Web Appendix for "The Seeds of Negativity: Knowledge and Money"

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1 Technical Details of Theoretical Model

The first two subsections of this appendix focuses on the second stage of the game (i.e., the negativity decision) and the next two on the first stage (i.e., the choice of budget).

1.1 Negativity Decision and Second Order Condition

In this subsection we identify sufficient conditions to ensure that the objective function is concave.

Recall that the first order condition is

$$\begin{aligned} \lambda_j &\equiv p_j(e_j^*)\left[1-p_j(e_j^*)\right]F(e_j^*)=0 \qquad (1)\\ \text{where } F(e_j^*) &\equiv \left(\alpha\gamma\left(e_j^*\right)^{\gamma-1}\left[a_{j0}+\left(e_j^*\right)^{\gamma}\right]^{\alpha-1}-\beta\gamma\left(E_j-e_j^*\right)^{\gamma-1}\left[b_{-j0}+\left(E_j-e_j^*\right)^{\gamma}\right]^{\beta-1}\right) \end{aligned}$$

Since $F(e_j^*)$ is always zero, to ensure concavity of the objective function we only need to verify that $\frac{\partial F(e_j^*)}{\partial e_j^*}<0$. It is easy to show that

$$\begin{aligned} \frac{\partial F(e_j^*)}{\partial e_j^*} &= A_j+B_j \qquad (2)\\ \text{where } A_j &\equiv \alpha\gamma\left(e_j^*\right)^{\gamma-2}\left(a_{j0}+\left(e_j^*\right)^{\gamma}\right)^{\alpha-2}\left[(\alpha\gamma-1)\left(e_j^*\right)^{\gamma}+(\gamma-1)a_{j0}\right]\\ \text{and } B_j &\equiv \beta\gamma\left(E_j-e_j^*\right)^{\gamma-2}\left(b_{-j0}+\left(E_j-e_j^*\right)^{\gamma}\right)^{\beta-2}\left[(\beta\gamma-1)\left(E_j-e_j^*\right)+(\gamma-1)b_{-j0}\right] \end{aligned}$$

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We note that α, β, γ, e_j^*, $\left(E_j - e_j^*\right)$, a_{j0}, and b_{-j0}, are all non-negative. Hence, the two bracketed terms can be used to create a sufficient condition for concavity of the objective function. Specifically, since $\gamma < 1$, $(\gamma - 1)$ is negative, and thus the only requirement is that the terms $(\alpha\gamma - 1)$ and $(\beta\gamma - 1)$ are each less than 1. Thus, the *sufficient* conditions to ensure concavity are $\alpha\gamma < 1$ and $\beta\gamma < 1$. We assume that these conditions are met and verify it in the estimation.

1.2 Negativity Decision and Role of Knowledge and Budget

We first analyze the effect of the knowledge variables a_{j0} and b_{-j0} noting that the variables a_{-j0} and b_{j0} have no effect on j's decisions. Specifically, using the implicit function theorem we get:

$$\frac{\partial e_j^*}{\partial a_{j0}} = -\frac{1}{\frac{\partial F}{\partial e_j^*}} \frac{\partial F(e_j^*)}{\partial a_{j0}} \text{ and } \frac{\partial e_j^*}{\partial b_{-j0}} = -\frac{1}{\frac{\partial F}{\partial e_j^*}} \frac{\partial F(e_j^*)}{\partial b_{-j0}} \tag{3}$$

From the second order condition we know that $-\frac{1}{\frac{\partial F}{\partial e_j^*}} > 0$ (see the discussion in the previous subsection). Thus, the sign of $\frac{\partial e_j^*}{\partial a_{j0}}$ and $\frac{\partial e_j^*}{\partial b_{-j0}}$ depend on $\frac{\partial F}{\partial a_{j0}}$ and $\frac{\partial F}{\partial b_{-j0}}$, respectively, and these two are:

$$\begin{aligned} \frac{\partial F(e_j^*)}{\partial a_{j0}} &= \alpha(\alpha-1)\gamma\left(e_j^*\right)^{\gamma-1}\left[a_{j0} + \left(e_j^*\right)^{\gamma}\right]^{\alpha-2} \\ \frac{\partial F(e_j^*)}{\partial b_{-j0}} &= -\beta(\beta-1)\gamma\left(E_j - e_j^*\right)^{\gamma-1}\left[b_{-j0} + \left(E_j - e_j^*\right)^{\gamma}\right]^{\beta-2} \end{aligned} \tag{4}$$

Since knowledge and advertising are non-negative and $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ the sign of the derivatives above depend on $(\alpha - 1)$ and $(\beta - 1)$ respectively. Thus, when $\alpha > 1$ an increase in a_{j0} leads to an increase in e_j^* and when $\alpha < 1$ the opposite holds. Accordingly, for $\beta > 1$ an increase in b_{-j0} leads to a decrease in e_j^* and when $\beta < 1$ the opposite holds.

