Technical Appendix to Accompany
"Cross-Market Discounts"

Marcel Goić

Kinshuk Jerath

Kannan Srinivasan†

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†Marcel Goić is a doctoral candidate at the Tepper School of Business, Carnegie Mellon University and Instructor at the Department of Industrial Engineering, University of Chile (email: mgoic@cmu.edu, phone: 412-268-6266). Kinshuk Jerath is Assistant Professor of Marketing at the Tepper School of Business, Carnegie Mellon University (email: kinshuk@cmu.edu, phone: 412-268-2215). Kannan Srinivasan is Rohet Tolani Distinguished Professor in International Business and H.J. Heinz II Professor of Management, Marketing and Information Systems at the Tepper School of Business, Carnegie Mellon University (email: kannans@cmu.edu, phone: 412-268-8840). All authors can be contacted at the following address: Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. We thank the editor, the associate editor, two anonymous reviewers, and seminar participants at the University of Pittsburgh and the 2010 Customer Insights Conference at Yale University for their helpful comments.

TA1 Extensions of the monopoly-monopoly scenario

TA1.1 Analysis with different rates of diminishing marginal utilities

We assume that firm 1 is the monopolist in two markets s and t, selling at prices p_{s1} and p_{t1} respectively.[TA1] We introduce more general rates of diminishing marginal utilities by assuming that the utility function of a representative consumer is given by

$$\mathcal{U}(q_{s1}, q_{t1}) = \alpha_s \left(q_{s1} - \psi_s \frac{q_{s1}^2}{2} \right) + \alpha_t \left(q_{t1} - \psi_t \frac{q_{t1}^2}{2} \right),$$

where q_{s1} and q_{t1} denote the quantities demanded in the markets s and t, respectively, and α_s and α_t denote the "importance" of consumption utility in the respective markets. We further assume that $\psi_s, \psi_t > 0$ implying that the marginal utility from consuming an extra unit of this product decreases with consumption in both markets. The introduction of more general rates of diminishing marginal utilites does not change the definition of the consumer expenditure function.

We model the game in three stages and we solve for the subgame-perfect equilibrium of the game using backward induction. In Stage 3, given the posted prices and the cross-market discount, the consumer decides the optimal quantities to purchase by maximizing her surplus. This gives the following functions for quantities consumed in each market:

$$\begin{aligned} q_{s1} &= \frac{\alpha_t \delta + \alpha_s \alpha_t \psi_t}{\alpha_s \alpha_t \psi_s \psi_t - \delta^2} - \frac{\delta}{\alpha_s \alpha_t \psi_s \psi_t - \delta^2} p_{t1} - \frac{\alpha_t \psi_t}{\alpha_s \alpha_t \psi_s \psi_t - \delta^2} p_{s1} \\ \text{and } q_{t1} &= \frac{\alpha_s \delta + \alpha_t \alpha_s \psi_s}{\alpha_t \alpha_s \psi_t \psi_t - \delta^2} - \frac{\delta}{\alpha_s \alpha_t \psi_s \psi_t - \delta^2} p_{s1} - \frac{\alpha_s \psi_s}{\alpha_s \alpha_t \psi_s \psi_t - \delta^2} p_{t1}. \end{aligned}$$

The model is well defined if $\delta < \sqrt{\alpha_s \alpha_t \psi_s \psi_t}$. Notice that if $\delta = 0$, the demand functions are given by $q_{s1} = 1/\psi_s - 1/(\alpha_s \psi_s) p_{s1}$ and $q_{t1} = 1/\psi_t - 1/(\alpha_t \psi_t) p_{t1}$, and the quantities demanded in the two markets are completely independent. In this case, we note that as the rate of satiation becomes larger, the base demand becomes smaller and the demand becomes less price sensitive in the market in question, all else equal. In Stage 2, the firm foresees the above response by the consumers and sets the optimal posted prices, p_{s1} and p_{t1}, by solving its profit maximization problem. The prices and the corresponding quantities consumed are given in Table TA1.

[TA1] We include the subscript 1 for the firm identity to be consistent with the notation in the main paper where we have competitor firms.

Prices	
p_{s1}	$\frac{\alpha_s\alpha_t\psi_s(\delta+2\alpha_s\psi_t)}{4\alpha_s\alpha_t\psi_s\psi_t-\delta^2}$
p_{t1}	$\frac{\alpha_s\alpha_t(\delta+2\alpha_t\psi_s)\psi_t}{4\alpha_s\alpha_t\psi_s\psi_t-\delta^2}$
$p_{t1}-\delta q_{s1}$	$\frac{\alpha_t\left(2\alpha_s\alpha_t\psi_s\psi_t-\delta^2-\delta\alpha_s\psi_t\right)}{4\alpha_s\alpha_t\psi_s\psi_t-\delta^2}$
Quantities	
q_{s1}	$\frac{\alpha_t(\delta+2\alpha_s\psi_t)}{4\alpha_t\psi_s\psi_t-\delta^2}$
q_{t1}	$\frac{\alpha_s(\delta+2\alpha_t\psi_s)}{4\alpha_s\alpha_t\psi_s\psi_t-\delta^2}$

Table TA1: Prices and quantities in Stage 2 in the monopoly-monopoly scenario.

To formally evaluate how the size of cross-market discount affects prices, demand and profits, we inspect the partial derivatives with respect to δ.

$$\begin{aligned}
\frac{\partial p_{s1}}{\partial\delta} &= \frac{\alpha_s\alpha_t\psi_s\left(\delta^2+4\alpha_s(\delta+\alpha_t)\psi_t\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}\\
\frac{\partial p_{t1}}{\partial\delta} &= \frac{\alpha_s\alpha_t\psi_t\left(\delta^2+4\alpha_t(\delta+\alpha_s)\psi_s\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}\\
\frac{\partial q_{s1}}{\partial\delta} &= \frac{\alpha_t\left(\delta^2+4\alpha_s(\delta+\alpha_t\psi_s)\psi_t\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}\\
\frac{\partial q_{t1}}{\partial\delta} &= \frac{\alpha_s\left(\delta^2+4\alpha_t(\delta+\alpha_s\psi_t)\psi_s\right)}{\left(\delta^2-4\alpha_t\alpha_s\psi_t\psi_s\right)^2}\\
\frac{\partial\Pi_{st1}}{\partial\delta} &= \frac{\alpha_s\alpha_t(\delta+2\alpha_t\psi_s)(\delta+2\alpha_s\psi_t)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}
\end{aligned}$$

Given $\alpha_s,\alpha_t,\psi_s,\psi_t>0$, we verify that posted prices and purchased quantities are increasing in δ and, therefore, the total profit of the firm is monotonically increasing in δ. (Notice that the rates of variation of prices, quantities and total profit as functions of δ are decreasing in ψ_s and ψ_t.) To find the optimal rate of the cross-market discount, the firm will keep increasing δ while ensuring two constraints — nonnegative consumer surplus and nonnegative effective price in target market. It turns out that both consumer surplus and effective price in the target market are decreasing in δ (as shown below) and, therefore, the optimal is achieved when one of those constraints is binding.

$$\begin{aligned}
\frac{\partial\mathcal{CS}}{\partial\delta} &= \frac{\alpha_s\alpha_t\delta\left(\delta^3+12\alpha_s\alpha_t\delta\psi_s\psi_t+3\delta^2(\alpha_t\psi_s+\alpha_s\psi_t)+4\alpha_s\alpha_t\psi_s\psi_t(\alpha_t\psi_s+\alpha_s\psi_t)\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^3}\\
\frac{\partial(p_{t1}-\delta q_{s1})}{\partial\delta} &= -\frac{\alpha_s\alpha_t\psi_t\left(\delta^2+4\alpha_t\delta\psi_s+4\alpha_s\alpha_t\psi_s\psi_t\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}
\end{aligned}$$

From the above, notice that the rates at which consumer surplus and effective price in the target market decrease with δ are decreasing in the rates of satiation. Consequently, larger values of ψ_s and ψ_t expand the set of feasible values of δ. The optimal values of δ that complete the characterization of the equilibrium depend on which constraint is binding. Let δ^{ep} and δ^{cs} be the smallest positive roots of the equations enforcing the conditions that effective price in target market is zero and consumer surplus across the two markets is zero, respectively. To find the regions of the parameter space corresponding to the case in which the nonnegativity of the effective price in the target market is the binding constraint, we impose $\delta^{ep} \leq \delta^{cs}$.

