G_3} = \frac{1}{2}p_{1A} + \frac{1}{2}$; if $p_{1A} > 4z - 1$, the optimal price is $p_{1T}^{G_3} = p_{1A} + 1 - 2z$.

Comparing the profits corresponding to prices $p_{1T}^{G_1}$, $p_{1T}^{G_2}$,and $p_{1T}^{G_3}$, one can show that Third-party's optimal Segment-1 price is as follows.

$$\hat{p}_{1T}^{FS,\circ} = \begin{cases} \frac{1}{2}p_{1A} + \frac{1}{2} & \text{if} \quad 0 \leq p_{1A} \leq 4z - 1 \\ p_{1A} + 1 - 2z & \text{if} \quad 4z - 1 \leq p_{1A} \leq 2\frac{1+w}{w}z - 1 \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \text{if} \quad 2\frac{1+w}{w}z - 1 \leq p_{1A} \leq 3 + 2\frac{1-w}{w}z \\ p_{1A} - 1 & \text{if} \quad p_{1A} \geq 3 + 2\frac{1-w}{w}z \end{cases}. \tag{C3}$$

Using $\hat{p}_{1T}^{FS,\circ}$, we can obtain the location of the indifferent customer $\tilde{x} = \frac{\hat{p}_{1T}^{FS,\circ} - \hat{p}_{1A}}{2} + \frac{1}{2}$. Depending on the location of $\tilde{x}$ and the fuzzy segmentation rule $1 - z$, Platform's demand in Segment 1 is

$$\hat{D}_{1A} = \begin{cases} -\frac{1}{4}p_{1A} + \frac{3}{4} - (1-w)(1-z) & \text{if} \quad 0 \leq p_{1A} \leq 4z - 1 \\ 1 - z & \text{if} \quad 4z - 1 \leq p_{1A} \leq 2\frac{1+w}{w}z - 1 \\ w\left(-\frac{1}{4}p_{1A} + \frac{3}{4} + \frac{1}{2}\frac{1-w}{w}z\right) & \text{if} \quad 2\frac{1+w}{w}z - 1 \leq p_{1A} \leq 3 + 2\frac{1-w}{w}z \\ 0 & \text{if} \quad p_{1A} \geq 3 + 2\frac{1-w}{w}z \text{ (trivially dominated)} \end{cases}.$$

Platform maximizes its profit, $\hat{\pi}_{1A} = \hat{D}_{1A}\ p_{1A}$. Note that Platform's profit function is also continuous and piecewise concave on each interval. Therefore, to find the optimal p_{1A}, we first find the locally optimal p_{1A} within each of the intervals $F_1 \equiv (0, 4z-1]$, $F_2 \equiv [4z-1, 2\frac{1+w}{w}z - 1]$, and $F_3 \equiv [2\frac{1+w}{w}z - 1, 3 + 2\frac{1-w}{w}z]$. Then, comparing the profits under each local optimum, we can find the globally optimal price $\hat{p}_{1A}^{FS,\bullet}$.

- Suppose $p_{1A} \in F_1$, we have $1 - z \leq \tilde{x} \leq 1$. If $0 \leq z < \frac{1+4w}{4+4w}$, one can show that the optimal price is $p_{1A}^{F_1} = \max\{0, 4z-1\}$; if $\frac{1+4w}{4+4w} \leq z \leq 1$, the optimal price is $p_{1A}^{F_1} = \frac{3}{2} - 2(1-w)(1-z)$.
- Suppose $p_{1A} \in F_2$, we have $\tilde{x} = 1 - z$. The objective function increases in p_{1A}, so the optimal price is $p_{1A}^{F_2} = 2\frac{1+w}{w}z - 1$.
- Suppose $p_{1A} \in F_3$, we have $0 \leq \tilde{x} \leq 1-z$. If $0 \leq z \leq \frac{5w}{2+6w}$, the optimal price is $p_{1A}^{F_3} = \frac{3}{2} + \frac{1-w}{w}z$; if $\frac{5w}{2+6w} < z \leq 1$, the optimal price is $p_{1A}^{F_3} = 2\frac{1+w}{w}z - 1$.

Comparing the profits corresponding to prices $p_{1A}^{F_1}$, $p_{1A}^{F_2}$, and $p_{1A}^{F_3}$, one can show that Platform's optimal Segment-1 price is as follows.

$$\hat{p}_{1A}^{FS,\bullet} = \begin{cases} \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad 0 \leq z \leq z_a \\ 2\frac{1+w}{w}z - 1 & \text{if} \quad z_a \leq z \leq z_b \\ \frac{3}{2} - 2(1-w)(1-z) & \text{if} \quad z_b \leq z \leq 1 \end{cases},$$

where $z_a = \frac{5w}{2+6w}$ and $z_b = \frac{2w^2 + \frac{1}{2}w + \frac{5}{2} + \sqrt{-2w^2 - 7w/2 + 11/2}}{2w^2+6}$.

We summarize Segment 1's optimal seller prices and the corresponding market outcomes under any given fuzzy-segmentation rule z, in Table A7.

Table A7: Summary of Equilibrium Prices and Profits in Segment 1
(Under Fuzzy Segmentation with Segmentation Rule z, without CP Constraint)

Region	z Range	$\tilde{x}_1$ Location	Price	Profit
1	$z \leq z_a$	$0 \leq \tilde{x}_1 \leq 1 - z$	$\hat{p}^{\bullet}_{1A} = \frac{3}{2} + \frac{1-w}{w} z$ $\hat{p}^{\bullet}_{1T} = \frac{5}{4} + \frac{3}{2} \frac{1-w}{w} z$	$\hat{\pi}^{\bullet}_{1A} = \frac{1}{4} w (\frac{3}{2} + \frac{1-w}{w} z)^2$ $\hat{\pi}^{\bullet}_{1T} = \frac{1}{2} w (\frac{5}{4} + \frac{3}{2} \frac{1-w}{w} z)^2$
2	$z_a < z \leq z_b$	$\tilde{x}_1 = 1 - z$	$\hat{p}^{\bullet}_{1A} = 2 \frac{1+w}{w} z - 1$ $\hat{p}^{\bullet}_{1T} = 2 \frac{1}{w} z$	$\hat{\pi}^{\bullet}_{1A} = w (2 \frac{1+w}{w} z - 1)(1 - z)$ $\hat{\pi}^{\bullet}_{1T} = 2 \frac{1}{w} z^2$
3	$z_b < z$	$1 - z \leq \tilde{x}_1 \leq 1$	$\hat{p}^{\bullet}_{1A} = \frac{3}{2} - 2(1-w)(1-z)$ $\hat{p}^{\bullet}_{1T} = \frac{5}{4} - (1-w)(1-z)$	$\hat{\pi}^{\bullet}_{1A} = \frac{1}{4} [\frac{3}{2} - 2(1-w)(1-z)]^2$ $\hat{\pi}^{\bullet}_{1T} = \frac{1}{2} [\frac{5}{4} - (1-w)(1-z)]^2$

where $z_a = \frac{5w}{2+6w}$ and $z_b = \frac{2w^2 + \frac{1}{2}w + \frac{5}{2} + \sqrt{-2w^2 - \frac{7}{2}w + \frac{11}{2}}}{2w^2+6}$.

Segment 2. The Segment-2 equilibrium outcome in the fuzzy segmentation case can be obtained through simple modifications from the Segment-1 equilibrium in hard segmentation case. We need to plug in $z_0 = 1 - z$ and rescale the total profit by $1 - w$. The equilibrium outcome is summarized in Table A8.

Table A8: Summary of Equilibrium Prices and Profits in Segment 2
(Under Fuzzy Segmentation with Segmentation Rule z)

Region	z range	Price	Profit
A	$0 < z \leq \frac{5}{6}$	$\hat{p}^{\bullet}_{2A} = \frac{3}{2} - z$ $\hat{p}^{\bullet}_{2T} = \frac{5}{4} - \frac{3}{2} z$	$\hat{\pi}^{\bullet}_{2A} = \frac{1}{4}(1-w)(z - \frac{3}{2})^2$ $\hat{\pi}^{\bullet}_{2T} = \frac{1}{8}(1-w)(3z - \frac{5}{2})^2$
B	$\frac{5}{6} < z \leq 1$	$\hat{p}^{\bullet}_{2A} = 2z - 1$ $\hat{p}^{\bullet}_{2T} = 0$	$\hat{\pi}^{\bullet}_{2A} = (1-w)(2z-1)(1-z)$ $\hat{\pi}^{\bullet}_{2T} = 0$

This finishes the proof of Lemma 3. □

C.1.5 Proof of Lemma 4

To determine Platform's optimal fuzzy segmentation rule, we solve for z that maximizes Platform's total profit, i.e., $z := \arg\max_z \hat{\pi}^{\bullet}_A = \hat{\pi}^{\bullet}_{1A} + \hat{\pi}^{\bullet}_{2A}$. Building on the results in Tables A7 and A8, we partition the potential range of z into four intervals: $H_1 \equiv [0, z_a]$, $H_2 \equiv [z_a, z_b]$, $H_3 \equiv [z_b, \frac{5}{6}]$, and $H_4 \equiv [\frac{5}{6}, 1]$. For an overview, we establish the following.

1. Any $z \in H_3 \cup H_4$ yields a subgame equilibrium strictly dominated by the no-segment (uniform-pricing) benchmark and therefore cannot be optimal; see Lemma C5.

2. There exists a $z \in H_1 \cup H_2$ that yields a subgame equilibrium exceeding the no-segment benchmark in terms of Platform profit; see Remark 2.

As a result, it suffices to focus on candidates of z in intervals H_1 or H_2.

Case 1: $z \in H_3 \cup H_4$**.** As shown in Table A7, if $z > z_b$, $\tilde{x}_1 > 1 - z$. We refer to such subgame equilibrium as a "right subgame", as the indifferent customer lies to the right of the segmentation threshold. We later demonstrate that Platform never has an incentive to choose z in a way that would result in a right subgame (summarized in Lemma C5), thereby proving that any $z \in H_3 \cup H_4$ is dominated.

Lemma C5 *Platform does not have the incentive to choose a strategy, defined by* $(z, \mathbf{p}_A)$*, with which it will sell to some of the data-sharing customers in Segment 1, i.e.,* $\tilde{x}_1 > 1 - z$*. Such strategy is always dominated by the no-segment equilibrium, where* $z = 0$ *and* $p_{1A}^{FS} = p_A^{NS,*}$*.*

Proof. See Online Appendix C.1.6.

Case 2: $z \in H_1 \cup H_2$**.** Now we turn our attention to segmentation rules where z in H_1 or H_2.

- Suppose $z \in H_1$. The total profit of Platform is $\frac{1}{4}w\left(\frac{3}{2} + \frac{1-w}{w}z\right)^2 + \frac{1}{4}(1-w)\left(z - \frac{3}{2}\right)^2$. Since the first-order condition is positive when $z > 0$, the optimal segmentation rule is the largest z in the range, i.e., $z^{H_1} = z_a$, coinciding with the left boundary of H_2.

- Suppose $z \in H_2$. The total profit of Platform is $w\left(4z - 1 + 2\frac{1-w}{w}z\right)(1-z) + \frac{1}{4}(1-w)\left(z - \frac{3}{2}\right)^2$. One can show that the optimal segmentation rule is $z^{H_2} = \frac{5+15w}{14+18w}$.

Comparing the the profits corresponding to segmentation rules z^{H_1} and z^{H_2}, one can show that Platform's optimal fuzzy segmentation rule is $\hat{z}^{FS,*} = \frac{5+15w}{14+18w}$.

Plugging $\hat{z}^{FS,*}$ into the expression for $\hat{p}_{1A}^{FS,\bullet}$, $\hat{p}_{1T}^{FS,\bullet}$, $\hat{p}_{2A}^{FS,\bullet}$, and $\hat{p}_{2T}^{FS,\bullet}$, we find that

$$\begin{cases} \hat{p}_{1A}^{FS,*} &= 2\frac{1+w}{w}\hat{z}^{FS,*} - 1 \\ \hat{p}_{1T}^{FS,*} &= 2\frac{1}{w}\hat{z}^{FS,*} \\ \hat{p}_{2A}^{FS,*} &= \frac{3}{2} - \hat{z}^{FS,*} \\ \hat{p}_{2T}^{FS,*} &= \frac{5}{4} - \frac{3}{2}\hat{z}^{FS,*} \end{cases}. \tag{C4}$$

Plugging $(z^{FS,*}, \mathbf{p}^{FS,*})$ into the expression for each seller's total profit, we solve each seller's total profit under the fuzzy segmentation equilibrium as follows.

$$
\begin{aligned}
\hat{\pi}_A^{FS,*} &= w\left(2\frac{1+w}{w}\hat{z}^{FS,*}-1\right)(1-\hat{z}^{FS,*})+\frac{1}{4}(1-w)\left(\hat{z}^{FS,*}-\frac{3}{2}\right)^2 \\
&= w\left(2\frac{1+w}{w}\frac{5+15w}{14+18w}-1\right)\left(1-\frac{5+15w}{14+18w}\right)+\frac{1}{4}(1-w)\left(\frac{5+15w}{14+18w}-\frac{3}{2}\right)^2 \\
&= \dots \\
&= \frac{7w+11}{18w+14} \qquad \text{(C5)} \\
\hat{\pi}_T^{FS,*} &= 2\frac{1}{w}\left(\hat{z}^{FS,*}\right)^2+\frac{1}{8}(1-w)\left(3\hat{z}^{FS,*}-\frac{5}{2}\right)^2 \\
&= \frac{2}{w}\left(\frac{5+15w}{14+18w}\right)^2+\frac{1}{8}(1-w)\left(3\frac{5+15w}{14+18w}-\frac{5}{2}\right)^2 \\
&= ... \\
&= \frac{50(8w^2+7w+1)}{w(18w+14)^2} \qquad \text{(C6)}
\end{aligned}
$$

This finishes the proof of Lemma 4. □

C.1.6 Proof of Lemma C5

Below, we prove that Platform never has an incentive to choose strategies in a way that would result in a right subgame. Recall from Online Appendix C.1.4, the right subgame equilibrium $\tilde{x}_1 > 1-z$ occurs only when $p_{1A} \in F_1$. Specifically, if $0 \le z < \frac{1+4w}{4+4w}$, $p_{1A}^{F_1} = \max\{0, 4z-1\}$ and $\hat{\pi}_{1A}^{FS} = 0$; if $\frac{1+4w}{4+4w} \le z \le 1$, $p_{1A}^{F_1} = \frac{3}{2} - 2(1-w)(1-z)$ and $\hat{\pi}_{1A}^{FS} = \frac{1}{4}\left[\frac{3}{2} - 2(1-w)(1-z)\right]$. Taking into account the subgame equilibrium prices in Segment 2 (as shown in Table A8), the total profit of Platform can be analyzed across three cases.

- Suppose $0 \le z < \frac{1+4w}{4+4w}$. We have $\hat{\pi}_A^{FS} = \frac{1}{4}(1-w)(z-\frac{3}{2})^2$, which is obviously less than the no-segment equilibrium profit of Platform, $\pi_A^{NS,*} = \frac{9}{16}$.
- Suppose $\frac{1+4w}{4+4w} \le z \le \frac{5}{6}$. We have $\hat{\pi}_A^{FS} = \frac{1}{4}\left[\frac{3}{2} - 2(1-w)(1-z)\right] + \frac{1}{4}(1-w)(z-\frac{3}{2})^2$. When $z \in \left[\frac{1+4w}{4+4w}, \frac{5}{6}\right] \subset \left(\frac{8w-5}{2(4w-5)} - \sqrt{\frac{w+5}{(4w-5)^2}}, \frac{8w-5}{2(4w-5)} + \sqrt{\frac{w+5}{(4w-5)^2}}\right)$ and $w \in (0,1)$, one can show that $\hat{\pi}_A^{FS} < \frac{9}{16}$ always holds.
- Suppose $\frac{5}{6} < z \le 1$. We have $\hat{\pi}_A^{FS} = \frac{1}{4}\left[\frac{3}{2} - 2(1-w)(1-z)\right] + (1-w)(2z-1)(1-z)$. When

$w \in (0,1)$, one can show that $\hat{\pi}_A^{FS} < \frac{9}{16}$ always holds.

In sum, when $w \in (0,1)$ and $z \in [0,1]$, it is always true that when $p_{1A} \in F_1$ ($\Leftrightarrow \tilde{x}_1 > 1 - z$), $\hat{\pi}_A^{FS} < \pi_A^{NS,*}$. Therefore, Platform does not have an incentive to choose a strategy, defined by $(z, \mathbf{p}_A)$, that will induce a right subgame (i.e., $\tilde{x}_1 > 1 - z$). $\square$

C.1.7 Proof of Remark 1

The proof of these observations follows directly from the results in Table 1 and Table 3. $\square$

C.1.8 Proof of Remark 2 and Remark 3

The proof of Remark 2 and Remark 3 follows directly from comparing sellers' equilibrium profits under fuzzy segment without consumer participation constraint in Lemma 4 and their benchmark profits under no segment in Lemma 1. Specifically, $\frac{7w+11}{18w+14} > \frac{9}{16}$ and $\frac{50(8w^2+7w+1)}{w(18w+14)^2} > \frac{25}{32}$ for all $w \in (0,1)$. $\square$

C.1.9 Proof of Proposition 2 and Proposition A1

Since Proposition 2 can be viewed as a special case of Proposition A1, we prove both by focusing on the more general result in Proposition A1. The proof is relatively extensive, as it involves a broader parameter space and more complex demand functions. To maintain coherence in the presentation of the main proofs, we defer this proof to Online Appendix D.1, while Online Appendix C contains the proofs of all other results.

C.1.10 Proof of Theorem 1

In this appendix, we prove Theorem 1 that both Platform and Third-party enjoy higher total profits in equilibrium under fuzzy segmentation compared to their uniform-pricing benchmark.

In the case where $w \geq \breve{w}$, recall from Proposition 2 that, the CP constraint is not binding in the equilibrium, resulting in the same equilibrium described in Lemma 3 under no CP constraint. Remark 2 and Remark 3 have stated that both sellers enjoy higher total profits at equilibrium.

Therefore, we focus on discussing the case where $0.1 \leq w \leq \breve{w}$. In this case, each seller's total profit is as follows.

$$
\begin{aligned}
\pi_A^{FS,*} &= w\left(1-\frac{w}{2+w}v\right)\left[2\frac{1+w}{w}\frac{w}{2+w}v-1\right]+\frac{1}{4}(1-w)\left(\frac{3}{2}-\frac{w}{2+w}v\right)^2\\
\pi_T^{FS,*} &= \frac{2}{w}\left(\frac{w}{2+w}v\right)^2+\frac{1}{2}(1-w)\left(\frac{5}{4}-\frac{3}{2}\frac{w}{2+w}v\right)^2
\end{aligned}
$$

First, we show that both $\pi_A^{FS,*}$ and $\pi_T^{FS,*}$ always increases in v when $v \geq 3$ and $0.1 \leq w \leq \breve{w}$.

$$
\begin{aligned}
\frac{\partial \pi_A^{FS,*}}{\partial v} &= \frac{w\left(-3w^2(6v-5)-7w(2v-5)+10\right)}{4(w+2)^2}\\
&\propto -3w^2(6v-5)-7w(2v-5)+10 \qquad \text{(C7)}\\
\frac{\partial \pi_T^{FS,*}}{\partial v} &= -\frac{w(3w^2(6v-5)-3w(6v+5)-32v+30)}{8(w+2)^2}\\
&\propto -\left(3w^2(6v-5)-3w(6v+5)-32v+30\right) \qquad \text{(C8)}
\end{aligned}
$$

Eq. (C7) ≥ 0 when $\frac{35-14v-\sqrt{196v^2-260v+625}}{2(18v-15)} \leq w \leq \breve{w}$, where $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)}$. Since $\frac{35-14v-\sqrt{196v^2-260v+625}}{2(18v-15)} < 0$ when $v \geq 3$, for the entire region where $v \geq 3$ and $0.1 \leq w \leq \breve{w}$, we have $\frac{\partial \pi_A^{FS,*}}{\partial v} \geq 0$.

Additionally, we can show that Eq. (C8) ≥ 0 when $v \geq 1$ and $0 < w < 1$. To see this, notice that $3w^2(6v-5)-3w(6v+5) < 3w^2(6v+5)-3w(6v+5) < 3(w^2-w)(6v+5)$ and $-32v+30 < 0$. Therefore, we have $\frac{\partial \pi_T^{FS,*}}{\partial v} \geq 0$ when $v \geq 3$ and $0.1 \leq w \leq \breve{w}$.

Next, we show that $\pi_A^{FS,*}(v,w) \geq \pi_A^{FS,*}(v,w)|_{v=3} \geq \frac{9}{16}$ and $\pi_T^{FS,*}(v,w) \geq \pi_T^{FS,*}(v,w)|_{v=3} \geq \frac{25}{32}$. To see that, we have

$$
\pi_A^{FS,*}(v,w)|_{v=3} = w\left(1-\frac{3w}{2+w}\right)\left(6\frac{1+w}{2+w}-1\right)+\frac{1}{4}(1-w)\left(\frac{3}{2}-\frac{3w}{2+w}\right)^2 \qquad \text{(C9)}
$$

$$
\pi_T^{FS,*}(v,w)|_{v=3} = \frac{2w}{(2+w)^2}\cdot 9+\frac{1}{2}(1-w)\left(\frac{5}{4}-\frac{3}{2}\frac{w}{2+w}v\right)^2 \qquad \text{(C10)}
$$

Since Eq. (C9) $\geq \frac{9}{16}$ when $w < 0.5$ and $\breve{w} < 0.5$ $\forall$ $v \geq 3$, we can conclude that $\pi_A^{FS,*}(v,w) \geq \frac{9}{16}$ $\forall$ $v \geq 3$ and $0.1 \leq w \leq \breve{w}$. One can also show that Eq. (C10) $\geq \frac{25}{32}$ $\forall w \in (0,1)$. Hence, both sellers enjoy profits higher than their uniform-pricing benchmarks. $\square$

C.1.11 Proof of Theorem 3

The total transportation cost under the no-segment (uniform-pricing) benchmark equals to $\frac{17}{64}$. Under fuzzy segmentation with segmentation rule z, the transportation cost equals to

$$
\begin{aligned}
TC^{FS} &= TC^{FS}_{\text{anon.}} + TC^{FS}_{\text{anon.}} \\
&= w\left[\left(z-\frac{1}{2}\right)^2+\frac{1}{4}\right]+(1-w)\left[\left(\frac{1}{4}z+\frac{1}{8}\right)^2+\frac{1}{4}\right] \\
&= \left(\frac{1}{16}+\frac{15}{16}w\right)z^2+\left(\frac{1}{16}-\frac{17}{16}w\right)z+\frac{17}{64}+\frac{15}{64}w.
\end{aligned} \tag{C11}
$$

Below, we show that the equilibrium transportation cost is higher under fuzzy segmentation compared to the uniform-pricing benchmark. Mathematically:

$$
TC^{FS} > \frac{17}{64} \Longleftrightarrow \left(\frac{1}{16}+\frac{15}{16}w\right)z^2+\left(\frac{1}{16}-\frac{17}{16}w\right)z+\frac{15}{64}w>0 \tag{C12}
$$

We demonstrate that this inequality always holds in our context by separately analyzing the following two cases.