Next, we examine the effect of budget on negativity. We start by showing that both e_j^* and $\left(E_j - e_j^*\right)$ increase in E_j and then show that the increase $\left(E_j - e_j^*\right)$ is larger (i.e., that negative ad spending increases faster than positive). We start with the effect on e_j^*. It is easy to show that:

$$\frac{\partial e_j^*}{\partial E_j} = \frac{1}{\frac{\partial F(e_j^*)}{\partial e_j^*}} B_j \tag{5}$$

Since we have shown above that $B_j < 0$ and $\frac{\partial F}{\partial e_j^*} < 0$, we get that $\frac{\partial e_j^*}{\partial E_j} > 0$. Next, we show that negativity also increases in the budget. It is easy to show that:

$$\frac{\partial(E_j - e_j^*)}{\partial E_j} = \frac{1}{\frac{\partial F}{\partial e_j^*}} \left(\frac{\partial F}{\partial e_j^*} + \frac{\partial F}{\partial E_j}\right) = \frac{1}{\frac{\partial F}{\partial e_j^*}} \left(A_j + B_j + (-B_j)\right) = \frac{1}{\frac{\partial F}{\partial e_j^*}} A_j \tag{6}$$

and since we have shown above that $A_j < 0$ and $\frac{\partial F}{\partial e_j^*} < 0$, we get that $\frac{\partial(E_j - e_j^*)}{\partial E_j} > 0$.

What remains to be shown is whether the portion of negativity grows with the budget. Hence, we are interested in $\frac{\partial(E_j - e_j^*)/E_j}{\partial E_j}$, which can be restated as simply $\frac{\partial\left(-e_j^*/E_j\right)}{\partial E_j}$. We first show by contradiction that $\frac{\partial\left(-e_j^*/E_j\right)}{\partial E_j} > 0$ for $\alpha < 1 < \beta$.

Assume that $\frac{e_j^*}{Ej}$ is a *non-negative* function of E_j. Thus, when E_j increases, $\left(\frac{e_j^*}{E_j - e_j^*}\right)^{\gamma-1} \frac{\alpha}{\beta}$ is non-increasing, since $\gamma < 1$ and $\frac{\alpha}{\beta} > 0$. Note that the first-order condition can be written as:

$$\left(\frac{e_j^*}{E_j - e_j^*}\right)^{\gamma-1} \frac{\alpha}{\beta} = \frac{\left(b_{-j0} + \left(E_j - e_j^*\right)^{\gamma}\right)^{\beta-1}}{\left(a_{j0} + \left(e_j^*\right)^{\gamma}\right)^{\alpha-1}} \tag{7}$$

Since $\left(\frac{e_j^*}{E_j - e_j^*}\right)^{\gamma-1} \frac{\alpha}{\beta}$ is non-increasing in E_j so is the right hand side (denoted h). Since $\beta > 1$ and (as just shown) $\left(E_j - e_j^*\right)$ is a positive function of E_j the numerator of h is increasing. Further $\alpha < 1$ and (as just shown) e_j^* is a positive function of E_j, the denominator must decrease. However, this implies that h is increasing. This produces the contradiction and proves $\frac{e_c^*}{E_c}$ is a negative function of E_c.

A similar argument can be made to show that $\frac{\partial\left(-e_j^*/E_j\right)}{\partial E_j} < 0$ for $\alpha > 1 > \beta$.

When either both parameters are bigger than one ($\alpha > 1$ and $\beta > 1$) or both are smaller than one ($\alpha < 1$ and $\beta < 1$) the logic above cannot assist us in determining the sign of $\frac{\partial\left(-\frac{e_j^*}{E_j}\right)}{\partial E_j}$. Furthermore, it is easy to show that under such conditions this derivative can be either positive or negative.

