Technical Appendix
Limited Edition Products: When and When Not to Offer Them?

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Proof of proposition 1

(i) Brand M sells only the regular product:

Given p_M and q_M, the utility of a consumer located at x is given by:

$$U = u - k\frac{(q_M - 1)}{(N-1)} - p_M - t\,x \tag{1}$$

We analyze the case when the market is not fully served at the optimal solution. Therefore, there exists a consumer z for whom $U = 0$. Because of rational expectation, we set $z = q_M/N$ in (1) and compute p_M as follows:

$$p_M = u - k\frac{(q_M - 1)}{(N-1)} - \frac{tq_M}{N}$$

Brand M maximizes its profit function $\Pi_M = p_M q_M$ with respect to q_M. By solving the first-order conditions we obtain the optimal values as follows:

$$\begin{aligned}
p_M^* &= \frac{u}{2} + \frac{k}{2(N-1)} \\
q_M^* &= \frac{N(k+(N-1)u)}{2(kN+(N-1)t)} \\
\Pi_M^* &= \frac{N(k+(N-1)u)^2}{4(N-1)(kN+(N-1)t)}
\end{aligned}$$

The second-order condition for a maximum is satisfied. (Note that the same solution would be obtained if brand M were to commit to the quantity of the product, which would have made it a LE product by our model assumptions.) For the market to be not fully served, we need $q_M^* < N$, which leads to $u < u^*$, where $u^* = [k(2N-1) + 2(N-1)t]/(N-1)$. We obtain the following comparative statics.

$$\begin{aligned}
\frac{\partial p_M^*}{\partial k} &= \frac{1}{2(N-1)} > 0 \\
\frac{\partial q_M^*}{\partial k} &= -\frac{(N-1)N(Nu-t)}{2(kN+(N-1)t)^2} < 0
\end{aligned}$$

To understand why $\partial q_M^*/\partial k < 0$ based on the above expression, note that we need $u > t/N$ for brand M to sell a positive quantity since t/N is the disutility for the consumer closest to 0. We note that $q_M^* >$

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1 unless k is very large. Lastly, by the envelope theorem, we obtain $d\Pi^*_M/dk =$ - $q^*_M(q^*_M - 1)/(N-1) < 0$ when $q^*_M > 1$.

(ii) and (iii) Brand M sells both products

Note that a consumer's preference between brand M's LE product and regular product is independent of x. If brand M has to sell its LE product, the consumer at any x should not obtain a lower surplus from the LE product in comparison to the regular product. Thus, we have the following necessary condition for positive sales of the LE product for brand M:

$$\begin{aligned} u - k\frac{(q_M - 1)}{(N-1)} - p_M - t\,x &\leq u - k\frac{(l_M - 1)}{(N-1)} - p_{lM} - t\,x \\ p_{lM} &\leq p_M + k\frac{(q_M - l_M)}{(N-1)} \end{aligned}$$

We assume that the brand makes positive sales of both products. (Otherwise, this case becomes indistinguishable from the case discussed in (i) above). In this case, there are two possibilities for consumer preferences of the products: (1) all consumers strictly prefer the LE product to the regular product, or (2) all consumers are indifferent between the regular and LE product. First, consider case (1). In this case, the brand will charge the maximum possible for its LE product. Thus, we should have for $\varepsilon > 0$ and arbitrarily small,

$$p_{lM} = p_M + \frac{k(q_M - l_M)}{N-1} - \varepsilon \tag{2}$$

As we did earlier, we analyze the situation where the market is not fully served at the optimal solution. Because of rational expectation, we set $q_M = Nz-\ l_M$ in (1) and compute the location z of the consumer for whom $U = 0$ as follows:

$$z = \frac{k(1 + l_M) + (N-1)(u - p_M)}{kN + (N-1)t}$$

The brand maximizes profit $\Pi_{lM} = p_M(Nz - l_M) + p_{lM}l_M$ by choosing l_M in the first stage, and p_M and p_{lM}, in the second stage. Considering the second stage first, we substitute for p_{lM} from (2) and solve for the first-order conditions with respect to p_M to yield the following:

$$p^*_M = \frac{k + (N-1)u}{2(N-1)}$$

The second order conditions are satisfied. Next, considering stage 2, we solve the first-order conditions with respect to l_M and obtain the following:

$$\begin{aligned} l^*_M &= \frac{N(k + (N-1)u)}{2kN + 4(N-1)t} \\ \Pi^*_{lM} &= \frac{N(k + (N-1)u)^2}{2(N-1)(kN + 2(N-1)t)} \end{aligned}$$

Once again, the second-order conditions are satisfied. Π^*_{lM} is brand M's optimal profit before introduction cost. We further obtain $q^*_M = l^*_M$, and $p^*_{lM} = p^*_M$ at the optimal solution. The condition for the market to be not fully served is $u < \overline{u} = k + 2t$. We can show that $\overline{u} < u^*$.