$$\delta^{ep} = \frac{1}{2}\left(-\alpha_s\psi_t + \sqrt{\alpha_s\psi_t(8\alpha_t\psi_s + \alpha_s\psi_t)}\right)$$

$$\delta^{cs} = \frac{1}{2}\left(-\alpha_s\psi_t - \alpha_t\psi_s + (\Delta_1 + 4\Delta_2)^{1/3} + (\Delta_1 - 4\Delta_2)^{1/3}\right)$$

where

$$\Delta_1 = -\alpha_t^3\psi_s^3 + 5\alpha_s\alpha_t^2\psi_s^2\psi_t + 5\alpha_s^2\alpha_t\psi_s\psi_t^2 - \alpha_s^3\psi_t^3$$

$$\Delta_2 = \sqrt{-\alpha_s\alpha_t\psi_s\psi_t\left(\alpha_t^2\psi_s^2 - \alpha_s^2\psi_t^2\right)^2}$$

To shed further light on these conditions, in Figure TA1 we display for several values of ψ_s and ψ_t the region of values of α_s and α_t where the positive effective price in the target market is the binding condition (the shaded regions). We conclude that, in general, when the target market is more important than the source market, the monopolist can fully extract consumer surplus.

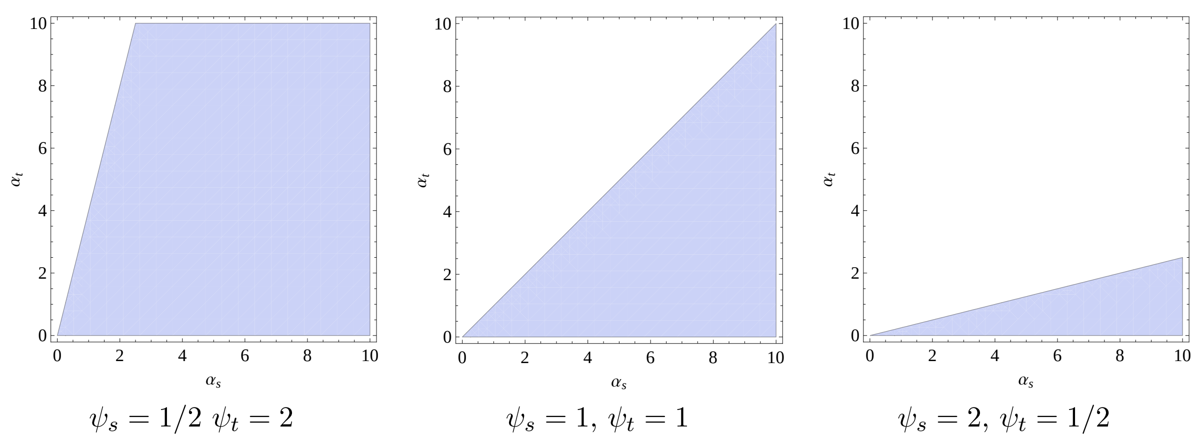


Figure TA1: The shaded regions correspond to the parameter space where the nonnegativity of the effective price in target market is the binding constraint.

TA1.2 Analysis of quantity discounts limited to the source market

In this section, we present a detailed analysis of the case in which discounts are redeemable in the same market in which they are accumulated. For simplicity, we use the monopoly-monopoly scenario to study this strategy and compare it against cross-market discounts. For this comparison, it is useful to use the extended model with "satiation parameters" $\psi_s > 0$ and $\psi_t > 0$ as we did in Appendix TA1.1. Then, if δ_s denotes the rate of the "self-market discount," the consumer utility and expenditure are given by

$$\mathcal{U}(q_{s1}, q_{t1}) = \alpha_s \left(q_{s1} - \psi_s \frac{q_{s1}^2}{2} \right) + \alpha_t \left(q_{t1} - \psi_t \frac{q_{t1}^2}{2} \right),$$

$$\mathcal{E}(q_{s1}, q_{t1} | p_{s1}, p_{t1}, \delta_s) = (p_{s1} - \delta_s q_{s1}) q_{s1} + p_{t1} q_{t1}.$$

Notice that under this scheme, the second market t is rather irrelevant. However, we keep it in the analysis to make proper comparisons of profits with the cross-market discount scheme. The profit of the firm is given by $\Pi_{st,1}(p_{s1}, p_{t1}, \delta_s) = (p_{s1} - \delta_s q_{s1}) q_{s1} + p_{t1} q_{t1}$.

As before, we solve the game in three stages and find the equilibrium by backward induction. In Stage 3, consumers maximize their surplus. In Stage 2, the firm sets its posted prices by maximizing its profit given the discount rate δ_s. The resulting prices, demand and profits in Stage 2 are displayed on the left side of Table TA2. We ensure nonnegative and finite prices, demand, consumer surplus and profits. Note that since purchase decisions can be made separately, the relevant consumer surplus is that associated with market s only. This surplus decreases in δ_s while firm profit increases in δ_s. Therefore, the optimal value of δ_s in Stage 3 will be the value at which the surplus becomes zero, which is given by $\delta_s = \alpha_s \psi_s / 2$. The right side of Table TA2 displays the prices, demand and profits at the global optimum.

	Conditional on δ_s		Global	
Quantity	Market s	Market t	Market s	Market t
Price	$\frac{\alpha_s^2 \psi_s}{-2\delta_s + 2\alpha_s \psi_s}$	$\frac{\alpha_t}{2}$	α_s	$\frac{\alpha_t}{2}$
Demand	$\frac{\alpha_s}{-2\delta_s + 2\alpha_s \psi_s}$	$\frac{1}{2\psi_t}$	$\frac{1}{\psi_s}$	$\frac{1}{2\psi_t}$
Profits	$\frac{\alpha_s^2}{-4\delta_s + 4\alpha_s \psi_s}$	$\frac{\alpha_t}{4\psi_t}$	$\frac{\alpha_s}{2\psi_s}$	$\frac{\alpha_t}{4\psi_t}$

Table TA2: Results for self-market discounts.

As is well documented in nonlinear pricing literature, a self-market discount (basically, a quantity discount) could result in higher prices and higher profits than uniform monopoly prices, as we also find in our model. However, the relevant question here is whether a firm will be better off by offering a cross-market discount or a self market discount. We compare against the corresponding cross-market discount monopoly-monopoly model with parameters ψ_s and ψ_t added as we described in Appendix TA1.1. Recall that according to that appendix, the exact expression of optimal profits using cross-market discounts depends on which constraint is binding. For simplicity we restrict our description to the case in which the effective price in the target market is zero. The resulting prices, demand and profits are displayed in Table TA3.

	Conditional on δ		Global	
Quantity	Market s	Market t	Market s	Market t
Price	$\frac{\alpha_s\alpha_t\psi_s\delta+2\alpha_s\psi_t}{-\delta^2+4\alpha_s\alpha_t\psi_s\psi_t}$	$\frac{\alpha_s\alpha_t(\delta+2\alpha_t\psi_s)\psi_t}{-\delta^2+4\alpha_s\alpha_t\psi_s\psi_t}$	$\frac{\alpha_s}{4}+\frac{\sqrt{\alpha_s}\sqrt{8\alpha_t\psi_s+\alpha_s\psi_t}}{4\sqrt{\psi_t}}$	α_t
Demand	$\frac{\alpha_t(\delta+2\alpha_s\psi_t)}{-\delta^2+4\alpha_s\alpha_t\psi_s\psi_t}$	$\frac{\alpha_s(\delta+2\alpha_t\psi_s)}{-\delta^2+4\alpha_s\alpha_t\psi_s\psi_t}$	$\frac{1}{4\psi_s}+\frac{\sqrt{8\alpha_t\psi_s+\alpha_s\psi_t}}{\sqrt{\alpha_s\psi_t}}$	$\frac{1}{\psi_t}$
Profits	$\frac{\alpha_s\alpha_t^2\psi_s(\delta+2\alpha_s\psi_t)^2}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}$	$-\frac{\alpha_s\alpha_t(\delta+2\alpha_t\psi_s)\left(\delta^2+\alpha_s(\delta-2\alpha_t\psi_s)\psi_t\right)}{\left(\delta^2-4\alpha_s\alpha_t\psi_s\psi_t\right)^2}$	$\frac{4\alpha_t\psi_s+\alpha_s\psi_t+\sqrt{\alpha_s\psi_t 8\alpha_t\psi_s+\alpha_s\psi_t}}{8\psi_s\psi_t}$	0

Table TA3: Results for cross-market discounts.

