WEB APPENDIX

A Viral Branching Model for Predicting the Spread of Electronic Word-of-Mouth

Ralf van der Lans
Gerrit van Bruggen
Jehoshua Eliashberg
Berend Wierenga

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Ralf van der Lans is Assistant Professor of Marketing at Rotterdam School of Management, Erasmus University, Rotterdam, PO BOX 1738, The Netherlands (email: RLans@rsm.nl), Gerrit van Bruggen is Professor of Marketing at Rotterdam School of Management, Erasmus University (email: GBruggen@rsm.nl), Jehoshua Eliashberg is Sebastian S. Kresge Professor of Marketing and Professor of Operations and Information Management at The Wharton School, University of Pennsylvania (email: Eliashberg@wharton.upenn.edu), and Berend Wierenga is Professor of Marketing at Rotterdam School of Management, Erasmus University (email: BWierenga@rsm.nl).
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WEB APPENDIX A

Derivation of the Viral Branching Process Variables: $M(t)$, $V(t)$, and $N(t)$

Web Appendix A derives the expectations of the three stochastic processes $M(t)$, $V(t)$, and $N(t)$ of the viral branching model. The process denoted by $M(t)$ captures the number of unopened seeding emails. The process $V(t)$ captures the number of unopened viral emails and it depends on $M(t)$, and includes immigration that is the number of viral emails may also increase due to consumers that participate because of other sources $q \in Q$ than seeding or viral emails, such as banners and traditional advertising. Finally, the process $N(t)$ denotes the number of participants in the viral campaign and depends on both processes $M(t)$ and $V(t)$**.** Since the viral branching model, represented by the processes $M(t)$, $V(t)$, and $N(t)$, is a continuous time Markov process, we can derive the Kolmogorov forward equations. This is done in the first Section of Web Appendix A. These differential equations represent the probability distributions that the three stochastic processes should satisfy. Since these differential equations do not have a closed form solution, we use them in the second section to derive the differential equations of the probability generating functions. In the final section we use these probability generating functions to derive closed-form solutions for the first moments of $M(t)$, $V(t)$, and $N(t)$.

1. Derivation of the Kolmogorov Forward Equations

Let $P_{\mathbf{ik}}(t)$ denote the transition probability of switching from state $\mathbf{i} = (i_m, i_v, i_n)'$ to $\mathbf{k} = (k_m, k_v, k_n)'$ in time *t* (i.e., $P_{\mathbf{ik}}(t) = P\big(Z(t+s) = \mathbf{k} \mid Z(s) = \mathbf{i}\big)$, with $s > 0$ and $Z(t) = \{M(t), V(t), N(t)\}$, (see Ross 1997)), where $\mathbf{i} = (i_m, i_v, i_n)'$ and $\mathbf{k} = (k_m, k_v, k_n)'$ are nonnegative integers counting respectively the number of unopened seeding emails (indicated by subscript *m*), unopened viral emails (indicated by subscript *v*), and number of participants (indicated by subscript *n*). The Kolmogorov forward equations are defined as follows (Ross 1997):

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{\mathbf{ik}}(t) = \sum_{\mathbf{j} \neq \mathbf{k}} h_{\mathbf{jk}} P_{\mathbf{ij}}(t) - w_{\mathbf{k}} P_{\mathbf{ik}}(t), \tag{A1}$$

for all $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, with $\mathbf{j} = (j_m, j_v, j_n)'$. In (A1), $w_{\mathbf{k}}$ indicates the rate at which the process makes a transition given it is in state $\mathbf{k}$. This transition occurs due to the three types of sources $b \in \{m, v, Q\}$, i.e. when 1) a customer opens a seeding email (*m*), 2) a customer opens a viral email (*v*), and 3) a customer participates in the viral campaign by accepting an invitation from another source $q \in Q$. Because of the assumptions that the time between receiving a seeding or viral email and participating in the campaign is exponentially distributed with parameters λ_m and λ_v respectively, a transition from state *k* due to a seeding email occurs at rate $k_m \lambda_m$ and a transition due to a viral email happens at rate $k_v \lambda_v$ (i.e., the number of unopened seeding and viral emails multiplied by the speed in which seeding and viral emails are opened respectively[1]). We model the third possibility, i.e. the process making a transition due to other sources *Q* given it is in state **k**, using an immigration process (Harris 1963). This allows consumers to participate in the viral campaign at a given exponentially distributed rate, without being invited by seeding or viral emails. Since a customer participates in the viral campaign due to source $q \in Q$ at rate $\pi_q \beta_q$, where β_q is the exponentially distributed rate at which customers are invited by seeding tool *q* and π_q is the probability that such a customer subsequently participates in the campaign, given that it is invited by source *q*. Hence, given seeding sources *Q*, transitions from state **k** due to these sources occur at rate $\sum_{q=1}^{Q} \pi_q \beta_q$. Because all rates are independent and exponentially distributed, we add these three possibilities of making a transition from state **k**, to arrive at the overall rate $w_{\mathbf{k}}$ at which the process makes a transition equals from state **k**:

$$w_{\mathbf{k}} = k_m \lambda_m + k_v \lambda_v + \sum_{q=1}^{Q} \pi_q \beta_q\ . \tag{A2}$$

In (A1), $h_{\mathbf{jk}}$ represents the instantaneous transition rates that equal $h_{\mathbf{jk}} = w_{\mathbf{j}} r_{\mathbf{jk}}$ (Ross 1997),

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[1] Note that if X_1, X_2, .., X_k are independent exponentially distributed random variables with parameter λ, than the minimum of these random variables, i.e. $\min\{X_1, X_2, .., X_k\}$, is exponentially distributed with parameter $k\lambda$.

where $r_{\mathbf{jk}}$ denotes the probability that a transition will occur into state $\mathbf{k}$ given that the process is currently in state $\mathbf{j}$. To derive $r_{\mathbf{jk}}$, note that transitions may occur due to three types of sources of invitation $b \in \{m, v, Q\}$. Therefore, we define $p_{j_z k_z}^{z,b}$ to denote the transition probability of process $z \in \{m, v, n\}$, representing respectively the number of unopened seeding emails (*m*), number of unopened viral emails (*v*), and number of participants (*n*), due to invitation source type $b \in \{m, v, Q\}$. Using these definitions, the probability that the process switches from state **j** to state **k** due to invitation source *b* equals: $r_{\mathbf{jk}} = r_{(j_m, j_v, j_n),(k_m, k_v, k_n)} = \prod_{z \in \{m,v,n\}} p_{j_z k_z}^{z,b} = p_{j_m k_m}^{m,b} \, p_{j_v k_v}^{v,b} \, p_{j_n k_n}^{n,b}$. Hence, given the three types of seeding sources $b \in \{m, v, Q\}$, and the fact that $h_{\mathbf{jk}} = w_{\mathbf{j}} r_{\mathbf{jk}}$ and using (A2), we get:

$$h_{\mathbf{jk}} = j_m \lambda_m p_{j_m k_m}^{mm} \, p_{j_v k_v}^{vm} \, p_{j_n k_n}^{nm} + j_v \lambda_v p_{j_m k_m}^{mv} \, p_{j_v k_v}^{vv} \, p_{j_n k_n}^{nv} + \sum_{q=1}^{Q} \pi_q \beta_q p_{j_m k_m}^{mq} \, p_{j_v k_v}^{vq} \, p_{j_n k_n}^{nq} \quad . \tag{A3}$$