Case 1: $w \in [0.1, \frac{49+\sqrt{2145}}{128}]$. In this case, inequality (C12) holds for any $z \in (0,1)$. Consequently, it must also hold for the equilibrium segmentation $z^{FS,*}$ as specified in Proposition 2.

Case 2: $w \in (\frac{49+\sqrt{2145}}{128}, 1)$. In this case, inequality (C12) holds for $z < \frac{17w-1-\sqrt{64w^2-49w+1}}{2(1+15w)}$ or $z > \frac{17w-1+\sqrt{64w^2-49w+1}}{2(1+15w)}$. Since $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)} < \frac{49+\sqrt{2145}}{128}$ when $v \geq 3$[11], we have $z^{FS,*} = \frac{5+15w}{14+18w}$ (as shown in Proposition 2). It is straightforward to verify that $z^{FS,*} = \frac{5+15w}{14+18w} > \frac{17w-1+\sqrt{64w^2-49w+1}}{2(1+15w)}$ in this case. And as such, the inequality (C12) is satisfied.

□

[11]To prove this, first see that the value of $\breve{w}$ decreases in v. Next, one can show that that $\breve{w}(v)|_{v=3} < \frac{49+\sqrt{2145}}{128}$.

C.1.12 Proof of Table 6

Transportation cost. For privacy-preserving customers, $TC_{\text{anon.}} = \left(\tilde{x}_1^{FS,*} - \frac{1}{2}\right)^2 + \frac{1}{4}$, where $\tilde{x}_1^{FS,*} = 1 - z^{FS,*}$ and $z^{FS,*}$ are as specified in Proposition 2. As reasoned in Section 5.2,

$$TC_{\text{anon.}} \begin{cases} \text{decreased} & \text{if} \quad \tilde{x}_1^{FS,*} \in (\frac{3}{8}, \frac{5}{8}) \\ \text{unchanged} & \text{if} \quad \tilde{x}_1^{FS,*} \in \{\frac{3}{8}, \frac{5}{8}\} \\ \text{increased} & \text{if} \quad \tilde{x}_1^{FS,*} \in [0, \frac{3}{8}) \cup (\frac{5}{8}, 1] \end{cases} \tag{C13}$$

Since $\tilde{x}_1^{FS,*} = 1 - z^{FS,*}$ could fall in each of the three regions, the direction of change in $TC_{anon.}$ is indefinite.

For data-sharing customers who prefer Third-party (those who fall in Segment 1), they still buy from Third-party, so the transportation cost for them is unchanged.

For data-sharing customers who prefer Platform (those who fall in Segment 2), the indifferent customer $\tilde{x}_2^{FS,*} = \frac{3}{8} - \frac{1}{4} z^{FS,*}$, so more customers will buy from a farther seller (switching from Platform to Third-party despite being located closer to Platform). Therefore, the average transportation cost for them increases. □

Price Paid. For privacy-preserving customers,

- When $w \geq \breve{w}$, $p_{1A}^{FS,*} = 2\frac{1+w}{2+w}v - 1 \geq 2$ when $v \geq 3$ and $p_{1T}^{FS,*} = 2\frac{2}{2+w}v - 1 \geq 2$ when $v \geq 3$. Since $p_A^{NS,*} = \frac{3}{2}$ and $p_T^{NS,*} = \frac{5}{4}$, privacy-preserving customers always face a higher price no matter who they are buying from.

- When $\breve{w} < w$, $p_{1A}^{FS,*} = 2\frac{1+w}{w}\frac{5+15w}{14+18w} - 1 \geq p_A^{NS,*}$ and $p_{1T}^{FS,*} = \frac{2}{w}\frac{5+15w}{14+18w} \geq p_A^{NS,*}$. Therefore, privacy-preserving customers who stay with the same seller after the adoption of fuzzy segmentation by Platform always pay higher prices. Furthermore, notice that $\tilde{x}_1^{FS,*} = 1 - z^{FS,*} = \frac{9+3w}{14+18w} \geq \frac{3}{8} = \tilde{x}^{NS,*}$, it is always the case that some customers who used to buy from Third-party will switch to Platform under fuzzy segmentation. Since $p_A^{NS,*} > p_T^{NS,*}$, these customers face a higher price as well.

In sum, all privacy-preserving customers will face higher prices and the average price paid by privacy-preserving customers increases.

For data-sharing customers who prefer Third-party (those who fall in Segment 1), they still buy from Third-party, but now faces a higher price after the adoption of fuzzy segmentation.

For data-sharing customers who prefer Platform (those who fall in Segment 2), the equilibrium price under fuzzy segmentation by each seller is lower than the no-segment counterpart, i.e. $p_{2A}^{FS,*} = \frac{3}{2} - z \leq \frac{3}{2}$ and $p_{2T}^{FS,*} = \frac{5}{4} - \frac{3}{2}z \leq \frac{5}{4}$. Therefore, those who stay with the same seller after the adoption of fuzzy segmentation by Platform always pay lower prices. Furthermore, notice that $\tilde{x}_2^{FS,*} = \frac{3}{8} - \frac{1}{4}z^{FS,*} \leq \frac{3}{8} = \tilde{x}^{NS,*}$, it is always the case that some customers who used to buy from Platform will switch to Third-party under fuzzy segmentation. Since $p_A^{NS,*} > p_T^{NS,*}$, these customers also face a lower price. In sum, all data-sharing customers who prefer Platform will face lower prices and the average price paid by privacy-preserving customers decreases. □

Total Cost (TC + PC). For privacy-preserving customers, the total cost under uniform-pricing (before the adoption of fuzzy segmentation) is

$$\begin{aligned} TC_{\text{anon.}}^{NS} + PC_{\text{anon.}}^{NS} &= w\left(\frac{17}{64} + \frac{9}{16} + \frac{25}{32}\right) \\ &= \frac{103}{63}w \end{aligned} \tag{C14}$$

The total cost after the adoption of fuzzy segmentation is calculated as follows.

$$\begin{aligned} TC_{\text{anon.}}^{FS} &= w\left[\left(\tilde{x}_1 - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \\ &= w\left[\left(1 - z^{FS,*} - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \\ &= w\left[(z^{FS,*})^2 - z^{FS,*} + \frac{1}{2}\right] \\ PC_{\text{anon.}}^{FS} &= w\big[\tilde{x}_1 \cdot p_{1A} + (1 - \tilde{x}_1)p_{1T}\big] \\ &= w\left[(1 - z^{FS,*}) \cdot (2\frac{1+w}{w}z^{FS,*} - 1) + (z^{FS,*}) \cdot \left(\frac{2}{w}z^{FS,*}\right)\right] \\ &= w\left[2\frac{1+w}{w}z^{FS,*} - 2(z^{FS,*})^2 - 1 + z^{FS,*}\right] \\ TC_{\text{anon.}}^{FS} + PC_{\text{anon.}}^{FS} &= w\left[-(z^{FS,*})^2 + 2\frac{1+w}{w}z^{FS,*} - \frac{1}{2}\right] \end{aligned} \tag{C15}$$

Below, we show that Eq. (C15) > Eq. (C14) is always true in the relevant parameter space and under the optimal choice of the fuzzy segmentation rule $z^{FS,*}$. First we solve for which region of z does the inequality hold, and next we show that the choice of $z^{FS,*}$ always fall in this region.

First, we have

$$\begin{aligned} & z^2 - 2\frac{1+w}{w}z + \frac{135}{64} < 0 \\ \Longleftrightarrow\ & \frac{1+w}{w} - \sqrt{\frac{(1+w)^2}{w^2} - \frac{135}{64}} < z < \frac{1+w}{w} + \sqrt{\frac{(1+w)^2}{w^2} - \frac{135}{64}} \end{aligned} \tag{C16}$$

Next, we want to show $z^{FS,*}$ falls in the inequality in Eq. (C16) for all a and v in the parameter space. Notice that

$$z^{FS,*} = \begin{cases} \frac{w}{2+w}v & \text{if} \quad \frac{w}{2+w}v < \frac{5+15a}{14+18a} \\ \frac{5+15a}{14+18a} & \text{o.w.} \end{cases}. \tag{C17}$$

- Suppose $z^{FS,*} = \frac{5+15w}{14+18w}$, it always satisfies Eq. (C16) as $\left(\frac{1+w}{w} - \frac{5+15w}{14+18w}\right)^2 < \frac{(1+w^2)}{w^2} - \frac{135}{64}$ always hold when $w < 1$.
- Support $z^{FS,*} = \frac{w}{2+w}v \geq \frac{3w}{2+w}$ (and $\frac{w}{2+w}v < \frac{5+15w}{14+18w}$), it always satisfies Eq. (C16) as $\left(\frac{1+w}{w} - \frac{3w}{2+w}\right)^2 < \frac{(1+w^2)}{w^2} - \frac{135}{64}$ always holds when $\frac{1}{109}(102-14\sqrt{73}) < w < \frac{1}{109}(102+14\sqrt{73})$, including $0 < w < 1$.

In sum, we always have $z^{FS,*}$ satisfies Eq. (C16) for $0.1 < w < 1$ and $v \geq 3$. Therefore, the total cost for privacy-preserving customers always increases after Platform adopts the optimal segmentation rule.

For data-sharing customer who prefer Third-party (those who fall in Segment 1), since the transportation cost does not change and the price paid always increases, the total cost for them always increases after Platform adopts the optimal segmentation rule.

For data-sharing customer who prefer Platform (those who fall in Segment 2), the total cost before the adoption of fuzzy segmentation is

$$TC^{NS}_{\text{data-share,L}} + PC^{NS}_{\text{data-share,L}} = (1-w)\left[\frac{17}{64} - \frac{1}{2}z^2 + \frac{9}{16} + \frac{25}{32} - \frac{5}{4}z)\right] \tag{C18}$$

The total cost after the adoption of fuzzy segmentation is

$$
\begin{aligned}
TC^{FS}_{\text{data-share,L}} + PC^{FS}_{\text{data-share,L}} &= (1-w)\left[\left(\frac{1}{4}z+\frac{1}{8}\right)^2 + \frac{1}{4} - \frac{1}{2}z^2 + \frac{1}{4}\left(\frac{3}{2}-z\right)^2 + \frac{1}{2}\left(\frac{5}{4}-\frac{3}{2}z\right)^2\right] \\
&= (1-w)\left[\frac{1}{16}z^2 + \frac{1}{16}z + \frac{17}{64} - \frac{1}{2}z^2 + \frac{9}{16} - \frac{3}{4}z + \frac{1}{4}z^2 + \frac{25}{32} - \frac{15}{8}z + \frac{9}{8}z^2\right] \qquad \text{(C19)}
\end{aligned}
$$

Taking the difference of Eq. (C19) and Eq. (C18), we have $Eq.\ \text{(C19)} - Eq.\ \text{(C18)} \propto \frac{23}{16}z^2 - \frac{21}{16}z^2$ and the different is negative when $0 < z < \frac{21}{23}$. Since $z^{FS,*}$ as shown in Eq. (C17) is always less than $\frac{5}{8}$, it always falls in this range. Therefore, the total cost of data-sharing customers in Segment 2 always decreases after the adoption of fuzzy segmentation. □

C.1.13 Proof of Theorem 2

With the results from Table 6, we can infer the direction of the change in consumer surplus, which is the opposite of the change in total cost.

The total consumer surplus is lower because the total TC increases (as shown in Theorem 3) and the total PC increases (as shown in Theorem 1, which states that both sellers enjoy higher total profits). □

C.2 Model Extension (1)—When Third-party Serves as Stackelberg Leader

C.2.1 No-segment benchmark

As done in the analysis for the baseline no-segment scenario, if Third-party is the price leader, it should capture $\frac{3}{8}$ of the total demand at a price of $\frac{3}{2}$, resulting in a profit of $\pi_T^{NS,TL} = \frac{9}{16}$. Platform, as the follower, captures $\frac{5}{8}$ of the total demand at a price of $\frac{5}{4}$, resulting in a profit of $\pi_A^{NS,TL} = \frac{25}{32}$. Notice that these results are consistent with the fact that the Hotelling model is symmetric when there is no segmentation imposed. These equilibrium outcomes mirror those of the baseline no-segmentation case with the platform as the price leader, reflecting the inherent symmetry of the Hotelling model with no segmentation.

C.2.2 Hard-segment benchmark

The derivation of the hard segmentation case when the third party acts as the price leader closely resembles that of the baseline hard-segmentation scenario. For brievity, we present the equilibrium prices and profits, under any given segmentation rule z_0, in Tables A9 and A10, without going into the detailed derivations.[12]

Table A9: Sellers' Optimal Prices Under Hard Segment z_0 (Third-party as Price Leader)

Segment	Platform Price	Third-Party Price	Range of z_0
Segment 1	$p^{\bullet}_{1A} = 1 - 2z_0$	$p^{\bullet}_{1T} = 0$	$0 \leq z_0 < \frac{1}{4}$
	$p^{\bullet}_{1A} = \frac{5}{4} - (1 - z_0)$	$p^{\bullet}_{1T} = \frac{3}{2} - 2(1 - z_0)$	$\frac{1}{4} \leq z_0 \leq 1$
Segment 2	$p^{\bullet}_{2A} = \frac{5}{4} - \frac{3}{2}z_0$	$p^{\bullet}_{2T} = \frac{3}{2} - z_0$	$0 < z_0 < \frac{5}{6}$
	$p^{\bullet}_{2A} = 0$	$p^{\bullet}_{2T} = 2z_0 - 1$	$\frac{5}{6} \leq z_0 \leq 1$

Table A10: Sellers' Profits in Hard-Segment Pricing Equilibrium Under z_0 (Third-party as Price Leader)

	Segment 1 Profits	Segment 2 Profits	Total Profits
$0 \leq z_0 \leq \frac{1}{4}$	$\pi_{1A} = (1 - 2z_0)z_0$	$\pi_{2A} = \frac{1}{8}(3z_0 - \frac{5}{2})^2$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = 0$	$\pi_{2T} = \frac{1}{4}(z_0 - \frac{3}{2})^2$	$\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{1}{4} \leq z_0 \leq \frac{5}{6}$	$\pi_{1A} = \frac{1}{2}(z_0 + \frac{1}{4})^2$	$\pi_{2A} = \frac{1}{8}(3z_0 - \frac{5}{2})^2$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = (z_0 - \frac{1}{4})^2$	$\pi_{2T} = \frac{1}{4}(z_0 - \frac{3}{2})^2$	$\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{5}{6} \leq z_0 \leq 1$	$\pi_{1A} = \frac{1}{2}(z_0 + \frac{1}{4})^2$	$\pi_{2A} = 0$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = (z_0 - \frac{1}{4})^2$	$\pi_{2T} = (2z_0 - 1)(1 - z_0)$	$\pi_T = \pi_{1T} + \pi_{2T}$

It is straightforward to show that the non-trivial hard segmentation still results in lower equilibrium prices and profits. Therefore, Proposition 1 continues to hold when the third-party is the price leader.

C.2.3 Fuzzy-segment analysis

The pricing game. Segment 1. For a given fuzzy segmentation rule- z and Third-party price p_{1T}, Platform chooses its price p_{1A} to maximize its profit in Segment 1, $\pi_{1A} = D_{1A} \cdot p_{1A}$. Platform's demand function in Segment 1 is as follows.

[12] As a consistency check, note that this scenario can be viewed as reversing the axis labels of the Hoteling line: Third-party locates at 0, Platform locates at 1, and the segmentation rule applies at $1 - z_0$. To maintain clarity, all the results are still presented using the original axis setup shown in Figure 1.

$$D_{1A} = \begin{cases} 1-(1-w)(1-z) & \text{if} \quad p_{1A} \le p_{1T}-1 \\ \frac{p_{1T}-p_{1A}}{2}+\frac{1}{2}-(1-w)(1-z) & \text{if} \quad p_{1T}-1 \le p_{1A} \le p_{1T}+2z-1 \\ w\left(\frac{p_{1T}-p_{1A}}{2}+\frac{1}{2}\right) & \text{if} \quad p_{1T}+2z-1 \le p_{1A} \le p_{1T}+1 \\ 0 & \text{if} \quad p_{1A} > p_{1T}+1 \text{ (trivially dominated)} \end{cases}$$

Note that Platform's profit function is continuous and piecewise concave on each interval. We first obtain the optimal prices within each of the following three intervals $G_1 \equiv (0, p_{1T}-1]$, $G_2 \equiv [p_{1T}-1, p_{1T}+2z-1]$, and $G_3 \equiv [p_{1A}+2z-1, p_{1A}+1]$; among the three, the price that yields the highest profit is Platform's optimal Segment-1 price. Let $p_{1A}^{G_i}$ denote Platform's optimal price within the interval G_i.

- Suppose $p_{1A} \in G_1$. One can show that the optimal price is $p_{1A}^{G_1} = p_{1T}-1$.
- Suppose $p_{1A} \in G_2$. If $p_{1T} < 2w-2z-2wz+1$, the optimal price is $p_{1A}^{G_2} = p_{1T}+2z-1$; if $2w-2z-2wz+1 \le p_{1T} \le 2w+2z-2wz+1$, the optimal price is $p_{1A}^{G_2} = \frac{1}{2}p_{1T}+\frac{1}{2}-(1-w)(1-z)$ (the interior solution); if $p_{1T} > 2w+2z-2wz+1$, the optimal price is $p_{1A}^{G_2} = p_{1T}-1$.
- Suppose $p_{1A} \in G_3$. If $p_{1T} \le 3-4z$, the optimal price is $p_{1T}^{G_3} = \frac{1}{2}p_{1T}+\frac{1}{2}$ (the interior solution); if $p_{1T} > 3-4z$, the optimal price is $p_{1A}^{G_3} = p_{1T}+2z-1$.

Comparing the profits corresponding to prices $p_{1A}^{G_1}$, $p_{1A}^{G_2}$,and $p_{1A}^{G_3}$, one can show that Platform's optimal Segment-1 price is as follows.

- If $\zeta_b \le z \le 1$, where $\zeta_b = \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)}$,[13]

$$p_{1A}^{FS,TL,\circ} = \begin{cases} \frac{1}{2}p_{1T}+\frac{1}{2} & \text{if} \quad p_{1T} \in T_1 \\ \frac{1}{2}p_{1T}+\frac{1}{2}-(1-w)(1-z) & \text{if} \quad p_{1T} \in T_2 \\ p_{1T}-1 & \text{if} \quad p_{1T} \in T_3 \end{cases}, \tag{C20}$$

where $T_1 \equiv [0, 1-2z+2\sqrt{w-2wz+wz^2}]$, $T_2 \equiv [1-2z+2\sqrt{w-2wz+wz^2}, 2w+2z-2wz+1]$, and $T_3 \equiv [2w+2z-2wz+1, \infty)$.

[13]This region is equivalent to the region where $1-2z+2\sqrt{w-2wz+wz^2} \le 2w+2z-2wz+1$.

- If $0 \le z \le \zeta_b$,[14]

$$p_{1A}^{FS,TL,\circ} = \begin{cases} \frac{1}{2}p_{1T} + \frac{1}{2} & \text{if} \quad p_{1T} \in T_4 \\ \\ p_{1T} - 1 & \text{if} \quad p_{1T} \in T_5 \end{cases}, \tag{C21}$$

where $T_4 \equiv [0, p_h]$ and $T_5 \equiv [p_h, \infty)$, with $p_h = \frac{-4wz+3w+4z}{w} - \frac{4}{w}\sqrt{w^2z^2 - w^2z - 2wz^2 + wz + z^2}$.

Observe that Platform's best response price function is a piecewise correspondence and increases in p_{1T} on each interval, but it is not continuous. The discontinuity occurs at $p_{1T} = 1 - 2z + 2\sqrt{w - 2wz + wz^2}$ when $\zeta_b \le z \le 1$ and at $p_{1T} = p_h$ when $0 \le z \le \zeta_b$. The underlying mechanism is as follows: As Third-party's price rises, Platform initially follows by increasing its own price. In this range, the indifferent customer lies to the left of $1-z$ (i.e., Platform only sells to privacy-preserving customers). However, once Third-party's price reaches a threshold, Platform finds it more profitable to substantially lower its price and capture the additional demand to the right of $1-z$ (i.e., selling to both privacy-preserving and data-sharing customers). Notice that such discontinuity does not exist in Third-party's best response function when Platform is a price leader, because Third-party will face a smaller marginal demand as it lowers the price (i.e., trying to compete for the privacy-preserving customers to the right of $1-z$).

To resolve the indifference at the discontinuity, we adopt the tie-breaking convention that Platform selects the higher-price best response. This rule pins down a unique equilibrium at a knife-edge parameter value. This can also be interpreted as the outcome of an arbitrarily small cost perturbation that makes the higher-price choice weakly preferred when profits are otherwise equal.

Using $p_{1A}^{FS,TL,\circ}$, we can obtain the location of the indifferent customer $\tilde{x} = \frac{p_{1T}^{FS,TL,\circ} - p_{1A}}{2} + \frac{1}{2}$. Depending on the location of $\tilde{x}$ and the fuzzy segmentation rule $1-z$, Third-party's demand in Segment 1 is

[14] This region is equivalent to the region where $p_h \le 2w + 2z - 2wz + 1$.

- When $\zeta_b \leq z \leq 1$,

$$D_{1T} = \begin{cases} w(\frac{3}{4} - z - \frac{1}{4}p_{1T}) + z & \text{if} \quad p_{1T} \in T_1 \\ \frac{3}{4} - \frac{1}{4}p_{1T} - \frac{1}{2}(1-w)(1-z) & \text{if} \quad p_{1T} \in T_2 \setminus \min(T_2) \\ 0 & \text{if} \quad p_{1T} \in T_3 \text{ (trivially dominated)} \end{cases},$$

- When $0 \leq z \leq \zeta_b$,

$$D_{1T} = \begin{cases} w(\frac{3}{4} - z - \frac{1}{4}p_{1T}) + z & \text{if} \quad p_{1T} \in T_4 \\ 0 & \text{if} \quad p_{1T} \in T_5 \setminus \min(T_5) \text{ (trivially dominated)} \end{cases},$$

Third-party maximizes its profit, $\pi_{1T} = D_{1T} \cdot p_{1T}$. Note that Third-party's profit function is piecewise concave on each interval, but not continuous. Therefore, to find the optimal p_{1T}, we first find the locally optimal p_{1T} within the closure of each interval. Then, comparing the profits under each local optimum, we can find the globally optimal price $p_{1T}^{FS,\bullet}$.