1.3 Budget Decision and Second Order Condition

Recall that the first order condition for the budget decision is:

$$p_j\left(E_j, E_{-j}\right)\left[1 - p_j\left(E_j, E_{-j}\right)\right]\delta_j\left(E_j\right) - \frac{\partial C_j}{\partial E_j} = 0 \tag{8}$$

where $\delta_j\left(E_j\right) \equiv \beta\gamma\left(E_j - e_j^*\left(E_j\right)\right)^{\gamma-1}\left[b_{-j0} + \left(E_j - e_j^*\left(E_j\right)\right)^{\gamma}\right]^{\beta-1}$, and thus the second order condition is:

$$\lambda_{EE} = \frac{\partial p_j p_{-j} \delta_j}{\partial E_j} - \frac{\partial^2 C(E_j)}{\partial E_j^2} \text{ where} \tag{9a}$$

$$\frac{\partial p_j p_{-j} \delta_j}{\partial E_j} = p_j p_{-j}\left[(1-2p_j)\delta_j^2 + \frac{\partial \delta_j}{\partial E_j}\right] \tag{9b}$$

$$\frac{\partial \delta_j}{\partial E_j} = \xi \frac{\partial\left(E_j - e_j^*(E_j)\right)}{\partial E_j}[(\gamma-1)b_{0-j} + (\beta\gamma-1)(E_j - e_j^*)^{\gamma}] \tag{9c}$$

$$\xi = \beta\gamma(b_{0j} + (E_j - e_j^*)^{\gamma})^{\beta-2}(E_j - e_j^*)^{\gamma-2} \tag{9d}$$

We have shown above that $\frac{\partial\left(E_j - e_j^*(E_j)\right)}{\partial E_j} > 0$ and required that $\beta\gamma < 1$ and thus it is immediate that $\frac{\partial \delta_j}{\partial E_j} < 0$. Further we note that $p_j p_{-j}$, δ_j^2, and $\frac{\partial^2 C(E_j)}{\partial E_j^2}$ are always positive by definition. Thus, the only potential positive value comes from $(1-2p_j)$. This will always be negative for the candidate with the larger share and positive for her opponent. However, it is immediate to see that for any given value of $\frac{\partial p_j p_{-j} \delta_j}{\partial E_j}$ with sufficiently convex costs (i.e., if $\frac{\partial^2 C(E_j)}{\partial E_j^2}$ is high enough), λ_{EE} will be less than zero and the objective function is concave. We assume that the cost function is convex enough in the theoretical model and enforce it in the estimation.

1.4 Proof of proposition 1

First, we show that the optimal budget is always positive.

Lemma 1 $E_j^*(E_{-j}) > 0$ *for any* E_{-j}.

Proof. We want to show that the first order condition for E_j is always positive at $E_j = 0$. Specifically, $p_j\left(0, E_{-j}\right)\left[1 - p_j\left(0, E_{-j}\right)\right]\delta_j(0) - C_j'(0) > 0$. Of course, we know that $p_j\left(0, E_{-j}\right)\left[1 - p_j\left(0, E_{-j}\right)\right] \leq 0.25 < \infty$ for every E_{-j}. We also know that $C_j'(0) < \infty$. Finally, we will show that $\delta_j(0) = \infty$. For $E_j = 0$ we know that $e_j^*(0) = 0$ and since $\gamma < 1$ we get that $\delta_j(0) = \infty$. ■

Proposition 2 *There exists an equilibrium in pure strategies. Specifically, there exists a pair* $\left(\dot{E}_j, \dot{E}_{-j}\right)$ *such that* $\dot{E}_j = E_j^*\left(\dot{E}_{-j}\right)$ *and* $\dot{E}_{-j} = E_{-j}^*\left(\dot{E}_j\right)$.

Proof. We will show that there exists an E_j^0 such that $E_j^0 < E_j^*(E_{-j}^*(E_j^0))$ and an E_j^1 such that $E_j^1 \geq E_j^*(E_{-j}^*(E_j^1))$. Since the reaction functions (i.e., $E_j^*(E_{-j})$ and $E_{-j}^*(E_j)$) are continuous this immediately means that there exists a $\dot{E}_j$ such that $E_j^* = E_j^*(E_{-j}^*(\dot{E}_j))$ which is what we want to show (note that we can denote $\dot{E}_{-j} = E_{-j}^*(\dot{E}_j)$).

Let $E_j^0 = 0$. Since we know from the Lemma that $E_j^*(E_{-j}) > 0$ for any E_{-j} we know that $E_j^*(E_{-j}^*(E_j^0)) > 0 = E_j^0$.