Note that the process $M(t)$ only decreases when a customer opens a seeding email of the company, i.e. $p_{j_m k_m}^{mm} = 1$ when $j_m = k_m + 1$, zero otherwise, and does not change due to other sources $b = \{v, Q\}$, i.e. $p_{j_m k_m}^{mv} = p_{j_m k_m}^{mq} = 1$ for all $q \in Q$ when $j_m = k_m$, zero otherwise. On the other hand, $V(t)$ may change due to all three types of sources *b*. First, due to opening a seeding email (*m*), a customer decides to send one or more viral emails after participating in the viral campaign due to opening a seeding email. Second, due to opening a viral email (*v*) a customer decides to forward viral emails to two or more friends, i.e. $V(t)$ increases, or a customer decides not to invite any friend and $V(t)$ decreases by one. Third, due to source $q \in Q$, a customer participates in the campaign and decides to invite one ore more friends by sending a viral email. When the change is due to company activities, i.e. seeding (*m*) or other sources $q \in Q$, $V(t)$ cannot decrease. Hence, given that a consumer participates in the campaign with probability π_m due to opening a seeding email, $p_{j_v k_v}^{vm} = \begin{cases} 0 & \text{if } k_v < j_v \\ \pi_m \phi_{k_v - j_v} & \text{if } k_v \geq j_v \end{cases}$, where $\phi_{k_v - j_v}$ indicates the probability that a consumer sends $k_v - j_v$ viral emails to friends that have not been invited or did not participate yet. Similarly $p_{j_v k_v}^{vq} = \begin{cases} 0 & \text{if } k_v < j_v \\ \pi_q \phi_{k_v - j_v} & \text{if } k_v \geq j_v \end{cases}$ when the change is due to source $q \in Q$ with probability π_q. However, as described above, when a customer participates with probability π_v after receiving a viral email, $V(t)$ may also decrease which gives the following:

$p_{j_v k_v}^{vv} = \begin{cases} 0 & \text{if } k_v < j_v \\ \pi_v \phi_{k_v - j_v + 1} & \text{if } k_v \geq j_v - 1 \end{cases}$. Next, since $N(t)$ counts the number of participants that participated in the viral campaign, and at most one participant can start participating in the viral campaign, $p_{j_n k_n}^{nb} = 1$ if $j_n = k_n - 1$, and zero otherwise for all sources $b = \{m, v, Q\}$.

Using these derivations of the transition probabilities $p_{j_z k_z}^{z,b}$ in combination with (A3), the Kolmogorov forward equations (A1) of a viral marketing campaign become:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P_{\mathbf{ik}}(t) = \; & (k_m + 1)\lambda_m \left(\pi_m \sum_{j_v=0}^{k_v} \phi_{k_v - j_v} P_{(i_m, i_v, i_n)(k_m+1, j_v, k_n-1)}(t) + (1 - \pi_m) P_{(i_m, i_v, i_n)(k_m+1, k_v, k_n)}(t) \right) \\ & + \lambda_v \left(\pi_v \sum_{j_v=1}^{k_v+1} j_v \cdot \phi_{k_v - j_v + 1} P_{(i_m, i_v, i_n)(k_m, j_v, k_n-1)}(t) + (1 - \pi_v)(k_v + 1) P_{(i_m, i_v, i_n)(k_m, k_v+1, k_n)} \right) \\ & + \sum_{q=1}^{Q} \pi_q \beta_q \sum_{j_v=0}^{k_v} \phi_{k_v - j_v} P_{(i_m, i_v, i_n)(k_m, j_v, k_n-1)}(t) \\ & - \left(k_m \lambda_m + k_v \lambda_v + \sum_{q=1}^{Q} \pi_q \beta_q \right) P_{(i_m, i_v, i_n)(k_m, k_v, k_n)}(t) \end{aligned} \quad , \quad \text{(A4)}$$

Equation (A4) consists of four parts (corresponding to the four lines at the right-hand-side of the equation). Recalling that the first part of (A4) denotes:

$$(k_m + 1)\lambda_m \left(\overbrace{\pi_m \sum_{j_v=0}^{k_v} \phi_{k_v - j_v} P_{(i_m, i_v, i_n)(k_m+1, j_v, k_n-1)}(t)}^{\text{Customer accepts seeding invitation}} + (1 - \pi_m) \overbrace{P_{(i_m, i_v, i_n)(k_m+1, k_v, k_n)}(t)}^{\text{Customer rejects seeding invitation}} \right) \text{(A4.1)}$$

accounts for two situations. In the first situation, the customer opens the seeding invitation and participates in the campaign with probability π_m, and the process $V(t)$ changes from j_v to k_v if this customer forwards $k_v - j_v$ viral emails which happens with probability $\phi_{k_v - j_v}$. Furthermore, $N(t)$ increases by 1 and hence j_n should equal k_{n-1} in order to switch to k_n. In the second situation when the customer opens a seeding email but decides not to participate in the viral campaign, which happens with probability $1-\pi_m$, only the process $M(t)$ changes and decreases by one, $V(t)$ and $N(t)$ are left unchanged. Similarly, recalling that the second part of (A4) also denotes two situations:

$$\lambda_v\left(\pi_v \overbrace{\sum_{j_v=1}^{k_v+1} j_v \cdot \phi_{k_v-j_v+1} P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}(t)}^{\text{Customer accepts viral invitation}} + (1-\pi_v)\overbrace{(k_v+1)P_{(i_m,i_v,i_n)(k_m,k_v+1,k_n)}}^{\text{Customer rejects viral invitation}}\right). \quad \text{(A4.2)}$$

In the first situation, the customer opens a viral email and participates the viral campaign with probability π_v. If the process switches from state **j** to state **k**, this customer needs to send $k_v - j_v + 1$ viral emails which happens with probability $\pi_v \phi_{k_v-j_v+1}$, and the arrival rate of such a customer equals $\lambda_v j_v$. In this situation, $N(t)$ increases by 1 and hence j_n should equal k_{n-1} in order to switch to k_n. In the second situation of (A4.2), the customer decides to reject the viral invitation and $V(t)$ decreases by 1 (so $j_v = k_v + 1$), leaving the other two process $M(t)$ and $N(t)$ unchanged. This situation occurs with probability $(1-\pi_v)$ and at speed $\lambda_v j_v = \lambda_v (k_v + 1)$.

The third part of (A4):

$$\sum_{q=1}^{Q} \pi_q \beta_q \sum_{j_v=0}^{k_v} \phi_{k_v-j_v} P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}(t), \quad \text{(A4.3)}$$

represents participation due to seeding sources $q \in Q$ at rate $\sum_{q=1}^{Q} \beta_q$ with probabilities π_q. In this case $M(t)$ remains the same, $N(t)$ increases by one, hence $j_n = k_n - 1$. Furthermore, the process $V(t)$ may increase from state j_v to k_v if the customer forwards $k_v - j_v$ viral emails which occurs with probability $\phi_{k_v-j_v}$. Finally, recalling that part four of (A4):

$$-\left(k_m\lambda_m + k_v\lambda_v + \sum_{q=1}^{Q} \pi_q \beta_q\right) P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t), \quad \text{(A4.4)}$$

incorporates the rate $w_{\mathbf{k}}$ at which the process makes a transition (see also A2).

Solving equation (A4) for arbitrary combinations of **i**, **k**, and *t* results in the complete probability distribution of the viral marketing campaign over time. However, the computations are highly cumbersome, as there is generally no analytical solution that expresses its probability distribution, except for very special cases such as the birth and death process (Athreya and Ney 1972). However, it is possible to derive the differential equation of the probability generating function of the process using equation (A4) (Athreya and Ney 1972; Harris 1963), which we describe in the following Section.

2. Derivation of the Probability Generating Function

Each probability distribution has a unique probability generating function from which we are able to derive its moments. Therefore, probability generating functions are popular representations of distributions especially when analytical representations are unknown. The probability generating function $F_{\mathbf{i}}(\mathbf{s},t)$ of the viral branching process is defined as follows:

$$F_{\mathbf{i}}(\mathbf{s},t) = F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) = \sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} P_{(i_m,i_v,i_n),(k_m,k_v,k_n)}(t)\, s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n}, \quad \text{with } |\mathbf{s}| \le \mathbf{1}. \qquad \text{(A5)}$$

To derive the conditional moments of the corresponding distribution, we only need to differentiate to **s** and evaluate the resulting equation in $\mathbf{s} = \mathbf{1}$. For example $E\left(N(t) \mid N(t') = i_n\right) = \frac{\mathrm{d}}{\mathrm{d}\, s_n} F_{\mathbf{i}}(\mathbf{s}=\mathbf{1},t)$, and $E\left(M(t) \mid M(t') = i_m\right) = \frac{\mathrm{d}}{\mathrm{d}\, s_m} F_{\mathbf{i}}(\mathbf{s}=\mathbf{1},t)$. To obtain the differential equation that $F_{\mathbf{i}}(\mathbf{s},t)$ must satisfy, we multiply (A4) by $s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n}$, and sum the resulting equation over k_m, k_v and k_n. For (A4.1) this leads to:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n} (k_m+1)\lambda_m \left(\pi_m \sum_{j_v=0}^{k_v} p_{k_v-j_v} P_{(i_m,i_v,i_n)(k_m+1,j_v,k_n-1)}(t) + (1-\pi_m) P_{(i_m,i_v,i_n)(k_m+1,k_v,k_n)}(t) \right). \text{ (A6)}$$