We find that the parameters ψ_s and ψ_t have two effects. They affect the rate at which firm profit increases with the discounts and the domain in which the model is well defined. Given that profits are increasing in δ these boundaries directly affect the global optimum of the model. We find that while the self-market discount scheme is very sensitive to the parameter ψ_s, cross-market discounts can sustain relatively higher profits. This leads to the interesting insight that consumption is greater in cross-market discounts because they distribute the additional consumption (motivated by the price discount) in two markets and, therefore, they can lead to higher profits for the firm. Figure TA2 displays this pattern by plotting profit for both schemes as a function of discounts where the function is well defined, for several values of ψ_s and ψ_t (for clarity of exposition we have fixed $\alpha_s = \alpha_t = 1$).

We can also verify that a cross-market discount could generate higher profits. To derive conditions under which this happens, we need to compare optimal profits for each strategy. In this case, in addition to the conditions above characterizing the solutions, we note that the condition under which the optimal profit from a cross-market discount strategy ($\Pi^{CMD}_{st,1}$) is greater than the

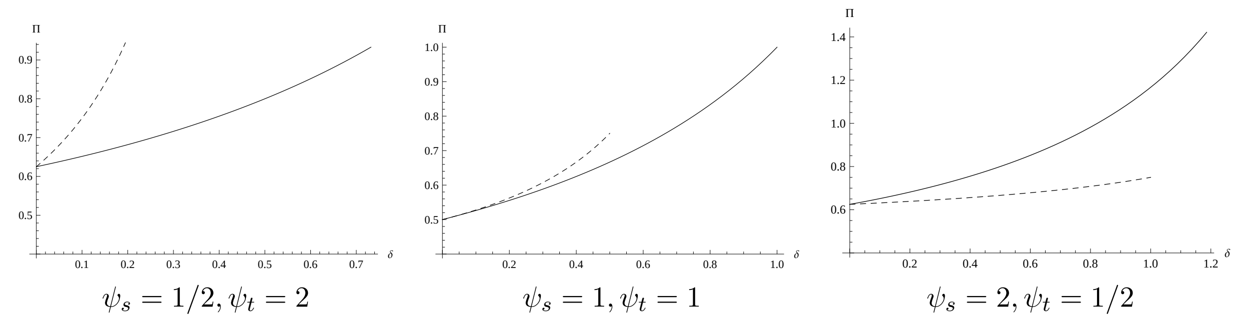


Figure TA2: In the above plots, the solid line shows total profit for the firm from a cross-market discount scheme, and the dashed line shows total profit for the firm from a self-market discount scheme.

optimal profit from a self-market discount strategy ($\Pi_{st,1}^{SMD}$) is given by

$$\Pi_{st,1}^{CMD} - \Pi_{st,1}^{SMD} = \frac{2\alpha_t\psi_s - 3\alpha_s\psi_t + \sqrt{(\alpha_s\psi_t)8\alpha_t\psi_s + \alpha_s\psi_t}}{8\psi_s\psi_t} > 0.$$

To gain further insight into this, in Figure TA3 we plot the regions in the ψ_s-ψ_t plane (left panel) and the α_s-α_t plane (right panel) where a cross-market discount is more profitable than a self-market discount. We conclude that a cross-market discount dominates a self-market discount except for the cases in which the source market is significantly more important or saturates at a significantly lower rate than target market. In particular, for equal rates of diminishing returns ($\psi_s = \psi_t = \psi$) and equal market importance ($\alpha_s = \alpha_t = \alpha$), a cross-market discount is always the preferred strategy.

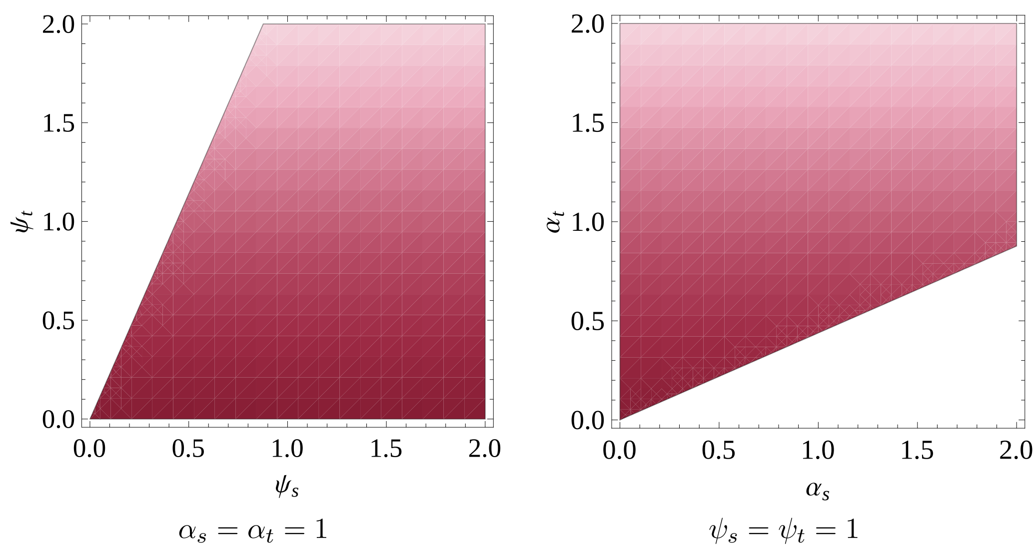


Figure TA3: In the shaded regions, a cross-market discount is prefered over a self-market discount.

TA1.3 Analysis with nonzero marginal costs

In this section, we provide details for the analysis of the monopoly-monopoly scenario with nonzero marginal costs. Let c_s and c_t be the marginal costs for the source and the market target, respectively. By adding such costs, there is no change in consumer utility maximization problem. The firm's total profit is given by $\Pi_{st,1}(p_{s1}, p_{t1}, \delta) = (p_{s1} - c_s)q_{s1} + (p_{t1} - c_t - \delta q_{s1})q_{t1}$.

We again solve the model in the same three stages as the basic model in Section **??** in the paper. The resulting prices, demand and profits as functions of δ after Stage 2 are displayed in Table TA4. We verify that the main characteristics of this model are the same as of the basic model — the optimal posted prices in both markets increase in δ, the optimal effective price in the target market decreases in δ, the quantities demanded in both markets increase in δ, and the total profit of the firm from the two markets increases in δ.

Quantity	Market s	Market t
Price	$c_s + \frac{\alpha_s(-c_t\delta+\alpha_t(-2cs+2\alpha_s+\delta))}{4\alpha_s\alpha_t-\delta^2}$	$c_t + \frac{\alpha_t(-c_s\delta+\alpha_s(-2c_t+2\alpha_t+\delta))}{4\alpha_s\alpha_t-\delta^2}$
Demand	$\frac{2c_s\alpha_t+c_t\delta-\alpha_t(2\alpha_s+\delta)}{-4\alpha_s\alpha_t+\delta^2}$	$\frac{2c_t\alpha_s+c_s\delta-\alpha_s(2\alpha_t+\delta)}{-4\alpha_s\alpha_t+\delta^2}$
Profits	$\frac{\alpha_s(2c_s\alpha_t+c_t\delta-\alpha_t(2\alpha_s+\delta))^2}{(-4\alpha_s\alpha_t+\delta^2)^2}$	$\frac{\left(2\alpha_s(c_t-\alpha_t)\alpha_t+(-c_s+\alpha_s)\alpha_t\delta+(-c_t+\alpha_t)\delta^2\right)(2c_t\alpha_s+c_s\delta-\alpha_s(2\alpha_t+\delta))}{(-4\alpha_s\alpha_t+\delta^2)^2}$

Table TA4: Stage 2 results for the monopoly-monopoly scenario with positive marginal costs.

The validity of the model requires positive and finite prices, demand, consumer surplus and profits. Interestingly, given that a cross-market discount couples the demand in the two markets, it could generate positive demand in a market that would not exist independently. By inspection, finite quantities are guaranteed if $\delta_s < \sqrt{\alpha_s\alpha_t}$. To check positiveness, given the model is well defined at $\delta = 0$, we need to only monitor those quantities that decrease as δ increases.