Now, consider case (2) in which consumers are made indifferent between the brand's LE and regular products. In this case, by our consumer behavior assumptions, the brand sells equal quantities of the LE and regular products, if the former is available in sufficient quantity. Else, the brand sells a lesser quantity of the LE product. Both of these outcomes in terms of quantity sold of the two products are also possible under case (1). Further, since p_M and p_{lM} are related by equation (2) in any optimal solution in case (1) with $\varepsilon \to 0$, the brand's profits for these outcomes are identical under both cases (1) and (2). Thus, the profit outcomes for case (2) are subsumed in case (1), suggesting that the above solution is the overall optimum. Note that since the quantities sold for both products are identical, the optimal solution in case (1) can also be attained by making the consumers indifferent between the LE and regular products.

Let $\widehat{F} = \Pi^*_{lM} - \Pi^*_M$. $\widehat{F}$ simplifies to the following:

$$\widehat{F} = \frac{kN^2(k+(N-1)u)^2}{4(N-1)(kN+(N-1)t)(kN+2(N-1)t)} > 0$$

Thus, brand M makes a higher profit before product introduction costs when offering both LE and regular products. Further, it introduces a LE product if and only if $F \leq \widehat{F}$. ■

Vertical Differentiation Model: Derivation for Price Subgames

Subgame S_{11}

We analyze the case where the market is fully covered in equilibrium. In any equilibrium, in which both brands have positive sales, the market is divided into two contiguous segments of consumers, with the consumer segment with higher values of θ buying from the high-quality brand. This is because if a consumer θ' prefers the high-quality brand, so do all consumers with $\theta > \theta'$. Thus, the consumer θ who is just indifferent between purchasing the high- and the low-quality products can be computed as:

$$\theta - k\frac{(q_L-1)}{N-1} - p_L = (1+s)\theta - k\frac{(q_H-1)}{N-1} - p_H$$

Note that the consumer's utility function for both brands incorporates an exclusivity penalty that depends on anticipated q_H and q_L. Assuming rational expectations, we substitute $q_H = N(\overline{\theta}-\theta)/(\overline{\theta}-\underline{\theta})$ and $q_L = N(\theta-\underline{\theta})/(\overline{\theta}-\underline{\theta})$ to obtain:

$$\theta = \frac{(N-1)(p_H-p_L)(\overline{\theta}-\underline{\theta}) + kN(\overline{\theta}+\underline{\theta})}{(N-1)s(\overline{\theta}-\underline{\theta})+2kN}$$

The profit functions of the low- and high quality sellers are $\Pi_H(p_H) = p_H N(\overline{\theta}-\theta)/(\overline{\theta}-\underline{\theta})$ and $\Pi_L(p_L) = p_L N(\theta-\underline{\theta})/(\overline{\theta}-\underline{\theta})$, respectively. By solving the first order conditions we obtain the equilibrium prices presented in Table 2. The second order sufficient conditions for a maximum are satisfied at these prices. Our assumption that $\overline{\theta} > 2\underline{\theta}$ is sufficient for $q^*_L > 0$. Moreover, it can be shown with the help of some tedious algebra that $k < k_1$ is sufficient for the market to be covered in equilibrium for the overall game, where k_1 is given as follows:

$$k_1 = \frac{3(N-1)(2N\underline{\theta} + s(\overline{\theta} - 2N\overline{\theta} + (3N-1)\underline{\theta}))}{6N(3N-2)} + \frac{\sqrt{3}}{6N(3N-2)}$$
$$* \sqrt{(N-1)^2((3-4N)s^2\overline{\theta}^2 + 6(N-1)s(2N+s)\overline{\theta}\underline{\theta} + (2N(6-s)s + 3s^2 + 3N^2(4+s^2))\underline{\theta}^2)}$$

Subgames S_{12}, S_{21} and S_{22}

In this section we derive the equilibrium for subgame S_{22}. The equilibrium results of subgames S_{12} and S_{21} can be derived in a similar fashion.[1]. To derive the equilibrium for subgame S_{22}, we note first that a consumer's preference between a brand's LE product and regular product is independent of his type θ since both these products have the same quality. Thus, if a brand makes positive sales of its LE product, the surplus to a consumer θ from the LE product cannot be less than his surplus from the brand's regular product. Thus, we have the following condition for brand H:

$$\begin{aligned}(1+s)\theta - k\frac{(l_H-1)}{N-1} - p_{lH} \quad &\geq \quad (1+s)\theta - k\frac{(q_H-1)}{N-1} - p_H \\ &\implies \quad p_{lH} \leq p_H + \frac{k(q_H-l_H)}{N-1}\end{aligned}$$

A similar condition obtains for positive sales of LE product in the case of brand L. In any stage 3 equilibrium in which a brand makes positive sales of both its LE and regular product, there are two possibilities for consumer preferences of the brand's products in equilibrium: (a) all consumers strictly prefer the LE product of the brand to its regular product, or (b) all consumers are indifferent between the

[1] Details are available from the authors.