Letting k_m run from 1 to infinity, and recognizing that $P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)} = 0$ for $k_n = 0$, leads to the following result for the first part of (A6):

$$\begin{aligned} &\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n} (k_m+1)\lambda_m \pi_m \sum_{j_v=0}^{k_v} p_{k_v-j_v} P_{(i_m,i_v,i_n)(k_m+1,j_v,k_n-1)}(t) = \\ &\qquad \sum_{k_m=1}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m-1} s_v{}^{k_v} s_n{}^{k_n+1} k_m \lambda_m \pi_m \sum_{j_v=0}^{k_v} p_{k_v-j_v} P_{(i_m,i_v,i_n)(k_m,j_v,k_n)}(t) \end{aligned} \text{. (A7.1)}$$

Noting that $\sum_{k_v=0}^{\infty} s_v^{k_v} \sum_{j_v=0}^{k_v} \phi_{k_v-j_v} P_{(i_m,i_v,i_n)(k_m,j_v,k_n)}(t) = \sum_{k_v=0}^{\infty} s_v^{k_v} \sum_{k=0}^{\infty} \phi_k s_v^k P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t)$ in (A7.1), leads to:

$$\sum_{k_m=1}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m-1} s_v{}^{k_v} s_n{}^{k_n+1} k_m \lambda_m \pi_m \sum_{k=0}^{\infty} \phi_k s_v{}^{k} P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t). \tag{A7.2}$$

Note that $\sum_{k_m=1}^{\infty} k_m s_m{}^{k_m-1} = \sum_{k_m=0}^{\infty} k_m s_m{}^{k_m-1} = \frac{\mathrm{d}}{\mathrm{d}\, s_m}\sum_{k_m=0}^{\infty} s_m{}^{k_m}$, and taking into account (A5), this leads to the following result for the first part of (A6):

$$s_n \lambda_m \pi_m \sum_{k=0}^{\infty} \phi_k s_v{}^{k} \frac{\mathrm{d}\, F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\, s_m}. \tag{A7.3}$$

Similarly, by letting k_m run from 1 to infinity, the second part of (A6) becomes:

$$\begin{aligned}&\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n} (k_m+1)\lambda_m(1-\pi_m) P_{(i_m,i_v,i_n)(k_m+1,k_v,k_n)}(t) = \\ &\qquad \sum_{k_m=1}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m-1} s_v{}^{k_v} s_n{}^{k_n} k_m\lambda_m(1-\pi_m) P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t)\end{aligned}. \tag{A8.1}$$

Similar to step from (A7.2) to (A7.3), we observe that $\sum_{k_m=1}^{\infty} k_m s_m{}^{k_m-1} = \frac{\mathrm{d}}{\mathrm{d}\, s_m}\sum_{k_m=0}^{\infty} s_m{}^{k_m}$. Combining this with definition (A5), (A8.1) equals:

$$\lambda_m(1-\pi_m)\frac{\mathrm{d}\, F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\, s_m} \tag{A8.2}$$

Multiplying (A4.2) by $s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n}$, and summing the resulting equation over k_m , k_v and k_n , leads to:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n} \lambda_v\left(\pi_v \sum_{j_v=1}^{k_v+1} j_v\cdot\phi_{k_v-j_v+1} P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}(t) + (1-\pi_v)(k_v+1) P_{(i_m,i_v,i_n)(k_m,k_v+1,k_n)}\right). \tag{A9}$$

Noting that $P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)} = 0$ for $k_n = 0$ leads to the following for the first part of (A9):

$$\begin{aligned}&\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n} \lambda_v \pi_v \sum_{j_v=1}^{k_v+1} j_v\cdot\phi_{k_v-j_v+1} P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}(t) = \\ &\qquad \sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m} s_v{}^{k_v} s_n{}^{k_n+1} \lambda_v \pi_v \sum_{j_v=1}^{k_v+1} j_v\cdot\phi_{k_v-j_v+1} P_{(i_m,i_v,i_n)(k_m,j_v,k_n)}(t)\end{aligned}. \tag{A10.1}$$

Noting that $\sum_{k_v=0}^{\infty} s_v^{k_v} \sum_{j_v=1}^{k_v+1} j_v \phi_{k_v-j_v+1} P_{(i_m,i_v,i_n)(k_m,j_v,k_n)} = \sum_{k_v=0}^{\infty}(k_v+1) s_v^{k_v} \sum_{k=0}^{\infty}\phi_k s_v^{k} P_{(i_m,i_v,i_n)(k_m,k_v+1,k_n)}$ in (A10.1), leads to:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m}\left(k_v+1\right)s_v{}^{k_v}s_n{}^{k_n+1}\lambda_v\pi_v\sum_{k=0}^{\infty}\phi_k s_v^k P_{(i_m,i_v,i_n)(k_m,k_v+1,k_n)}\left(t\right). \qquad \text{(A10.2)}$$

Letting k_v run from 1 to infinity and observing that (A10.3) is equal to zero if $k_v=0$, (A10.2) can be written as:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m}k_v s_v{}^{k_v-1}s_n{}^{k_n+1}\lambda_v\pi_v\sum_{k=0}^{\infty}\phi_k s_v^k P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}\left(t\right). \qquad \text{(A10.3)}$$

Note again that $\sum_{k_v=0}^{\infty}k_v s_v{}^{k_v-1}=\frac{\mathrm{d}}{\mathrm{d}\,s_v}\sum_{k_v=0}^{\infty}s_v{}^{k_v}$, which leads to the following expression for (A10.3):

$$\lambda_v s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v^k\,\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v}. \qquad \text{(A10.4)}$$

By letting k_v run from 1 to infinity, and observing that the rhs of (A11.1) equals zero if $k_v=0$, the second part of (A9) becomes:

$$\begin{aligned}&\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m}s_v{}^{k_v}s_n{}^{k_n}\lambda_v\left(1-\pi_v\right)\left(k_v+1\right)P_{(i_m,i_v,i_n)(k_m,k_v+1,k_n)}=\\&\qquad\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m}s_v{}^{k_v-1}s_n{}^{k_n}\lambda_v\left(1-\pi_v\right)k_v P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}\end{aligned}. \qquad \text{(A11.1)}$$

Note again that $\sum_{k_v=0}^{\infty}k_v s_v{}^{k_v-1}=\frac{\mathrm{d}}{\mathrm{d}\,s_v}\sum_{k_v=0}^{\infty}s_v{}^{k_v}$, which leads to the following expression for (A11.1):

$$\lambda_v\left(1-\pi_v\right)\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v}. \qquad \text{(A11.2)}$$

The multiplication of (A4.3) by $s_m{}^{k_m}s_v{}^{k_v}s_n{}^{k_n}$, and summing the resulting equation over k_m , k_v and k_n , leads to:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m{}^{k_m}s_v{}^{k_v}s_n{}^{k_n}\sum_{q=1}^{Q}\pi_q\beta_q\sum_{j_v=0}^{k_v}\phi_{k_v-j_v}P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}\left(t\right). \qquad \text{(A12)}$$

Taking into account that $\sum_{k_v=0}^{\infty}s_v^{k_v}\sum_{j_v=0}^{k_v}\phi_{k_v-j_v}P_{(i_m,i_v,i_n)(k_m,j_v,k_n-1)}\left(t\right)=\sum_{k_v=0}^{\infty}s_v^{k_v}\sum_{k=0}^{\infty}\phi_k s_v^k P_{(i_m,i_v,i_n)(k_m,k_v,k_n-1)}\left(t\right)$ as noted above, and recognizing that $P_{(i_m,i_v,i_n)(k_m,k_v,k_n-1)}=0$ for $k_n=0$, (A12) can be written as:

$$\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m^{\ k_m} s_v^{\ k_v} s_n^{\ k_n+1} \sum_{q=1}^{Q} \pi_q \beta_q \sum_{k=0}^{\infty} \phi_k s_v^k P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t). \tag{A13.1}$$