Let E_j^1 be the solution to the following equation $\frac{1}{4}\delta_j(E_j) = C_j'(E_j)$. Since $p_j p_{-j} \leq \frac{1}{4}$ we know that $E_j^* \leq E_j^1$. In other words, the optimal E_j cannot be bigger than E_j^1. Thus, $E_j^1 \geq E_j^*(E_{-j}^*(E_j^1))$. ■

Next, we show that this equilibrium is unique. We need the following for the proof. Let $\hat{E}_{-j} = \arg\max_{E_{-j}} E_j^*(E_{-j})$. It is immediate to show that $p_j\left(E_j^*(\hat{E}_{-j}), \hat{E}_{-j}\right) = 0.5$. Furthermore, it is easy to show that $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} = \left(\frac{-1}{\lambda_{EE}}\right)\delta_j\left[\frac{\partial p_j p_{-j}}{\partial E_{-j}}\right]$ and that $\frac{\partial p_j p_{-j}}{\partial E_{-j}} = (1-2p_{-j})\frac{\partial p_{-j}}{\partial E_{-j}}$ and notice that we know that $\delta_j > 0$ and (from the first and second order conditions) that $\frac{\partial p_{-j}}{\partial E_{-j}} > 0$ and $\left(\frac{-1}{\lambda_{EE}}\right) > 0$. Accordingly, we know that $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ and $\frac{\partial p_j p_{-j}}{\partial E_{-j}} > 0$ for $E_{-j} < \hat{E}_{-j}$ and $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} < 0$ and $\frac{\partial p_j p_{-j}}{\partial E_{-j}} < 0$ for $E_{-j} > \hat{E}_{-j}$.

Proposition 3 *The pair $\left(\dot{E}_j, \dot{E}_{-j}\right)$ is unique. Specifically, for any $E_j' \neq \dot{E}_j$ we have $E_j^*\left(E_{-j}^*\left(E_j'\right)\right) \neq E_j'$.*

Proof. We show this by contradiction. We start by showing that the derivatives $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}}$ and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j}$ cannot have the same sign in equilibrium. Consider, for example, the case where both of them are positive (i.e., $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} > 0$). If $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ we know that $\frac{\partial p_j p_{-j}}{\partial E_{-j}} > 0$ and thus $(1-2p_{-j}) > 0$. However, if $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} > 0$ we also know that $\frac{\partial p_j p_{-j}}{\partial E_j} > 0$ and thus $(1-2p_j) > 0$ (which means that $(1-2p_{-j}) < 0$). Thus, a contradiction. Based on the same logic, it is easy to show that both derivatives cannot be negative in equilibrium.

Next, assume that there is more than one equilibrium. Specifically, points A and B are equilibria (i.e., $E_j^A = E_j^*(E_{-j}^A)$ and $E_{-j}^A = E_{-j}^*(E_j^A)$ and the same holds for B). Based on the above, we know that the only possible pairs to consider (without loss of generality) are (i) for both A and B we have $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} < 0$ and (ii) for A we have $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} < 0$ and for B we have $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} < 0$ and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} > 0$).

We proceed by showing that case (i) leads to a contradiction. From $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ we get that $\left(E_j^A - E_j^B\right)\left(E_{-j}^A - E_{-j}^B\right) > 0$. Specifically, when $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$, an increase in E_{-j} leads to an increase in E_j. Thus if $E_{-j}^A > E_{-j}^B$ we also have $E_j^A > E_j^B$ and thus $\left(E_j^A - E_j^B\right)\left(E_{-j}^A - E_{-j}^B\right) > 0$. Of course, the case $E_{-j}^A < E_{-j}^B$ also leads to $\left(E_j^A - E_j^B\right)\left(E_{-j}^A - E_{-j}^B\right) > 0$. However, from $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} < 0$ we get that $\left(E_j^A - E_j^B\right)\left(E_{-j}^A - E_{-j}^B\right) < 0$. Thus, a contradiction.