regular and LE products of the brand. In any stage 3 equilibrium in which all consumers strictly prefer the LE product of brand H to its regular product, a brand will charge the maximum possible for its LE product. Thus, we should have for $\varepsilon > 0$ and arbitrarily small,

$$p_{lH} = p_H + \frac{k(q_H - l_H)}{N-1} - \varepsilon \tag{3}$$

Similarly, we compute the following relationship between the prices for the LE product and the regular product for brand L, for $\varepsilon > 0$ and arbitrarily small:

$$p_{lL} = p_L + \frac{k(q_L - l_L)}{N-1} - \varepsilon \tag{4}$$

Therefore, note that when $l_i \leq q_i$, $i \in \{H, L\}$, both cases (a) and (b) result in identical profits to brand i since ε is arbitrarily small ($\varepsilon \to 0$) in equations (6) and (7).

Next, we claim that $l_i < q_i$, $i \in \{H, L\}$, in the unique subgame perfect equilibrium of S_{22}. We establish this claim in three steps. First, we identify the necessary conditions for an equilibrium with $l_i < q_i$, $i \in \{H, L\}$. These necessary conditions allows us to compute the equilibrium strategies. Second, we show that no brand can improve its profit by deviating in either stage 3 (pricing subgame) or in stage 2. Lastly, we show that no other equilibrium exists.

Necessary Conditions for the Claimed Equilibrium

Since $l_i < q_i$, $i \in \{H, L\}$ in the claimed equilibrium, the equilibrium profits are identical whether we assume case (b) or case (a) with $\varepsilon \to 0$. We will assume case (a) with $\varepsilon \to 0$, since the necessary conditions are easier to work with under this assumption. Since consumer preference between a brand's LE product and regular product in the claimed equilibrium is independent of the consumer's location, the market is composed of two contiguous segments with the consumer segment located closer to $\overline{\theta}$ ($\underline{\theta}$) purchasing the LE product of brand H (brand L) if it is available, and settling for the same brand's regular product otherwise. In order to derive each brand's demand, we solve for the consumer θ_2 who is indifferent between brand H's and brand L's products as follows (In the rest of the proof, we ignore the ε for ease of exposition):

$$\begin{aligned}
(1+s)\theta_2 - k\frac{(q_H - 1)}{N-1} - p_H &= \theta_2 - k\frac{(q_L - 1)}{N-1} - p_L \\
\implies \theta_2 &= \frac{(N-1)(p_H - p_L) + k(q_H - q_L)}{(N-1)s}
\end{aligned}$$

Assuming rational expectations, we substitute $q_H = N(\frac{\overline{\theta} - \theta_2}{\overline{\theta} - \underline{\theta}}) - l_H$ and $q_L = N(\frac{\theta_2 - \underline{\theta}}{\overline{\theta} - \underline{\theta}}) - l_L$ in the above expression for θ_2 to obtain:

$$\theta_2 = \frac{(N-1)(p_H - p_L)(\overline{\theta} - \underline{\theta}) + k((l_L - l_H)(\overline{\theta} - \underline{\theta}) + N(\overline{\theta} + \underline{\theta}))}{(N-1)s(\overline{\theta} - \underline{\theta}) + 2kN} \tag{5}$$

The profit functions for the high- and low quality sellers are $\Pi_H(p_H, l_H) = p_H q_H + p_{lH} l_H$ and $\Pi_L(p_L, l_L) = p_L q_L + p_{lL} l_L$, respectively, where p_{lH} and p_{lL} are given by equations (3) and (4) respectively. Further, we substitute $q_H = N(\frac{\overline{\theta} - \theta_2}{\overline{\theta} - \underline{\theta}}) - l_H$ and $q_L = N(\frac{\theta_2 - \underline{\theta}}{\overline{\theta} - \underline{\theta}}) - l_L$, where θ_2 is given by equation (5). A necessary condition for equilibrium is that each brand's prices in stage 3 be a best response to the other brand's prices, assuming both brands follow the proposed equilibrium strategy. Therefore, we solve the first order conditions in prices for p_H and p_L as follows:

$$\begin{aligned}
p_H^* &= \frac{k(3N - l_H - 2l_L) + (N-1)s(2\overline{\theta} - \underline{\theta})}{3(N-1)} \\
p_L^* &= \frac{k(3N - 2l_H - l_L) + (N-1)s(\overline{\theta} - 2\underline{\theta})}{3(N-1)}
\end{aligned}$$

The second order sufficient conditions for a local maximum are satisfied. Now, considering stage 2, a necessary condition for equilibrium is that l_H and l_L are best responses to each other. Therefore, we solve the first order conditions with respect to l_H and l_L to yield the equilibrium values of l_H^* and l_L^* given in Table 3. It can be shown that $l_i^* < q_i^*$, $i \in \{H, L\}$, as claimed.