Given the definition in (A5), (A13.1) can be written as:

$$s_n \sum_{q=1}^{Q} \pi_q \beta_q \sum_{k=0}^{\infty} \phi_k s_v^k \cdot F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t). \tag{A13.2}$$

Finally, the multiplication of (A4.4) by $s_m^{\ k_m} s_v^{\ k_v} s_n^{\ k_n}$, and summing the resulting equation over k_m, k_v and k_n, leads to:

$$\begin{aligned} -\sum_{k_m=0}^{\infty}\sum_{k_v=0}^{\infty}\sum_{k_n=0}^{\infty} s_m^{\ k_m} s_v^{\ k_v} s_n^{\ k_n} \left(k_m\lambda_m + k_v\lambda_v + \sum_{q=1}^{Q}\pi_q\beta_q \right) P_{(i_m,i_v,i_n)(k_m,k_v,k_n)}(t) = \\ -\left(k_m\lambda_m + k_v\lambda_v + \sum_{q=1}^{Q}\pi_q\beta_q \right) \cdot F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t) \end{aligned} \tag{A14}$$

Given these derivations, the differential equation of the probability generating function of the viral branching model is equal to the sum of equations (A7.3), (A8.2), (A10.4), (A11.2), (A13.2), and (A14), which equals:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t) \quad = \quad & \lambda_m \left((1-\pi_m) + s_n \pi_m \sum_{k=0}^{\infty} \phi_k s_v^{\ k} - s_m \right) \frac{\mathrm{d}\, F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t)}{\mathrm{d}\, s_m} \\ & + \lambda_v \left((1-\pi_v) + s_n \pi_v \sum_{k=0}^{\infty} \phi_k s_v^{\ k} - s_v \right) \frac{\mathrm{d}\, F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t)}{\mathrm{d}\, s_v} . \\ & + \sum_{q=1}^{Q} \pi_q \beta_q \left(s_n \sum_{k=0}^{\infty} \phi_k s_v^{\ k} - 1 \right) F_{(i_m,i_v,i_n)}(s_m, s_v, s_n, t) \end{aligned} \tag{A15}$$

Using (A15), we are now able to derive the moments $E\left(M(t) \,|\, M(t')\right)$, $E\left(V(t) \,|\, V(t')\right)$, and $E\left(N(t) \,|\, N(t')\right)$, with $0 \le t' \le t$, of the viral marketing processes in the next Section.

3. Derivation of the moments of the Viral Branching Model

Derivation of $E\left(M(t) \,|\, M(t') = i_m\right)$:

Let, $M(i_m, t) = E\left(M(t) \,|\, M(t') = i_m\right) = \dfrac{\mathrm{d}}{\mathrm{d}\, s_m} F_{(i_m,i_v,i_n)}(s_m = 1, s_v = 1, s_n = 1, t)$. Differentiating (A15) to s_m leads to the following equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s_m}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \quad = \quad & -\lambda_m \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m} \\ & +\lambda_m\left((1-\pi_m)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k - s_m\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_m} \\ & +\lambda_v\left((1-\pi_v)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k - s_v\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_m} \\ & +\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v{}^k - 1\right)\frac{F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m}\end{aligned} \quad . \qquad \text{(A16)}$$

Setting $s_m = s_v = s_n = 1$ in (A16), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v{}^k = 1$ if $s_v = 1$, we get $M(i_m,t)$ by solving the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(i_m,t) = -\lambda_m \cdot M(i_m,t). \qquad \text{(A17)}$$

Using the fact that $M(i_m,t') = i_m$, we get:

$$M(i_m,t) = i_m e^{-\lambda_m(t-t')}. \qquad \text{(A18)}$$

Clearly, as λ_m is always positive, $M(t)$ decreases exponentially over time and reaches zero as time passes by. A marketer, however, may increase $M(t)$ by sending an additional set of seeding emails to a list of customers, i.e., a marketer controls the value i_m directly.

Derivation of $E\left(V(t) \mid V(t') = i_v\right)$:

Let $V(i_v,t) = E\left(V(t) \mid V(t') = i_v\right) = \frac{\mathrm{d}}{\mathrm{d}s_v}F_{(i_m,i_v,i_n)}(s_m = 1, s_v = 1, s_n = 1, t)$. Differentiating (A15) to s_v leads to the following equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\mathrm{d}}{\mathrm{d}\,s_v}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right) \quad = \quad & \lambda_m s_n \pi_m \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m} \\
& +\lambda_m \left(\left(1-\pi_m\right) + s_n \pi_m \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - s_m \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_v} \\
& +\lambda_v \left(s_n \pi_v \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} - 1 \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v} \\
& +\lambda_v \left(\left(1-\pi_v\right) + s_n \pi_v \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - s_v \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_v} \\
& +\sum_{q=1}^{Q} \pi_q \beta_q s_n \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right) \\
& +\sum_{q=1}^{Q} \pi_q \beta_q \left(s_n \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - 1 \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v}
\end{aligned}
\quad . \qquad \text{(A19)}
$$

Setting $s_m = s_v = s_n = 1$ in (A19), and by observing that $\sum_{k=0}^{\infty} \phi_k {s_v}^{k} = 1$, and $\sum_{k=0}^{\infty} k\phi_k {s_v}^{k} = \mu$, i.e. the expected number of forwarded viral emails to friends that did not participate or have not been invited yet. Note that in the paper, $\mu = \mu^*\left(1-\theta\right)$, where μ^* denotes the average number of forwarded viral emails, and θ denotes the probability of sending a viral email to a friend that has already received an invitation or that already participated. Furthermore, note that if $s_v = 1$, and , $\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m=1,s_v=1,s_n=1,t\right)}{\mathrm{d}\,s_m} = M\left(i_m,t\right)$ we get $V\left(i_v,t\right)$ by solving the following differential equation:

$$
\frac{\mathrm{d}}{\mathrm{d}\,t}V\left(i_v,t\right) = \lambda_m \pi_m \mu \cdot M\left(i_m,t\right) + \lambda_v\left(\pi_v \mu - 1\right)\cdot V\left(i_v,t\right) + \sum_{q=1}^{Q} \pi_q \beta_q \mu\,, \qquad \text{(A20)}
$$

Using the fact that $V\left(i_v,t'\right) = i_v$, and $M\left(i_m,t\right) = i_m e^{-\lambda_m\left(t-t'\right)}$, we get:

$$
V\left(i_v,t\right) = i_v e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)} + K_1\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)} - e^{-\lambda_m\left(t-t'\right)}\right) + K_2\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)} - 1\right), \qquad \text{(A21)}
$$

after solving (A20), with $K_1 = \frac{\lambda_m \pi_m \mu i_m}{\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}$, and $K_2 = \frac{\sum_{q=1}^{Q}\pi_q\beta_q\mu}{\lambda_v\left(\pi_v\mu-1\right)}$. (A21) consists of three components. The first component, not directly under the marketer's control, depends on the number of unopened viral emails at $t = t'$, i.e. i_v. These customers may invite new customers by opening their emails and forwarding it to their friends. When $\pi_v \mu < 1$, this process dies out as time passes by. The second component depends on the number of unopened seeding emails i_m at time t' and the subsequent viral process, and is therefore under marketers control. Because $\left(e^{\lambda_v(\pi_v\mu-1)(t-t')} - e^{-\lambda_m(t-t')}\right)$ goes to zero when *t* gets very large, the second component goes to zero as well. The third component is also under marketers control and depends on seeding activities Q and the subsequent viral process. Interestingly, this component reaches an equilibrium larger than zero that equals $-K_2$, which is nonnegative when $\pi_v \mu - 1 < 0$. However, when marketers quit their seeding activities (i.e. $\beta_q = 0$ for all $q \in Q$), K_2 becomes zero, and the process dies out.