- If $\zeta_b \leq z \leq 1$, we need to compare the locally optimal p_{1T} within each of the intervals, T_i.
  - Suppose $p_{1T} \in T_1$. If $z \in \mathbb{Z}_1$, the optimal price is $p_{1T}^{T_1} = \frac{3}{2} + 2\frac{1-w}{w}z$ (the interior solution). If $z \in \mathbb{Z}_2$, the optimal price is $p_{1T}^{T_1} = 1 - 2z + 2\sqrt{w - 2wz + wz^2}$. We denote $\mathbb{Z}_1 \equiv \{z | \frac{3}{2} + 2\frac{1-w}{w}z \leq 1 - 2z + 2\sqrt{w - 2wz + wz^2}, \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)} \leq z \leq 1, 0 \leq z \leq 1\} \equiv \{z | \zeta_b \leq z \leq \zeta_c\}$ and $\mathbb{Z}_2 \equiv \{z | \frac{3}{2} + 2\frac{1-w}{w}z > 1 - 2z + 2\sqrt{w - 2wz + wz^2}, \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)} \leq z \leq 1, 0 \leq z \leq 1\} \equiv \{z | \max\{\zeta_b, \zeta_c\} \leq z \leq 1\}$, where $\zeta_b = \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)}$, and $\zeta_c = \frac{4w^3+w-w(w+4)\sqrt{w}}{4(w^3-1)}$.
  - Suppose $p_{1T} \in T_2$. If $z \in \mathbb{Z}_3$, the optimal price is $p_{1T}^{T_2} = \frac{3}{2} - (1-w)(1-z)$ (the interior solution). If $z \in \mathbb{Z}_4$, the optimal price is $p_{1T}^{T_2} = 1 - 2z + 2\sqrt{w - 2wz + wz^2}$. We denote $\mathbb{Z}_3 \equiv \{z | \frac{3}{2} - (1-w)(1-z) > 1 - 2z + 2\sqrt{w - 2wz + wz^2}, \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)} \leq z \leq 1, 0 \leq z \leq 1\} \equiv \{z | \zeta_d \leq z \leq 1\}$ and $\mathbb{Z}_4 \equiv \{z | \frac{3}{2} - (1-w)(1-z) \leq 2(1+\sqrt{w})(1-z) - 1, \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)} \leq z \leq 1, 0 \leq z \leq 1\} \equiv \{z | \zeta_b \leq z \leq \zeta_d\}$, where $\zeta_d = \frac{2w^2-15w+3+10\sqrt{w}}{2(w^2-10w+9)}$.

Comparing the profits corresponding to prices $p_{1T}^{T_1}$ and $p_{1T}^{T_2}$, one can show that Third-party's optimal Segment-1 price when $\zeta_b \le z \le 1$ is as follows.

$$p_{1T}^{FS,\bullet} = \begin{cases} \frac{3}{2} + 2\frac{1-w}{w}z & \text{if} \quad \zeta_b \le z \le \zeta_c \\ 1 - 2z + 2\sqrt{w - 2wz + wz^2} & \text{if} \quad \max\{\zeta_b, \zeta_c\} \le z \le \zeta_d \\ \frac{3}{2} - (1-w)(1-z) & \text{if} \quad \zeta_d \le z \le 1 \end{cases},$$

where $\zeta_b = \frac{2\sqrt{w} - w(3-w)}{(1-w)(4-w)}$, $\zeta_c = \frac{4w^3 + w - w(w+4)\sqrt{w}}{4(w^3-1)}$, and $\zeta_d = \frac{2w^2 - 15w + 3 + 10\sqrt{w}}{2(w^2 - 10w + 9)}$.

- If $0 \le z \le \zeta_b$, we need to find the optimal p_{1T} within $T_4 \equiv [0, p_h]$.
  - Suppose $p_{1T} \in T_4$. If $0 \le z \le \zeta_a$, the optimal price is $p_{1T}^{T_4} = \frac{3}{2} + 2\frac{1-w}{w}z$ (the interior solution). If $\zeta_a \le z \le \zeta_b$, the optimal price is $p_{1T}^{T_4} = p_h$. We denote $\zeta_a = \frac{-5+2\sqrt{13}}{12}\frac{w}{1-w}$.

Therefore, Third-party's optimal Segment-1 price when $0 \le z \le \zeta_b$ is as follows.

$$p_{1T}^{FS,\bullet} = \begin{cases} \frac{3}{2} + 2\frac{1-w}{w}z & \text{if} \quad 0 \le z \le \min\{\zeta_a, \zeta_b\} \\ p_h & \text{if} \quad \zeta_a \le z \le \zeta_b \end{cases},$$

In sum, Third-party's optimal Segment-1 price is

$$p_{1T}^{FS,\bullet} = \begin{cases} \frac{3}{2} + 2\frac{1-w}{w}z & \text{if} \quad 0 \le z \le \min\{\zeta_a, \zeta_b\} \text{ or } \zeta_b \le z \le \zeta_c \\ p_h & \text{if} \quad \zeta_a \le z \le \zeta_b \\ 1 - 2z + 2\sqrt{w - 2wz + wz^2} & \text{if} \quad \max\{\zeta_b, \zeta_c\} \le z \le \zeta_d \\ \frac{3}{2} - (1-w)(1-z) & \text{if} \quad \zeta_d \le z \le 1 \end{cases},$$

where $\zeta_a = \frac{-5+2\sqrt{13}}{12}\frac{w}{1-w}$, $\zeta_b = \frac{2\sqrt{w} - w(3-w)}{(1-w)(4-w)}$, $\zeta_c = \frac{4w^3 + w - w(w+4)\sqrt{w}}{4(w^3-1)}$, and $\zeta_d = \frac{2w^2 - 15w + 3 + 10\sqrt{w}}{2(w^2 - 10w + 9)}$.

We summarize Segment 1's optimal seller prices and the corresponding market outcomes under any given fuzzy-segmentation rule z, in Table A11. The partition of the parameter space is illustrated in Figure A13.

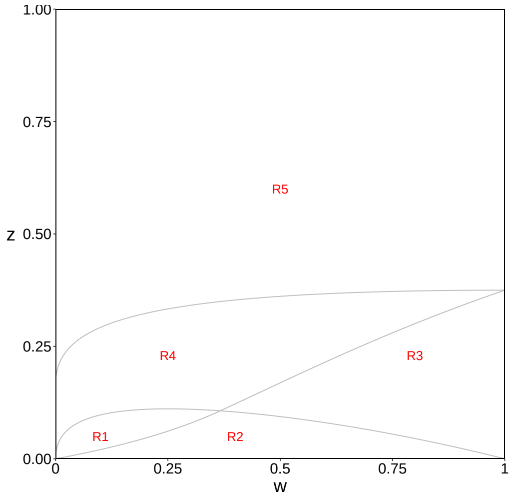


Figure A13: Illustration of the Parameter Space Partition, When Third-Party is Price Leader

Table A11: Summary of Equilibrium Prices and Profits in Segment 1
(Under Fuzzy Segmentation with Segmentation Rule z, When Third-Party is Price Leader)

Region	z Range	Price	Profit	$\tilde{x}_1$ Location
1	$\zeta_a \le z \le \zeta_b$	$p_{1T}^\bullet = p_h$ $p_{1A}^\bullet = \frac{1}{2}p_h + \frac{1}{2}$	To simplify the notation, we keep $p_{1T}^\bullet$ and $p_{1A}^\bullet$ in the expression for profit: $\pi_{1A}^\bullet = \frac{1}{8}w(p_{1T}^\bullet + 1)^2 = \frac{1}{2}w(p_{1A}^\bullet)^2$ $\pi_{1T}^\bullet = [w(\frac{3}{4} - z - \frac{1}{4}p_{1T}^\bullet) + z]p_{1T}^\bullet$	$0 \le \tilde{x}_1 \le 1 - z$
2	$0 < z \le \min\{\zeta_a, \zeta_b\}$	$p_{1T}^\bullet = \frac{3}{2} + 2\frac{1-w}{w}z$ $p_{1A}^\bullet = \frac{5}{4} + \frac{1-w}{w}z$		
3	$\zeta_b \le z \le \zeta_c$			
4	$\zeta_c \le z \le \zeta_d$	$p_{1T}^\bullet = 1 - 2z + 2\sqrt{w - 2wz + wz^2}$ $p_{1A}^\bullet = 1 - z + \sqrt{w - 2wz + wz^2}$		
5	$\zeta_d < z$	$p_{1T}^\bullet = \frac{3}{2} - (1-w)(1-z)$ $p_{1A}^\bullet = \frac{5}{4} - \frac{3}{2}(1-w)(1-z)$	$\pi_{1A}^\bullet = [z - \frac{1}{4}(1-w)(1-z) - \frac{3}{8} + w(1-z)]p_{1A}^\bullet$ $\pi_{1T}^\bullet = [\frac{1}{4}(1-w)(1-z) + \frac{3}{8}]p_{1T}^\bullet$	$1 - z < \tilde{x}_1 \le 1$

where $\zeta_a = \frac{-5+2\sqrt{13}}{12}\frac{w}{1-w}$, $\zeta_b = \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)}$, $\zeta_c = \frac{4w^3+w-w(w+4)\sqrt{w}}{4(w^3-1)}$, and $\zeta_d = \frac{2w^2-15w+3+10\sqrt{w}}{2(w^2-10w+9)}$; $p_h = \frac{-4wz+3w+4z}{w} - \frac{4}{w}\sqrt{w^2z^2 - w^2z - 2wz^2 + wz + z^2}$.

Segment 2. The Segment-2 equilibrium outcome in the fuzzy segmentation case can be obtained through simple modifications from the Segment-1 equilibrium in hard segmentation case. We plug in $z_0 = 1 - z$ and rescale the total profit by $1 - w$. The equilibrium outcome is summarized in Table A12.

Table A12: Summary of Equilibrium Prices and Profits in Segment 2
(Under Fuzzy Segmentation with Segmentation Rule z, When Third-Party is Price Leader)

Region	z range	Price	Profit
A	$0 < z \leq \frac{3}{4}$	$p^{\bullet}_{2A} = \frac{5}{4} - z$ $p^{\bullet}_{2T} = \frac{3}{2} - 2z$	$\pi^{\bullet}_{2A} = \frac{1}{2}(1-w)(\frac{5}{4} - z)^2$ $\pi^{\bullet}_{2T} = \frac{1}{4}(1-w)(\frac{3}{2} - 2z)^2$
B	$\frac{3}{4} < z \leq 1$	$p^{\bullet}_{2A} = 2z - 1$ $p^{\bullet}_{2T} = 0$	$\pi^{\bullet}_{2A} = (1-w)(2z-1)(1-z)$ $\pi^{\bullet}_{2T} = 0$

The segmentation stage. In the segmentation game, we need to combine Platform's profit from Segment 1 and Segment 2 and solve for the segmentation rule z that yield the highest Platform revenue. Specifically, deriving from Table A11 (Segment 1) and Table A12 (Segment 2), We need to discuss five different regions: R1, {R2 and R3}, R4, R5 while $z \leq \frac{3}{4}$, and R5 while $\frac{3}{4} < z \leq 1$.[15] For each region, we specify Platform's total profit and determine the optimal z within that region. If there are multiple regions correspond to the same w value, we compare the profits associated with the optimal z in each region and select the one that yields the highest profit. The discussion is as follows.

1. When $\zeta_a \leq z \leq \zeta_b$ and $0 < z \leq \frac{3}{4}$, Platform's total profit is

$$\pi_A(z) = \frac{1}{8}w(p^{\bullet}_{1T} + 1)^2 + \frac{1}{2}(1-w)\left(\frac{5}{4} - z\right)^2$$

   where $p^{\bullet}_{1T} = p_h = \frac{-4wz+3w+4z}{w} - \frac{4}{w}\sqrt{w^2z^2 - w^2z - 2wz^2 + wz + z^2}$.

   The F.O.C. is always negative when $z \in (0, \frac{3}{4}]$ and $w \in (0,1)$, so the optimal z in this range takes the smallest z in the feasible region. Therefore, $z^{1,*} = \zeta_a$.

2. When $0 \leq z \leq \min\{\zeta_a, \zeta_b\}$ or $\zeta_b \leq z \leq \zeta_c$, Platform's total profit is

$$\pi_A(z) = \frac{1}{8}w(p^{\bullet}_{1T} + 1)^2 + \frac{1}{2}(1-w)\left(\frac{5}{4} - z\right)^2$$

   where $p^{\bullet}_{1T} = \frac{3}{2} + 2\frac{1+w}{w}z$.

[15] Observe from Figure A13, none of R1, R2, R3, and R4 interacts with $[\frac{3}{4}, 1]$.

The F.O.C. equals to $\frac{1-w}{w} > 0$ when $w \in (0,1)$, so the optimal z in this range takes the largest z in the feasible region. Therefore,

$$z^{\{2,3\},*} = \begin{cases} \zeta_a & \text{if} \quad w \in \{w|\zeta_a(w) < \zeta_b(w)\} \\ \zeta_c & \text{o.w.} \end{cases}$$

3. When $\zeta_c \leq z \leq \zeta_d$ or $\zeta_b \leq z \leq \zeta_c$, Platform's total profit is

$$\pi_A(z) = \frac{1}{8}w(p_{1T}^{\bullet} + 1)^2 + \frac{1}{2}(1-w)\left(\frac{5}{4} - z\right)^2$$

where $p_{1T}^{\bullet} = 1 - 2z + 2\sqrt{w - 2wz + wz^2}$.

The F.O.C. is always negative when $w \in (0,1)$ and $z \in (0, \frac{3}{4}]$, so the optimal z in this range takes the smallest z in the feasible region. Therefore,

$$z^{4,*} = \begin{cases} \zeta_b & \text{if} \quad w \in \{w|\zeta_a(w) < \zeta_b(w)\} \\ \zeta_c & \text{o.w.} \end{cases}$$

4. When $z_d < z \leq 1$, Platform's total profit is

$$\pi_A(z) = \begin{cases} \left(z - \frac{1}{4}(1-w)(1-z) - \frac{3}{8} + w(1-z)\right) p_{1A}^{\bullet} + \frac{1}{2}(1-w)\left(\frac{5}{4} - z\right)^2 & \text{if} \quad 0 \leq z \leq \frac{3}{4} \\ \left(z - \frac{1}{4}(1-w)(1-z) - \frac{3}{8} + w(1-z)\right) p_{1A}^{\bullet} + (1-w)(2z-1)(1-z) & \text{if} \quad \frac{3}{4} < z \leq 1 \end{cases},$$

where $p_{1A}^{\bullet} = \frac{5}{4} - \frac{3}{2}(1-w)(1-z)$. The profit is always less than the no-segment banchmark $\pi_A^{NS,TL} = \frac{25}{32}$. Therefore, we eliminate them.

Comparing all the cases above, we derive the optimal fuzzy segmentation rule $z^{FS,TL,*}$ under a given w is

$$z^{FS,TL,*} = \begin{cases} \zeta_a & \text{if} \quad 0 < w \leq \frac{2\left(16\sqrt{13}+59\right)}{28\sqrt{13}+101} - \frac{2\sqrt{880\sqrt{13}+3173}}{28\sqrt{13}+101} \\ \zeta_c & \text{o.w.} \end{cases},$$

where $\zeta_a = \frac{-5+2\sqrt{13}}{12}\frac{w}{1-w}$ and $\zeta_c = \frac{4w^3+w-w(w+4)\sqrt{w}}{4(w^3-1)}$.

The equilibrium prices are: $p_{1A}^{FS,TL,*} = \frac{5}{4} + \frac{1-w}{w}z^{FS,TL,*}$, $p_{1T}^{FS,TL,*} = \frac{3}{2} + 2\frac{1-w}{w}z^{FS,TL,*}$, $p_{2A}^{FS,TL,*} =$

$\frac{5}{4} - z^{FS,TL,*}$, and $p_{2T}^{FS,TL,*} = \frac{3}{2} - 2z^{FS,TL,*}$.

C.3 Model Extension (2)—When Both Sellers Move Simultaneously

When sellers set price simultaneously in the pricing stage (sometimes denoted with the superscript "S" to distinguish from the baseline analysis), the sequence of the game is modified as follows.

- Stage 1: When applicable, Platform will choose the segmentation rule z_0 or z.
- Stage 2: Platform and Third-party decide their respective price in each segment (when applicable) at the same time.

We adopt backward induction as our solution method. We first solve the pricing game under a given segmentation rule (when applicable), and then search for the optimal segmentation rule on behalf of Platform. When solving for the pricing game, we adopt the solution concept of the Nash equilibrium (NE).

C.3.1 No-segment benchmark

When there is no segmentation, both sellers can only set a uniform price for the entire customer base. We denote them as p_A and p_T. The indifferent customer locates at $\frac{p_T - p_A}{2} + \frac{1}{2}$. Platform's and Third-party's demands are specified as follows.

$$D_A = \begin{cases} 0 & \text{if} \quad p_A \geq p_T + 1 \\ \frac{p_T - p_A}{2} + \frac{1}{2} & \text{if} \quad p_T - 1 \leq p_A \leq p_T + 1 \\ 1 & \text{if} \quad p_A \leq p_T - 1 \end{cases}, \quad D_T = \begin{cases} 1 & \text{if} \quad p_T \leq p_A - 1 \\ \frac{p_A - p_T}{2} + \frac{1}{2} & \text{if} \quad p_A - 1 \leq p_T \leq p_A + 1 \\ 0 & \text{if} \quad p_T \geq p_A + 1 \end{cases}.$$

The profit functions for Platform and Third-party are $\pi_A = D_A \cdot p_A$ and $\pi_T = D_T \cdot p_T$, respectively.

Solving each seller's profit maximization game separately, we have the best response function for each seller as follows.

$$p_A^{br} = \begin{cases} \frac{1}{2}p_T + \frac{1}{2} & \text{if} \quad p_T \leq 3 \\ p_T - 1 & \text{if} \quad p_T > 3 \end{cases}, \quad p_T^{br} = \begin{cases} \frac{1}{2}p_A + \frac{1}{2} & \text{if} \quad p_A \leq 3 \\ p_A - 1 & \text{if} \quad p_A > 3 \end{cases}.$$

Substituting Third-party's best response into Platform's best response, we solve for the equilibrium prices: $p_A^* = 1$ and $p_T^* = 1$. The constraints are satisfied. Therefore, we conclude that $p_A^{NS,S,*} = 1$ and $p_T^{NS,S,*} = 1$ is a pair of pure-strategy Nash equilibrium (PSNE). We use S in the superscript to denote that this is an equilibrium under simultaneous pricing game. The corresponding profits for both sellers are $\pi_A^{NS,S,*} = 0.5$ and $\pi_T^{NS,S,*} = 0.5$.

C.3.2 Hard-segment benchmark

The derivation in the hard-segmentation case (when segmentation rule is specified by z_0 as illustrated in Figure 1b) closely resembles the one in the no-segment case, except that demand functions need to be modified. Since the prices in Segment 1 and Segment 2 are independent, we solve them separately.

Segment 1. Platform's and Third-party's demands are specified as follows.

$$D_{1A} = \begin{cases} 0 & \text{if} \quad p_{1A} \geq p_{1T} + 1 \\ \frac{p_{1T}-p_{1A}}{2} + \frac{1}{2} & \text{if} \quad p_{1T} - 2z_0 + 1 \leq p_{1A} \leq p_{1T} + 1 \\ z_0 & \text{if} \quad p_{1A} \leq p_{1T} - 2z_0 + 1 \end{cases}, \quad D_{1T} = \begin{cases} 1 & \text{if} \quad p_{1T} \leq p_{1A} - 1 \\ z_0 - \frac{p_{1T}-p_{1A}}{2} - \frac{1}{2} & \text{if} \quad p_{1A} - 1 \leq p_{1T} \leq p_{1A} + 2z_0 - 1 \\ 0 & \text{if} \quad p_{1T} \geq p_{1A} + 2z_0 - 1 \end{cases}.$$

The profit functions for Platform and Third-party are $\pi_{1A} = D_{1A} \cdot p_{1A}$ and $\pi_{1T} = D_{1T} \cdot p_{1T}$, respectively.

Solving each seller's profit maximization game separately, we have the best response function for each seller as follows.

$$p_{1A}^{br} = \begin{cases} \frac{1}{2}p_{1T} + \frac{1}{2} & \text{if} \quad p_{1T} \leq 4z_0 - 1 \\ p_{1T} - 2z_0 + 1 & \text{if} \quad p_{1T} > 4z_0 - 1 \end{cases}, \quad p_{1T}^{br} = \begin{cases} \frac{1}{2}p_{1A} + z_0 - \frac{1}{2} & \text{if} \quad p_{1A} \leq 2z_0 + 1 \\ p_{1A} + 2z_0 - 1 & \text{if} \quad p_{1A} > 2z_0 + 1 \end{cases}.$$

The feasible region depends on z_0.

- When $z_0 \geq \frac{1}{4}$, we have an interior PSNE — $p_{1A}^* = \frac{2}{3}z_0 + \frac{1}{3}$ and $p_{1T}^* = \frac{4}{3}z_0 - \frac{1}{3}$. The corresponding indifferent customer lies at $\tilde{x}_1^* = \frac{1}{3}z_0 + \frac{1}{6}$.
- When $z_0 < \frac{1}{4}$, Third-party's price reaches the zero lower bound. Therefore, $p_{1A}^* = 1 - 2z_0$ and $p_{1T}^* = 0$ is a boundary PSNE.