Check for Deviations from Equilibrium Strategy

In stage 3, we need to consider two possibilities for a brand deviating from the equilibrium prices: (i) deviation prices that result in all consumers strictly preferring the regular product of the brand to its LE product (so that the brand sells no LE product), and (ii) deviation prices that result in a brand selling only the LE product (since the price of the regular product is so high that the consumer who cannot obtain the LE product either chooses not to purchase or purchases a product from the competing brand).

Consider case (i) first. Assume that brand H [2], after choosing l_H^* in stage 2, unilaterally deviates in its pricing strategy in stage 3 consistent with this case. Assuming that brand L follows its equilibrium strategy and solving the appropriate first-order conditions for brand H, we find the optimal deviation price for its regular product and thus the optimal deviation profit. However, we find that brand $H's$ deviation profit does not exceed its equilibrium profit. Next, we consider case (ii). It can be shown that this price deviation by either brand is unprofitable or infeasible. The intuition in both cases is that an offering of two products results in greater exclusivity value.

Now, consider a brand's deviations in stage 2 to a quantity of LE product (chosen from $(0, N]$) other than the equilibrium quantity. If both brands continue to price in the resulting stage 3 equilibrium such that case (a) holds with $\varepsilon \to 0$, the necessary conditions derived above ensures that such quantity deviations are unprofitable. However, if a brand, say brand H, unilaterally deviates to a sufficiently high $l_H > l_H^*$, it becomes no longer optimal for brand H to sell out its LE product before its regular product in stage 3. In this case, the equilibrium in which both brands price to sell their LE products before their regular products does not exist. However, an equilibrium in which at least one brand adopts a different pricing strategy may exist. Specifically, equilibrium may entail pricing strategies that induce one of three possibilities for consumer preference with respect to a brand's products: (i) all consumers strictly prefer the LE product of the brand to its regular product, (ii) all consumers are indifferent between the regular and LE products of the brand and (iii) all consumers strictly prefer the regular product of the brand to its LE product. With two brands, we thus have a total of 3^2 or nine possibilities of potential equilibria. These cases are characterized in the table A.2. The claimed equilibrium is of type I according to this table.

If brand H unilaterally deviates to a sufficiently high $l_H > l_H^*$, we can show that the only possible equilibrium is of type II. However, brand H's profit from this deviation is less then its equilibrium profit. [3]

Checking for Other Equilibria

We check to see if the equilibrium can be of types II through IX. First, we can show that equilibrium types III, V, VI, VIII and IX do not exist. For equilibria III, V to exist, brand H has to announce a sufficiently high l_H in stage 2. However, with such a high l_H, brand H finds it profitable to deviate to offering equal surplus to consumers from both its LE and regular products, causing non-existence of the equilibrium in each case. Furthermore, for equilibria VIII and IX to exist, brand L has to announce a sufficiently high l_L in stage 2. However, with such a high l_L, brand L finds it profitable to deviate to offering equal surplus to consumers from both its LE and regular products, causing non-existence of the equilibrium in each case. Moreover, for equilibrium VI to exist, both brand H and brand L have to announce a sufficiently high limited edition quantity, l_H and l_L respectively, in stage 2. However, with such high limited edition quantities both brands find it profitable to deviate to offering equal surplus to consumers from both their LE and regular products, causing non-existence of the equilibrium in each case.

Existence of equilibrium types II, IV and VII requires that the brand offering equal surplus to consumers from its LE and regular product announce a sufficiently high quantity of LE product in the first stage. However, we can show that the same brand (both brands, in the case of equilibrium type IV) can profitably deviate from this strategy by announcing a smaller quantity of LE product in stage 2 causing non-existence of these equilibrium types. Thus, the claimed equilibrium, which is of type I, is the only possible equilibrium. We also consider the possible existence of the following variation of the Type I equilibrium. In this equilibrium, one or both brands price in stage 3 such that they sell only the LE product (by pricing the regular

[2] The structure of the proof for brand L is similar and is available from the authors upon request.

[3] When brand H deviates to certain values of l_H, both type I and type II equilibria are possible. However, the deviation is unprofitable, no matter which equilibrium is played.

product very high). However, we find that such an equilibrium cannot exist. Thus, the claimed equilibrium is unique. ■

Proof of Proposition 2

$$\frac{\partial p_H^*}{\partial k} = \frac{\partial p_L^*}{\partial k} = \frac{N}{N-1} > 0$$

$$\frac{\partial q_H^*}{\partial k} = -\frac{s(\overline{\theta}+\underline{\theta})(N-1)N^2}{3((N-1)s(\overline{\theta}-\underline{\theta})+2kN)} < 0$$

$$\frac{\partial q_L^*}{\partial k} = \frac{s(\overline{\theta}+\underline{\theta})(N-1)N^2}{3((N-1)s(\overline{\theta}-\underline{\theta})+2kN)} > 0$$

$$\frac{\partial \Pi_H^*}{\partial k} = \frac{\partial \Pi_L^*}{\partial k} = \frac{2N^2((N-1)s(\overline{\theta}-2\underline{\theta})+3kN)((N-1)s(2\overline{\theta}-\underline{\theta})+3kN)}{9(N-1)((N-1)s(\overline{\theta}-\underline{\theta})+2kN)^2} > 0$$