Derivation of $E\left(N(t) \mid N(t') = i_n\right)$:

Let $N(i_n, t) = E\left(N(t) \mid N(t') = i_n\right) = \frac{d}{ds_n} F_{(i_m, i_v, i_n)}\left(s_m = 1, s_v = 1, s_n = 1, t\right)$. Differentiating (A15) to s_n leads to the following equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t\,\mathrm{d}s_n} F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right) \;=\; & \lambda_m \pi_m \sum_{k=0}^{\infty} \phi_k s_v{}^k \frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right)}{\mathrm{d}s_m} \\ & + \lambda_m \left((1-\pi_m) + s_n \pi_m \sum_{k=0}^{\infty} \phi_k s_v{}^k - s_m \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right)}{\mathrm{d}s_m\,\mathrm{d}s_n} \\ & + \lambda_v \pi_v \sum_{k=0}^{\infty} \phi_k s_v{}^k \frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right)}{\mathrm{d}s_v} \\ & + \lambda_v \left((1-\pi_v) + s_n \pi_v \sum_{k=0}^{\infty} \phi_k s_v{}^k - s_v \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right)}{\mathrm{d}s_v\,\mathrm{d}s_n} \\ & + \sum_{q=1}^{Q} \pi_q \beta_q \sum_{k=0}^{\infty} \phi_k s_v{}^k F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right) \\ & + \sum_{q=1}^{Q} \pi_q \beta_q \left(s_n \sum_{k=0}^{\infty} \phi_k s_v{}^k - 1 \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m, s_v, s_n, t\right)}{\mathrm{d}s_n} \end{aligned} \quad . \qquad \text{(A22)}$$

Setting $s_m = s_v = s_n = 1$ in (A22), and by observing that $\sum_{k=0}^{\infty} \phi_k s_v{}^k = 1$ if $s_v = 1$,

$\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m=1,s_v=1,s_n=1,t\right)}{\mathrm{d}\,s_m}=M\left(i_m,t\right)$, and $\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m=1,s_v=1,s_n=1,t\right)}{\mathrm{d}\,s_v}=V\left(i_m,t\right)$ we get $N\left(i_n,t\right)$ by solving the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}N\left(i_n,t\right)=\lambda_m\pi_m\cdot M\left(i_m,t\right)+\lambda_v\pi_v\cdot V\left(i_v,t\right)+\sum_{q=1}^{Q}\pi_q\beta_q\,. \tag{A23}$$

Using the fact that $N\left(i_n,t'\right)=i_n$, and the solutions for $M\left(i_m,t\right)$ and $V\left(i_v,t\right)$, we get:

$$N\left(i_n,t\right)=i_n+K_3\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-1\right)+K_4\left(e^{-\lambda_m\left(t-t'\right)}-1\right)+K_5\left(t-t'\right), \tag{A24}$$

with: $K_3=\frac{\pi_v}{\left(\pi_v\mu-1\right)}\left(K_1+K_2+i_v\right)$, $K_4=\frac{i_m\pi_m\left(\lambda_v-\lambda_m\right)}{\lambda_m+\lambda_v\left(\pi_v\mu-1\right)}$, and $K_5=-\frac{\sum_{q=1}^{Q}\pi_q\beta_q}{\pi_v\mu-1}$. Equation (A24) consists of 4 components. Because the cumulative number of participants in the viral campaign is strictly increasing, the first component represents the number of participants at time t', i.e. $N\left(t'\right)=i_n$. The second and third components are a mix of both participants opening seeding and viral emails, because $V\left(t\right)$ depends on $M\left(t\right)$. When time passes by, these two components do not generate additional participants and the total number of participants generated by these two processes equals K_3+K_4. As discussed previously, a marketer may directly influence this sum by sending out additional seeding emails to a list of customers. The fourth component increases linearly in time with coefficient K_5, which depends on seeding sources $q\in Q$ and the subsequent viral process. Again, when marketers quit their seeding activities Q, K_5 gets equal to zero and $N\left(t\right)$ does not increase further.

WEB APPENDIX B

Derivation of Confidence Intervals using Second-Order Moments of the Viral Branching Process Variables

Web Appendix B describes how to obtain confidence intervals as presented in Figure 6 of the paper. For the derivation of these confidence intervals, we take into account two sources of stochasticity: 1) stochasticity due to parameter uncertainty, and 2) stochasticity due to uncertainty of the viral branching process itself. We solve the first source of stochasticity by simulating repeatedly from the distribution of the estimated parameters, and subsequently computing the expected number of participants $N(t)$ over time. However, the resulting distribution of $N(t)$ underestimates the true variation in the process, because $N(t)$ is stochastic as well. In order to take this stochasticity into account, we derive the second-order moments $M(t)(M(t)-1)$, $V(t)(V(t)-1)$, and $N(t)(N(t)-1)$ of the viral branching process variables which we denote respectively by $M_2(t)$, $V_2(t)$, and $N_2(t)$. Using these second order moments, we are able to derive the variance of the number of participants in the viral campaign which equals:

$$\operatorname{var}(N(t)) = N_2(t) + N(t) - N(t)^2 . \tag{B1}$$

Because of the large number of participants in viral marketing campaigns, we apply the Central Limit Theorem, which states that the distribution of the number of participants at time *t* is approximately normal with mean $N(t)$ and variance $N_2(t)$.

Using the above procedure, we simulate the distribution of $N(t)$ by repeatedly executing the following steps[2]:

Step 1) Randomly draw each of the parameters from their estimated distributions.

Step 2) Using the parameter draws from Step 1), compute the expected mean and variance of the process variable $N(t)$.

Step 3) Draw $N(t)$ from a normal distribution with mean and variance as computed in Step 2).

[2] In the empirical application we used 20,000 draws to simulate the 95 percent prediction intervals in Figure 6.

Using the draws generated in Step 3, it is straightforward to compute confidence intervals as we presented in Figure 6 of our paper. However, to execute these three steps repeatedly, we need a closed-form expression of the second-order moment of $N_2(t)$, which we derive next.

Second-order moments

Similar to the first-order moments in Web Appendix A, we derive the second-order moments using the differential equation of the probability generating function (A15). Using the notation in Web Appendix A, the second-order moment of the process $N(t)$ can be computed as follows:

$N_2(i_n,t) = E\left(N(t)\left(N(t)-1\right) \mid N(t') = i_n\right) = \frac{\mathrm{d}}{\mathrm{d}\, s_n \, \mathrm{d}\, s_n} F_{(i_m,i_v,i_n)}(s_m = 1, s_v = 1, s_n = 1, t)$. Using (A15) and (A22), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \quad = \quad & 2\lambda_m \pi_m \sum_{k=0}^{\infty} \phi_k {s_v}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_n} \\
& + \lambda_m \left((1-\pi_m) + s_n \pi_m \sum_{k=0}^{\infty} \phi_k {s_v}^k - s_m \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} \\
& + 2\lambda_v \pi_v \sum_{k=0}^{\infty} \phi_k {s_v}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_n} \\
& + \lambda_v \left((1-\pi_v) + s_n \pi_v \sum_{k=0}^{\infty} \phi_k {s_v}^k - s_v \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} \\
& + 2\sum_{q=1}^{Q} \pi_q \beta_q \sum_{k=0}^{\infty} \phi_k {s_v}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_n} \\
& + \sum_{q=1}^{Q} \pi_q \beta_q \left(s_n \sum_{k=0}^{\infty} \phi_k {s_v}^k - 1 \right) \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_n\,\mathrm{d}\,s_n}
\end{aligned} \qquad . \qquad \text{(B1)}$$

Note that (B1) depends, among others, on $\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_n}$, which equals $E\left(M(t)N(t) \mid M(t') = i_m, N(t') = i_n\right)$ and represents the interaction between process variables $M(t)$ and $N(t)$. Hence, to derive second-order moment of the process $N(t)$, we also need to derive the second-order moments of the other processes, $M(t)$ and $V(t)$, and its interactions. We first derive $M_2(t)$ next.

Derivation of $E\left(M\left(t\right)\left(M\left(t\right)-1\right)|\,M\left(t'\right)=i_m\right)$:

Let, $M_2\left(i_m,t\right)=E\left(M\left(t\right)\left(M\left(t\right)-1\right)|\,M\left(t'\right)=i_m\right)=\dfrac{d}{ds_m ds_m}F_{\left(i_m,i_v,i_n\right)}\left(s_m=1,s_v=1,s_n=1,t\right)$.