Given that the profit of the focal firm is increasing in δ, and the effective price in target market and the consumer surplus are decreasing in δ, by figuring out which one of the latter two is smaller and limiting it at zero, we can directly identify the optimal discount in Stage 1.

The main new insight that we obtain from this extension is that the firm can find it advantageous to set a cross-market discount large enough that the effective price in the target market is below marginal cost (but still nonnegative). The dark-shaded region in Figure TA4 shows the region in the α_s-α_t plane (with $c_s = c_t = 1$) in which the optimal cross-market discount implies an effective price below marginal cost in the target market. Broadly speaking, when the target market is

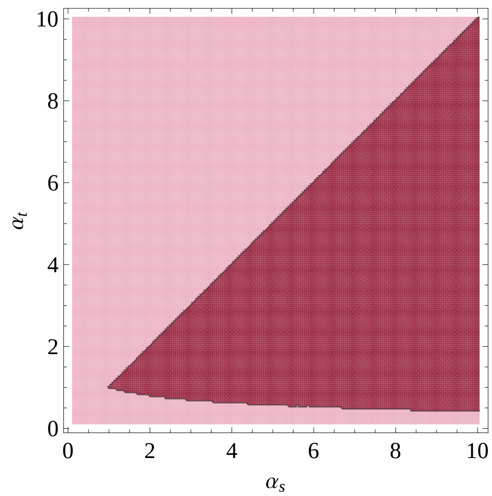


Figure TA4: In the dark-shaded region, the price in the target market is below the marginal cost in this market.

less important than the source market in the consumption utility function, the firm can have the incentive to take a loss in the target market to make overall higher total profit by selling more in the source market. On the other hand, if the target market is more important than the source market, it is not profitable to sacrifice this source of profit and the firm chooses a less aggressive discount policy.

TA2 Analysis of the marginal effects of parameters in the monopoly-duopoly scenario

To analyze how each of the parameters of the model affects the cross-market discount decisions, we computed the optimal rate δ for different points of the parameter space. To conduct a comprehensive analysis we need to define a fine grid for each of the parameters of the demand. The parameter θ_t is naturally bounded between 0 and 1, but the parameters α_s and α_t could take any positive value. Therefore, we reparametrized them as $\alpha_s = a_s/(1-a_s)$ and $\alpha_t = a_t/(1-a_t)$, where $a_s, a_t \in [0,1)$. Therefore, by varying each of θ_t, α_s and α_t on a grid of values between 0 and 1, we can conduct a complete numerical analysis. Figures TA5, TA6 and TA7 display the optimal rate of the cross-market discount as a function of θ_t, α_s and α_t, respectively, for various representative values of the other parameters. We conclude that when competition is present in the target market, the optimal rate of the cross-market discount decreases in the intensity of competition, and increases in the importance of the source and target markets.

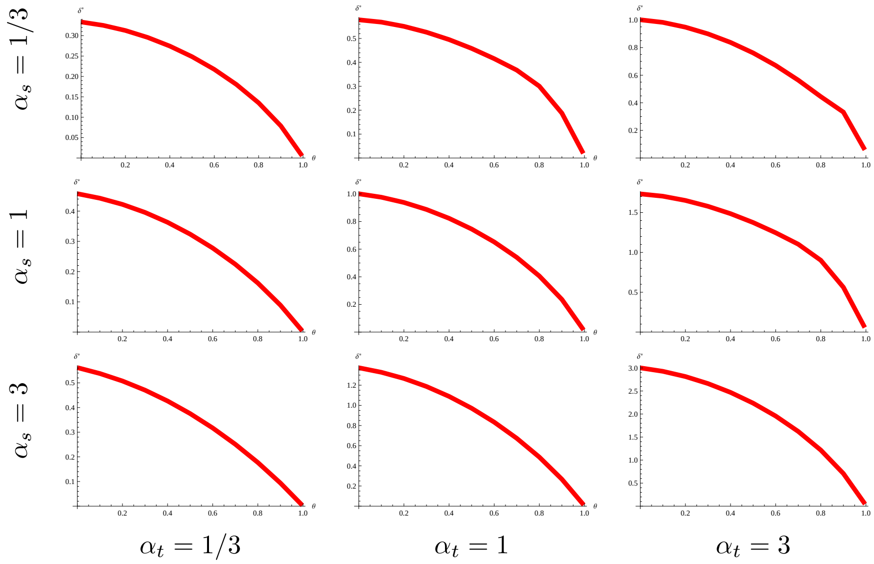


Figure TA5: Optimal rate of the cross-market discount as a function of θ_t.

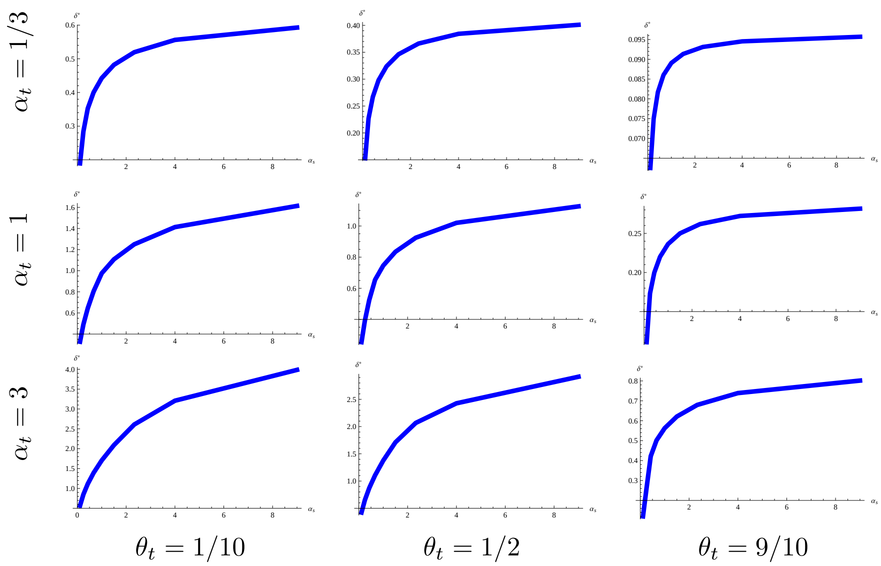


Figure TA6: Optimal rate of the cross-market discount as a function of α_s.

TA3 Analysis of the duopoly-duopoly scenario

TA3.1 Without competing cross-market discount program

In this section, we present a detailed analysis of the case in which the focal firm faces competition in both markets. We assume that Firm 1 is the focal firm which operates in both markets, Firm 2

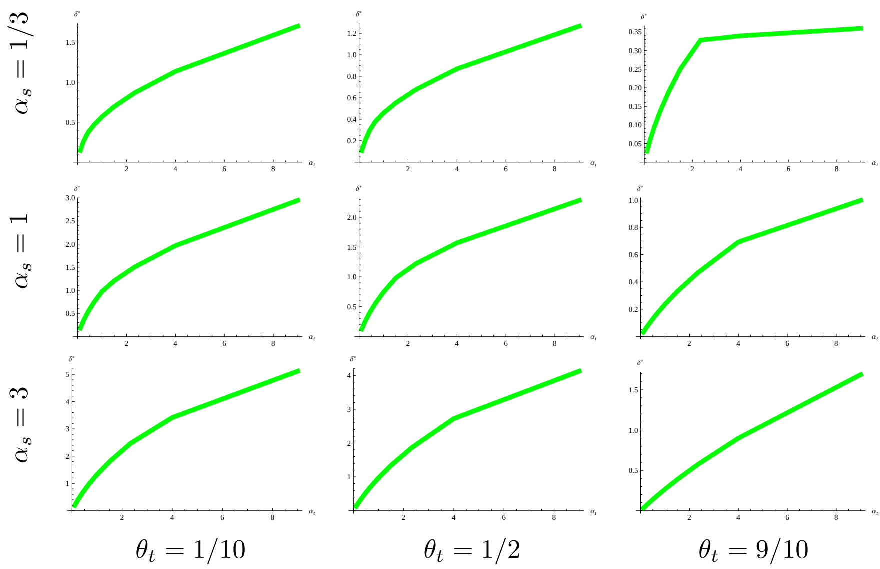


Figure TA7: Optimal rate of the cross-market discount as a function of α_t.