With slight algebra rearrangement, the equilibrium prices in Segment 1 are shown below.

$$p_{1A}^{HS,S,*} = \begin{cases} 1 - \frac{2}{3}(1-z_0) & \text{if} \quad \frac{1}{4} \le z_0 \le 1 \\ 1 - 2z_0 & \text{if} \quad 0 \le z_0 \le \frac{1}{4} \end{cases}, \quad p_{1T}^{HS,S,*} = \begin{cases} 1 - \frac{4}{3}(1-z_0) & \text{if} \quad \frac{1}{4} \le z_0 \le 1 \\ 0 & \text{if} \quad 0 \le z_0 \le \frac{1}{4} \end{cases}.$$

Segment 2. Platform's and Third-party's demands are specified as follows.

$$D_{2A} = \begin{cases} 0 & \text{if} \quad p_{2A} \ge p_{2T} - 2z_0 + 1 \\ \frac{p_{2T}-p_{2A}}{2} + \frac{1}{2} - z_0 & \text{if} \quad p_{2T} - 1 \le p_{2A} \le p_{2T} - 2z_0 + 1 \\ 1 - z_0 & \text{if} \quad p_{2A} \le p_{2T} - 1 \end{cases}, \quad D_{1T} = \begin{cases} 1 - z_0 & \text{if} \quad p_{2T} \le p_{2A} + 2z_0 - 1 \\ \frac{p_{2A}-p_{2T}}{2} - \frac{1}{2} & \text{if} \quad p_{2A} + 2z_0 - 1 \le p_{2T} \le p_{2A} + 1 \\ 0 & \text{if} \quad p_{2T} \ge p_{2A} + 1 \end{cases}.$$

The profit functions for Platform and Third-party are $\pi_{2A} = D_{2A} \cdot p_{2A}$ and $\pi_{2T} = D_{2T} \cdot p_{2T}$, respectively.

Solving each seller's profit maximization game separately, we have the best response function for each seller as follows.

$$p_{2A}^{br} = \begin{cases} \frac{1}{2}p_{2T} + \frac{1}{2} - z_0 & \text{if} \quad p_{2T} \le 3 - 2z_0 \\ p_{2T} - 1 & \text{if} \quad p_{2T} > 3 - 2z_0 \end{cases}, \quad p_{2T}^{br} = \begin{cases} \frac{1}{2}p_{2A} + \frac{1}{2} & \text{if} \quad p_{1A} \le 3 - 4z_0 \\ p_{2A} - 1 & \text{if} \quad p_{1A} > 3 - 4z_0 \end{cases}.$$

The feasible region depends on z_0.

- When $z_0 \le \frac{3}{4}$, we have an interior PSNE — $p_{2A}^* = 1 - \frac{4}{3}z_0$ and $p_{2T}^* = 1 - \frac{2}{3}z_0$. The corresponding indifferent customer lies at $\tilde{x}_2^* = \frac{1}{3}z_0 + \frac{1}{2}$.
- When $z_0 > \frac{3}{4}$, Platform's price research the zero lower bound. Therefore, $p_{2A}^* = 0$ and $p_{2T}^* = 2z_0 - 1$ is a boundary PSNE.

The equilibrium prices in Segment 2 are shown below.

$$p_{2A}^{HS,S,*} = \begin{cases} 1 - \frac{4}{3}z_0 & \text{if} \quad 0 \le z_0 \le \frac{3}{4} \\ 0 & \text{if} \quad \frac{3}{4} \le z_0 \le 1 \end{cases}, \quad p_{2T}^{HS,S,*} = \begin{cases} 1 - \frac{2}{3}z_0 & \text{if} \quad 0 \le z_0 \le \frac{3}{4} \\ 2z_0 - 1 & \text{if} \quad \frac{3}{4} \le z_0 \le 1 \end{cases}.$$

We summarize the equilibrium prices and profits, under any given segmentation rule z_0, in Tables A13 and A14. It is straightforward to show that the non-trivial hard segmentation still results in lower equilibrium prices and profits. Therefore, Proposition 1 is valid despite the sellers engaging in simultaneous pricing.

Table A13: Sellers' Optimal Prices Under Hard Segment z_0 (Under Simultaneous Pricing)

Segment	Platform Price	Third-Party Price	Range of z_0
Segment 1	$p_{1A}^{\bullet} = 1 - \frac{2}{3}(1 - z_0)$	$p_{1T}^{\bullet} = 1 - \frac{4}{3}(1 - z_0)$	$0 \leq z_0 < \frac{1}{4}$
	$p_{1A}^{\bullet} = 1 - 2z_0$	$p_{1T}^{\bullet} = 0$	$\frac{1}{4} \leq z_0 \leq 1$
Segment 2	$p_{2A}^{\bullet} = 1 - \frac{2}{3}z_0$	$p_{2T}^{\bullet} = 1 - \frac{4}{3}z_0$	$0 < z_0 < \frac{3}{4}$
	$p_{2A}^{\bullet} = 0$	$p_{2T}^{\bullet} = 2z_0 - 1$	$\frac{3}{4} \leq z_0 \leq 1$

Table A14: Sellers' Profits in Hard-Segment Pricing Equilibrium Under z_0 (Under Simultaneous Pricing)

	Segment 1 Profits	Segment 2 Profits	Total Profits
$0 \leq z_0 \leq \frac{1}{4}$	$\pi_{1A} = (1 - 2z_0)z_0$	$\pi_{2A} = \frac{1}{2}(1 - \frac{4}{3}z_0)^2$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = 0$	$\pi_{2T} = \frac{1}{2}(1 - \frac{2}{3}z_0)^2$	$\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{1}{4} \leq z_0 \leq \frac{3}{4}$	$\pi_{1A} = \frac{1}{2}(\frac{2}{3}z_0 + \frac{1}{3})^2$	$\pi_{2A} = \frac{1}{2}(1 - \frac{4}{3}z_0)^2$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = \frac{1}{2}(\frac{4}{3}z_0 - \frac{1}{3})^2$	$\pi_{2T} = \frac{1}{2}(1 - \frac{2}{3}z_0)^2$	$\pi_T = \pi_{1T} + \pi_{2T}$
$\frac{3}{4} \leq z_0 \leq 1$	$\pi_{1A} = \frac{1}{2}(\frac{2}{3}z_0 + \frac{1}{3})^2$	$\pi_{2A} = 0$	$\pi_A = \pi_{1A} + \pi_{2A}$
	$\pi_{1T} = \frac{1}{2}(\frac{4}{3}z_0 - \frac{1}{3})^2$	$\pi_{2T} = (2z_0 - 1)(1 - z_0)$	$\pi_T = \pi_{1T} + \pi_{2T}$

C.3.3 Fuzzy-segment Analysis

As done in the main analysis, we separately discuss the pricing game in Segment 1 and Segment 2. Furthermore, as shown in the main analysis, the analysis in Segment 1 is more involved, and is hence our focus below.

The pricing game Segment 1. We start by presenting each seller's best response function under a fuzzy segmentation rule z. Each of them have been derived in the two sequential pricing case, so we directly use the results. Third-party's best response function is:

$$p_{1T}^{br} = \begin{cases} \frac{1}{2}p_{1A} + \frac{1}{2} & \text{if} \quad p_{1A} \in A_1 \\ p_{1A} + 1 - 2z & \text{if} \quad p_{1A} \in A_2 \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \text{if} \quad p_{1A} \in A_3 \\ p_{1A} - 1 & \text{if} \quad p_{1A} \in A_4 \end{cases},$$

where $A_1 \equiv [0, 4z - 1]$, $A_2 \equiv [4z - 1, 2\frac{1+w}{w}z - 1]$, $A_3 \equiv [2\frac{1+w}{w}z - 1, 3 + 2\frac{1-w}{w}z]$, and $A_4 \equiv$

$[3+2\frac{1-w}{w}z,\infty)$.

Platform's best response function is:

- If $\zeta_b \le z \le 1$, where $\zeta_b = \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)}$,

$$p_{1A}^{br} = \begin{cases} \frac{1}{2}p_{1T} + \frac{1}{2} & \text{if} \quad p_{1T} \in T_1 \\ \frac{1}{2}p_{1T} - (1-w)(1-z) + \frac{1}{2} & \text{if} \quad p_{1T} \in T_2 \\ p_{1T} - 1 & \text{if} \quad p_{1T} \in T_3 \end{cases},$$

where $T_1 \equiv [0, 1-2z+2\sqrt{w-2wz+wz^2}]$, $T_2 \equiv [1-2z+2\sqrt{w-2wz+wz^2}, 2w+2z-2wz+1]$, and $T_3 \equiv [2w+2z-2wz+1, \infty)$.

- If $0 \le z \le \zeta_b$,

$$p_{1A}^{br} = \begin{cases} \frac{1}{2}p_{1T} + \frac{1}{2} & \text{if} \quad p_{1T} \in T_4 \\ p_{1T} - 1 & \text{if} \quad p_{1T} \in T_5 \end{cases},$$

where $T_4 \equiv [0, p_h]$ and $T_5 \equiv [p_h, \infty)$, with $p_h = \frac{-4wz+3w+4z}{w} - \frac{4}{w}\sqrt{w^2z^2 - w^2z - 2wz^2 + wz + z^2}$.

First, we note that the pricing subgame does not always have a pure-strategy Nash equilibrium for every pair (z, w). This arises because Platform's best response function is always discontinuous as discussed in Section C.2.3. For the purpose of this study, we focus on discussing the pure-strategy Nash equilibrium (PSNE).

Next, to solve for PNSE, we obtain the intersection of the two best-response schedules. We evaluate all possible combination of regions (A_k, T_l) and retain those solutions that satisfy their respective boundaries. We summarize each combination in Table A15 and illustrate the partition of the parameter space in Figure A14.

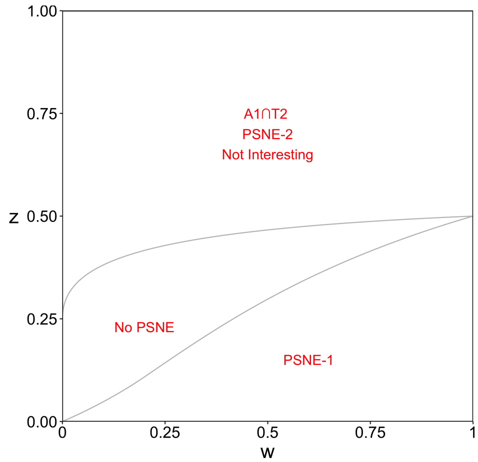


Figure A14: Partition of the Parameter Space

Table A15: Summary of PSNE under Simultaneous Pricing Subgame

Equilibrium Type	Range	Equilibrium Price	Conditions Needed to Sustain the Equilibrium
PSNE-2	$A_1 \cap T_2$	$p_{1A} = 1 - \frac{4}{3}(1-w)(1-z)$ $p_{1T} = 1 - \frac{2}{3}(1-w)(1-z)$	$\zeta_e \leq z \leq 1$
No PSNE	$A_2 \cap T_1$	$p_{1A} = 2(1-z)$ $p_{1T} = 3 - 4z$	N.A.
	$A_2 \cap T_2$	$p_{1A} = 2w(1-z)$ $p_{1T} = 1 + 2w - 2(w+1)z$	N.A.
PSNE-1	$A_3 \cap T_1$	$p_{1A} = 1 + \frac{2}{3}\frac{1-w}{w}z$ $p_{1T} = 1 + \frac{4}{3}\frac{1-w}{w}z$	$\zeta_b \leq z \leq \zeta_f$
	$A_3 \cap T_4$	$p_{1A} = 1 + \frac{2}{3}\frac{1-w}{w}z$ $p_{1T} = 1 + \frac{4}{3}\frac{1-w}{w}z$	$0 \leq z \leq \zeta_b$ and $0 \leq z \leq \zeta_g$

Note: $\zeta_b = \frac{2\sqrt{w}-w(3-w)}{(1-w)(4-w)}$, $\zeta_e = \frac{w^2-14w+4+9\sqrt{w}}{w^2-17w+16}$, $\zeta_f = \frac{9w^3-3w(w+2)\sqrt{w}}{9w^3-w^2-4w-4}$, $\zeta_g = \frac{3\sqrt{6}-3}{10}\frac{w}{1-w}$.

Lastly, we discuss which equilibrium is of interest. Similar to Lemma C5, Platform never has an incentive to induce an equilibrium where the pricing equilibrium falls in range $A_1 \cap T_2$ (i.e. choosing $z \in [\zeta_e, 1]$), as the total profit is always less than the no-segment benchmark price.[16] Therefore, we focus on the subgame price equilibrium where

$$\begin{cases} p_{1A}^{FS,S,*} = 1 + \frac{2}{3}\frac{1+w}{w}z \\ p_{1T}^{FS,S,*} = 1 + \frac{4}{3}\frac{1+w}{w}z \end{cases}. \tag{C22}$$

[16]When $z \in [\zeta_e, 1]$, Platform's Segment-1 profit is $\pi_{1A}^R = \frac{1}{2}[1 - \frac{4}{3}(1-w)(1-z)]^2$. Platform's Segment-2 profit is $\pi_{2A} = \begin{cases} \frac{1}{2}(1-w)[1-\frac{2}{3}z]^2 & \text{if } 0 \leq z \leq \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} \leq z \leq 1 \end{cases}$. With some mathematical simplification, one can show that $\pi_{1A}^R + \pi_{2A} \leq \frac{1}{2}$ for any $z \in [\zeta_e, 1]$, with the equality only achieved at the trivial segmentation case when $z = 1$.

The range of z where this equilibrium can be sustained is $0 < z \leq \bar{z}$, where $\bar{z} = \max\{\zeta_f, \zeta_g\}$.[17] The equilibrium Segment-1 profit for each seller when $0 \leq z \leq \bar{z}$ is

$$\begin{cases} \pi_{1A}^{FS,S,*}(z) = \frac{1}{2}w\left(1 + \frac{2}{3}\frac{1-w}{w}z\right)^2 \\ \pi_{1T}^{FS,S,*}(z) = \frac{1}{2}w\left(1 + \frac{4}{3}\frac{1-w}{w}z\right)^2 \end{cases} . \tag{C23}$$

Segment 2. For the pricing game in Segment 2, we substitute $z_0 = 1 - z$ into the Segment-1 equilibrium in the hard-segment case. We focus on $0 \leq z \leq \bar{z}$, where $\bar{z} \leq \frac{3}{4}$. The equilibrium prices are

$$\begin{cases} p_{2A}^{FS,S,*}(z) = 1 - \frac{2}{3}z \\ p_{2T}^{FS,S,*}(z) = 1 - \frac{4}{3}z \end{cases} . \tag{C24}$$

The equilibrium Segment-2 profit for each seller when $0 \leq z \leq \bar{z}$ is

$$\begin{cases} \pi_{2A}^{FS,S,*}(z) = \frac{1}{2}(1-w)\left(1 - \frac{2}{3}z\right)^2 \\ \pi_{2T}^{FS,S,*}(z) = \frac{1}{2}(1-w)\left(1 - \frac{4}{3}z\right)^2 \end{cases} . \tag{C25}$$

The segmentation stage In the first-stage segmentation decision, we subsequently show that for any given w there exists a non-trivial segmentation rule $z \in [0, \bar{z}]$ such that the platform's profit is strictly higher than the one under uniform pricing (the no-segmentation benchmark). To do so, Platform solves a profit-maximization problem where the total profit equals to the sum of Segment-1 and Segment-2 profits. Our analysis continues to focus on $z \in [0, \bar{z}]$, where, as established above, the pricing subgame has a pure-strategy Nash equilibrium.

$$\begin{aligned} \max_{z} \quad & \frac{1}{2} \cdot w \left(1 + \frac{2}{3}\ \frac{1-w}{w} z\right)^2 + \frac{1}{2} \cdot (1-w)\left(1 - \frac{2}{3}\ z\right)^2 \\ \text{s.t.} \quad & 0 \leq z \leq \bar{z} \end{aligned}$$

The F.O.C. of the objective function equals to $\frac{4}{9} \cdot \frac{1-w}{w} z$ and is always positive when $0 < w < 1$. Therefore, the optimal solution is attained at the largest z in the feasible region. This implies that the platform's profit is strictly increasing in z over the interval $0 \leq z \leq \bar{z}$. Furthermore, when

[17] When $0 < w < \breve{w}'''$, $\zeta_f < \zeta_b$ and $\zeta_f < \zeta_g$, so $\bar{z} = \zeta_g$. When $\breve{w}''' < w < 1$, $\zeta_b < \zeta_f$ and $\zeta_g < \zeta_f$, so $\bar{z} = \zeta_f$. When $w = \breve{w}'''$, $\zeta_b = \zeta_f = \zeta_g$. The value of $\breve{w}'''$ is $10 - 4\sqrt{6}$.

$z = 0$, the outcome converges to the no-segmentation, yielding $\pi_A = 0.5$. Hence, there exists a segmentation rule z that improves the platform's profit beyond the uniform-pricing benchmark.

C.4 Model Extension (3)—Revenue Sharing

C.4.1 No-segment benchmark (Proof of Lemma B3)

We solve the game by backward induction with Platform as a price leader. For a given Platform price p_A, Third-party chooses its price p_T to maximize its profit, $\pi_T^{rs,NS} = D_T \cdot p_T$. Similarly to the proof of the main model, we use $\tilde{x} = \frac{1}{2}(p_T - p_A) + \frac{1}{2}$ to help illustrate the demand, where $\tilde{x}$ solves $v - p_A - x = v - p_T - (1-x)$. Furthermore, since revenue-sharing softens the price competition (will be elaborated in the following), we need to actively consider consumer participation constraints in the analysis. Therefore, we also use $\bar{x}_T$ to help illustrate the demand, where $\bar{x}_T$ represents the customer who is indifferent between Third-party and the outside option. We use rs in the superscript to denote the "revenue-sharing" scenario. Third-party's demand function is as follows.

$$D_T^{rs,NS} = \begin{cases} 1 & \text{if} \quad \max\{\tilde{x}, \bar{x}_{1T}\} < 0 \Longleftrightarrow p_T \in E_0 \\ 1 - \tilde{x} = \frac{1}{2} + \frac{1}{2}(p_A - p_T) & \text{if} \quad \max\{\bar{x}_{1T}, 0\} \leq \tilde{x} \leq 1 \Longleftrightarrow p_T \in E_1 \\ 1 - \bar{x}_T = v - p_T & \text{if} \quad \max\{\tilde{x}_{1T}, 0\} \leq \bar{x}_T \leq 1 \Longleftrightarrow p_T \in E_2 \\ 0 & \text{if} \quad 1 \leq \min\{\tilde{x}, \bar{x}_T\} \Longleftrightarrow p_T \in E_3 \end{cases}$$

The feasible regions for each condition are $E_0 \equiv \{p_T | p_T < \min\{p_A - 1, v - 1\}\}$; $E_1 \equiv \{p_T | p_A - 1 \leq p_T \leq \min\{2v - p_A - 1, p_A + 1\}\}$; $E_2 \equiv \{p_T | \min\{2v - p_A - 1, v - 1\} \leq p_T \leq v\}$; $E_3 \equiv \{p_T | p_T \geq \{p_A + 1, v\}\}$. One can show that the case where $\max\{\tilde{x}, \bar{x}_T\} < 0$ is dominated by the case where $\max\{\tilde{x}, \bar{x}_T\} = 0$ and the case where $1 < \min\{\tilde{x}, \bar{x}_T\}$ is dominated by the case where $\min\{\tilde{x}, \bar{x}_T\} = 1$. Therefore, we only discuss the remaining two regions (i.e., E_1 and E_2).

Third-party's demand function is continuous and piecewise-linear on each interval, and its profit function, $\pi_T^{rs,NS} = (1 - s) \cdot D_T^{rs,NS} \cdot p_T$, is continuous and piecewise concave. We first obtain the optimal prices within each of the following two intervals E_1 and E_2; among the two, the price that yields the highest profit is Third-party's best response price. Let $p_T^{E_i}$ denote Third-party's optimal price within the interval E_i.

- Suppose that $p_T \in E_1$. The interior solution is $\frac{1}{2}(p_A+1)$. When $v \geq 3$, one can show that if $p_A \leq 3$, the optimal price is $p_T^{E_1} = \frac{1}{2}(p_A+1)$; if $p_A > 3$, one can show that the optimal price is $p_T^{E_1} = p_A - 1$.

- Suppose that $p_T \in E_2$. The interior solution is $\frac{1}{2}v$. It is always true that $v-1 \leq 2v - p_A - 1$ when $p_A \leq v$. Furthermore, it is always true that $\frac{1}{2}v \leq v-1$ when $v \geq 2$. Therefore, the left boundary is always attained and $p_T^{E_2} = 2v - p_A - 1$.

Comparing the profits corresponding to prices $p_T^{E_1}$ and $p_T^{E_2}$, one can show that Third-party's best response function is as follows.

$$p_T^{rs,NS,\circ}(p_A) = \begin{cases} \frac{1}{2}(p_A+1) & \text{if} \quad p_A \leq 3 \\ p_A - 1 & \text{if} \quad 3 \leq p_A \leq v \end{cases}$$

Using $p_T^{rs,NS,\circ}$, we then solve for Platform's optimal price to maximize its profit, i.e., $\pi_A^{rs,NS} = D_A^{rs,NS} p_A + s \cdot R_T^{\circ}(p_A)$, where $R_T^{\circ}(p_A) = D_T^{rs,NS} \cdot p_T^{rs,NS,\circ}(p_A)$. Platform's demand function is as follows.

$$D_A^{rs,NS} = \begin{cases} -\frac{1}{4}p_A + \frac{3}{4} & \text{if} \quad p_A \leq 3 \\ 0 & \text{if} \quad 3 \leq p_A \leq v \end{cases}.$$

The corresponding Third-party's demand function is:

$$D_T^{rs,NS} = \begin{cases} \frac{1}{4}p_A + \frac{1}{4} & \text{if} \quad p_A \leq 3 \\ 1 & \text{if} \quad 3 \leq p_A \leq v \end{cases}.$$

Notice that due to revenue sharing, even when Platform has zero demand, its profit is non-negative. Therefore, we cannot eliminate the range where $3 \leq p_A \leq v$. In fact, $p_A = v$ could be a dominating strategy under certain parameter space.

Platform's demand function is continuous and piecewise linear; its profit function is continuous and piecewise concave (including the revenue sharing portion from Third-party). We first obtain the optimal price within each of the following three intervals $F_1 \equiv (0, 3]$, and $F_2 \equiv [3, v]$; among the two, the price that yields the higher profit is Platform's optimal price. Let $p_A^{F_i}$ denote Platform's optimal price within the interval F_i.

- Suppose that $p_A \in F_1$, Platform's profit is $\pi_A^{rs,NS,F_1} = \left(-\frac{1}{4}p_A + \frac{3}{4}\right)p_A + s\left(\frac{1}{4}p_A + \frac{1}{4}\right)\frac{1}{2}(p_A+1)$. One can show that the optimal price is $p_A^{F_1} = \frac{3+s}{2-s}$ when $s \leq \frac{3}{4}$ and $p_A^{F_1} = 3$ when $s > \frac{3}{4}$.

- Suppose that $p_A \in F_2$, Platform's profit is $\pi_A^{rs,NS,F_2} = s(p_A - 1)$, increasing in p_A. Therefore, the optimal price is $p_A^{F_2} = v$.