Proof of Proposition 3

We use the notation Π_{bij}^* to denote a brand's equilibrium profit in a subgame, where the subscript b denotes the brand and i, $j \in [1, 2]$ refer to the subscript of the subgame. For example, Π_{H22}^* refers to the profit of brand H in subgame S_{22}. Define $F^* = \Pi_{L22}^* - \Pi_{L21}^*$. F^* simplifies to the following:

$$F^* = \frac{2kN^2\left(1+\frac{2kN(44kN+27(N-1)s(\overline{\theta}-\underline{\theta}))}{(14kN+9(N-1)s(\overline{\theta}-\underline{\theta}))^2}\right)(7kN+3(N-1)s(\overline{\theta}-2\underline{\theta}))^2}{9(N-1)(16kN+9(N-1)s(\overline{\theta}-\underline{\theta}))^2}$$

Since we can show that $\Pi_{H22}^* - \Pi_{H12}^* > F^*$, we have a subgame perfect equilibrium with both brands offering a LE product when $F \leq F^*$. Now, define $F^{**} = \Pi_{H21}^*$ - Π_{H11}^*. F^{**} simplifies to the following:

$$F^{**} = \frac{2kN^2(3kN+(N-1)s(2\overline{\theta}-\underline{\theta}))^2}{9(N-1)(2kN+(N-1)s(\overline{\theta}-\underline{\theta}))(16kN+9(N-1)s(\overline{\theta}-\underline{\theta}))}$$

We can show that $F^{**} > F^*$ for $\overline{\theta} > 2\underline{\theta}$. Thus, if $F^* < F \leq F^{**}$, we have a subgame perfect equilibrium in which only brand H offers a LE product. It follows that if $F > F^{**}$, we have a subgame perfect equilibrium in which neither brand introduces a LE product. Finally, we can show that $\Pi_{H22}^* - \Pi_{H12}^* > \Pi_{L12}^* - \Pi_{L11}^*$ for $\overline{\theta} > 2\underline{\theta}$. Thus, if $\Pi_{L12}^* - \Pi_{L11}^* > F$, then $\Pi_{H22}^* - \Pi_{H12}^*$ will also exceed F. Thus, we cannot have a subgame perfect equilibrium in which only brand L introduces a LE product. Since, each of the subgame perfect equilibria identified above exist for different values of F, the equilibria are unique. ■

Proof of Proposition 4

We can show algebraically that $\Pi_{L22}^* - \Pi_{L11}^* < 0$ when $\overline{\theta} > 2\underline{\theta}$. Next, define $\Delta = \Pi_{H22}^* - \Pi_{H11}^*$. Next, we can show that $\Delta < 0$ for $s = 0$ and $\partial\Delta/\partial s > 0$ for $s \geq 0$. Since Δ is continuous in s, there exists some sufficiently large $s > 0$ at which $\Delta > 0$. If the boundary conditions for the equilibrium are satisfied for this value of s, there would exist an equilibrium in which $\Delta > 0$. The following example shows that the boundary conditions for equilibrium can be satisfied for s such that $\Delta > 0$. Assume $N = 100$, $\overline{\theta} = 2.5$, $\underline{\theta} = 1$, $s = 1$, $k = 0.5$. $k < k_1 \approx 0.64$ thereby satisfying the market coverage condition. Thus, the equilibrium for these parameters satisfy the boundary conditions. Moreover $\Delta \approx 2.12 > 0$ showing existence of an equilibrium where the high-quality brand can make a higher profit than in the benchmark case for F sufficiently small. ■

Proof of Proposition 5

$$\frac{\partial(q_L^* + l_L^*)}{\partial k} = \frac{21N^2(N-1)s(\overline{\theta}+\underline{\theta})}{(14kN+9(N-1)s(\overline{\theta}-\underline{\theta}))^2} > 0$$

$$\frac{\partial(q_H^* + l_H^*)}{\partial k} = -\frac{21N^2(N-1)s(\overline{\theta}+\underline{\theta})}{(14kN+9(N-1)s(\overline{\theta}-\underline{\theta}))^2} < 0$$

$$\frac{\partial(q_L^*/l_L^*)}{\partial k} = \frac{\partial(q_H^*/l_H^*)}{\partial k} = 0$$

■

Proof of Proposition 6

In the subgame perfect equilibrium in which both brands introduce a LE product, we have the following results:

$$\frac{\partial(p_{lL}^* - p_L^*)}{\partial k} = \frac{N(98k^2N^2 + 126k(N-1)Ns(\overline{\theta} - \underline{\theta}) + 27(N-1)^2 s^2 (\overline{\theta} - 2\underline{\theta})(\overline{\theta} - \underline{\theta}))}{3(N-1)(14kN + 9(N-1)s(\overline{\theta} - \underline{\theta}))^2} > 0$$

$$\frac{\partial(p_{lH}^* - p_H^*)}{\partial k} = \frac{N(98k^2N^2 + 126k(N-1)Ns(\overline{\theta} - \underline{\theta}) + 27(N-1)^2 s^2 (2\overline{\theta} - \underline{\theta})(\overline{\theta} - \underline{\theta}))}{3(N-1)(14kN + 9(N-1)s(\overline{\theta} - \underline{\theta}))^2} > 0$$