competes in market t, and a new firm, Firm 3, competes in market s. We keep all notation used in previous scenarios and introduce a new parameter θ_s that corresponds to the degree of competition in market s. Also, p_{s3} and q_{s3} denote the price and the quantity demanded from Firm 3 in the source market. Then, in the duopoly-duopoly scenario the consumer utility and expenditure are given by

$$\mathcal{U}(q_{s1}, q_{t1}) = \alpha_s \left(q_{s1} - \frac{q_{s1}^2}{2} + q_{s3} - \frac{q_{s3}^2}{2} - \theta_s q_{s1} q_{s3} \right) + \alpha_t \left(q_{t1} - \frac{q_{t1}^2}{2} + q_{t2} - \frac{q_{t2}^2}{2} - \theta_t q_{t1} q_{t2} \right),$$

$$\mathcal{E}(q_{s1}, q_{s3}, q_{t1}, q_{t2} \big| p_{s1}, p_{s3}, p_{t1}, p_{t2}, \delta) = p_{s1} q_{s1} + p_{s3} q_{s3} + (p_{t1} - \delta q_{s1}) q_{t1} + p_{t2} q_{t2}.$$

Quantities demanded as a function of posted prices and the cross-market discount are obtained by solving the consumer surplus maximization problem and given by

$$q_{s1} = \frac{\alpha_t(p_{s1} - p_{s3}\theta_s + \alpha_s\theta_s - \alpha_s - \delta) + \alpha_t\theta_t^2(-p_{s1} + p_{s3}\theta_s - \alpha_s\theta_s + \alpha_s) + p_{t1}\delta + \delta\theta_t(\alpha_t - p_{t2})}{\delta^2 - \alpha_s\alpha_t(\theta_s^2 - 1)(\theta_t^2 - 1)},$$

$$q_{s3} = \frac{\alpha_s(p_{s1}\alpha_t\theta_s(1 - \theta_t^2) + \delta\theta_s(p_{t1} - p_{t2}\theta_t + \alpha_t(\theta_t - 1)) + \alpha_s\alpha_t(\theta_s - 1)(\theta_t^2 - 1) - \delta^2) + p_{s3}(\alpha_s\alpha_t(\theta_t^2 - 1) + \delta^2)}{\alpha_s(\alpha_s\alpha_t(\theta_s^2 - 1)(\theta_t^2 - 1) - \delta^2)},$$

$$q_{t1} = \frac{\delta(p_{s1} - p_{s3}\theta_s) + p_{t1}(\alpha_s - \alpha_s\theta_s^2) + \alpha_s(\theta_s^2 - 1)\theta_t(p_{t2} - \alpha_t) + \alpha_s(\theta_s - 1)(\alpha_t\theta_s + \alpha_t + \delta)}{\delta^2 - \alpha_s\alpha_t(\theta_s^2 - 1)(\theta_t^2 - 1)},$$

$$q_{t2} = \frac{\alpha_t\theta_t(\delta(p_{s1} - p_{s3}\theta_s) + p_{t1}(\alpha_s - \alpha_s\theta_s^2) + \alpha_s(\theta_s - 1)(\alpha_t\theta_s + \alpha_t + \delta)) + (p_{t2} - \alpha_t)(\alpha_s\alpha_t(\theta_s^2 - 1) + \delta^2)}{\alpha_t(\alpha_s\alpha_t(\theta_s^2 - 1)(\theta_t^2 - 1) - \delta^2)}.$$

Foreseeing consumer responses, all firms simultaneously determine their posted prices to maximize their respective profits which are given by

$$\Pi_{st,1}(p_{s1}, p_{t1}, \delta) = p_{s1}q_{s1} + (p_{t1} - \delta q_{s1})q_{t1}, \quad \Pi_{t2}(p_{t2}) = p_{t2}q_{t2}, \quad \Pi_{s3}(p_{s3}) = p_{s1}q_{s3}.$$

The model is well defined for $\delta < \sqrt{\alpha_s \alpha_t (1-\theta_s^2)(1-\theta_t^2)}$, but the nature of the resulting equilibrium now depends on four parameters in a nonlinear fashion. For the special case in which $\alpha_s = \alpha_t = 1$ and $\theta_s = \theta_t = \theta$, Table TA5 displays the posted prices, quantities demanded and profits for the focal and competitor firms.[TA2]

Quantity	Focal Firm	Competitor
Price	$\frac{(\theta^2-1)(\delta^2(\theta-2)+(\theta-1)^2(1+\theta)(2+\theta))}{2(2-\delta)(\delta-1)(1+\delta)-3(\delta^2+\delta-3)\theta^2+(\delta-6)\theta^4+\theta^6}$	$\frac{(\delta+1-\theta^2)(\delta-1+\theta^2)(\delta-2+\theta+\theta^2)}{2(\delta-2)(\delta-1)(1+\delta)+3(\delta^2-3+\delta)\theta^2+(\delta-6)\theta^4-\theta^6}$
Demand	$\frac{\delta^2(\theta-2)+(\theta-1)^2(1+\theta)(2+\theta)}{2(\delta-2)(\delta-1)(1+\delta)+3(\delta^2+\delta-3+)\theta^2(6-\delta)\theta^4-\theta^6}$	$\frac{(\delta^2-1+\theta^2)(\delta-2+\theta+\theta^2)}{2(\delta-2)(\delta-1)(1+\delta)+3(\delta^2+\delta-3+)\theta^2(6-\delta)\theta^4-\theta^6}$
Profits	$\frac{(2-\delta-2\theta^2)(\delta^2(\theta-2)+(\theta-1)^2(1+\theta)(2+\theta))^2}{(2(2-\delta)(1-\delta)(1+\delta)+3(3-\delta-\delta^2)\theta^2+(\delta-6)\theta^4+\theta^6)^2}$	$\frac{(\delta+1-\theta^2)(\delta-1+\theta^2)(\delta^2-1+\theta^2)(\delta-2+\theta+\theta^2)^2}{(2(2-\delta)(1-\delta)(1+\delta)+3(3-\delta-\delta^2)\theta^2+(\delta-6)\theta^4+\theta^6)^2}$

Table TA5: Results for the duopoly-duopoly scenario.

Because of the more intense competition, posted prices and quantities demanded for the focal firm are not monotonically increasing in the rate of the cross-market discount. For moderate values of δ, prices and quantities demanded increase. However, when facing deeper discounts, both competitors react stronger, hurting the profitability of the strategy. As a consequence we observe that prices, quantities and therefore profits exhibit an inverted-U shape as a function of the cross-market discount.

We then proceed to numerically compute the optimal discount. As in the monopoly-duopoly scenario, the optimal cross-market discount is increasing in the importance of both markets (α_s and α_t), but decreasing in the degree of competition in the target market (θ_t). Increasing the degree of competition in the source market (θ_s) leads to a similar decreasing trend. Figure TA8 displays contour plots summarizing these relationships.

Extending previous results, we find that competition in any market prevents firms from fully extracting consumer surplus. A situation in which consumers have no surplus only occurs when competition parameters in both markets are close to zero, resembling the monopoly-monopoly

[TA2] For this special case, the equilibrium in the two markets is identical. This is a direct consequence of the simplifying assumption that markets have the same importance and intensity of competition.

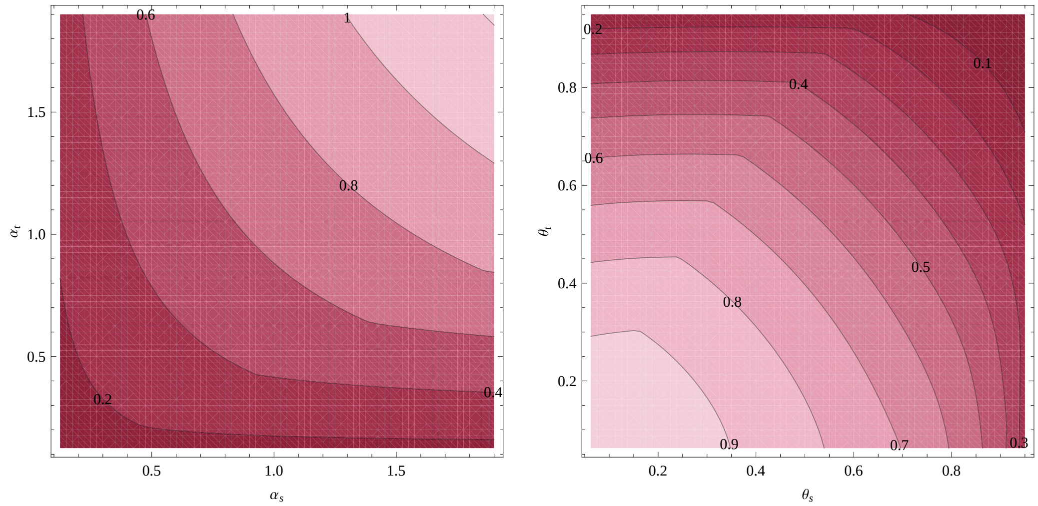


Figure TA8: Contour plots for optimal discount as a function of market importance (left panel: $\theta_s = \theta_t = 1/2$) and degree of competition (right panel: $\alpha_s = \alpha_t = 1$).

scenario. Also, we verify that competitor firms in both markets make positive profits in the interior of the parameter space. The profit of Firm 2, which competes in the target market, approaches zero when its market has no attractiveness to the consumer ($\alpha_t \to 0$), or asymptotically when the importance of the source market dominates the consumer's decision ($\alpha_s \to \infty$). Similarly, the profit of Firm 3, which competes in the source market, asymptotically goes to zero when the importance of the target market dominates consumer's decision ($\alpha_t \to \infty$). Regarding the degree of competition, profits approach zero when the corresponding parameter approaches its maximum (θ_s or $\theta_t \to 1$).