Comparing the profits corresponding to prices $p_A^{F_1}$ and $p_A^{F_2}$, one can show that Platform's optimal price when $v \geq 3$ is,

$$p_A^{rs,NS,\bullet} = \begin{cases} \frac{3+s}{2-s} & \text{if} \quad 0 \leq s \leq \bar{s} \\ v & \text{if} \quad \bar{s} < s \leq 1 \end{cases},$$

where $\bar{s} = \frac{1}{4}\left(\frac{4v-6}{v-1} - \sqrt{2}\sqrt{\frac{8v^2-33v+27}{(v-1)^2}}\right)$. The corresponding Third-party's price is

$$p_T^{rs,NS,\bullet} = \begin{cases} \frac{5}{4-2s} & \text{if} \quad 0 \leq s \leq \bar{s} \\ v-1 & \text{if} \quad \bar{s} < s \leq 1 \end{cases}.$$

Both sellers' profit at equilibrium follows directly from the equilibrium prices and the corresponding demand functions, as specified in this proof. □

C.4.2 Hard-segment benchmark (Proof of Lemma B4)

Similar to the discussion on the no-segment case with revenue-sharing, we analyze the hard-segment pricing subgame with the consideration of the consumer value v. We solve the pricing subgames by backward induction. For a given segmentation rule z_0, both sellers engage in a sequential pricing game in each segment, with Platform as price leader. Since the pricing subgame in each section is independent from each other, we solve the subgames in Segment 1 and Segment 2 separately.

Segment 1. For a given hard segmentation rule z_0 and Platform price p_{1A}, Third-party chooses its price p_{1T} to maximize its profit, $\pi_{1T}^{rs,HS} = D_{1T} \cdot p_{1T}$. Similarly to the proof of the main model, we use $\tilde{x}_1 = \frac{1}{2}(p_{1T} - p_{1A}) + \frac{1}{2}$ to help illustrate the demand, where $\tilde{x}_1$ solves $v - p_{1A} - x = v - p_{1T} - (1-x)$. Furthermore, since revenue-sharing softens the price competition (will be elaborated in the following), we need to actively consider consumer participation constraints in the analysis.

Therefore, we also use $\bar{x}_{1T}$ to help illustrate the demand, where $\bar{x}_{1T}$ represents the customer who is indifferent between Third-party and the outside option. We use rs in the superscript to denote the "revenue-sharing" scenario. Third-party's demand function is as follows.

$$D_{1T}^{rs,HS} = \begin{cases} z_0 & \text{if} \quad \max\{\tilde{x}_1, \bar{x}_{1T}\} < 0 \Longleftrightarrow p_{1T} \in G_0 \\ z_0 - \tilde{x}_1 = z_0 - \frac{1}{2} + \frac{1}{2}(p_{1A} - p_{1T}) & \text{if} \quad \max\{\bar{x}_{1T}, 0\} \le \tilde{x}_1 \le z_0 \Longleftrightarrow p_{1T} \in G_1 \\ z_0 - \bar{x}_{1T} = z_0 - p_{1T} + v - 1 & \text{if} \quad \max\{\tilde{x}_{1T}, 0\} \le \bar{x}_{1T} \le z_0 \Longleftrightarrow p_{1T} \in G_2 \\ 0 & \text{if} \quad z_0 \le \min\{\tilde{x}_1, \bar{x}_{1T}\} \Longleftrightarrow p_{1T} \in G_3 \end{cases}$$

The feasible regions for each condition are $G_0 \equiv \{p_{1T} | p_{1T} < \min\{p_{1A} - 1, v - 1\}\}$; $G_1 \equiv \{p_{1T} | p_{1A} - 1 \le p_{1T} \le \min\{2v - p_{1A} - 1, p_{1A} + 2z_0 - 1\}\}$; $G_2 \equiv \{p_{1T} | \min\{2v - p_{1A} - 1, v - 1\} \le p_{1T} \le v + z_0 - 1\}$; $G_3 \equiv \{p_{1T} | p_{1T} \ge \{p_{1A} + 2z_0 - 1, v + z_0 - 1\}\}$. One can show that the case where $\max\{\tilde{x}, \bar{x}_{1T}\} < 0$ is dominated by the case where $\max\{\tilde{x}_1, \bar{x}_{1T}\} = 0$ and the case where $1 < \min\{\tilde{x}_1, \bar{x}_{1T}\}$ is dominated by the case where $\min\{\tilde{x}_1, \bar{x}_{1T}\} = 1$. Therefore, we only discuss the remaining two regions (i.e., G_1 and G_2).

Third-party's demand function is continuous and piecewise-linear on each interval, and its profit function, $\pi_{1T}^{rs,HS} = (1 - s) \cdot D_{1T}^{rs,HS} \cdot p_{1T}$, is continuous and piecewise concave. We first obtain the optimal prices within each of the following two intervals G_1 and G_2; among the two, the price that yields the highest profit is Third-party's best response price. Let $p_T^{G_i}$ denote Third-party's optimal price within the interval G_i.

- Suppose that $p_{1T} \in G_1$. The interior solution is $\frac{1}{2}(p_{1A} + 2z_0 - 1)$.
  - When $v \ge 2z_0 + 1$, one can show that if $p_{1A} \le 2z_0 + 1$, the optimal price is $p_T^{G_1} = \frac{1}{2}(p_A + 2z_0 - 1)$; if $p_A > 2z_0 + 1$, one can show that the optimal price is $p_T^{G_1} = p_A - 1$.
  - When $2 \le v < 2z_0 + 1$, one can show that if $p_{1A} \le \frac{4}{3}v - \frac{1}{3} - \frac{2}{3}z_0$, the optimal price is $p_T^{G_2} = \frac{1}{2}(p_A + 2z_0 - 1)$; if $p_A > \frac{4}{3}v - \frac{1}{3} - \frac{2}{3}z_0$, one can show that the optimal price is $p_T^{G_1} = 2v - p_{1A} - 1$.
- Suppose that $p_{1T} \in G_2$. The interior solution is $\frac{1}{2}(v + z_0 - 1)$. It is always true that $v - 1 \le 2v - p_{1A} - 1$ when $p_{1A} \le v$. Furthermore, it is always true that $\frac{1}{2}(v + z_0 - 1) \le v - 1$ when $v \ge 2 \ge z_0 + 1$. Therefore, the left boundary is always attained and $p_{1T}^{G_2} = 2v - p_{1A} - 1$.

Comparing the profits corresponding to prices $p_{1T}^{G_1}$ and $p_{1T}^{G_2}$, one can show that Third-party's Segment-1 best response function is as follows.

- When $v \geq 2z_0 + 1$,

$$p_{1T}^{rs,HS,\circ}(p_{1A}) = \begin{cases} 0 & \text{if} \quad 0 \leq p_{1A} \leq 1 - 2z_0 \\ \frac{1}{2}(p_{1A} + 2z_0 - 1) & \text{if} \quad 1 - 2z_0 \leq p_{1A} \leq 2z_0 + 1 \\ p_{1A} - 1 & \text{if} \quad 2z_0 + 1 \leq p_{1A} \leq v \end{cases}$$

- When $v \leq 2z_0 + 1$,

$$p_{1T}^{rs,HS,\circ}(p_{1A}) = \begin{cases} 0 & \text{if} \quad 0 \leq p_{1A} \leq 1 - 2z_0 \\ \frac{1}{2}(p_{1A} + 2z_0 - 1) & \text{if} \quad 1 - 2z_0 \leq p_{1A} \leq \frac{4}{3}v - \frac{1}{3} - \frac{2}{3}z_0 \\ 2v - p_{1A} - 1 & \text{if} \quad \frac{4}{3}v - \frac{1}{3} - \frac{2}{3}z_0 \leq p_{1A} \leq v \end{cases}$$

Focusing on the results when $v \geq 2z_0 + 1$ (due to our focus on $v \geq 3$), we have proven Eq. (B1) in Lemma B4. $\square$

In what follows, we illustrate the attempt to solve the pricing game analytically and demonstrate the challenge of doing so for completeness. Readers can skip this part and refer to the numerical routine as outlined in Algorithm 1 for numerical results.

Using $p_{1T}^{rs,HS,\circ}$, we then solve for Platform's optimal price to maximize its profit, i.e., $\pi_{1A}^{rs,HS} = D_{1A}^{rs,HS} p_{1A} + s \cdot R_{1T}^{\circ}(p_{1A})$, where $R_{1T}^{\circ}(p_{1A}) = D_{1T}^{rs,HS} \cdot p_{1T}^{rs,HS,\circ}(p_{1A})$. Platform's demand function is as follows.

$$D_{1A}^{rs,HS} = \begin{cases} -\frac{1}{4}p_A + \frac{1}{2}z_0 + \frac{1}{4} & \text{if} \quad 0 \leq p_{1A} \leq 2z_0 + 1 \\ 0 & \text{if} \quad 2z_0 + 1 \leq p_{1A} \leq v \end{cases}.$$

The corresponding Third-party's demand function is:

$$D_{1T}^{rs,HS} = \begin{cases} \frac{1}{4}p_{1A} + \frac{1}{2}z_0 - \frac{1}{4} & \text{if} \quad 0 \leq p_{1A} \leq 2z_0 + 1 \\ z_0 & \text{if} \quad 2z_0 + 1 \leq p_{1A} \leq v \end{cases}.$$

Notice that due to revenue sharing, even when Platform has zero demand, its profit is non-negative. Therefore, we cannot eliminate the range where $2z_0 + 1 \leq p_{1A} \leq v$. In fact, $p_A = v$ could be a dominating strategy under certain parameter space.

Platform's demand function is continuous and piecewise linear; its profit function is continuous and piecewise concave (including the revenue sharing portion from Third-party). We first obtain the optimal price within each of the following three intervals $H_1 \equiv (0, 2z_0+1]$ and $H_2 \equiv [2z_0+1, v]$; between the two, the price that yields the highest profit is Platform's optimal price. Let $p_{1A}^{H_i}$ denote Platform's Segment-1 optimal price within the interval H_i.

- Suppose that $p_{1A} \in H_1$, Platform's Segment-1 profit is $\pi_{1A}^{rs,HS,H_1} = \left(-\frac{1}{4}p_{1A} + \frac{1}{2}z_0 + \frac{1}{4}\right)p_{1A} + s\left(\frac{1}{4}p_{1A} + \frac{1}{2}z_0 - \frac{1}{4}\right)\frac{1}{2}(p_{1A} + 2z_0 - 1)$. The interior solution is $\frac{3+s}{2-s} - \frac{2+2s}{2-s}(1-z_0)$. One can show that when $\frac{1}{6} \leq z_0 \leq \bar{z}_0$, the interior solution can be attained and $p_{1A}^{H_1} = \frac{3+s}{2-s} - \frac{2+s}{2-s}(1-z_0)$; when $z_0 > \bar{z}_0$, $p_{1A}^{H_1} = 2z_0 + 1$; when $z_0 < \frac{1}{6}$, Third-party's price reaches the zero-lower bound, so Platform has an incentive to charge $p_{H_1} = 1 - 2z_0$ rather than the interior solution. Specifically,

$$p_{1A}^{rs,HS,H_1} = \begin{cases} 1-2z_0 & \text{if} \quad z_0 < \frac{1}{6} \\ \frac{3+s}{2-s} - \frac{2+2s}{2-s}(1-z_0) & \text{if} \quad \frac{1}{6} \leq z_0 \leq \bar{z}_0 \\ 1+2z_0 & \text{if} \quad z_0 > \bar{z}_0 \end{cases}, \quad p_{1T}^{rs,HS,H_1} = \begin{cases} 0 & \text{if} \quad z_0 < \frac{1}{6} \\ \frac{5}{4-2s} - \frac{6}{4-2s}(1-z_0) & \text{if} \quad \frac{1}{6} \leq z_0 \leq \bar{z}_0 \\ 2z_0 & \text{if} \quad z_0 > \bar{z}_0 \end{cases}, \tag{C26}$$

where $\bar{z}_0 = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq \frac{3}{4} \\ \frac{1}{-2+4s} & \text{if} \quad \frac{3}{4} \leq s \leq 1 \end{cases}.$

- Suppose that $p_{1A} \in H_2$, Platform's profit is $\pi_{1A}^{rs,HS,H_2} = s(p_{1A} - 1)z_0$, increasing in p_{1A}. Therefore, the optimal price is $p_{1A}^{H_3} = v$. Specifically,

$$p_{1A}^{rs,HS,H_2} = v, \quad p_{1T}^{rs,HS,H_2} = v - 1 \tag{C27}$$

Platform's optimal Segment-1 price is defined as follows:

$$p_{1A}^{rs,HS,\bullet} = \begin{cases} p_{1A}^{rs,HS,H_1} & \text{if} \quad \pi_{1A}^{rs,HS,H_1}\big|_{p_{1A}=p_{1A}^{rs,HS,H_1}} \geq \pi_{1A}^{rs,HS,H_2}\big|_{p_{1A}=p_{1A}^{rs,HS,H_2}} \\ p_{1A}^{rs,HS,H_2} & \text{otherwise} \end{cases}. \tag{C28}$$

Notice that while Eq. (C28) is well defined given any realization of the model primitives (v, s), solving the explicitly expression conditional of (v, s) is very involved and does not add much to our main discussion. Therefore, we employ the numerical routine as outlined in Algorithm 1 to characterize the equilibrium.

Both sellers' Segment-1 profit at equilibrium follows directly from the equilibrium prices and the corresponding demand functions, as specified in this proof.

Segment 2. Similar to the analysis for Segment 1, we solve the pricing equilibrium in segment 2 using backward induction. Conditional on the segmentation rule z_0 and Platform price p_{1A}, Third-party's demand function is as follows.

$$D_{2T}^{rs,HS} = \begin{cases} 1 - z_0 & \text{if} \quad \max\{\tilde{x}_2, \bar{x}_{2T}\} < z_0 \iff p_{2T} \in M_0 \\ 1 - \tilde{x}_2 = \frac{1}{2} + \frac{1}{2}(p_{2A} - p_{2T}) & \text{if} \quad \max\{\bar{x}_{2T}, z_0\} \leq \tilde{x}_2 \leq 1 \iff p_{2T} \in M_1 \\ 1 - \bar{x}_{2T} = v - p_{2T} & \text{if} \quad \max\{\tilde{x}_{2T}, z_0\} \leq \bar{x}_{2T} \leq 1 \iff p_{2T} \in M_2 \\ 0 & \text{if} \quad 1 \leq \min\{\tilde{x}_2, \bar{x}_{2T}\} \iff p_{2T} \in M_3 \end{cases}$$

The feasible regions for each condition are $M_0 \equiv \{p_{2T} | p_{2T} < \min\{p_{2A} + 2z_0 - 1, v + z_0 - 1\}\}$; $M_1 \equiv \{p_{2T} | p_{2A} + 2z_0 - 1 \leq p_{2T} \leq \min\{2v - p_{2A} - 1, p_{2A} + 1\}\}$; $M_2 \equiv \{p_{2T} | \max\{2v - 1 - p_{2A}, v + z_0 - 1\} \leq p_{2T} \leq v\}$; $M_3 \equiv \{p_T | p_T \geq \max\{p_{2A} + 1, v\}\}$. One can show that the case where $\max\{\tilde{x}_2, \bar{x}_{2T}\} < z_0$ is dominated by the case where $\max\{\tilde{x}_2, \bar{x}_{2T}\} = z_0$ and the case where $1 < \min\{\tilde{x}_2, \bar{x}_{2T}\}$ is dominated by the case where $\min\{\tilde{x}_2, \bar{x}_{2T}\} = 1$. Therefore, we only discuss the remaining two regions (i.e., M_1 and M_2).

Third-party's demand function is continuous and piecewise-linear on each interval, and its profit function, $\pi_{2T}^{rs,HS} = (1 - s) \cdot D_{2T}^{rs,NS} \cdot p_T$, is continuous and piecewise concave. We first obtain the optimal prices within each of the following two intervals M_1 and $M2$; among the two, the price that yields the highest profit is Third-party's Segment-2 best response price. Let $p_{2T}^{M_i}$ denote Third-party's optimal Segment-2 price within the interval M_i.

- Suppose that $p_T \in M_1$.
  - When $p_{2A} < v - z_0$, $M_1 \neq \emptyset$, the interior solution is $\frac{1}{2}(p_{2A}+1)$.
    * When $v \geq 3(1-z_0)$, one can show that if $p_{2A} \leq 3-4z_0$, the optimal price is $p_{2T}^{M_1} = \frac{1}{2}(p_{2A}+1)$; if $p_{2A} > 3-4z_0$, the optimal price is $p_{2T}^{M_1} = p_{2A}+2z_0-1$.
    * When $v < 3(1-z_0)$, one can show that if $p_{2A} \leq \frac{4}{3}v-1$, the optimal price is $p_{2T}^{M_1} = \frac{1}{2}(p_{2A}+1)$; if $p_{2A} > \frac{4}{3}v-1$, the optimal price is $p_{2T}^{M_1} = 2v-p_{2A}-1$.
  - When $p_{2A} \geq v - z_0$, $M_1 = \emptyset$.
- Suppose that $p_{2T} \in M_2$. The interior solution is $\frac{1}{2}v$. It is always true that $\frac{1}{2}v \leq v$, so we focus on discussing the left boundary.
  - When $p_{2A} \leq v - z_0$, the left boundary takes $2v - p_{2A} - 1$. It is always true that $p_{2A} \leq v - z_0 \leq \frac{3}{2}v - 1$, since $v \geq 2$. Therefore, the optimal price takes the left boundary: $p_{2T}^{M_2} = 2v - p_{2A} - 1$.
  - When $v - z_0 \leq p_{2A}$, the left boundary takes $v + z_0 - 1$. The left boundary is always violated when $v \geq 2$, so the optimal price still takes the left boundary: $p_{2T}^{M_2} = v + z_0 - 1$.

Comparing the profits corresponding to prices $p_{2T}^{M_1}$ and $p_{2T}^{M_2}$, one can show that Third-party's Segment-2 best response function is as follows.

- When $v \geq 3(1-z_0)$,

$$p_{2T}^{rs,HS,\circ}(p_{2A}) = \begin{cases} \frac{1}{2}(p_{2A}+1) & \text{if} \quad 0 \leq p_{2A} \leq 3-4z_0 \\ p_{2A}+2z_0-1 & \text{if} \quad 3-4z_0 \leq p_{2A} \leq v-z_0 \\ v+z_0-1 & \text{if} \quad v-z_0 \leq p_{2A} \leq v \end{cases} .$$

- When $v < 3(1-z_0)$,

$$p_{2T}^{rs,HS,\circ}(p_{2A}) = \begin{cases} \frac{1}{2}(p_{2A}+1) & \text{if} \quad 0 \leq p_{2A} \leq \frac{4}{3}v - 1 \\ 2v - p_{2A} - 1 & \text{if} \quad \frac{4}{3}v - 1 \leq p_{2A} \leq v - z_0 \\ v + z_0 - 1 & \text{if} \quad v - z_0 \leq p_{2A} \leq v \end{cases}.$$

Focusing on the results when $v \geq 3(1-z_0)$ (due to our focus on $v \geq 3$), we have proven Eq. (B2) in Lemma B4. □

Similar to the discussion for Segment 1, in what follows, we illustrate the attempt to solve the pricing game analytically and demonstrate the challenge of doing so for completeness. Readers can skip this part and refer to the numerical routine as outlined in Algorithm 1 for numerical results.

Using $p_{2T}^{rs,HS,\circ}$, we then solve for Platform's optimal price to maximizes its profit, i.e., $\pi_{2A}^{rs,HS} = D_{2A}^{rs,HS} p_{2A} + s \cdot R_{2T}^{\circ}(p_{2A})$, where $R_{2T}^{\circ}(p_{2A}) = D_{2T}^{rs,HS} \cdot p_{2T}^{rs,HS,\circ}(p_{2A})$. Platform's demand function is as follows.

$$D_{2A}^{rs,HS} = \begin{cases} -\frac{1}{4}p_{2A} + \frac{3}{4} - z_0 & \text{if} \quad 0 \leq p_{2A} \leq 3 - 4z_0 \\ 0 & \text{if} \quad 3 - 4z_0 \leq p_{2A} \leq v - z_0 \\ 0 & \text{if} \quad v - z_0 \leq p_{2A} \leq v \end{cases}.$$

The corresponding Third-party's demand function is:

$$D_{2T}^{rs,HS} = \begin{cases} \frac{1}{4}p_{2A} + \frac{1}{4} & \text{if} \quad 0 \leq p_{2T} \leq 3 - 4z_0 \\ 1 - z_0 & \text{if} \quad 3 - 4z_0 \leq p_{2A} \leq v - z_0 \\ 1 - z_0 & \text{if} \quad v - z_0 \leq p_{2A} \leq v \end{cases}.$$

Again, due to revenue sharing, Platform could obtain a positive profit even when its demand is 0, so we cannot eliminate the range where $3 - 4z_0 \leq p_{2A} \leq v$.

Platform's demand function is continuous and piecewise linear; its profit function is continuous and piecewise concave (including the revenue sharing portion from Third-party). We first obtain the optimal price within each of the following two intervals $N_1 \equiv (0, 3-4z_0]$ and $N_2 \equiv [3-4z_0, v]$;

among the two, the price that yields the highest profit is Platform's optimal price. Let $p_{2A}^{N_i}$ denote Platform's optimal price within the interval N_i.

- Suppose that $p_{2A} \in N_1$, Platform's profit is $\pi_{2A}^{rs,HS,N_1} = \left(-\frac{1}{4}p_{2A} + \frac{3}{4} - z_0\right) p_{2A} + s(\frac{1}{4}p_{2A} + \frac{1}{4})\frac{1}{2}(p_{2A}+1)$. The interior solution is $\frac{3+s}{2-s} - \frac{4}{2-s}z_0$. When $z_0 \le \underline{z}_0 \le \frac{3+s}{4}$, where $\underline{z}_0 = \begin{cases} \frac{3-4s}{4-4s} & \text{if} \quad 0 \le s \le \frac{3}{4} \\ 0 & \text{if} \quad \frac{3}{4} < s \le 1 \end{cases}$, the optimal solution is attained at $p_{2A}^{rs,HS,N_1} = \frac{3+s}{2-s} - \frac{4}{2-s}z_0$. When $\underline{z}_0 \le z_0 \le \frac{3}{4}$, p_{2A}^{rs,HS,N_1} reaches the right boundary, so $p_{2A}^{rs,HS,N_1} = 3 - 4z_0$. When $z_0 > \frac{3}{4}$, p_{2A}^{rs,HS,N_1} reaches the zero-lower bound, so $p_{2A}^{rs,HS,N_1} = 0$. Specifically,

$$p_{2A}^{rs,HS,N_1} = \begin{cases} \frac{3+s}{2-s} - \frac{4}{2-s}z_0 & \text{if} \quad z_0 < \underline{z}_0 \\ 3 - 4z_0 & \text{if} \quad \underline{z}_0 \le z_0 \le \frac{3}{4} \\ 0 & \text{if} \quad z_0 > \frac{3}{4} \end{cases}, \quad p_{2T}^{rs,HS,N_1} = \begin{cases} \frac{5}{4-2s} - \frac{2}{4-2s}z_0 & \text{if} \quad z_0 < \underline{z}_0 \\ 2 - 2z_0 & \text{if} \quad \underline{z}_0 \le z_0 \le \frac{3}{4} \\ 2z_0 - 1 & \text{if} \quad z_0 > \frac{3}{4} \end{cases}, \tag{C29}$$

where $\underline{z}_0 = \begin{cases} \frac{3-4s}{4-4s} & \text{if} \quad 0 \le s \le \frac{3}{4} \\ 0 & \text{if} \quad \frac{3}{4} \le s \le 1 \end{cases}$.