These results also hold in the subgame perfect equilibrium in which only brand H introduces a LE product. ■

Proof of Proposition 7

We prove this by showing algebraically that $\partial F^*/\partial s < 0$ and $\partial F^*/\partial k > 0$ when $\overline{\theta} > 2\underline{\theta}$. ■

Horizontal Differentiation Model: Derivation for Price Subgames

Subgame S_{RR}

We assume that the reservation price u is sufficiently high that the market is fully covered in equilibrium. If both brands enjoy positive sales in equilibrium, there exists a consumer $z \in [0, 1]$ who is indifferent between the brands, with consumers located to the left and right of z purchasing brands A and B respectively. Given the indifference of consumer z between the brands, we have:

$$u - k\frac{q_A - 1}{N-1} - p_A - t\ z = u - k\ \frac{q_B - 1}{N-1} - p_B - t(1-z)$$

Note that the exclusivity penalty in the consumer's utility function depends anticipated quantities, q_A and q_B. Assuming rational expectations we substitute $q_A = Nz$ and $q_B = N(1-z)$ to obtain:

$$z\ = \frac{kN - (N-1)(p_A - p_B - t)}{2Nk + 2t(N-1)}$$

The profit functions of firm A and firm B are $\Pi_A(p_A) = p_A N z$ and $\Pi_B(p_B) = p_B N(1-z)$, respectively. By solving the first order conditions we obtain the equilibrium prices presented in Table 4. The second order sufficient conditions for a local maximum are fulfilled at these prices.

Subgames S_{LL} and S_{LR}

We derive the equilibrium for subgame S_{LL} in this section. The equilibrium results of subgame S_{LR} can be obtained in a similar fashion.[4]. When deriving the equilibrium for subgame S_{LL} we note that a consumer's preference between a brand's LE product and regular product is independent of his location relative to the brand since both these products have the same horizontal product characteristic. Thus, if a brand enjoys positive sales of its LE product, the surplus from the brand's LE product to a consumer at x cannot be less than his surplus from the brand's regular product. Hence, we have the following necessary condition for positive sales of the LE product for brand A:

$$\begin{aligned} u - k\frac{l_A - 1}{N-1} - p_{lA} - tx &\geq u - k\frac{q_A - 1}{N-1} - p_A - tx \\ p_{lA} &\leq p_A + \frac{k(q_A - l_A)}{N-1} \end{aligned}$$

[4] Details are available from the authors.

A similar condition obtains for positive sales of LE product in the case of brand B. In any stage 3 equilibrium in which a brand makes positive sales of both its LE and regular product, there are two possibilities for consumer preferences of the brand's products in equilibrium: (a) all consumers strictly prefer the LE product of the brand to its regular product, or (b) all consumers are indifferent between the regular and LE products of the brand. In any stage 3 equilibrium in which all consumers strictly prefer the LE product of brand A to its regular product, a brand will charge the maximum possible for its LE product. Thus, we should have for $\varepsilon > 0$ and arbitrarily small,

$$p_{lA} = p_A + \frac{k(q_A - l_A)}{N-1} - \varepsilon \tag{6}$$

Similarly, we compute the following relationship between the prices for the LE product and the regular product for brand B, for $\varepsilon > 0$ and arbitrarily small:

$$p_{lB} = p_B + \frac{k(q_B - l_B)}{N-1} - \varepsilon \tag{7}$$

Therefore, note that when $l_i \leq q_i$, $i \in \{A, B\}$, both cases (a) and (b) result in identical profits to brand i since ε is arbitrarily small ($\varepsilon \to 0$) in equations (6) and (7).

Next, we claim that $l_i < q_i$, $i \in \{A, B\}$, in the unique subgame perfect equilibrium of S_{LL}. We establish this claim in three steps. First, we identify the necessary conditions for an equilibrium with $l_i < q_i$, $i \in \{A, B\}$. These necessary conditions allows us to compute the equilibrium strategies. Second, we show that no brand can improve its profit by deviating in either stage 3 (pricing subgame) or in stage 2. Lastly, we show that no other equilibrium exists.

