Differences in Dynamic Brand Competition across Markets: An Empirical Analysis: Technical Appendix

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Appendix

Solving the Bertrand price equilibrium for a triopoly

In this appendix, we derive the Bertrand equilibrium prices, conditional on advertising. Recall that prices do not impact the future profits of firms and, hence, we solve for the static period-profit-maximizing prices. Substituting the equilibrium price expressions back into the profit function allows us to re-parameterize the problem in terms of advertising. Recall that the firms pricing problem is based on maximizing the profit function gross of advertising costs:

$$\pi_{jt} = (p_{jt} - mc_{jt})\, Q_{jt}, j = 1, \cdots, 3 \tag{1}$$

where demand has the form:

$$Q_{jt} = \alpha_j + \sum_{k=1}^{J} \beta_{jk} p_{kt} + \phi_j F_{jt} + \sum_{k=1}^{J} \delta_{jk} G_{kt} + \gamma_j y_t. \tag{2}$$

In the empirical application there are three firms, $J = 3$. The optimal price satisfies the following system of first-order conditions:

$$p_{jt} = mc_{jt} + \frac{1}{\beta_{jj}} Q_{jt}, j = 1, ..., J. \tag{3}$$

To simplify the analysis, let $A_{jt} = \phi_j F_{jt} + \sum_{k=1}^{J} \delta_{jk} G_{kt} + \gamma_j y_t$. Substituting (2), we derive the reduced-form to the system (3):

$$\begin{aligned}
Bp_1^* &= (4\beta_{33}\beta_{22} - \beta_{23}\beta_{32})A_1 + (\beta_{13}\beta_{32} + 2\beta_{33}\beta_{12})A_2 + (2\beta_{13}\beta_{22} + \beta_{12}\beta_{23})A_3 \\
&\quad + (4\beta_{33}\beta_{11}\beta_{22} - \beta_{23}\beta_{11}\beta_{32})mc_1 + (2\beta_{33}\beta_{12}\beta_{22} + \beta_{22}\beta_{13}\beta_{32})mc_2 + (2\beta_{33}\beta_{13}\beta_{22} + \beta_{33}\beta_{12}\beta_{23})mc_3 \\
Bp_2^* &= (2\beta_{21}\beta_{33} + \beta_{23}\beta_{31})A_1 + (4\beta_{33}\beta_{11} - \beta_{31}\beta_{13})A_2 + (2\beta_{11}\beta_{23} + \beta_{21}\beta_{13})A_3 \\
&\quad + (2\beta_{21}\beta_{33}\beta_{11} + \beta_{23}\beta_{31}\beta_{11})mc_1 + (4\beta_{33}\beta_{11}\beta_{22} - \beta_{22}\beta_{31}\beta_{13})mc_2 + (2\beta_{33}\beta_{11}\beta_{23} + \beta_{21}\beta_{33}\beta_{13})mc_3 \\
Bp_3^* &= (2\beta_{31}\beta_{22} + \beta_{21}\beta_{32})A_1 + (2\beta_{11}\beta_{32} + \beta_{12}\beta_{31})A_2 + (4\beta_{11}\beta_{22} - \beta_{12}\beta_{21})A_3 \\
&\quad + (2\beta_{11}\beta_{31}\beta_{22} + \beta_{11}\beta_{21}\beta_{32})mc_1 + (2\beta_{11}\beta_{32}\beta_{22} + \beta_{12}\beta_{31}\beta_{22})mc_2 + (4\beta_{11}\beta_{33}\beta_{22} - \beta_{12}\beta_{21}\beta_{33})mc_3
\end{aligned} \tag{4}$$

where $B = (-2\beta_{11}\beta_{32}\beta_{23} - 2\beta_{12}\beta_{21}\beta_{33} - \beta_{12}\beta_{31}\beta_{23} + 8\beta_{11}\beta_{33}\beta_{22} - 2\beta_{13}\beta_{31}\beta_{22} - \beta_{13}\beta_{21}\beta_{32})$. For simplicity, we re-write the system (4) as:

$$p_{jt}^* = \rho_j + \sum_{k=1}^{J} \Theta_{jk} G_{kt} + \Phi F_j + \sum_{k=1}^{J} \Lambda_{kt} mc_{jk}, j = 1, ..., J \tag{5}$$

to focus on the relationship between optimal prices and the state variables, where Θ, Φ, Λ, and ρ are just functions of the model parameters. Note that while the prices obtained in equilibrium are the outcome of a static decision, they still exhibit dynamic properties due to advertising.

In particular, if firm k's current advertising adjustment is Δad_{kt}, then firm j's current prices will differ from those of the previous period by $p_{jt} - p_{j(t-1)} = \Theta_{jk}\Delta G_{kt}$, *ceteribus paribus* and setting all other goodwill levels to zero.

Substituting the optimal prices and demand back into the per-period profit function we, we can now re-parameterize the profit function in terms of advertising adjustments:

$$
\begin{aligned}
\pi_{jt} &= \left(\rho_j + \sum_{k=1}^{J} \Theta_{jk} G_{kt} + \sum_{k=1}^{J} (\Lambda_{jk} - 1) mc_{kt}\right) \qquad (6) \\
&\quad \left(\begin{array}{c} \alpha_j + \sum_{k=1}^{J} \beta_{jk} \left[\rho_k + \sum_{n=1}^{J} \Theta_{kn} G_{nt} + \Phi F_{kt} + \sum_{n=1}^{J} \Lambda_{kn} mc_{nt}\right] \\ + \phi_j F_{jt} + \sum_{k=1}^{J} \delta_{jk} G_{kt} + \gamma_j y_t \end{array} \right) - p_t^{GRP} ad_{jt} - \frac{1}{2} g \left(\Delta ad_{jt}\right)^2 \\
&= \rho_j \left(\alpha_j + \sum_{k=1}^{J} \beta_{jk} \left[\rho_k + \Phi F_{kt}\right] + \phi F_{jt}\right) + \rho_j \left(\sum_{k=1}^{J} \beta_{jk} \sum_{n=1}^{J} \left[\Theta_{kn} G_{nt} + \Lambda_{kn} mc_{kt}\right] + \sum_{k=1}^{J} \delta_{jk} G_{kt}\right) \\
&\quad + \left(\alpha_j + \sum_{k=1}^{J} \beta_{jk} \left[\rho_k + \Phi F_{kt}\right] + \phi_j F_{jt} + \gamma_j y_t\right) \left(\sum_{k=1}^{J} \Theta_{jk} G_{kt} + \sum_{k=1}^{J} (\Lambda_{jk} - 1) mc_{kt}\right) \\
&\quad + \left(\sum_{k=1}^{J} \beta_{jk} \left[\sum_{n=1}^{J} \Theta_{kn} G_{nt} + \sum_{n=1}^{J} \Lambda_{kn} mc_{nt}\right] + \sum_{k=1}^{J} \delta_{jk} G_{kt}\right) \left(\sum_{k=1}^{J} \Theta_{jk} G_{kt} + \sum_{k=1}^{J} (\Lambda_{jk} - 1) mc_{kt}\right) \\
&\quad - p_t^{GRP} AD_{jt} - \frac{1}{2} g \left(\Delta ad_{jt}\right)^2 . \qquad (7)
\end{aligned}
$$

Note the above expression consists of terms that are invariant to the policy or the state, along with terms that are linear and quadratic in the state and policy variables respectively. Ignoring the terms that are invariant to the state and the policy variables, we can re-write the profit equation in matrix notation:

$$
\pi_{jt} = S_t' \Omega_j S_t - \chi_j S_t - \frac{1}{2} g \left(\Delta ad_{jt}\right)^2 \qquad (8)
$$

where Ω_j and χ_j are functions of model parameters and are not time-dependent.