- Suppose that $p_{2A} \in N_2$, Platform's profit is $\pi_{2A}^{rs,HS,N_2} = s \cdot p_{2T}^{rs,HS,\circ}(p_{2A}) \cdot (1 - z_0)$, relying only on Third-party's profit sharing, with Third-party's demand is fixed. Observe that $p_{2T}^{rs,HS,\circ}(p_{2A})$ increases in p_{2A} up to $v + z_0 - 1$. Therefore, Platform is indifferent in choosing $p_{2A}^{rs,HS,N_2} \in [v - z_0, v]$. Without loss of generality, we take $p_{2A}^{rs,HS,N_2} = v - z_0$ and the corresponding $p_{2T}^{rs,HS,\circ} = v + z_0 - 1$. Specifically,

$$p_{2T}^{rs,HS,N_2} = v - z_0, \quad p_{2T}^{rs,HS,N_2} = v + z_0 - 1 \quad . \tag{C30}$$

Platform's optimal Segment-2 price is defined as follows:

$$p_{2A}^{rs,HS,\bullet} = \begin{cases} p_{2A}^{rs,HS,N_2} & \text{if} \quad \pi_{2A}^{rs,HS,N_1}\big|_{p_{2A}=p_{2A}^{rs,HS,N_1}} \ge \pi_{2A}^{rs,HS,N_2}\big|_{p_{2A}=p_{2A}^{rs,HS,N_2}} \\ p_{2A}^{rs,HS,H_2} & \text{otherwise} \end{cases} . \tag{C31}$$

Similar to the analysis in Segment 1, while Eq. (C31) is well defined given any realization of the model primitives (v, s), solving the explicitly expression conditional of (v, s) is very involved and does not add much to our main discussion. Therefore, we employ the numerical routine as outlined in Algorithm 1 to characterize the equilibrium.

Both sellers' profit at equilibrium follows directly from the equilibrium prices and the corresponding demand functions, as specified in this proof. We denote them by $\pi_{ij}^{rs,HS,\bullet}$ for $i = 1, 2$ and $j = A, T$.

Optimal Segmentation Rule Under Hard Segmentation. Platform chooses the optimal hard segmentation rule z_0 that maximizes its total profit from both segments, including both its revenue and the revenue-sharing from Third-party. Specifically, $z_0^{rs,HS,*} = \arg\max_{z_0} \; \pi_A^{rs,HS,\bullet}(z_0)$. The hard-segment equilibrium under revenue sharing is characterized by $\left(z_0^{rs,HS,*}, \mathbf{p}(z_0^{rs,HS,*})\right)$. As discussed in the derivation of the second-stage pricing equilibrium, we use a numerical routine to determine the equilibrium price by comparing two equilibrium candidates. Furthermore, we use a grid search of z_0 to illustrate the optimal hard-segmentation rule, since our goal is to determine whether there exists a non-trivial hard-segmentation rule z_0 to strictly improve the platform's profit from the no-segment benchmark profit under each pair of model primitives, (v, s).

C.4.3 Fuzzy Segmentation Results

The analysis for the fuzzy segmentation case follows the hard segmentation analysis closely, except that one more model primitive needs to be considered—w, the portion of customers choosing to be privacy-preserving. Nevertheless, the Third-party's best response function has already been derived in the proof of Propositions 2 and A1.[18]

In Lemma C6, we provide Third-party's best response prices analytically, and in Algorithm 2, we provide the numerical routine to find (1) Platform's optimal prices $p_{1A}^{rs,FS,\bullet}$ and $p_{2A}^{rs,FS,\bullet}$ conditional on a fuzzy-segmentation rule z, and compare the corresponding market outcomes across z to determine the optimal segmentation strategy.

Lemma C6 ***Fuzzy-Segmentation Third-party Best-Response Prices with Revenue-Sharing.*** *Third-party's best response price, conditional on Platform's price and a fuzzy-segmentation rule, when it pays a share* s *of its total revenue as a commission to Platform and the consumer valuation is at least* 3*, is characterized by* $(p_{1T}^{rs,FS,\circ}, p_{2T}^{rs,FS,\circ})$ *as summarized in Equations C32, C33, and C34.*

[18] Adding the revenue sharing scheme only rescale Third-party's profit by $(1-s)$. It does not impact Third-party's best response price conditional on Platform price and segmentation rule in either segment.

- *When* $\frac{w}{2+w}v \leq 1$,

$$p_{1T}^{rs,FS,\circ} = \begin{cases} \frac{1}{2}p_{1A} + \frac{1}{2} & \textit{if} \quad 0 \leq p_{1A} \leq p_a & \textit{and} \quad 0 \leq z \leq 1 \\ p_{1A} - 2z + 1 & \textit{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} & \textit{and} \quad 0 \leq z \leq 1 \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \textit{if} \quad (p_b \leq p_{1A} \leq \min\{p_d, p_c\} & \textit{and} \quad 0 \leq z \leq 1) \\ & \textit{or} \quad \big(p_c \leq p_{1A} \leq p_d & \textit{and} \quad 0 \leq z \leq z_c\big) \\ & \textit{or} \quad \big(p_c \leq p_{1A} \leq p_e & \textit{and} \quad z_c \leq z \leq \min\{z_d, 1\}\big) \\ 2v - p_{1A} - 1 & \textit{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq \min\{p_f, v\} & \textit{and} \quad z_c \leq z \leq \min\{z_d, 1\} \\ p_{1A} - 1 & \textit{if} \quad \big(p_d \leq p_{1A} \leq p_c & \textit{and} \quad 0 \leq z \leq 1\big) \\ & \textit{or} \quad \big(\max\{p_c, p_d\} \leq p_{1A} \leq v & \textit{and} \quad 0 \leq z \leq z_c\big) \end{cases} . \tag{C32}$$

where $p_a = 4z - 1$, $p_b = 2\frac{1+w}{w}z - 1$, $p_c = v + z - 1$, $p_d = 3 + 2\frac{1-w}{w}z$, $p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$, $p_f = \frac{3}{2}v - \frac{1}{2}\frac{1-w}{w}z - 1$; $z_c = \frac{1}{2}\frac{w}{1-w}(v-3)$, $z_d = \frac{w}{1+w}v$.

- *When* $1 \leq \frac{w}{2+w}v$,

$$p_{1T}^{rs,FS,\circ} = \begin{cases} \frac{1}{2}p_{1A} + \frac{1}{2} & \textit{if} \quad 0 \leq p_{1A} \leq p_a & \textit{and} \quad 0 \leq z \leq 1 \\ p_{1A} - 2z + 1 & \textit{if} \quad p_a \leq p_{1A} \leq p_b & \textit{and} \quad 0 \leq z \leq 1 \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \textit{if} \quad p_b \leq p_{1A} \leq p_d & \textit{and} \quad 0 \leq z \leq 1 \\ p_{1A} - 1 & \textit{if} \quad p_d \leq p_{1A} \leq v & \textit{and} \quad 0 \leq z \leq 1 \end{cases} . \tag{C33}$$

$$p_{2T}^{rs,FS,\circ} = \begin{cases} 0 & \textit{if} \quad 0 \leq p_{2A} \leq 2z - 1 \\ \frac{1}{2}(p_{1A} + 2z_0 - 1) & \textit{if} \quad 2z - 1 \leq p_{1A} \leq 3 - 2z \\ p_{1A} - 1 & \textit{if} \quad 3 - 2z \leq p_{1A} \leq v \end{cases} \tag{C34}$$

The pseudocode for the numerical routine is described in Algorithm 2.

Algorithm 2: Grid Search for Optimal $(z^{rs,FS,*}, \mathbf{p}^{rs,FS,*})$ Given (s, v, w).

Input : Model primitives (s, v, w); grid step sizes for z and p_A.

Construct grids $\mathcal{Z} \subset [0,1]$ and $\mathcal{P}_\mathcal{A} \subset [0, v]$

Output: Set of $(z^{rs,FS,*}, \{p_{ij}^{rs,FS,*}\})$, with $i = 1, 2$ and $j = A, T$.

Construct grids:

Choose grids $\mathcal{Z} \subset [0,1]$ and $\mathcal{P}_\mathcal{A} \subset [0, v]$

for $z \in \mathcal{Z}$ **do**

`// Initialize a grid over potential segmentation rules`

foreach $i \in \{1, 2\}$ **do**

`// Initialize a grid over potential Platform prices in Segment` i

foreach $p_{iA} \in \mathcal{P}_A$ **do**

Compute Third-party's best response price, p_{iT}, according to Lemma C6

Compute the location of the indifferent customer, $\tilde{x}_i$, $\bar{x}_{iA}$, $\bar{x}_{iT}$

Compute the demand for each seller, q_{iA} and q_{iT}

Compute the profit for each seller, $\pi_{iA} \leftarrow p_{iA} \cdot q_{iA} + s \cdot p_{iT} \cdot q_{iT}$,

$\pi_{iT} \leftarrow (1 - s) \cdot p_{iT} \cdot q_{iT}$

end

`// Check constraint` $p_{iA} < v$

Define feasible price grid:

$$\mathcal{P}_{iA} \leftarrow \{p_{iA}^{feas} \in \mathcal{P}_A : p_{iA} < v\}$$

`// Determine the optimal segment-specific Platform price,` p_{iA}

$p_{iA}^{rs,FS,\bullet}(z) \leftarrow argmax_{p_{iA} \in \mathcal{P}_{1A}^{feas}} \pi_{iA}(p_{iA}|z)$

$\pi_{iA}^{rs,FS,\bullet}(z) \leftarrow \pi_{iA}\left(p_{iA}^{rs,FS,\bullet}(z)|z\right)$

end

end

$z^* \leftarrow argmax_z \pi_{1A}^{rs,FS,\bullet}(z) + \pi_{2A}^{rs,FS,\bullet}(z)$

return $(z^{rs,FS,*}, \{p_{ij}^{rs,FS,*}\}),\ with\ i = 1, 2\ and\ j = A, T$

D Additional Technical Analysis

D.1 Proof of Proposition 2 and Proposition A1

In this appendix, we provide the proof of Proposition 2 in Section 4.2 and Proposition A1 in Appendix A. Since Proposition 2 can be viewed as a special case of Proposition A1, we prove both by focusing on the more general result in Proposition A1.

To analyze the fuzzy-segmentation equilibrium, we apply backward induction. First, we derive the second-stage subgame equilibrium in the sellers' pricing games for Segments 1 and 2. Next, we solve for Platform's first-stage optimal segmentation strategy as well as the corresponding full equilibrium for all players.

Second-Stage Pricing Equilibrium in Segment 1. Third-party chooses p_{1T} to maximize its Segment-1 profit, $\pi_{1T} = D_{1T}\, p_{1T}$, where D_{1T} is as follows:

$$D_{1T} = \begin{cases} (1-z)w + z & \text{if} \quad \max\{\tilde{x}_1, \bar{x}_{1T}\} < 0 \Longleftrightarrow p_{1T} \in \Omega_1 \\ [1 - z - \tilde{x}_1]w + z & \text{if} \quad \max\{\bar{x}_{1T}, 0\} \le \tilde{x}_1 \le 1 - z \Longleftrightarrow p_{1T} \in \Omega_2 \\ [1 - z - \bar{x}_{1A}]w + z & \text{if} \quad \max\{\tilde{x}_1, 0\} \le \bar{x}_{1T} \le 1 - z \Longleftrightarrow p_{1T} \in \Omega_3 \\ 1 - \tilde{x}_1 & \text{if} \quad \max\{1 - z, \bar{x}_{1T}\} \le \tilde{x}_1 \le 1 \Longleftrightarrow p_{1T} \in \Omega_4 \\ 1 - \bar{x}_{1T} & \text{if} \quad \max\{1 - z, \tilde{x}_1\} \le \bar{x}_{1T} \le 1 \Longleftrightarrow p_{1T} \in \Omega_5 \\ 0 & \text{if} \quad 1 < \min\{\tilde{x}_1, \bar{x}_{1T}\} \Longleftrightarrow p_{1T} \in \Omega_6 \end{cases},$$

where $\tilde{x}_1 = \frac{p_{1T} - p_{1A}}{2} + \frac{1}{2}$ represents the customer that is indifferent between Platform and Third-party, and $\bar{x}_{1T} = 1 - v + p_{1T}$ represents the customer that is indifferent between Third-party and the outside option.[19] Furthermore, the feasible regions for each condition are $\Omega_1 \equiv \{p_{1T} | p_{1T} < \min\{p_{1A} - 1, v - 1\}\}$; $\Omega_2 \equiv \{p_{1T} | p_{1A} - 1 \le p_{1T} \le \min\{2v - 1 - p_{1A}, p_{1A} - 2z + 1\}\}$; $\Omega_3 \equiv \{p_{1T} | \max\{2v - p_{1A} - 1, v - 1\} \le p_{1T} \le v - z\}$; $\Omega_4 \equiv \{p_{1T} | p_{1A} - 2z + 1 \le p_{1T} \le \min\{p_{1A} + 1, 2v - p_{1A} - 1\}\}$; $\Omega_5 \equiv \{p_{1T} | \max\{2v - p_{1A} - 1, v - z\} \le p_{1T} \le v\}$; $\Omega_6 \equiv \{p_{1T} | p_{1T} > \min\{p_{1A} + 1, v\}\}$.

One can show that the case where $\max\{\tilde{x}_1, \bar{x}_{1T}\} < 0$ is dominated by the case where $\max\{\tilde{x}_1, \bar{x}_{1T}\} =$

[19] Similarly, we can define $\bar{x}_{1A} = v - p_{1A}$ as the customer that is indifferent between Platform and the outside option. Since this is not used in Third-party's best response price derivation, we do not state it in the main text.

0 and the case where $1 < \min\{\tilde{x}_1, \bar{x}_{1T}\}$ is dominated by the case where $\min\{\tilde{x}_1, \bar{x}_{1T}\} = 1$. Therefore, we only discuss the remaining four regions (i.e., Ω_i, where $i = 2, 3, 4, 5$).

Third-party's demand function is continuous and piecewise-linear on each interval and its profit function is continuous and piecewise-concave on each interval. To find its optimal price, we first obtain the optimal prices within each of the four intervals, $\Omega_2, \Omega_3, \Omega_4$, and Ω_5; among the four, the price that yields the highest profit is Third-party's optimal price in Segment 1. To streamline our analysis, we focus only on analyzing Third-party's best response to *on-equilibrium* Platform strategies—namely, the subset of (p_{1A},z) that could arise in the final equilibrium. Specifically, this leads to the omitting of Platform's strategy where $p_{1A} < p_a$ when solving for Third-party's best response (as it would lead to "right equilibrium", where the indifferent customers lies to the right of $1-z$, which is proved to be dominated by the no-segment equilibrium. See Lemma C5 in the Online Appendix C.1.9). We summarize these *on-equilibrium* Third-party's best responses in Lemma D7 and provide the formal proof in Online Appendix D.2.

Lemma D7 ***On-Equilibrium Third-party's Best Response.*** *Third-party's optimal price in response to Platform's on-equilibrium strategy is as follows.*

- *When* $\frac{w}{2+w} v \leq 1$,

$$p_{1T}^{\circ} = \begin{cases} p_{1A} - 2z + 1 & \textit{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} \ \textit{and} \ 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \textit{if} \quad p_b \leq p_{1A} \leq \min\{p_c, p_d\} \ \textit{and} \ 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ & \quad p_c \leq p_{1A} \leq \min\{p_d, p_e\} \ \textit{and} \ 0 \leq z \leq z_c \\ & \quad p_c \leq p_{1A} \leq \min\{p_d, p_e\} \ \textit{and} \ z_c \leq z \leq \min\{z_d, \frac{1}{3}v, 1\} \\ 2v - p_{1A} - 1 & \textit{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq \min\{p_f, v\} \ \textit{and} \ z_c \leq z \leq \min\{z_d, \frac{1}{3}v, 1\}; \\ & \quad p_{1A} = p_c \ \textit{and} \ z_d < z \leq \min\{\frac{1}{3}v, 1\}; \\ & \quad p_c \leq p_{1A} \leq p_f \ \textit{and} \ \frac{1}{3}v \leq z \leq z_d \end{cases}, \tag{D35}$$

where $p_a = 4z - 1$, $p_b = 2\frac{1+w}{w}z - 1$, $p_c = v + z - 1$, $p_d = 3 + 2\frac{1-w}{w}z$, $p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$, $p_f = \frac{3}{2}v - \frac{1}{2}\frac{1-w}{w}z - 1$; $z_c = \frac{1}{2}\frac{w}{1-w}(v-3)$, $z_d = \frac{w}{1+w}v$.

- *When* $\frac{w}{2+w}v > 1$,

$$p_{1T}^{\circ} = \begin{cases} p_{1A} - 2z + 1 & \text{if} \quad p_a \le p_{1A} \le p_b \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \text{if} \quad p_b \le p_{1A} \le p_d \end{cases}. \tag{D36}$$

Notice that when $\frac{w}{2+w}v > 1$, Third-party's best response function is identical to the one without the CP constraint in Section 3.1, indicating that the CP constraint is not binding in this region. We do not repeat the proof for the analysis without the CP constraint and refer readers to Online Appendix C.1.9 for it. When $\frac{w}{2+w}v \le 1$, consumer valuation constraint could be active; therefore Third-party's best response function is more complex, incorporating consumer valuation v when $p_{1T}^{\circ} = 2v - p_{1A} - 1$. This region is our main focus when solving for Platform's optimal pricing rule.

We then solve for Platform's optimal price under a given fuzzy-segment rule z when $\frac{w}{2+w}v \le 1$. Recall from Lemma C5 in the Online Appendix C.1.9 that Platform never has an incentive to adopt strategies leading to a right subgame equilibrium. Accordingly, we need only analyze Platform strategies that would result in the potential "left subgame" equilibrium, where the indifferent customer lies at or to the left of the threshold $1 - z$. Hence, we do not need to consider Platform's pricing strategy where $p_{1A} < p_a$ in the pricing game.

Platform anticipates Third-party's best response p_{1T}° to p_{1A} (as stated in Eq. (D35)) and chooses its optimal left subgame price to maximize its profit $\pi_{1A} = D_{1A}\, p_{1A}$, where D_{1A} is as follows:

$$D_{1A} = \begin{cases} (1-z)\; w & \text{if} \quad p_{1A} \in \Lambda_1 \\ \left(-\frac{1}{4}p_{1A} + \frac{3}{4} + \frac{1}{2}\frac{1-w}{w}z\right)\; w & \text{if} \quad p_{1A} \in \Lambda_2 \\ (v - p_{1A})\; w & \text{if} \quad p_{1A} \in \Lambda_3 \end{cases}, \tag{D37}$$

where $\Lambda_1 \equiv \{p_{1A} | p_{1T}^{\circ} = p_{1A} - 2z + 1\}$, $\Lambda_2 \equiv \{p_{1A} | p_{1T}^{\circ} = \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z\}$, and $\Lambda_3 \equiv \{p_{1A} | p_{1T}^{\circ} = 2v - p_{1A} - 1\}$.Notice that Λ_i are, in fact, functions of z, but we omit this in the notice for conciseness. We will first find Platform's optimal price $p_{1A}^{\Lambda_i}$ within each of the sets Λ_i. The price corresponding to the highest profit will be Platform's left-subgame equilibrium price in Segment 1. We use a triangle superscript to denote the optimal solution to the profit maximization problem in the left subgame.

- Suppose the $p_{1A} \in \Lambda_1$. Platform's profit increases in p_{1A}, so the optimal choice of p_{1A} should be the largest one in the feasible region. Since it coincides with the left boundary of Λ_2, this scenario is weakly dominated and can be eliminated.

- Suppose the $p_{1A} \in \Lambda_3$. The optimal solution is always the left boundary solution, as the interior solution $\frac{v}{2} \leq v+z-1 = p_c \leq \max\{p_c, p_e\}$ for all z when $v \geq 2$. Therefore, the optimal solution coincides with the right boundary of Λ_2, so this scenario is weakly dominated and can be eliminated.

Consequently, we focus on discussing the interval Λ_2. The objective function is concave and the interior solution is $p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w} z$. The boundary solutions for this interval depends on the values of model primitives (v, w) as well as the segmentation rule z. We discuss them separately below.

1. When $\frac{w}{2+w} v > 1$ and $0 \leq z \leq 1$, the constraint is $p_b \leq p_{1A} \leq p_d$. Therefore, the optimal solution is

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w} z & \text{if} \quad 0 \leq z \leq \frac{5w}{2+6w} \\ p_b = 2\frac{1+w}{w} z - 1 & \text{if} \quad \frac{5w}{2+6w} \leq z \leq 1 \end{cases} \tag{D38}$$

2. When $\frac{w}{2+w} v \leq 1$ and $0 \leq z \leq z_c$, the constraint is also $p_b \leq p_{1A} \leq p_d$. Furthermore, when $v \leq \frac{8+4w}{1+3w} \leq \frac{2+w}{w}$, we have $z \leq z_c \leq \frac{5w}{2+6w}$. Therefore,

   (a) when $v \leq \frac{8+4w}{1+3w} \leq \frac{2+w}{w}$, the optimal solution is

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w} z & \text{if} \quad 0 \leq z \leq z_c \end{cases} \tag{D39}$$

   (b) when $\frac{8+4w}{1+3w} \leq v \leq \frac{2+w}{w}$, the optimal solution is

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w} z & \text{if} \quad 0 \leq z \leq \frac{5w}{2+6w} \\ p_b = 2\frac{1+w}{w} z - 1 & \text{if} \quad \frac{5w}{2+6w} \leq z \leq z_c \end{cases} \tag{D40}$$

3. When $\frac{w}{2+w} v \leq 1$ and $z_c \leq z \leq \min\{z_d, \frac{1}{3} v, 1\}$, the constraint is $p_b \leq p_{1A} \leq p_e$, which is nonempty when $z \leq \frac{w}{2+w} v$. The left constraint holds when $z \leq \frac{5w}{2+6w}$ and the right constraint holds when $z \leq \frac{w}{1-w}(\frac{4}{5} v - \frac{3}{2})$.