Necessary Conditions for the Claimed Equilibrium

Since $l_i < q_i$, $i \in \{A, B\}$ in the claimed equilibrium, the equilibrium profits are identical whether we assume case (b) or case (a) with $\varepsilon \to 0$. We will assume case (a) with $\varepsilon \to 0$, since the necessary conditions are easier to work with under this assumption. Since consumer preference between a brand's LE product and regular product in the claimed equilibrium is independent of the consumer's location, the market is composed of two contiguous segments with the consumer segment located closer to 0 (1) purchasing the LE product of brand A (brand B) if it is available, and settling for the same brand's regular product otherwise. We solve for the consumer z who is indifferent between brand A's and brand B's products as follows (In the rest of the proof, we ignore the ε for ease of exposition since $\varepsilon \to 0$) :

$$\begin{aligned} u - k\frac{q_A - 1}{N-1} - tz - p_A &= u - k\frac{q_B - 1}{N-1} - t(1-z) - p_B \\ &\Longrightarrow \quad z = \frac{k(q_B - q_A) - (N-1)(p_A - p_B) + (N-1)t}{2(N-1)t} \end{aligned}$$

Assuming rational expectations, we substitute $q_A = Nz - l_A$ and $q_B = N(1-z) - l_B$ in the above expression for z to obtain:

$$z = \frac{k(l_A - l_B + N) - (N-1)(p_A - p_B - t)}{-2t + 2N(k+t)} \tag{8}$$

The profit functions for firm A and firm B are $\Pi_A(p_A, l_A) = p_A\, q_A + p_{lA} l_A$ and $\Pi_B(p_B, l_B) = p_B q_B + p_{lB} l_B$, respectively, where p_{lA} and p_{lB} are given by equations (6) and (7) respectively. Moreover, we substitute $q_A = Nz - l_A$ and $q_B = N(1-z) - l_B$ where z is given by equation (8). A necessary condition for equilibrium is that each brand's prices in stage 3 be a best response to the other brand's prices, assuming both brands follow the proposed equilibrium strategy. Therefore, solving the first order conditions in prices for p_A and p_B as follows:

$$\begin{aligned} p_A^* &= \frac{k(3N - l_A - 2l_B)}{3(N-1)} + t \\ p_B^* &= \frac{k(3N - 2l_A - l_B)}{3(N-1)} + t \end{aligned}$$

The second order sufficient conditions for a local maximum are satisfied. Now, considering stage 2, a necessary condition for equilibrium is that l_A and l_B are best responses to each other. Therefore, we solve the first order conditions with respect to l_A and l_B to yield the equilibrium values of l_A^* and l_B^* given in Table 4. It can be shown that $l_i^* < q_i^*$, $i \in \{A, B\}$, as claimed.

Check for Deviations from Equilibrium Strategy

In stage 3, we need to consider two possibilities for a brand deviating from the equilibrium prices: (i) deviation prices that result in all consumers strictly preferring the regular product of the brand to its LE product (so that the brand sells no LE product), and (ii) deviation prices that result in a brand selling only the LE product (since the price of the regular product is so high that the consumer who cannot obtain the LE product either chooses not to purchase or purchases a product from the competing brand). Consider case (i) first. Assume that brand A, after choosing l_A^* in stage 2, unilaterally deviates in its pricing strategy in stage 3 consistent with this case. Assuming that brand B follows its equilibrium strategy and solving the appropriate first-order conditions for brand A, we find the optimal deviation price for its regular product and thus the optimal deviation profit. However, we find that brand A's deviation profit does not exceed its equilibrium profit. Next, we consider brand A deviating consistent with case (ii). In this case, there are two possibilities. Brand A can sell all of its LE product at the highest possible price it can get given brand B's equilibrium prices. We can show that such a deviation is not profitable. Alternately, brand A can consider selling less than l_A^* at higher prices. However, such a strategy is not profitable if u is sufficiently high (as we assume for market coverage).

Now, consider a brand's deviations in stage 2 to a quantity of LE product (chosen from $(0, N]$) other than the equilibrium quantity. If both brands continue to price in the resulting stage 3 equilibrium such that case (a) holds with $\varepsilon \to 0$, the necessary conditions derived above ensures that such quantity deviations are unprofitable. However, if a brand, say brand A, unilaterally deviates to a sufficiently high $l_A > l_A^*$, it becomes no longer optimal for brand A to sell out its LE product before its regular product in stage 3. In this case, the equilibrium in which both brands price to sell their LE products before their regular products does not exist. However, an equilibrium in which at least one brand adopts a different pricing strategy may exist. Specifically, equilibrium may entail pricing strategies that induce one of three possibilities for consumer preference with respect to a brand's products: (i) all consumers strictly prefer the LE product of the brand to its regular product, (ii) all consumers are indifferent between the regular and LE products of the brand and (iii) all consumers strictly prefer the regular product of the brand to its LE product. With two brands, we thus have a total of 3^2 or nine possibilities of potential equilibria. Due to the symmetry between the brands, the number of unique cases reduces to six, of which one is the equilibrium claimed above. These unique cases are characterized in the table A.1. The claimed equilibrium is of type I according to this table.