Figure TA9 displays total change in profits for all firms with respect to the case in which no cross-market discount is used, as a function of all four parameters of the model. Not surprisingly, the focal firm is always better off by offering a cross-market discount. At the same time, competitors in both markets are worse off when facing such a strategy.

Not obvious are the results derived from observing how the differential profit gains are affected by variations in the parameters of the model. When looking at importance parameters, we see that the focal firm benefits from and increase in the importance parameter in *either* market, and this also leads to a greater reduction in profits of the competitors in *both* markets. Variations derived from different degrees of competition also demonstrate the competitive advantage of the cross-market strategy, but the effect of competition is not symmetric between markets as we show

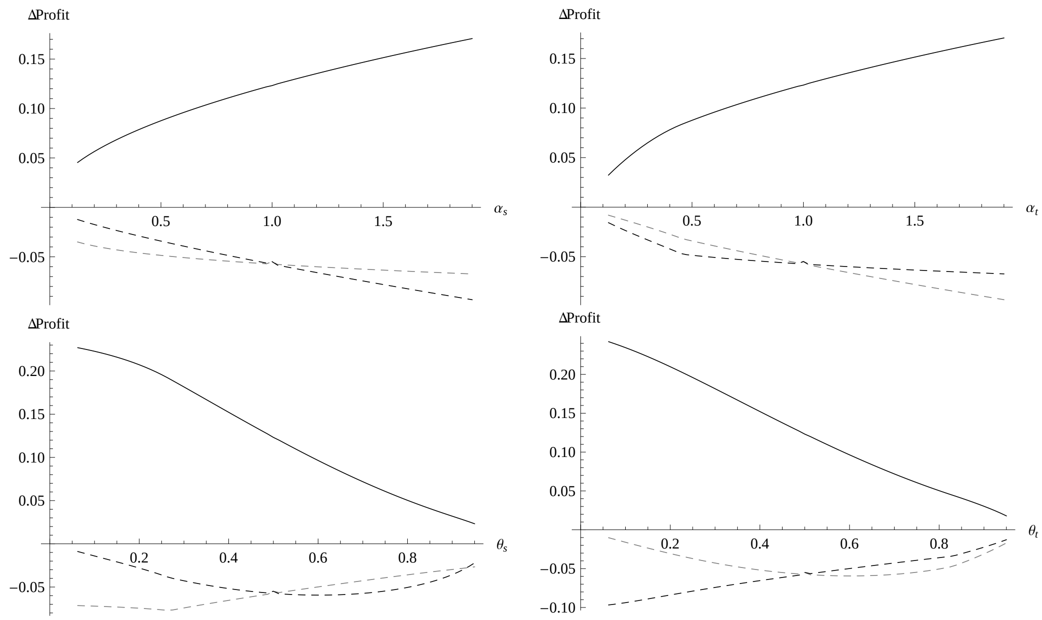


Figure TA9: Variation of profits with respect to the case in which no cross-market discount is used. In each panel, full black lines represent profit of firm 1 (focal firm), black-dashed lines represent profit of firm 3 (competitor in source market) and gray-dashed lines represent profit of firm 2 (competitor in target market).

with the impact of parameter θ_s. The largest loss that a cross-market discount could cause to the competitor in the source market is achieved for medium degrees of competition. When θ_s approaches 0, there is almost no interaction between the firms and what the focal firm does barely affects its competitors. When θ_s approaches 1, the competition is so strong and prices are so low to start with that neither firm can make any profits. Interestingly, the degree of competition in the source market also affect profits of the competitor in the target market. The smaller θ_s is, the larger is the reduction in profit for Firm 2, because the focal firm can leverage its cross-market discount strategy without worrying about Firm 3's competitive reactions. The impact of the parameter θ_t is analogous.

TA3.2 With competing cross-market discount program

In this section, we present a detailed analyzed of the case in which the firms compete in both source (s) and target (t) markets. We use the indices 1 and 2 to denote the two firms. Both firms have the possibility to implement cross-market discounts and, therefore, the game is symmetric. We keep all notation used in previous scenarios except for the need introduce an index on the cross-market discount offered by each firm, which we denote by δ_1 and δ_2. Then, in the symmetric duopoly-duopoly scenario the consumer utility and expenditure are given by

$$\mathcal{U}(q_{s1}, q_{t1}) = \alpha_s \left(q_{s1} - \frac{q_{s1}^2}{2} + q_{s2} - \frac{q_{s2}^2}{2} - \theta_s q_{s1} q_{s2} \right) + \alpha_t \left(q_{t1} - \frac{q_{t1}^2}{2} + q_{t2} - \frac{q_{t2}^2}{2} - \theta_t q_{t1} q_{t2} \right),$$

$$\mathcal{E}(q_{s1}, q_{s2}, q_{t1}, q_{t2} | p_{s1}, p_{s2}, p_{t1}, p_{t2}, \delta) = p_{s1} q_{s1} + p_{s2} q_{s2} + (p_{t1} - \delta_1 q_{s1}) q_{t1} + (p_{t2} - \delta_2 q_{s2}) q_{t2}.$$

Quantities demanded as a function of posted prices and cross-market discounts are obtained by solving the consumer surplus maximization problem. Demand functions for the special case in which $\alpha_s = \alpha_t = 1$ and $\theta_s = \theta_t = \theta$ are given by

$$q_{s1} = \frac{(p_{s1}-1+(p_{t1}-1)\delta_1)\left(\delta_2^2-1\right)+(p_{s2}-1-\delta_1+p_{t2}\delta_1+(p_{t2}-1+(p_{s2}-1)\delta_1)\delta_2)\theta+(p_{s1}-1+\delta_2-p_{t1}\delta_2)\theta^2-(p_{s2}-1)\theta^3}{\left(\delta_1^2-1\right)\left(\delta_2^2-1\right)-2(1+\delta_1\delta_2)\theta^2+\theta^4},$$

$$q_{s2} = \frac{\left(\delta_1^2-1\right)(p_{s2}-1+(p_{t2}-1)\delta_2)+(p_{s1}-1-\delta_1+p_{t1}\delta_1+(p_{t1}-1+(p_{s1}-1)\delta_1)\delta_2)\theta+(p_{s2}-1+\delta_1-p_{t2}\delta_1)\theta^2-(p_{s1}-1)\theta^3}{\left(\delta_1^2-1\right)\left(\delta_2^2-1\right)-2(1+\delta_1\delta_2)\theta^2+\theta^4},$$

$$q_{t1} = \frac{(p_{t1}-1+(p_{s1}-1)\delta_1)\left(\delta_2^2-1\right)+(p_{t2}-1-\delta_1+p_{s2}\delta_1+(p_{s2}-1+(p_{t2}-1)\delta_1)\delta_2)\theta+(p_{t1}-1+\delta_2-p_{s1}\delta_2)\theta^2-(p_{t2}-1)\theta^3}{\left(\delta_1^2-1\right)\left(\delta_2^2-1\right)-2(1+\delta_1\delta_2)\theta^2+\theta^4},$$

$$q_{t2} = \frac{\left(\delta_1^2-1\right)(p_{t2}-1+(p_{s2}-1)\delta_2)+(p_{t1}-1-\delta_1+p_{s1}\delta_1+(p_{s1}-1+(p_{t1}-1)\delta_1)\delta_2)\theta+(p_{t2}-1+\delta_1-p_{s2}\delta_1)\theta^2-(p_{t1}-1)\theta^3}{\left(\delta_1^2-1\right)\left(\delta_2^2-1\right)-2(1+\delta_1\delta_2)\theta^2+\theta^4}.$$