(a) When $2 \leq v \leq \frac{10+5w}{2+6w}$, we have $\frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2}) \leq \frac{w}{2+w}v \leq \frac{5w}{2+6w}$. Therefore,

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda} = \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad z_c \leq z \leq \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2}) \\ p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1 & \text{if} \quad \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2}) \leq z \leq \frac{w}{2+w}v \end{cases} \tag{D41}$$

(b) When $\frac{10+5w}{2+6w} \leq v \leq \frac{2+w}{w}$, we have $\frac{5w}{2+6w} \leq \frac{w}{2+w}v \leq \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2})$. Therefore,

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda} = \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad z_c \leq z \leq \frac{5a}{2+6a} \\ p_b = 2\frac{1+a}{a}z - 1 & \text{if} \quad \frac{5a}{2+6a} \leq z \leq \frac{a}{2+a}v \end{cases} \tag{D42}$$

Combining Equations (D38)–(D42), we have the full solution for Platform's optimal pricing in the left subgame.

1. When $2 \leq v \leq \frac{10+5w}{2+6w}$,

$$p_{1A}^{\triangle}(z) = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad 0 \leq z \leq z_e \\ p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1 & \text{if} \quad z_e \leq z \leq \frac{w}{2+w}v \end{cases} \tag{D43}$$

2. When $\frac{10+5w}{2+6w} < v \leq \frac{2+w}{w}$,

$$p_{1A}^{\triangle}(z) = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad 0 \leq z \leq \frac{5w}{2+6w} \\ p_b = 2\frac{1+w}{w}z - 1 & \text{if} \quad \frac{5w}{2+6w} \leq z \leq \frac{w}{2+w}v \end{cases} \tag{D44}$$

3. When $v \geq \frac{2+w}{w}$

$$p_{1A}^{\triangle} = \begin{cases} p_{1A}^{int,\Lambda_2} = \frac{3}{2} + \frac{1-w}{w}z & \text{if} \quad 0 \leq z \leq \frac{5w}{2+6w} \\ p_b = 2\frac{1+w}{w}z - 1 & \text{if} \quad \frac{5w}{2+6w} \leq z \leq 1 \end{cases} \tag{D45}$$

Plugging $p_{1A}^{\triangle}$ into the corresponding expression for Third-party's best response price in Segment 1, we derive the on-equilibrium pricing equilibrium for both sellers in Segment 1, as shown in Table A16.

Second-Stage Pricing Equilibrium in Segment 2. Since the competition is intensified in

Table A16: Sellers' Optimal Prices Under Fuzzy Segment z, When v Is General

Segment	Case	Platform Price	Third-Party Price	Range of z	Range of v
Segment 1	1	$p_{1A}^{\triangle} = \frac{3}{2} + \frac{1-w}{w}z$ $= \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$	$p_{1T}^{\triangle} = \frac{5}{4} + \frac{3}{2}\frac{1-w}{w}z$ $= \frac{2}{3}v + \frac{2}{3}\frac{1-w}{w}z$	$0 < z < z_e$ $z_e \le z \le z_f$	$2 \le v \le v_1$
	2	$p_{1A}^{\triangle} = \frac{3}{2} + \frac{1-w}{w}z$ $= 2\frac{1+w}{w}z - 1$	$p_{1T}^{\triangle} = \frac{5}{4} + \frac{3}{2}\frac{1-w}{w}z$ $= 2\frac{1}{w}z$	$0 < z < z_a$ $z_a \le z \le z_f$	$v_1 < v \le v_2$
	3	$p_{1A}^{\triangle} = \frac{3}{2} + \frac{1-w}{w}z$ $= 2\frac{1+w}{w}z - 1$	$p_{1T}^{\triangle} = \frac{5}{4} + \frac{3}{2}\frac{1-w}{w}z$ $= 2\frac{1}{w}z$	$0 < z < z_a$ $z_a \le z \le 1$	$v > v_2$
Segment 2	n.a.	$p_{2A}^{\bullet} = \frac{3}{2} - z$ $p_{2A}^{\bullet} = 2z - 1$	$p_{2T}^{\bullet} = \frac{5}{4} - \frac{3}{2}z$ $p_{2T}^{\bullet} = 0$	$0 < z < \frac{5}{6}$ $\frac{5}{6} \le z \le 1$	$v \ge 2$

Notes:
1. $z_a = \frac{5w}{2+6w}$, $z_e = \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2})$, $z_f = \frac{w}{2+w}v$, $v_1 = \frac{10+5w}{2+6w}$ and $v_2 = \frac{2+w}{w}$.
2. The triangle superscript in Segment 1 indicate that the optimal prices are for left subgame only.

Segment 2, the CP constraint is not binding and, hence, does not effectively impact the pricing equilibrium, as long as $v \ge 2$. Therefore, the pricing equilibrium in Segment 2 is identical to those in Section 4.2. The results are summarized in Table A16.

The corresponding seller profit in each segment is summarized in Table A17.

Table A17: Sellers' Optimal Profits Under Fuzzy Segment z, When v Is General

Segment	Case	Platform Profit	Third-Party Profit	Range of z	Range of v
Segment 1	1	$\pi_{1A}^{\triangle} = A_1$ $= A_2$	$\pi_{1T}^{\triangle} = B_1$ $= B_2$	$0 < z < z_e$ $z_e \le z \le z_f$	$2 \le v \le v_1$
	2	$\pi_{1A}^{\triangle} = A_1$ $= A_3$	$\pi_{1T}^{\triangle} = B_1$ $= B_3$	$0 < z < z_a$ $z_a \le z \le z_f$	$v_1 < v \le v_2$
	3	$\pi_{1A}^{\triangle} = A_1$ $= A_3$	$\pi_{1T}^{\triangle} = B_1$ $= B_3$	$0 < z < z_a$ $z_a \le z \le 1$	$v > v_2$
Segment 2	n.a.	$\pi_{2A}^{\bullet} = C_1$ $\pi_{2A}^{\bullet} = C_2$	$\pi_{2T}^{\bullet} = D_1$ $\pi_{2T}^{\bullet} = D_2$	$0 < z < \frac{5}{6}$ $\frac{5}{6} \le z \le 1$	$v \ge 2$

Notes:
[1] $z_a = \frac{5w}{2+6w}$, $z_e = \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2})$, $z_f = \frac{w}{2+w}v$, $\bar{z} = \min\{\frac{1}{3}v, 1\}$; $v_1 = \frac{10+5w}{2+6w}$ and $v_2 = \frac{2+w}{w}$.
[2] The triangle superscript in Segment 1 indicate that the optimal profits are for left subgame only.
[3] $A_1 = \frac{1}{4}w\left(\frac{3}{2} + \frac{1-w}{w}z\right)^2$, $B_1 = \frac{1}{2}w\left(\frac{5}{4} + \frac{3}{2}\frac{1-w}{w}z\right)^2$.
[4] $A_2 = w\left(\frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1\right)\left(\frac{2}{3}\frac{1-w}{w}z - \frac{1}{3}v + 1\right)$, $B_2 = \frac{1}{2}w\left(\frac{2}{3}v + \frac{2}{3}\frac{1-w}{w}z\right)^2$.
[5] $A_3 = w(1-z)\left(2\frac{1+w}{w}z - 1\right)$, $B_3 = 2\frac{1}{w}z^2$.
[6] $C_1 = \frac{1}{4}(1-w)(\frac{3}{2} - z)^2$, $D_1 = \frac{1}{2}(1-w)(\frac{5}{4} - \frac{3}{2}z)^2$.
[7] $C_2 = (1-w)(2z-1)(1-z)$, $D_2 = 0$.

First-Stage Optimal Segmentation Rule and Full Equilibrium. We consider the three ranges of model primitives: (1) $2 \le v < v_1$ (or $0 < w < \frac{10-2v}{6v-5}$), (2) $v_1 \le v < v_2$ (or $\frac{10-2v}{6v-5} \le w < \frac{2}{v-1}$), and (3) $v \le v_2$ (or $\frac{2}{v-1} \le w < 1$), respectively. In each range, Platform finds the optimal segmentation rule $z^{FS,*}$ that maximizes its total profit in two segments, i.e., $z^{FS,*} \equiv$

$\arg\max_z \pi_{1A}^{FS} + \pi_{2A}^{FS}$, where $\pi_{\cdot A}^{FS}$ is defined in Table A17. Since Platform's objective function is a continuous piecewise function (but not necessarily concave), we first find the optimal segmentation rule z within each interval and then find the segmentation rule that yields the highest profit among the candidates. Specifically, these intervals are:

When $2 \leq v < v_1$,[20] define: $S_1 := (0,\, z_e]$ and $S_2 := [z_e,\, z_f]$.

When $v_1 \leq v \leq v_2$,[21] define: $T_1 := (0,\, z_a],\ T_2 := [z_a,\, \min\{\, z_f,\, \frac{5}{6}\}],\ T_3 := [\frac{5}{6},\, z_f] \cap [\frac{5}{6},\, 1]$.

When $v > v_2$, define: $W_1 := (0,\, z_a],\ W_2 := [z_a,\, \frac{5}{6}],\ W_3 := [\frac{5}{6},\, 1]$.

Let z^{I_j} denote Platform's optimal segmentation rule within the interval I_j, where $I \in \{S, T, W\}$ and $j \in \{1, 2, 3\}$.

First, observe that some intervals can be omitted from formal analysis. Specifically, when $z \in S_1$ and $2 \leq v < v_1$, $z \in T_1$ and $v_1 \leq v \leq v_2$, or $z \in W_1$ and $v > v_2$, $p_{1A}^{\triangle} = \frac{3}{2} + \frac{1-w}{w}z$, $p_{1T}^{\triangle} = \frac{5}{4} + \frac{3}{2}\frac{1-w}{w}z$, $p_{2A}^{\bullet} = \frac{3}{2} - z$, and $p_{2T}^{\bullet} = \frac{5}{4} - \frac{3}{2}z$. The total profit of Platform is $\frac{1}{4}w\left(\frac{3}{2} + \frac{1-w}{w}z\right)^2 + \frac{1}{4}(1-w)\left(z - \frac{3}{2}\right)^2$. Since the first-order condition is positive when $z > 0$, the optimal segmentation rule is the largest z in each range, coinciding with the left boundary of the next range in each scenario. Therefore, we omit these intervals when solving for optimal segmentation.

Next, we analyze the remaining z-intervals under each of the three parameter ranges (defined by v or, equivalently, w) separately.

Range 1: When $2 \leq v < v_1$ (or $w < \frac{10-2v}{6v-5}$).

- Consider $z \in S_2$. Recall from Table A16 that the equilibrium prices are $p_{1A}^{\triangle} = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$, $p_{1T}^{\triangle} = \frac{2}{3}v + \frac{2}{3}\frac{1-w}{w}z$, $p_{2A}^{\bullet} = \frac{3}{2} - z$, and $p_{2T}^{\bullet} = \frac{5}{4} - \frac{3}{2}z$. The feasible region is $\frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2}) \leq z \leq \frac{w}{2+w}v$. Platform's profit function is $w(\frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1)(\frac{2}{3}\frac{1-w}{w}z - \frac{1}{3}v + 1) + \frac{1}{4}(1-w)(\frac{3}{2} - z)^2$. When $w < \frac{16}{25}$, the objective function is concave. Specifically, when $w \leq \frac{50-16v}{30v-25}$, $z^{S_2} = \frac{(40v-75)w}{32-50w}$; when $\frac{50-16v}{30v-25} \leq w \leq \frac{16}{25}$, $z^{S_2} = \frac{w}{2+w}v$. When $w = \frac{16}{25}$, the objective function is strictly increasing in z as the first order condition is positive. Therefore, $z^{S_2} = \frac{w}{2+w}v$. When $w > \frac{16}{25}$, the objective function is convex. The optimal segmentation rule is $z^{S_2} = \frac{w}{2+w}v$.

[20] When $2 \leq v < v_1$ (or $w < \breve{w}'$), it follows that $z_f \leq \frac{5}{6}$.
[21] Note that $z_a \leq \frac{5}{6}$ always holds.

In sum, the optimal segmentation rule in this parameter range is

$$z^{FS,*} = \begin{cases} \frac{(40v-75)w}{32-50w} & \text{if} \quad 0 < w \leq \max\{\frac{50-16v}{30v-25}, 0\} \\ \frac{w}{2+w}v & \text{if} \quad \max\{\frac{50-16v}{30v-25}, 0\} \leq w \leq \frac{10-2v}{6v-5} \end{cases} \tag{D46}$$

Range 2: When $v_1 \leq v \leq v_2$ (or $\frac{10-2v}{6v-5} \leq w \leq \frac{2}{v-1}$).

- Suppose $z \in T_2$. Recall from Table A16 that the equilibrium prices are $p_{1A}^{\triangle} = 2\frac{1+w}{w}z - 1$, $p_{1T}^{\triangle} = 2\frac{1}{w}z$, $p_{2A}^{\bullet} = \frac{3}{2} - z$, and $p_{2T}^{\bullet} = \frac{5}{4} - \frac{3}{2}z$. The feasible region is $\frac{5w}{2+6w} \leq z \leq \min\{z_f, \frac{5}{6}\}$. Platform's profit function is $w(2\frac{1+w}{w}z - 1)(1-z) + \frac{1}{4}(1-w)(\frac{3}{2} - z)^2$. The interior solution is $\frac{5+15w}{14+18w}$. When $\frac{w}{2+w}v \leq \frac{5+15w}{14+18w}$ (or $w \leq \breve{w}$, where $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)}$), the optimal solution is $z^{T_2} = \frac{w}{2+w}v$; when $\frac{w}{2+w}v > \frac{5+15w}{14+18w}$ (or $w < \breve{w}$), the optimal solution is $z^{T_2} = \frac{5+18w}{14+18w}$.

- Suppose $T_3 \neq \emptyset$ and $z \in T_3$. Recall from Table A16 that the equilibrium prices are $p_{1A}^{\triangle} = 2\frac{1+w}{w}z - 1$, $p_{1T}^{\triangle} = 2\frac{1}{w}z$, $p_{2A}^{\bullet} = 2z - 1$, and $p_{2T}^{\bullet} = 0$. The feasible region is $\frac{5}{6} \leq z \leq 1$. Platform's profit function is $w(2\frac{1+w}{w}z - 1)(1-z) + (1-w)(2z-1)(1-z)$. The optimal solution is $z^{T_3} = \frac{5}{6}$.

Comparing Platform's profit under z^{T_2} and z^{T_3}, the optimal segmentation rule in this parameter range is

$$z^{FS,*} = \begin{cases} \frac{w}{2+w}v & \text{if} \quad \frac{10-2v}{6v-5} \leq w \leq \breve{w} \\ \frac{5+15w}{14+18w} & \text{if} \quad \breve{w} < w \leq \frac{2}{v-1} \end{cases} \tag{D47}$$

Range 3: When $v > v_2$ (or $\frac{2}{v-1} \leq w \leq 1$). The analysis is identical to those in Section C.1.5, except that we did not explicitly consider the right subgame in Table A16. Therefore,

$$z^{FS,*} = \frac{5+15w}{14+18w} \quad \text{if} \quad \frac{2}{v-1} \leq w < 1 \tag{D48}$$

Combining these results for the three ranges of v, we have

$$z^{FS,*} = \begin{cases} \frac{(40v-75)w}{32-50w} & \text{if} \quad 0 < w \leq \breve{w}' \\ \frac{w}{2+w}v & \text{if} \quad \breve{w}' \leq a \leq \breve{w} \\ \frac{5+15w}{14+18w} & \text{if} \quad \breve{w} \leq w < 1 \end{cases}, \tag{D49}$$

where $\breve{w}' = \max\{\frac{50-16v}{30v-25}, 0\}$ and $\breve{w} = \frac{35-14v+\sqrt{196v^2-260v+625}}{2(18v-15)}$.

Substituting in the equilibrium prices in Table A16, we derive the full equilibrium $(\mathbf{z}^*, \mathbf{p}^*)$ in Proposition A1 in Appendix A.

When we consider only the region where $v \geq 3$ and $w \geq 0.1$, we no longer have the range where $0 < w \leq \breve{w}'$ and we are only left with the second and third case of Equation (D49). This leads us to the results of Proposition 2. $\square$

D.2 Proof of Lemma D7

We start with Third-party's demand function in Eq. (D34), which is stated again below.

$$D_{1T} = \begin{cases} [1-z-\tilde{x}_1]w + z & \text{if} \quad \max\{\bar{x}_{1T}, 0\} \leq \tilde{x}_1 \leq 1-z \iff p_{1T} \in \Omega_2 \\ [1-z-\bar{x}_{1A}]w + z & \text{if} \quad \max\{\tilde{x}_1, 0\} \leq \bar{x}_{1T} \leq 1-z \iff p_{1T} \in \Omega_3 \\ 1-\tilde{x}_1 & \text{if} \quad \max\{1-z, \bar{x}_{1T}\} \leq \tilde{x}_1 \leq 1 \iff p_{1T} \in \Omega_4 \\ 1-\bar{x}_{1T} & \text{if} \quad \max\{1-z, \tilde{x}_1\} \leq \bar{x}_{1T} \leq 1 \iff p_{1T} \in \Omega_5 \end{cases}, \tag{D50}$$

where $\tilde{x}_1 = \frac{p_{1T}-p_{1A}}{2} + \frac{1}{2}$ represents the customer that is indifferent between Platform and Third-party and $\bar{x}_{1T} = 1 - v + p_{1T}$ represents the customer that is indifferent between Third-party and the outside option. Furthermore, the feasible regions for each condition are $\Omega_2 \equiv \{p_{1T}|p_{1A} - 1 \leq p_{1T} \leq \min\{2v-1-p_{1A}, p_{1A}-2z+1\}\}$; $\Omega_3 \equiv \{p_{1T}|\max\{2v-p_{1A}-1, v-1\} \leq p_{1T} \leq v-z\}$; $\Omega_4 \equiv \{p_{1T}|p_{1A}-2z+1 \leq p_{1T} \leq \min\{p_{1A}+1, 2v-p_{1A}-1\}\}$; $\Omega_5 \equiv \{p_{1T}|\max\{2v-p_{1A}-1, v-z\} \leq p_{1T} \leq v\}$.

Third-party's profit function is

$$\pi_{1T}(p_{1T}|p_{1A}, z; w, v) = D_{1T}(p_{1T}) \cdot p_{1T}.$$

The profit function is continuous and piecewise concave in each of the four intervals. In addition, each interval is closed and convex. Therefore, there exists an optimal price for each interval. Among the four optimal price candidates, the one that generates the highest profit is the overall best response price. Since the intervals are compact and the profit function is concave in each interval, the optimal solution of each subproblem is either the interior solution (by solving the first order condition) or one of the boundary solutions. In the following paragraphs, we first solve the four individual constrained optimization problems and then compare the four candidates to choose the one that generates the highest profit.

To simplify the later discussion, we state a few notations up front. They are as follows.

- $p_a = 4z - 1$, $p_b = 2\frac{1+w}{w}z - 1$, $p_c = v + z - 1$, $p_d = 3 + 2\frac{1-w}{w}z$, $p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$, and $p_f = \frac{3}{2}v - \frac{1}{2}\frac{1-w}{w}z - 1$.
- $z_c = \frac{1}{2}\frac{w}{1-w}(v - 3)$, $z_d = \frac{w}{1+w}v$, $z_e = \frac{w}{1-w}(\frac{4}{5}v - \frac{3}{2})$, and $z_f = \frac{w}{2+w}v$.

First, we solve Third-party's profit maximization problem on each interval.

Suppose $p_{1T} \in \Omega_2$, the interior solution to Third-party's profit maximization problem is $p_{1T}^{\Omega_2,int} = \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z$; the left boundary solution is $p_{1T}^{\Omega_2,lbd} = p_{1A} - 1$ and the right boundary solution candidates are $p_{1T}^{\Omega_2,rbd-1} = 2v - p_{1A} - 1$ and $p_{1T}^{\Omega_2,rbd-2} = p_{1A} - 2z + 1$, depending on which value is smaller. Specifically, when $p_{1A} \leq p_c$, $p_{1T}^{\Omega_2,rbd-2} \leq p_{1T}^{\Omega_2,rbd-1}$, so the feasible region is simplified to $p_{1T}^{\Omega_2,lbd} \leq p_{1T} \leq p_{1T}^{\Omega_2,rbd-2}$. When $p_{1A} \geq p_c$, the inequality flips, so the feasible region is simplified to $p_{1T}^{\Omega_2,lbd} \leq p_{1T} \leq p_{1T}^{\Omega_2,rbd-1}$. Third-party's best response price in this interval is

$$
p_{1T}^{\Omega_2,*} = \begin{cases}
p_{1T}^{\Omega_2,rbd-2} & \text{if} \quad 0 \le p_{1A} \le \min\{p_b, p_c\} & \text{and} \quad 0 \le z \le 1 \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_b \le p_{1A} \le \min\{p_d, p_c\} & \text{and} \quad 0 \le z \le 1 \\
p_{1T}^{\Omega_2,lbd} & \text{if} \quad p_d \le p_{1A} \le p_c & \text{and} \quad 0 \le z \le 1 \\
--- & --\text{(split at } p_c\text{)}-- & \text{and} \quad ----- \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \le p_{1A} \le p_d & \text{and} \quad 0 \le z \le z_c \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \le p_{1A} \le p_e & \text{and} \quad z_c \le z \le 1 \\
p_{1T}^{\Omega_2,lbd} & \text{if} \quad \max\{p_c, p_d\} \le p_{1A} \le v & \text{and} \quad 0 \le z \le z_c \\
p_{1T}^{\Omega_2,rbd-1} & \text{if} \quad \max\{p_c, p_e\} \le p_{1A} \le v & \text{and} \quad z_c \le z \le 1
\end{cases} . \tag{D51}
$$

Notice that $p_d < p_e$ when $z < z_c$ and $p_d > p_e$ when $z > z_c$.