If brand A unilaterally deviates to a sufficiently high $l_A > l_A^*$, we can show that the only possible equilibrium is of type II. However, brand A's profit from this deviation is less then its equilibrium profit. [5]

Checking for Other Equilibria

We check to see if the equilibrium can be of types II through VI. First, we can show that equilibrium types III, V, and VI do not exist. For any of these equilibrium to exist, brand A (also brand B in the case of equilibrium VI) has to announce a sufficiently high l_A in stage 2. However, with such a high l_A, brand A finds it profitable to deviate to offering equal surplus to consumers from both its LE and regular products, causing non-existence of the equilibrium in each case. Existence of equilibrium types II and IV requires that the brand offering equal surplus to consumers from its LE and regular product announce a sufficiently high quantity of LE product in the first stage. However, we can show that the same brand can profitably deviate from this strategy by announcing a smaller quantity of LE product in stage 2 causing non-existence of these equilibrium types. Thus, the claimed equilibrium, which is of type I, is the only possible equilibrium.

We also consider the possible existence of the following variation of the Type I equilibrium. In this equilibrium, one or both brands price in stage 3 such that they sell only the LE product (by pricing the regular product very high). However, we find that such an equilibrium cannot exist. Thus, the claimed equilibrium is unique. ■

Proof of Proposition 8

[5] When brand A deviates to certain values of l_A, both type I and type II equilibria are possible. However, the deviation is unprofitable, no matter which equilibrium is played.

$$\frac{\partial p_A^*}{\partial k} = \frac{\partial p_B^*}{\partial k} = \frac{N}{N-1} > 0$$

$$\frac{\partial \Pi_H^*}{\partial k} = \frac{\partial \Pi_L^*}{\partial k} = \frac{N^2}{2(N-1)} > 0$$

■

Proof of Proposition 9

We use the notation Π_{bij}^* to denote a brand's equilibrium profit in a subgame, where the subscript b denotes the brand and i, $j \in [1,2]$ refer to the subscript of the subgame. For example, Π_{ALL}^* refers to the profit of brand A in subgame S_{LL}. Define $\underline{F} = \Pi_{BLL}^* - \Pi_{BLR}^*$. $\underline{F}$ simplifies to the following:

$$\underline{F} = \frac{kN^2(71k^2N^2 + 153k(N-1)Nt + 81(N-1)^2t^2)}{18(N-1)(8kN + 9(N-1)t)^2}$$

Thus, we have a subgame perfect equilibrium with both brands offering a LE product when $F \leq \underline{F}$. Now, define $\overline{F} = \Pi_{ALR}^* - \Pi_{ARR}^*$. We compute

$$\overline{F} = \frac{kN^2(kN + (N-1)t)}{2(N-1)(8kN + 9(N-1)t)}$$

We can show that $\overline{F} > \underline{F}$ for $k > 0$. Thus, if $\underline{F} < F \leq \overline{F}$, we have a subgame perfect equilibrium in which only one brand offers a LE product. It follows that if $F > \overline{F}$, we have a subgame perfect equilibrium in which neither brand introduces a LE product. The subgame perfect equilibria identified above for $F \leq \underline{F}$ and $F > \overline{F}$ are unique. With $\underline{F} < F \leq \overline{F}$, we have two subgame perfect equilibria, with a different brand offering a LE product in each equilibrium. ■

Proof of Proposition 10

$$\Pi_{ALL}^* - \Pi_{ARR}^* = \Pi_{BLL}^* - \Pi_{BRR}^* = -\frac{kN^2}{18(N-1)} < 0$$

■

Proof of comparative statics for horizontal differentiation case

$$\begin{aligned}
\frac{\partial \overline{F}}{\partial t} &= -\frac{k^2N^3}{2(8kN + 9(N-1)t)^2} < 0 \\
\frac{\partial \underline{F}}{\partial t} &= -\frac{3k^2N^3(2kN + 3(N-1)t)}{2(8kN + 9(N-1)t)^3} < 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \overline{F}}{\partial k} &= \frac{N^2(2kN + 3(N-1)t)(4kN + 3(N-1)t)}{2(N-1)(8kN + 9(N-1)t)^2} > 0 \\
\frac{\partial \underline{F}}{\partial k} &= \frac{N^2(568k^3N^3 + 1917k^2(N-1)N^2 + 2106k(N-1)^2Nt^2 + 729(N-1)^3t^3)}{18(N-1)(8kN + 9(N-1)t)^3} > 0
\end{aligned}$$

■

Table A.1. Possible Types of Equilibrium in subgame S_{LL}

		Consumer Preference for Brand *B*'s Products		
		Strictly Prefer LE	Indifferent	Strictly Prefer Regular
Consumer Preference for Brand A's Products	Strictly Prefer LE	I		
	Indifferent	II	IV	
	Strictly Prefer Regular	III	V	VI

Table A.2. Possible Types of Equilibrium in subgame S_{22}

		Consumer Preference for Brand L's Products		
		Strictly Prefer LE	Indifferent	Strictly Prefer Regular
Consumer Preference for Brand H's Products	Strictly Prefer LE	I	VII	VIII
	Indifferent	II	IV	IX
	Strictly Prefer Regular	III	V	VI