Differentiating (A16) to s_m leads to the following equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\mathrm{d}}{\mathrm{d}\,s_m\,\mathrm{d}\,s_m}F_{\left(i_m,i_v,i_n\right)}\left(s_m,s_v,s_n,t\right) \quad = \quad & -2\lambda_m\frac{\mathrm{d}\,F_{\left(i_m,i_v,i_n\right)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_m} \\
& +\lambda_m\left(\left(1-\pi_m\right)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k-s_m\right)\frac{\mathrm{d}\,F_{\left(i_m,i_v,i_n\right)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_m\,\mathrm{d}\,s_m} \\
& +\lambda_v\left(\left(1-\pi_v\right)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k-s_v\right)\frac{\mathrm{d}\,F_{\left(i_m,i_v,i_n\right)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_m\,\mathrm{d}\,s_m} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v{}^k-1\right)\frac{F_{\left(i_m,i_v,i_n\right)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_m}
\end{aligned}
\qquad \text{(B2)}
$$

Setting $s_m=s_v=s_n=1$ in (B2), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v{}^k=1$ if $s_v=1$, we get $M_2\left(i_m,t\right)$ by solving the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}M_2\left(i_m,t\right)=-\lambda_m\cdot M_2\left(i_m,t\right). \qquad \text{(B3)}$$

Using the fact that $M\left(i_m,t'\right)=i_m$, so that $M_2\left(i_m,t'\right)=i_m\left(i_m-1\right)$ we get:

$$M_2\left(i_m,t\right)=i_m\left(i_m-1\right)e^{-2\lambda_m\left(t-t'\right)}. \qquad \text{(B4)}$$

Derivation of $E\left(V\left(t\right)\left(V\left(t\right)-1\right)|\,V\left(t'\right)=i_v\right)$:

To derive the second-order moment of $V\left(t\right)$, we first need to have an expression for $E\left(M\left(t\right)\cdot V\left(t\right)|\,M\left(t'\right)=i_m,V\left(t'\right)=i_v\right)$, as $E\left(V\left(t\right)\left(V\left(t\right)-1\right)|\,V\left(t'\right)=i_v\right)$ depends on this. Let

$MV\left(i_m,i_v,t\right)=E\left(M\left(t\right)\cdot V\left(t\right)|\,M\left(t'\right)=i_m,V\left(t'\right)=i_v\right)=\dfrac{d}{ds_m ds_v}F_{\left(i_m,i_v,i_n\right)}\left(s_m=1,s_v=1,s_n=1,t\right)$.

Differentiating (A16) to s_v, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s_m\,\mathrm{d}s_v}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right) \;=\; & -\lambda_m\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_v} \\
& +\lambda_m s_n\pi_m\sum_{k=0}^{\infty}\phi_k k s_v^{\,k-1}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_m} \\
& +\lambda_m\left(\left(1-\pi_m\right)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v^{\,k}-s_m\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_m\,\mathrm{d}s_v} \\
& +\lambda_v\left(s_n\pi_v\sum_{k=0}^{\infty}\phi_k k s_v^{\,k-1}-1\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_v\,\mathrm{d}s_m} \\
& +\lambda_v\left(\left(1-\pi_v\right)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v^{\,k}-s_v\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_v\,\mathrm{d}s_m\,\mathrm{d}s_v} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q s_n\sum_{k=0}^{\infty}\phi_k k s_v^{\,k-1}\frac{F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v^{\,k}-1\right)\frac{F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_v}
\end{aligned}
\qquad (\text{B5})
$$

Setting $s_m = s_v = s_n = 1$ in (B5), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v^{\,k} = 1$, and $\sum_{k=0}^{\infty}k\phi_k s_v^{\,k} = \mu$, we get the following differential equation for $MV\left(i_m,i_v,t\right)$:

$$
\frac{d}{dt}MV\left(i_m,i_v,t\right)=\left(\lambda_v\left(\pi_v\mu-1\right)-\lambda_m\right)\cdot MV\left(i_m,i_v,t\right)+\lambda_m\pi_m\mu M_2\left(i_m,t\right)+\sum_{q=1}^{Q}\pi_q\beta_q\mu M\left(i_m,t\right). \quad (\text{B6})
$$

Using (A18), (B4) and the fact that $MV=\left(i_m,i_v,t'\right)=i_m i_v$ to solve (B6), we get:

$$
\begin{aligned}
MV\left(i_m,i_v,t\right) \;=\; & K_6\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}e^{-\lambda_m\left(t-t'\right)}-e^{-2\lambda_m\left(t-t'\right)}\right)+K_7\left(e^{\lambda_v\left(\pi_v\mu-1\right)t}e^{-\lambda_m\left(t-t'\right)}-e^{-\lambda_m\left(t-t'\right)}\right) \\
& +i_m i_v e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}e^{-\lambda_m\left(t-t'\right)}
\end{aligned}
, \quad (\text{B7})
$$

with $K_6=\dfrac{\lambda_m\pi_m\mu i_m\left(i_m-1\right)}{\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}$, and $K_7=\dfrac{\sum_{q=1}^{Q}\pi_q\beta_q\mu i_m}{\lambda_v\left(\pi_v\mu-1\right)}$.

Using (B7), we are able to derive the second-order moment

$V_2\left(i_v,t\right)=E\left(V\left(t\right)\left(V\left(t\right)-1\right)|V\left(0\right)=i_v\right)=\dfrac{d}{ds_v ds_v}F_{(i_m,i_v,i_n)}\left(s_m=1,s_v=1,s_n=1,t\right)$. Differentiating

(A19) to s_v, we get the following differential equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s_v\,\mathrm{d}s_v}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \quad = \quad & \lambda_m s_n \pi_m \sum_{k=0}^{\infty} k(k-1)\phi_k {s_v}^{k-2} \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m} \\
& +2\lambda_m s_n \pi_m \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_v} \\
& +\lambda_m \left((1-\pi_m) + s_n \pi_m \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - s_m \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_v\,\mathrm{d}s_v} \\
& +\lambda_v s_n \pi_v \sum_{k=0}^{\infty} k(k-1)\phi_k {s_v}^{k-2} \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v} \\
& +\lambda_v \left(s_n \pi_v \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} - 1 \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v} \\
& +\lambda_v \left(s_n \pi_v \sum_{k=0}^{\infty} \phi_k k {s_v}^{k-1} - 1 \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v} \qquad . \qquad \text{(B8)} \\
& +\lambda_v \left((1-\pi_v) + s_n \pi_v \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - s_v \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v\,\mathrm{d}s_v} \\
& +\sum_{q=1}^{Q} \pi_q \beta_q s_n \sum_{k=0}^{\infty} k(k-1)\phi_k {s_v}^{k-2} F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \\
& +\sum_{q=1}^{Q} \pi_q \beta_q s_n \sum_{k=0}^{\infty} k\phi_k {s_v}^{k-1} \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v} \\
& +\sum_{q=1}^{Q} \pi_q \beta_q s_n \sum_{k=0}^{\infty} \phi_k k {s_v}^{k-1} \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v} \\
& +\sum_{q=1}^{Q} \pi_q \beta_q \left(s_n \sum_{k=0}^{\infty} \phi_k {s_v}^{k} - 1 \right) \frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v}
\end{aligned}
$$

Setting $s_m = s_v = s_n = 1$ in (B8), and by observing that $\sum_{k=0}^{\infty} \phi_k {s_v}^{k} = 1$, $\sum_{k=0}^{\infty} k\phi_k {s_v}^{k} = \mu$, and $\sum_{k=0}^{\infty} k(k-1)\phi_k {s_v}^{k-2} = \mu_2$, where μ_2 is the second-order moment of forwarded viral emails to friends that did not participate or have not been invited yet[3], we get $V_2(i_v,t)$ by solving the following differential equation:

[3] Similar to the other parameters of the viral branching process, we estimate μ_2 directly from the individual-level data that readily comes available during a viral marketing campaign, i.e. $\mu_2 = \frac{1}{n_d} \sum_{d=1}^{D} \sum_{c=1}^{n_d} x_{cd}(x_{cd}-1)$.