Suppose $p_{1T} \in \Omega_3$, the interior solution to Third-party's profit maximization problem is $p_{1T}^{\Omega_3,int} = \frac{1}{2}v + \frac{1}{2}\frac{1-w}{w}z$. Since $v - 1 \le 2v - p_{1A} - 1$ always hold when $p_{1A} \le v$, the left boundary solution is $p_{1T}^{\Omega_3,lbd} = 2v - p_{1A} - 1$ and the right boundary solution is $p_{1T}^{\Omega_3,rbd} = v - z$. The feasible region is simplified to $p_{1T}^{\Omega_3,lbd} \le p_{1T} \le p_{1T}^{\Omega_3,rbd}$. Notice that when $p_{1A} < p_c$, $p_{1T}^{\Omega_3,rbd} < p_{1T}^{\Omega_3,lbd}$, and the feasible region is empty. Third-party's best response price in this interval is

$$
p_{1T}^{\Omega_3,*} = \begin{cases}
\text{NA} & \text{if} \quad p_{1A} < p_c & \text{and} \quad 0 \le z \le 1 \\
p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le v & \text{and} \quad 0 \le z \le \frac{w}{1-w}(v-2) \\
p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le p_f & \text{and} \quad \frac{w}{1-w}(v-2) \le z \le z_d \\
p_{1T}^{\Omega_3,int} & \text{if} \quad p_f \le p_{1A} \le v & \text{and} \quad \frac{w}{1-w}(v-2) \le z \le z_d \\
p_{1T}^{\Omega_3,rbd} & \text{if} \quad p_c \le p_{1A} \le v & \text{and} \quad z_d \le z \le 1
\end{cases} , \tag{D52}
$$

when $\frac{w}{1+w}v < 1$, so that $\frac{w}{1-w}(v-2) < z_d < 1$; or

$$p_{1T}^{\Omega_3,*} = \begin{cases} \text{NA} & \text{if} \quad p_{1A} < p_c \quad \text{and} \quad 0 \le z \le 1 \\ p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le v \quad \text{and} \quad 0 \le z \le 1 \end{cases}, \tag{D53}$$

when $\frac{w}{1+w}v \ge 1$, so that $1 \le z_d \le \frac{w}{1-w}(v-2)$.

Suppose $p_{1T} \in \Omega_4$, the interior solution to Third-party's profit maximization problem is $p_{1T}^{\Omega_4,int} = \frac{1}{2}p_{1A} + \frac{1}{2}$. Since $p_{1T}^{\Omega_4,int} \le p_{1A} + 1$ always hold, we only consider the remaining right boundary. The left boundary solution is $p_{1T}^{\Omega_4,lbd} = p_{1A} - 2z + 1$ and the right boundary solution is $p_{1T}^{\Omega_4,rbd} = 2v - p_{1A} - 1$. Notice that when $p_{1A} > p_c$, $p_{1T}^{\Omega_4,rbd} < p_{1T}^{\Omega_4,lbd}$, and the feasible region is empty. Third-party's best response price in this interval is

$$p_{1T}^{\Omega_4,*} = \begin{cases} p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \le p_{1A} \le \min\{p_a, \frac{4}{3}v - 1\} \quad \text{and} \quad 0 \le z \le 1 \\ p_{1T}^{\Omega_4,lbd} & \text{if} \quad p_a \le p_{1A} \le p_c \quad \text{and} \quad 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,rbd} & \text{if} \quad \frac{4}{3}v - 1 \le p_{1A} \le p_c \quad \text{and} \quad \min\{\frac{1}{3}v, 1\} \le z \le 1 \\ \text{NA} & \text{if} \quad p_c < p_{1A} \quad \text{and} \quad 0 \le z \le 1 \end{cases}, \tag{D54}$$

where $\min\{p_a, \frac{4}{3}v - 1\} = \begin{cases} p_a & \text{if } 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ \frac{4}{3}v - 1 & \text{if } \min\{\frac{1}{3}v, 1\} \le z \le 1 \end{cases}.$

Suppose $p_{1T} \in \Omega_5$, the interior solution to Third-party's profit maximization problem is $p_{1T}^{\Omega_5,int} = \frac{1}{2}v$. The left boundary solution candidates are $p_{1T}^{\Omega_5,lbd-1} = 2v - p_{1A} - 1$ and $p_{1T}^{\Omega_5,lbd-2} = v - z$, depending on which one is greater, and the right boundary solution is $p_{1T}^{\Omega_5,rbd} = v$. Notice that $p_{1T}^{\Omega_5,int} \le p_{1T}^{\Omega_5,rbd}$ always hold, but $p_{1T}^{\Omega_5,int} > p_{1T}^{\Omega_5,lbd-2}$ is always violated. Therefore, the optimal solution is always the left boundary solution (when the feasible region is non-empty). Third-party's best response price in this interval is

$$p_{1T}^{\Omega_5,*} = \begin{cases} \text{NA} & \text{if} \quad 0 \leq p_{1A} < v-1 \quad \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_5,lbd-1} & \text{if} \quad v-1 \leq p_{1A} \leq p_c \quad \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_5,lbd-2} & \text{if} \quad p_c \leq p_{1A} \leq v \quad \text{and} \quad 0 \leq z \leq 1 \end{cases} . \tag{D55}$$

Next, we compare the optimal solutions from the four intervals and determine which one generates the highest profit for Third-party. This comparison depends on the relative values of z_c, z_d, z_f, and 1, which depend on the parameter space. When $\frac{w}{1+w}v \leq 1$, one can show that $z_c \leq z_d \leq 1$; when $\frac{w}{2+w}v \leq 1 \leq \frac{w}{1+w}v$, one can show that $z_c \leq z_f \leq 1 \leq z_d$; when $1 \leq \frac{w}{2+w}v$, one can show that $1 \leq z_f \leq z_c$. We approach this comparison by first combining $p_{1T}^{\Omega_4,*}$ and $p_{1T}^{\Omega_5,*}$, then adding $p_{1T}^{\Omega_3,*}$ into comparison, and lastly adding $p_{1T}^{\Omega_2,*}$ into comparison. We use the property of concave functions and the fact that the four intervals are connected. As an intermediary step, we state the optimal solution from $\Omega_3 \cup \Omega_4 \cup \Omega_5$ as follows.

- When $\frac{w}{1+w}v < 1$

$$p_{1T}^{\Omega_3\cup\Omega_4\cup\Omega_5,*} = \begin{cases} p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \le p_{1A} \le p_a & \text{and} \quad 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,lbd} & \text{if} \quad p_a \le p_{1A} \le p_c & \text{and} \quad 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \le p_{1A} \le \frac{4}{3}v - 1 & \text{and} \quad \frac{1}{3}v \le z \le 1 \\ p_{1T}^{\Omega_4,rbd} & \text{if} \quad \frac{4}{3}v - 1 \le p_{1A} \le p_c & \text{and} \quad \frac{1}{3}v \le z \le 1 \\ --- & --\,(\text{split at } p_c)\,-- & \text{and} \quad ----- \\ p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le v & \text{and} \quad 0 \le z \le \frac{w}{1-w}(v-2) \\ p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le p_f & \text{and} \quad \frac{w}{1-w}(v-2) \le z \le z_d \\ p_{1T}^{\Omega_3,int} & \text{if} \quad p_f \le p_{1A} \le v & \text{and} \quad \frac{w}{1-w}(v-2) \le z \le z_d \\ p_{1T}^{\Omega_3,rbd} & \text{if} \quad p_c \le p_{1A} \le v & \text{and} \quad z_d \le z \le 1 \end{cases} . \quad \text{(D56)}$$

- When $\frac{w}{1+w}v \ge 1$

$$p_{1T}^{\Omega_3\cup\Omega_4\cup\Omega_5,*} = \begin{cases} p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \le p_{1A} \le p_a & \text{and} \quad 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,lbd} & \text{if} \quad p_a \le p_{1A} \le p_c & \text{and} \quad 0 \le z \le \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \le p_{1A} \le \frac{4}{3}v - 1 & \text{and} \quad \frac{1}{3}v \le z \le 1 \\ p_{1T}^{\Omega_4,rbd} & \text{if} \quad \frac{4}{3}v - 1 \le p_{1A} \le p_c & \text{and} \quad \frac{1}{3}v \le z \le 1 \\ --- & --\,(\text{split at } p_c)\,-- & \text{and} \quad ----- \\ p_{1T}^{\Omega_3,lbd} & \text{if} \quad p_c \le p_{1A} \le v & \text{and} \quad 0 \le z \le 1 \end{cases} . \quad \text{(D57)}$$

Lastly, we compared the optimal solution in Ω_2 (Eq. D51) and $\Omega_3 \cup \Omega_4 \cup \Omega_5$ (Eq. D56 or D57). We discuss the three ranges $\frac{w}{2+w}v \le \frac{w}{1+w}v \le 1$, $\frac{w}{2+w}v \le 1 \le \frac{w}{1+w}v$, and $1 \le \frac{w}{2+w}v \le \frac{w}{1+w}v$, respectively.

- When $\frac{w}{2+w}v \leq \frac{w}{1+w}v \leq 1$,[22]

$$
p_{1T}^{FS,\circ} = \begin{cases}
p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \leq p_{1A} \leq p_a & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\
p_{1T}^{\Omega_2,rbd-2} & \text{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_b \leq p_{1A} \leq \min\{p_d, p_c\} & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\
p_{1T}^{\Omega_2,lbd} & \text{if} \quad p_d \leq p_{1A} \leq p_c & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\
p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \leq p_{1A} \leq \frac{4}{3}v - 1 & \text{and} \quad \frac{1}{3}v \leq z \leq 1 \\
p_{1T}^{\Omega_4,rbd} & \text{if} \quad \frac{4}{3}v - 1 \leq p_{1A} \leq p_c & \text{and} \quad \frac{1}{3}v \leq z \leq 1 \\
--- & --\text{(split at } p_c\text{)}-- & \text{and} \quad ----- \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_d & \text{and} \quad 0 \leq z \leq z_c \\
p_{1T}^{\Omega_2,lbd} & \text{if} \quad \max\{p_c, p_d\} \leq p_{1A} \leq v & \text{and} \quad 0 \leq z \leq z_c \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_e & \text{and} \quad z_c \leq z \leq \frac{w}{1-w}(v-2) \\
p_{1T}^{\Omega_2,rbd-1} & \text{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq v & \text{and} \quad z_c \leq z \leq \frac{w}{1-w}(v-2) \\
p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_e & \text{and} \quad \frac{w}{1-w}(v-2) \leq z \leq z_d \\
p_{1T}^{\Omega_2,rbd-1} & \text{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq p_f & \text{and} \quad \frac{w}{1-w}(v-2) \leq z \leq z_d \\
p_{1T}^{\Omega_3,int} & \text{if} \quad p_f \leq p_{1A} \leq v & \text{and} \quad \frac{w}{1-w}(v-2) \leq z \leq z_d \\
p_{1T}^{\Omega_3,rbd} & \text{if} \quad p_c \leq p_{1A} \leq v & \text{and} \quad z_d \leq z \leq 1
\end{cases} . \quad \text{(D58)}
$$

[22] In this range of model primitives, $z_c \leq \frac{w}{1-w}(v-2) \leq z_d \leq 1$, when $w \in [0,1]$ and $v \geq 2$.

- When $\frac{w}{2+w}v \leq 1 \leq \frac{w}{1+w}v$,[23]

$$p_{1T}^{FS,\circ} = \begin{cases} p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \leq p_{1A} \leq p_a & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_2,rbd-2} & \text{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_2,int} & \text{if} \quad p_b \leq p_{1A} \leq \min\{p_d, p_c\} & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_2,lbd} & \text{if} \quad p_d \leq p_{1A} \leq p_c & \text{and} \quad 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \leq p_{1A} \leq \frac{4}{3}v - 1 & \text{and} \quad \frac{1}{3}v \leq z \leq 1 \\ p_{1T}^{\Omega_4,rbd} & \text{if} \quad \frac{4}{3}v - 1 \leq p_{1A} \leq p_c & \text{and} \quad \frac{1}{3}v \leq z \leq 1 \\ --- & --\text{(split at } p_c\text{)}-- & \text{and} \quad ----- \\ p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_d & \text{and} \quad 0 \leq z \leq z_c \\ p_{1T}^{\Omega_2,lbd} & \text{if} \quad \max\{p_c, p_d\} \leq p_{1A} \leq v & \text{and} \quad 0 \leq z \leq z_c \\ p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_e & \text{and} \quad z_c \leq z \leq 1 \\ p_{1T}^{\Omega_2,rbd-1} & \text{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq v & \text{and} \quad z_c \leq z \leq 1 \end{cases} . \tag{D59}$$

- When $1 \leq \frac{w}{2+w}v \leq \frac{w}{1+w}v$,[24]

$$p_{1T}^{FS,\circ} = \begin{cases} p_{1T}^{\Omega_4,int} & \text{if} \quad 0 \leq p_{1A} \leq p_a & \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_2,rbd-2} & \text{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} & \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_2,int} & \text{if} \quad p_b \leq p_{1A} \leq \min\{p_d, p_c\} & \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_2,lbd} & \text{if} \quad p_d \leq p_{1A} \leq p_c & \text{and} \quad 0 \leq z \leq 1 \\ --- & --\text{(split at } p_c\text{)}-- & \text{and} \quad ----- \\ p_{1T}^{\Omega_2,int} & \text{if} \quad p_c \leq p_{1A} \leq p_d & \text{and} \quad 0 \leq z \leq 1 \\ p_{1T}^{\Omega_2,lbd} & \text{if} \quad \max\{p_c, p_d\} \leq p_{1A} \leq v & \text{and} \quad 0 \leq z \leq 1 \end{cases} . \tag{D60}$$

[23] In this range of model primitives, $z_c \leq 1 \leq z_d \leq \frac{w}{1-w}(v-2)$, when $w \in [0,1]$ and $v \geq 2$.

[24] In this range of model primitives, $1 \leq \frac{w}{2+w}v \leq z_d \leq 1$ and $z_c \leq 1$, when $w \in [0,1]$ and $v \geq 2$. Furthermore, we always have $1 \leq \frac{w}{2+w}v \leq \frac{1}{3}v$.

According to Lemma D8, Lemma D9, and Lemma D10, as stated and proved in Online Appendix D.3, we can eliminate certain strategies for large values of p_{1A}. Furthermore, according to Lemma D11, as stated and proved in Online Appendix D.3, we can eliminate the strategy where $p_{1A} < \min\{p_a, \frac{4}{3}v - 1\}$. In sum, we get Third-party's best response as follows.

- When $\frac{w}{2+w}v \leq 1$

$$p_{1T}^{\circ} = \begin{cases} p_{1A} - 2z + 1 & \text{if} \quad p_a \leq p_{1A} \leq \min\{p_b, p_c\} \text{ and } 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \text{if} \quad p_b \leq p_{1A} \leq \min\{p_c, p_d\} \text{ and } 0 \leq z \leq \min\{\frac{1}{3}v, 1\} \\ & \quad p_c \leq p_{1A} \leq \min\{p_d, p_e\} \text{ and } 0 \leq z \leq z_c \\ & \quad p_c \leq p_{1A} \leq \min\{p_d, p_e\} \text{ and } z_c \leq z \leq \min\{z_d, \frac{1}{3}v, 1\} \\ 2v - p_{1A} - 1 & \text{if} \quad \max\{p_c, p_e\} \leq p_{1A} \leq \min\{p_f, v\} \text{ and } z_c \leq z \leq \min\{z_d, \frac{1}{3}v, 1\}; \\ & \quad p_{1A} = p_c \text{ and } z_d < z \leq \min\{\frac{1}{3}v, 1\}; \\ & \quad p_c \leq p_{1A} \leq p_f \text{ and } \frac{1}{3}v \leq z \leq z_d \end{cases},$$

where $p_a = 4z - 1$, $p_b = 2\frac{1+w}{w}z - 1$, $p_c = v + z - 1$, $p_d = 3 + 2\frac{1-w}{w}z$, $p_e = \frac{4}{3}v - \frac{2}{3}\frac{1-w}{w}z - 1$, $p_f = \frac{3}{2}v - \frac{1}{2}\frac{1-w}{w}z - 1$; $z_c = \frac{1}{2}\frac{w}{1-w}(v-3)$, $z_d = \frac{w}{1+w}v$.

- When $\frac{w}{2+w}v \geq 1$

$$p_{1T}^{\circ} = \begin{cases} p_{1A} - 2z + 1 & \text{if} \quad p_a \leq p_{1A} \leq p_b \\ \frac{1}{2}p_{1A} + \frac{1}{2} + \frac{1-w}{w}z & \text{if} \quad p_b \leq p_{1A} \leq p_d \end{cases}.$$

This concludes the proof of Lemma D7. □

D.3 Additional Lemmas and Proofs

Lemma D8 *Platform does not have the incentive to charge a price that $p_{1A} > p_f$, where $p_f = \frac{3}{2}v - 1 - \frac{1}{2}\frac{1-w}{w}z$. In other words, the strategy that $p_{1A} > p_f$ is dominated by $p_{1A} = p_f$.*

Proof: When $z_c < z \leq 1$, Third-party's best response to $p_{1A} \geq p_f$ is $\frac{1}{2}(v + \frac{1-w}{w}z)$ and the corresponding relative locations for indifferent customers are that $\bar{x}_{1A} \leq \tilde{x}_1 \leq \bar{x}_{1T} \leq 1 - z$.

Therefore, Platform's demand in Segment 1 is determined by $\bar{x}_{1A}$. Platform's profit in Segment 1 is $w(v-p_{1A})p_{1A}$. The F.O.C. is proportional to $v-2p_{1A}$, which is negative since $p_{1A} \geq p_f \geq p_c$ and $v \geq 2$. With the F.O.C non-positive, the objective function (weakly) decreases in p_{1A}. As a result, the choice of $p_{1A} = p_f$ (weakly) dominates other choices of p_{1A} that is greater than p_f.

When $0 \leq z \leq z_c$, Third-party's best response to $p_{1A} \geq p_f$ is either $p_{1A}-1$ or $\frac{1}{2}(v+\frac{1-w}{w}z)$. If its best response is $\frac{1}{2}(v+\frac{1-w}{w}z)$, our argument in the previous bullet point is still applicable. If its best response is $p_{1A}-1$, the indifferent customer $\tilde{x}_1$ lies at 0. Platform's profit is 0. This is clearly dominated by other cases where Platform could earn a positive profit in Segment 1 and therefore dominated. In sum, Platform does not have the incentive to choose a price strictly greater than p_f as well. □

Lemma D9 *Platform does not have the incentive to choose a strategy where $p_c < p_{1A} \leq v$ and $z_d < z \leq 1$, where $p_c = v+z-1$ and $z_d = \frac{a}{1+a}v$. In this range of z, the strategy that $p_c < p_{1A} \leq v$ is dominated by $p_{1A} = p_c$.* □

Proof: In this range of z, Third-party's best response to $p_{1A} \geq p_c$ is $v-z$. The corresponding relative locations for indifferent customers are that $\bar{x}_{1A} \leq \tilde{x}_1 \leq \bar{x}_{1T} \leq 1-z$. Therefore, Platform's demand in Segment 1 is determined by $\bar{x}_{1A}$. Platform's profit in Segment 1 is $w(v-p_{1A})p_{1A}$. The F.O.C. is $v-2p_{1A}$, which is non-positive since $p_{1A} \geq p_c$ and $v \geq 2$. With the F.O.C non-positive, the objective function (weakly) decreases in p_{1A}. As a result, the choice of $p_{1A} = p_c$ (weakly) dominates other choices of $p_{1A} > p_c$. □

Lemma D10 *Platform does not have the incentive to choose a strategy where $p_d \leq p_{1A} \leq p_c$ and $0 < z \leq \min\{\frac{1}{3}v, 1\}$, or where $\max\{p_c, p_d\} \leq p_{1A} \leq \min\{p_f, v\}$ and $0 \leq z \leq z_c$, where $p_c = v+z-1$, $p_d = 3+2\frac{1-w}{w}z$, $p_f = \frac{3}{2}v - \frac{1}{2}\frac{1-w}{w}z - 1$ and $z_c = \frac{1}{2}\frac{w}{1-w}(v-3)$.*

Proof: In this range of z and p_{1A}, Third-party's best response to p_{1A} is $p_{1A}-1$. As shown in the proof to D8, the corresponding indifferent customer $\tilde{x}_1$ lies at 0 and Platform's profit in Segment 1 is 0. This is clearly dominated by other cases where Platform could earn a positive profit in Segment 1 and therefore dominated. □

Lemma D11 *Platform does not have the incentive to choose a strategy, defined by $(z, \mathbf{p}_A)$, with which it will sell to some of the data-sharing customers in Segment 1, i.e., $\tilde{x}_1 > 1-z$.*

Proof: When $p_{1A} < \min\{4z-1, \frac{4}{3}v-1\}$, Third-party's best response function is $\frac{1}{2}p_{1A}+\frac{1}{2}$. Platform's optimal price in Segment 1 is $\min\{\max\{0, \frac{3}{2}-2(1-w)(1-z)\}\}$. As shown in the proof of Lemma C5, we always have $\hat{\pi}_A^{FS,R} \leq \frac{9}{16} = \pi^{NS,*} \quad \forall w \in (0,1)$ and $z \in [0,1]$. Therefore, $\pi_A^{FS,R} \leq \hat{\pi}_A^{FS,R} \leq \pi^{NS,*}$, where the first inequality follows from the fact that p_{1A} is more restricted in this case, because of the consumer participation constraint.

When $\frac{4}{3}v - 1 \leq p_{1A} \leq p_c$ and $\frac{1}{3}v \leq z \leq 1$, where $p_c = v + z - 1$, Third-party's best response is $2v-1-p_{1A}$. $\tilde{x}_1 = \bar{x}_{1A} = \bar{x}_{1T} = v - p_{1A} > 1-z$. In this case, Platform's profit function in Segment 1 is $\pi_{1A}^{FS,R} = [v-p_{1A}-(1-w)(1-z)]\cdot p_{1A}$, concave and maximized when $p_{1A} = \frac{1}{2}v - \frac{1}{2}(1-w)(1-z)$. However, notice that $\frac{1}{2}v - \frac{1}{2}(1-w)(1-z) < \frac{4}{3}v - 1$ always holds, since $-\frac{1}{2}(1-w)(1-z) < \frac{5}{6}v - 1$ when $v \geq 2$. Therefore, the optimal solution in this interval is $p_{1A} = \frac{4}{3}v - 1$, coinciding with the previous interval, and this interval is dominated.

In sum, we have shown that Platform still does not have the incentive to choose a strategy, defined by $(z, \mathbf{p}_A)$, with which a right subgame is entailed. $\square$

References

Yuxin Chen and Ganesh Iyer. Research note consumer addressability and customized pricing. *Marketing Science*, 21(2):197–208, 2002.

Sridhar Moorthy and Shervin Tehrani. Targeting advertising spending and price on the hotelling line. *Marketing Science*, 42(6):1057–1079, 2023.