Foreseeing the above consumer responses, all firms simultaneously determine their posted prices to maximize their respective profits which are given by

$$\Pi_{st,1}(p_{s1}, p_{t1}, \delta_1) = p_{s1} q_{s1} + (p_{t1} - \delta_1 q_{s1}) q_{t1}, \qquad \Pi_{st,2}(p_{s2}, p_{t2}, \delta_2) = p_{s2} q_{s2} + (p_{t2} - \delta_2 q_{s2}) q_{t2}.$$

After solving for the optimal posted prices of both firms in both markets, we can derive optimal profits as a function of cross-market discounts. The equilibrium levels of discounts are found by intersecting reaction functions derived from taking first order conditions for each firm:

$$\frac{\partial \Pi_{st,1}(\delta_1, \delta_2)}{\partial \delta_1} = 0 \qquad \text{and} \qquad \frac{\partial \Pi_{st,2}(\delta_1, \delta_2)}{\partial \delta_2} = 0.$$

As in other scenarios, we need to verify nonnegative consumer surplus, nonnegative profits for each firm and positive effective price that the consumer is paying in the target market (this is the price after the cross-market discount is applied). The problem is analytically intractable and therefore we proceed numerically to compute its solutions. For simplicity, we compute the equilibria for the case $\alpha_s = \alpha_t = \alpha$ and $\theta_s = \theta_t = \theta$ for several values of θ. We are interested in analyzing how the situation of the firms and the consumers changes in comparison to the case in which no cross-market discounts are implemented. In the left panel of Figure TA10, we plot the profit of the firms and the consumer surplus w.r.t. θ with and without cross-market discounts for the case of $\alpha = 1$. The right panel displays the equilibrium level of the discount δ, the difference in profit for the retailers and the difference in the consumer surplus.

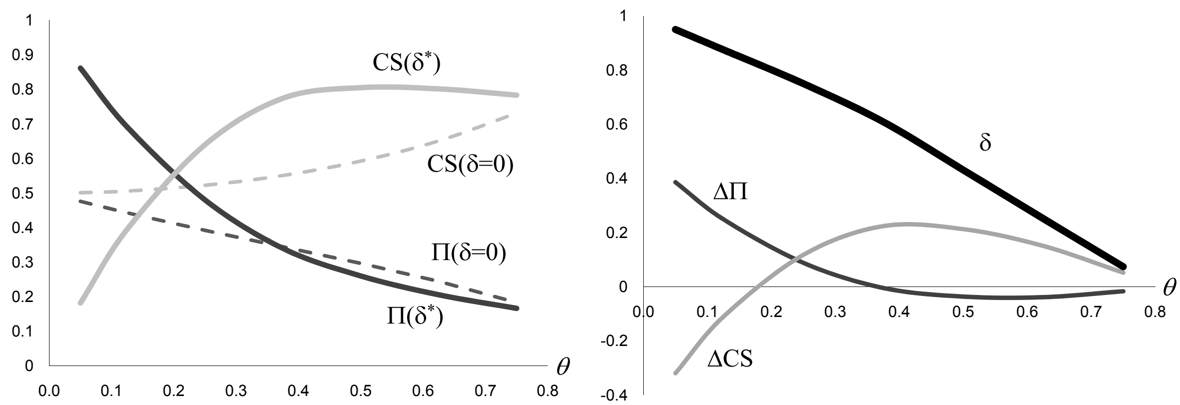


Figure TA10: Change in profits and surplus w.r.t. θ, compared to the case in which no cross-market discount is used.

We observe that the value of the cross-market discount rate (δ) decreases with the competitive intensity (θ). As is also expected, when the intensity of competition increases, consumer surplus increases while the firms' profits decrease, but at different rates depending on whether cross-market discounts are used or not.

For small values of θ, demand functions are almost independent and the model resembles the monopoly-monopoly scenario in which the retailers are better off when offering cross-market discounts because they can extract a larger surplus from the consumers. The introduction of cross-market discounts increases the rate at which firms' profits decrease with the competition parameter. As a consequence, in the other extreme, for relatively large values of θ, the introduction of cross-market discount leads to more intense competition, decreasing each firm's ability to extract surplus from the consumer, thus making the firms worse off with respect the case in which no cross-market discounts are allowed. A very interesting result is obtained when the intensity of competition is moderate. Here, all players, i.e., the consumer and both the firms, are better off with the introduction of cross-market discounts.

To confirm the generality of these results, we reparametrized $\alpha = a/(1-a)$, where $a \in [0, 1)$.

Therefore, by varying each of θ and α on a grid of values between 0 and 1, we can conduct a complete numerical analysis. In all of these cases, we found exactly the same patterns as above.

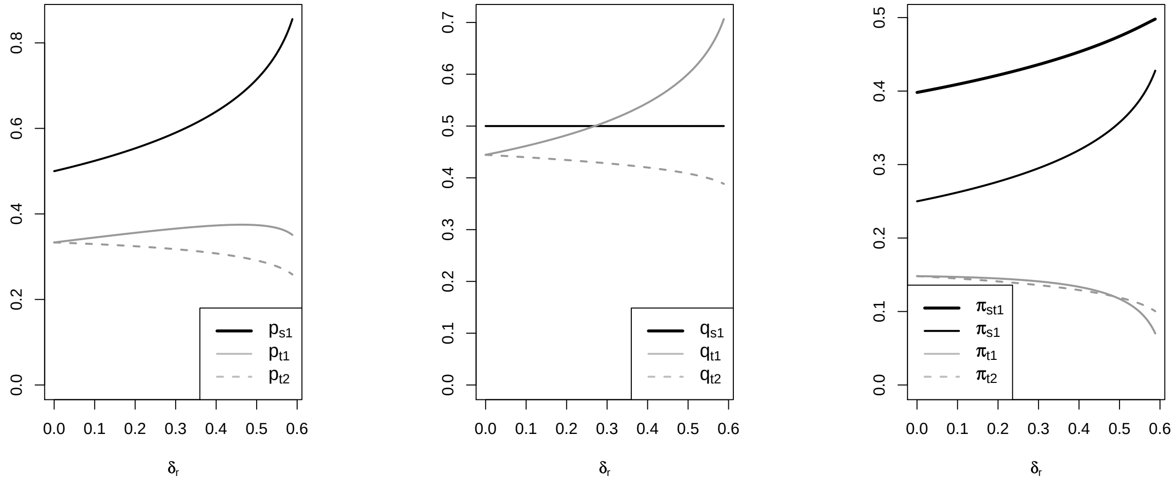
TA4 Analysis of cross-market discounts based on expenditure in the source market

In this section, we discuss the case in which discounts are based on the expenditure in the source market and not only on its quantity demanded. Our analysis here shows that, compared to this extension, the basic model is significantly more parsimonious and tractable, yet still captures most of the relevant insights of the discount scheme. In this extension, we base our analysis on the monopoly-duopoly scenario. The consumption utility remains unchanged, but the expenditure function changes. Let δ_r be the rate of cross-market discount. Then, the new expenditure function is given by $\mathcal{E}(q_{s1}, q_{t1}, q_{t2} | p_{s1}, p_{t1}, p_{t2}, \delta_r) = p_{s1}q_{s1} + (p_{t1} - \delta_r p_{s1} q_{s1})q_{t1} + p_{t2}q_{t2}$. In Stage 1, the demand functions are derived from consumer surplus maximization and are given by

$$
\begin{aligned}
q_{s1} &= \frac{\alpha_s\alpha_t\left(-1+\theta_t^2\right) + p_{s1}\left(\alpha_t(-1+\delta_r-\theta_t)(-1+\theta_t) + \delta_r(p_{t1}-p_{t2}\theta_t)\right)}{p_{s1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)},\\
q_{t1} &= \frac{p_{t1}\alpha_s + p_{s1}^2\delta_r - \alpha_s(\alpha_t + p_{s1}\delta_r) + \alpha_s(-p_{t2}+\alpha_t)\theta_t}{p_{s1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)},\\
q_{t2} &= \frac{(p_{t2}-\alpha_t)\left(\alpha_s\alpha_t - p_{s1}^2\delta_r^2\right) + \alpha_t(-p_{t1}\alpha_s + \alpha_s\alpha_t + p_{s1}(-p_{s1}+\alpha_s)\delta_r)\theta_t}{\alpha_t\left(p_{s1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)\right)}.
\end{aligned}
$$

In Stage 2, the analysis to find equilibrium prices is algebraically intractable and, therefore, we need to rely on numerical methods. We base our analysis on results computed on a grid of values of $\alpha_s, \alpha_t \in [0, 2]$ and $\theta_t \in [0, 1)$. To illustrate the nature of the resulting equilibrium, in Figure TA11 we display equilibrium prices, quantities demanded and profits as a function of the discount rate δ_r for the cases $\alpha_s = \alpha_t = 1$ and $\theta_t = 1/2$.