$$\frac{d}{dt}V_2\left(i_v,t\right) = K_8 \cdot V\left(i_v,t\right)+\lambda_m\pi_m\mu_2\cdot M\left(i_m,t\right)+2\lambda_m\pi_m\mu\cdot MV\left(i_m,i_v,t\right) +2\lambda_v\left(\pi_v\mu-1\right)V_2\left(i_v,t\right)+\sum_{q=1}^{Q}\pi_q\beta_q\mu_2 , \quad \text{(B9)}$$

with $K_8=\left(\lambda_v\pi_v\mu_2+2\sum_{q=1}^{Q}\pi_q\beta_q\mu\right)$. Using (A18), (A21), (B7) and $V_2\left(t'\right)=i_v\left(i_v-1\right)$, we get:

$$\begin{aligned} V_2\left(i_v,t\right) &= K_9\left(e^{2\lambda_v(\pi_v\mu-1)(t-t')}-e^{\lambda_v(\pi_v\mu-1)(t-t')}\right)+K_{10}\left(e^{2\lambda_v(\pi_v\mu-1)(t-t')}-e^{-\lambda_m(t-t')}\right) \\ &+K_{11}\left(e^{2\lambda_v(\pi_v\mu-1)(t-t')}-e^{-2\lambda_m(t-t')}\right)+K_{12}\left(e^{2\lambda_v(\pi_v\mu-1)(t-t')}-e^{\lambda_v(\pi_v\mu-1)(t-t')}e^{-\lambda_m(t-t')}\right), \quad \text{(B10)} \\ &+K_{13}\left(e^{2\lambda_v(\pi_v\mu-1)(t-t')}-1\right)+i_v\left(i_v-1\right)e^{2\lambda_v(\pi_v\mu-1)(t-t')} \end{aligned}$$

with $K_9=\dfrac{K_8\left(K_1+K_2+i_v\right)}{\lambda_v\left(\pi_v\mu-1\right)}$, $K_{10}=\dfrac{\lambda_m\pi_m\left(\mu_2 i_m-2\mu K_7\right)-K_1K_8}{2\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}$, $K_{11}=-\dfrac{2\lambda_m\pi_m\mu K_6}{2\lambda_v\left(\pi_v\mu-1\right)+2\lambda_m}$,

$K_{12}=\dfrac{2\lambda_m\pi_m\mu\left(K_6+K_7+i_m i_v\right)}{\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}$, and $K_{13}=\dfrac{\sum_{q=1}^{Q}\pi_q\beta_q\mu_2-K_2K_8}{2\lambda_v\left(\pi_v\mu-1\right)}$.

Derivation of $E\left(N\left(t\right)\left(N\left(t\right)-1\right)\mid N\left(t'\right)=i_n\right)$:

To derive the second-order moment of $N\left(t\right)$, we need, next to (B4), (B7) and (B10), expressions for $E\left(M\left(t\right)N\left(t\right)\mid M\left(t'\right)=i_m,N\left(t'\right)=i_n\right)$, and $E\left(V\left(t\right)N\left(t\right)\mid V\left(t'\right)=i_v,N\left(t'\right)=i_n\right)$, as $E\left(N\left(t\right)\left(N\left(t\right)-1\right)\mid N\left(t'\right)=i_n\right)$ depends on it. Let

$MN\left(i_m,i_n,t\right)=E\left(M\left(t\right)N\left(t\right)\mid M\left(t'\right)=i_m,N\left(t'\right)=i_n\right)=\dfrac{d}{ds_m ds_n}F_{\left(i_m,i_v,i_n\right)}\left(s_m=1,s_v=1,s_n=1,t\right)$.

Differentiating (A16) to s_n, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s_m\,\mathrm{d}s_n}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right) \;=\; & -\lambda_m\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_n} \\
& +\lambda_m\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k\,\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_m} \\
& +\lambda_m\left(\left(1-\pi_m\right)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k-s_m\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_m\,\mathrm{d}s_n} \\
& +\lambda_v\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k\,\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_v\,\mathrm{d}s_m} \\
& +\lambda_v\left(\left(1-\pi_v\right)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k-s_v\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_v\,\mathrm{d}s_m\,\mathrm{d}s_n} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q\sum_{k=0}^{\infty}\phi_k s_v{}^k\,\frac{F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v{}^k-1\right)\frac{F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}s_m\,\mathrm{d}s_n}
\end{aligned}
\qquad . \qquad \mathrm{B}(11)
$$

Setting $s_m = s_v = s_n = 1$ in (B11), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v{}^k = 1$ and $\sum_{k=0}^{\infty}k\phi_k s_v{}^k = \mu$, we get $MN\left(i_m,i_n,t\right)$ by solving the following differential equation:

$$
\begin{aligned}
\frac{d}{dt}MN\left(i_m,i_n,t\right) \;=\; & \lambda_v\pi_v\cdot MV\left(i_m,i_v,t\right)+\lambda_m\pi_m\cdot M_2\left(i_m,t\right)+\sum_{q=1}^{Q}\pi_q\beta_q\cdot M\left(i_m,t\right) \\
& -\lambda_m\cdot MN\left(i_m,i_n,t\right)
\end{aligned}
\qquad . \qquad \text{(B10)}
$$

Using (A18), (B4), (B7) and the fact that $MN\left(i_m,i_n,t'\right)=i_m i_n$, we get:

$$
\begin{aligned}
MN\left(i_m,i_n,t\right) \;=\; & K_{14}\left(t-t'\right)e^{-\lambda_m(t-t')}+K_{15}\left(e^{-\lambda_m(t-t')}-e^{-2\lambda_m(t-t')}\right) \\
& +K_{16}\left(e^{\lambda_v(\pi_v\mu-1)(t-t')}e^{-\lambda_m(t-t')}-e^{-\lambda_m(t-t')}\right)+i_m i_n e^{-\lambda_m(t-t')}
\end{aligned}
\qquad , \qquad \text{(B12)}
$$

with $K_{14}=\sum_{q=1}^{Q}\pi_q\beta_q i_m-\lambda_v\pi_v K_7$, $K_{15}=\dfrac{\lambda_m\pi_m i_m\left(i_m-1\right)-\lambda_v\pi_v K_6}{\lambda_m}$, and $K_{16}=\dfrac{\lambda_v\pi_v\left(K_6+K_7+i_m i_v\right)}{\lambda_v\left(\pi_v\mu-1\right)}$.

Let $VN\left(i_v,i_n,t\right)=E\left(V\left(t\right)N\left(t\right)|V\left(t'\right)=i_v,N\left(t'\right)=i_n\right)=\dfrac{d}{ds_v ds_n}F_{(i_m,i_v,i_n)}\left(s_m=1,s_v=1,s_n=1,t\right)$.

Differentiating (A19) to s_n, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s_v\,\mathrm{d}s_n}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \quad = \quad & \lambda_m\pi_m\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m} \\
&+\lambda_m s_n\pi_m\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_n} \\
&+\lambda_m\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^{k}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_v} \\
&+\lambda_m\left((1-\pi_m)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^{k}-s_m\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_m\,\mathrm{d}s_v\,\mathrm{d}s_n} \\
&+\lambda_v\pi_v\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v} \\
&+\lambda_v\left(s_n\pi_v\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}-1\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_n} \\
&+\lambda_v\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^{k}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v} \\
&+\lambda_v\left((1-\pi_v)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^{k}-s_v\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_v\,\mathrm{d}s_n} \\
&+\sum_{q=1}^{Q}\pi_q\beta_q\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t) \\
&+\sum_{q=1}^{Q}\pi_q\beta_q s_n\sum_{k=0}^{\infty}k\phi_k s_v{}^{k-1}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_n} \\
&+\sum_{q=1}^{Q}\pi_q\beta_q\sum_{k=0}^{\infty}\phi_k s_v{}^{k}\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v} \\
&+\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v{}^{k}-1\right)\frac{\mathrm{d}F_{(i_m,i_v,i_n)}(s_m,s_v,s_n,t)}{\mathrm{d}s_v\,\mathrm{d}s_n}
\end{aligned}
\quad . \qquad \text{(B13)}
$$

Setting $s_m = s_v = s_n = 1$ in (B13), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v{}^{k} = 1$ and $\sum_{k=0}^{\infty}k\phi_k s_v{}^{k} = \mu$, we get $VN(i_v,i_n,t)$ by solving the following differential equation:

$$
\begin{aligned}
\frac{d}{dt}VN(i_v,i_n,t) \quad = \quad & \left(\lambda_v\pi_v\mu+\sum_{q=1}^{Q}\pi_q\beta_q\right)V(i_v,t)+\lambda_m\pi_m\mu\cdot M(i_m,t)+\sum_{q=1}^{Q}\pi_q\beta_q\mu\cdot N(i_n,t)+ \\
& \lambda_m\pi_m\cdot MV(i_m,i_v,t)+\lambda_m\pi_m\mu MN(i_m,i_n,t)+\lambda_v\pi_v\cdot V_2(i_v,t)+\sum_{q=1}^{Q}\pi_q\beta_q\mu+ \\
& \lambda_v(\pi_v\mu-1)\cdot VN(i_v,i_n,t)
\end{aligned}
\quad . \quad \text{(B14)}
$$