First, we note that the optimal posted price in the source market increases in δ_r. The direction for the posted price in the target market depends on the competitive structure: p_{t1} decreases in δ_r if θ_t is small, but in the presence of more intense competition it exhibits an inverted-U shape. Quantities sold in the target market are increasing in δ_r, but constant in the source market. Profits follow exactly the same patterns as in the quantity-based discount scenario. Total profit for firm 1



Posted prices | Quantities demanded | Profits

Figure TA11: Prices, quantities and profits as functions of the discount rate δ_r.

is increasing if θ_t is small, but as θ_t increases, it exhibits an inverted-U shape if the target market is important enough with respect to the source market. Similarly, as in the quantity-based discount case, consumer surplus is decreasing in δ_r if θ_t is small, but increases if θ_t is large.

Inspecting the optimal rate of discount we verify that, as before, in the monopoly-duopoly scenario the competitor makes positive profits and the consumer has positive surplus. Finally, we analyze how the optimal value of the cross-market discount rate (δ_r) varies with each of the model parameters. Numerical results show that it increases with the importance of target market in consumption utility (α_t), but decreases with the importance of source-market in consumption utility (α_s) and with the competitive intensity in the target market (θ_t).[TA3] Although the directional effect of α_s on the *rate* of the discount is different from the one we observe in the basic model, its effect on the *monetary size* of the discount and effective price paid in the target market remains unchanged.

Therefore, we confirm that the key insights we derived from the basic model generalize to the case in which the discount is a function of the expenditure in the source market. This provides support to the idea that the properties that we described before using a quantity-based discount model are not an artifact of the specific implementation of the cross-market discount policy.

[TA3] For these numerical solutions, we confirm that we indeed have maximums and all second-order conditions are satisfied.

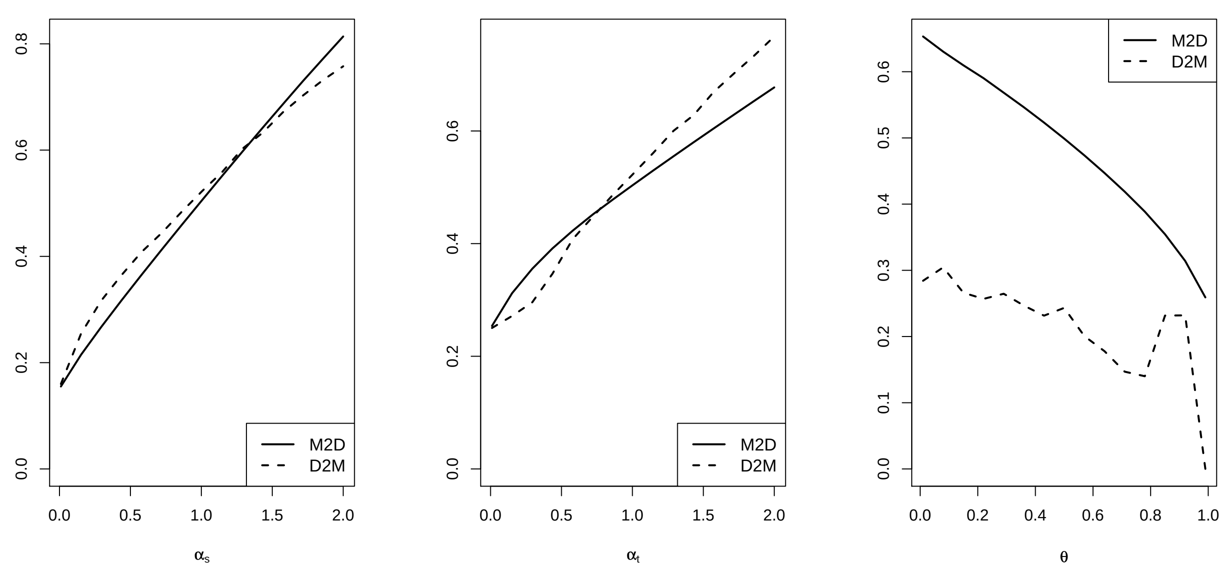
An interesting feature of this extension is that a discount from the monopoly to the duopoly market is not symmetric with respect to a discount from the duopoly to the monopoly market. The expenditure function when cross discounting from the duopoly to the monopoly market is given by $\mathcal{E}(q_{s1}, q_{t1}, q_{t2} \big| p_{s1}, p_{t1}, p_{t2}, \delta_r) = (p_{s1} - \delta_r p_{t1} q_{t1}) q_{s1} + p_{t1} q_{t1} + p_{t2} q_{t2}$. The resulting demand functions are given by

$$q_{s1} = \frac{p_{t1}\delta_r(p_{t1} + \alpha_t(-1+\theta_t) - p_{t2}\theta_t) + \alpha_s\alpha_t\left(-1+\theta_t^2\right) + p_{s1}\left(\alpha_t - \alpha_t\theta_t^2\right)}{p_{t1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)},$$

$$q_{t1} = \frac{p_{t1}(\alpha_s + p_{s1}\delta_r - \alpha_s\delta_r) + \alpha_s(\alpha_t(-1+\theta_t) - p_{t2}\theta_t)}{p_{t1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)},$$

$$q_{t2} = \frac{(p_{t2} - \alpha_t)\left(\alpha_s\alpha_t - p_{t1}^2\delta_r^2\right) + \alpha_t(\alpha_s\alpha_t + p_{t1}(\alpha_s(-1+\delta_r) - p_{s1}\delta_r))\theta_t}{\alpha_t\left(p_{t1}^2\delta_r^2 + \alpha_s\alpha_t\left(-1+\theta_t^2\right)\right)}.$$

Using these demand systems, we can explore which direction of the discount is more profitable for the focal firm by computing optimal profits under both strategies: cross discounting from monopoly to duopoly (M2D) and from duopoly to monopoly (D2M). Figure TA12 displays optimal profits for marginal variations of each parameter of the model around $(\alpha_s, \alpha_t, \theta_t) = (1, 1, 1/2)$.



$\alpha_t = 1, \theta_t = 1/2$ $\alpha_s = 1, \theta_t = 1/2$ $\alpha_s = 1, \alpha_t = 1$

Figure TA12: Variation in profits for discounts from monopoly to duopoly (M2D) and from duopoly to monopoly (D2M) as functions of model parameters.

First, we verify that if one of the market has no importance for the consumer ($\alpha_s = 0$ or $\alpha_t = 0$) the cross-market discount strategy has no value and there is no difference in using either direction.

Additionally, we note that profits for both strategies are increasing in market importance in any market. In other words, the focal firm could take advantage of a more valuable market regardless of whether it is cross discounting from the monopoly to the duopoly market, or from the duopoly to the monopoly market. Upon comparing profitability, we conclude that a cross-market discount should be offered from the more valuable to the less valuable market. For example, the leftmost panel of Figure TA12 displays profits as a function of importance of monopolistic market s (α_s), keeping constant the importance of duopoly market t ($\alpha_t = 1$) and competition intensity ($\theta_t = 1/2$). We observe that when α_s is small relative to α_t, profits of cross discounting from s to t are higher than discounting from t to s. This relation reverses when α_s gets larger relative to α_t. The center panel displays the marginal variations with respect to α_t and confirms this pattern. Finally, the rightmost panel illustrates how profits of the cross-market discount strategy in each direction vary with the degree of competition (θ_t) in the duopoly market t. We conclude that when importance of both markets is comparable, the focal firm should prefer to offer discounts that are redeemable in the market that faces competition.