Using (A18), (A21), (A24), (B7), (B10), (B12) and the fact that $VN(i_v,i_n,t') = i_v i_n$, we get:

$$
\begin{aligned}
VN\left(i_v,i_n,t\right) \quad = \quad & K_{17}\left(t-t'\right)e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}+K_{18}\left(t-t'\right)e^{-\lambda_m t}+K_{19}\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-e^{-\lambda_m\left(t-t'\right)}\right)\\
&+K_{20}\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-e^{-2\lambda_m\left(t-t'\right)}\right)+K_{21}\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}e^{-\lambda_m t}\right)\\
&+K_{22}\left(e^{2\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}\right)+K_{23}\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}-1\right)+K_{24}\left(t-t'\right)\\
&+i_v i_n e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}
\end{aligned}, \quad \text{(B15)}
$$

with $K_{17}=\left(\lambda_v\pi_v\mu+\sum_{q=1}^{Q}\pi_q\beta_q\right)\left(K_1+K_2+i_v\right)+\sum_{q=1}^{Q}\pi_q\beta_q\mu K_3-\lambda_v\pi_v K_9$, $K_{18}=-\frac{\lambda_m\pi_m\mu K_{14}}{\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}$,

$$
K_{19}=\frac{\lambda_m\pi_m\mu\left(K_{15}-K_{16}+i_m i_n+i_m\right)-\left(\lambda_v\pi_v\mu+\sum_{q=1}^{Q}\pi_q\beta_q\right)K_1+\sum_{q=1}^{Q}\pi_q\beta_q\mu K_4-\lambda_m\pi_m K_7-\lambda_v\pi_v K_{10}-K_{18}}{\lambda_v\left(\pi_v\mu-1\right)+\lambda_m}
$$

, $K_{20}=-\frac{\lambda_m\pi_m\left(K_6+\mu K_{15}\right)+\lambda_v\pi_v K_{11}}{\lambda_v\left(\pi_v\mu-1\right)+2\lambda_m}$, $K_{21}=\frac{\lambda_m\pi_m\left(K_6+K_7+i_m i_v+\mu K_{16}\right)-\lambda_v\pi_v K_{12}}{\lambda_m}$,

$$
K_{22}=\frac{\lambda_v\pi_v\left(K_9+K_{10}+K_{11}+K_{12}+K_{13}+i_v\left(i_v-1\right)\right)}{\lambda_v\left(\pi_v\mu-1\right)},
$$

$$
K_{23}=\frac{\sum_{q=1}^{Q}\pi_q\beta_q\mu-\left(\lambda_v\pi_v\mu+\sum_{q=1}^{Q}\pi_q\beta_q\right)K_2+\sum_{q=1}^{Q}\pi_q\beta_q\mu\left(i_x-K_3-K_4\right)-\lambda_v\pi_v K_{13}-K_{24}}{\lambda_v\left(\pi_v\mu-1\right)}, \text{ and}
$$

$$
K_{24}=-\frac{\sum_{q=1}^{Q}\pi_q\beta_q\mu K_5}{\lambda_v\left(\pi_v\mu-1\right)}.
$$

Let $N_2\left(i_n,t\right)=E\left(N\left(t\right)\left(N\left(t\right)-1\right)\mid N\left(t'\right)=i_n\right)=\frac{d}{ds_n ds_n}F_{\left(i_m,i_v,i_n\right)}\left(s_m=1,s_v=1,s_n=1,t\right)$.

Differentiating (A22) to s_n, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right) \quad = \quad & 2\lambda_m\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_n} \\
& +\lambda_m\left(\left(1-\pi_m\right)+s_n\pi_m\sum_{k=0}^{\infty}\phi_k s_v{}^k - s_m\right)\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_m\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} \\
& +2\lambda_v\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_n} \\
& +\lambda_v\left(\left(1-\pi_v\right)+s_n\pi_v\sum_{k=0}^{\infty}\phi_k s_v{}^k - s_v\right)\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_v\,\mathrm{d}\,s_n\,\mathrm{d}\,s_n} \\
& +2\sum_{q=1}^{Q}\pi_q\beta_q\sum_{k=0}^{\infty}\phi_k s_v{}^k \frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_n} \\
& +\sum_{q=1}^{Q}\pi_q\beta_q\left(s_n\sum_{k=0}^{\infty}\phi_k s_v{}^k - 1\right)\frac{\mathrm{d}\,F_{(i_m,i_v,i_n)}\left(s_m,s_v,s_n,t\right)}{\mathrm{d}\,s_n\,\mathrm{d}\,s_n}
\end{aligned} \quad . \qquad \text{(B16)}
$$

Setting $s_m = s_v = s_n = 1$ in (B16), and by observing that $\sum_{k=0}^{\infty}\phi_k s_v{}^k = 1$ we get $N_2\left(i_n,t\right)$ by solving the following differential equation:

$$
\frac{d}{dt}N_2\left(i_n,t\right) = 2\lambda_v\pi_v \cdot VN\left(i_v,i_n,t\right) + 2\lambda_m\pi_m \cdot MN\left(i_v,i_n,t\right) + 2\sum_{q=1}^{Q}\pi_q\beta_q \cdot N\left(i_n,t\right). \qquad \text{(B17)}
$$

Using (B12), (B15) and the fact that $N_2\left(i_n,t'\right) = i_n\left(i_n-1\right)$, we get:

$$
\begin{aligned}
N_2\left(i_n,t\right) \quad = \quad & i_n\left(i_n-1\right) + K_{25}\left(t-t'\right)e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)} + K_{26}\left(t-t'\right)e^{-\lambda_m\left(t-t'\right)} + K_{27}\left(1-e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}\right) \\
& +K_{28}\left(1-e^{-\lambda_m\left(t-t'\right)}\right) + K_{29}\left(1-e^{-2\lambda_m\left(t-t'\right)}\right) + K_{30}\left(e^{\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)}e^{-\lambda_m\left(t-t'\right)} - 1\right) \\
& +K_{31}\left(e^{2\lambda_v\left(\pi_v\mu-1\right)\left(t-t'\right)} - 1\right) + K_{32}\left(t-t'\right)^2 + K_{33}\left(t-t'\right)
\end{aligned} \quad , \quad \text{(B18)}
$$

with $K_{25} = \dfrac{2\lambda_v\pi_v K_{17}}{\left(\lambda_v\pi_v\mu-1\right)}$, $K_{26} = -\dfrac{2\lambda_v\pi_v K_{18} + 2\lambda_m\pi_m K_{14}}{\lambda_m}$,

$$
K_{27} = -\frac{2\lambda_v\pi_v\left(K_{19}+K_{20}+K_{21}-K_{22}+K_{23}+i_v i_n\right) + 2\sum_{q=1}^{Q}\pi_q\beta_q K_3 - K_{25}}{\lambda_v\left(\pi_v\mu-1\right)},
$$

$$
K_{28} = \frac{2\lambda_m\pi_m\left(K_{15}-K_{16}+i_m i_n\right) + 2\sum_{q=1}^{Q}\pi_q\beta_q K_4 - 2\lambda_v\pi_v K_{19} - K_{26}}{\lambda_m}, \quad K_{29} = -\frac{\lambda_v\pi_v K_{20} + \lambda_m\pi_m K_{15}}{\lambda_m},
$$

$$K_{30} = \frac{2\lambda_m \pi_m K_{16} - 2\lambda_v \pi_v K_{21}}{\lambda_v \left(\pi_v \mu - 1\right) - \lambda_m}, \; K_{31} = \frac{\pi_v K_{22}}{\pi_v \mu - 1}, \; K_{32} = \lambda_v \pi_v K_{24} + \sum_{q=1}^{Q} \pi_q \beta_q K_5, \text{ and}$$

$$K_{33} = 2\sum_{q=1}^{Q} \pi_q \beta_q \left(i_n - K_3 - K_4\right) - 2\lambda_v \pi_v K_{23}.$$

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