We complete by showing that case (ii) leads to a contradiction. Since $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} > 0$ for A and $\frac{\partial E_j^*(E_{-j})}{\partial E_{-j}} < 0$ for B, we know that $E_{-j}^A < \hat{E}_{-j} < E_{-j}^B$. This means that since δ decreases in E while C' increases in E, we get that $\frac{\delta_{-j}^A}{C'^A_{-j}} > \frac{\delta_{-j}^B}{C'^B_{-j}}$ and thus (from the first order conditions) we know that $p_j^A p_{-j}^A < p_j^B p_{-j}^B$. However, since $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} < 0$ for A and $\frac{\partial E_{-j}^*(E_j)}{\partial E_j} > 0$ for B, we know that

$E_j^B < \hat{E}_j < E_j^A$. This means that $\frac{\delta_{-j}^A}{C'^A_{-j}} < \frac{\delta_{-j}^B}{C'^B_{-j}}$ and thus $p_j^A p_{-j}^A > p_j^B p_{-j}^B$. Thus, a contradiction. ■

2 Markov Chain Monte Carlo Estimation

In this section, we discuss the likelihood, prior, and sampling method used for the structural estimation described briefly in the paper. In what follows we drop the market subscripts for ease of exposition and note that the likelihood for all markets is a product over all the pairs of candidates. We are using Bayesian estimation to recover the posterior distribution of $\theta = \{\alpha, \beta, \rho, \sigma_a^2, \sigma_b^2, \eta, \sigma_e^2, \sigma_w^2, a_x, b_x, c_x, e_x\}$. We also draw unobservables $(\nu_j^a, \nu_{-j}^a, \nu_j^b, \nu_{-j}^b)$. In the discussion below, we group these unobservables denoting the four of them together as ν_{jj}.

Our observed endogenous variables are budget (denoted as E_j^*) and negativity (denoted as N_j). Negativity has censored values as described in equation 12 in the main text. For ease of exposition, we group our exogenous variables $X_j = \{x_j^a, x_j^b, x_j^c, x_j^e\}$ and denote $e_j^*(E_j^*, x_j^a, x_{-j}^b, \nu_j^a, \nu_{-j}^b, \theta)$, the implicit function defined in equation 1 as simply e_j^* (but note that we condition on all the arguments wherever e_j^* appears). We use a one-dimensional solver to calculate the value of e_j^* in each iteration. We have the following likelihood of observing a single candidate's N_j:

$$\begin{aligned} p(N_j|X_j, e_j^*, E_j^*, \theta) \quad = \quad & \phi(N_j; \frac{E_j^* - e_j^*}{E_j^*} + \mu_{ej}, \sigma_e^2)^{1(0<N_j<1)} \\ & *\Phi(0; \frac{E_j^* - e_j^*}{E_j^*} + \mu_{ej}, \sigma_e^2)^{1(N_j=0)} \\ & *(1 - \Phi(1; \frac{E_j^* - e_j^*}{E_j^*} + \mu_{ej}, \sigma_e^2))^{1(N_j=1)} \end{aligned} \tag{10}$$

The likelihood of observing a pair of budgets, (E_j^*, E_{-j}^*) is derived from a variable transformation from $\omega = (\omega_j, \omega_{-j})$ to (E_j^*, E_{-j}^*). This variable transformation is well-defined because the model has a unique equilibrium when the second order condition (SOC) holds. This results in the following likelihood for the pair (E_j^*, E_{-j}^*):

$$p(E_j^*, E_{-j}^*|X_j, X_{-j}, e_j^*, e_{-j}^*, \nu_{jj}, \theta) \quad = \quad f_{MVN}(\omega; 0, I\sigma_\omega^2) \left| J_E(\omega(E_j^*, E_{-j}^*)) \right| \tag{11}$$

$$J_E(\omega) \quad = \quad \begin{vmatrix} \frac{\partial \omega_j}{\partial E_j^*} & \frac{\partial \omega_j}{\partial E_{-j}^*} \\ \frac{\partial \omega_{-j}}{\partial E_j^*} & \frac{\partial \omega_{-j}}{\partial E_{-j}^*} \end{vmatrix} \tag{12}$$

where f_{MVN} is a two-dimensional multivariate normal distribution.

Thus, we write the likelihood of the observed data for a pair of candidates, $y = (E_j^*, E_{-j}^*, N_j, N_{-j})$

given the parameters, θ, exogenous factors X_j, X_{-j}, and the unobservables, ν_{jj}, as

$$L(y|\theta) = p(N_j|X_j, e_j^*, E_j^*, \theta)p(N_{-j}|X_{-j}, e_{-j}^*, E_{-j}^*, \theta)p(E_{j,}^* E_{-j}^*|X_j, X_{-j}, e_j^*, e_{-j}^*, \nu_{jj}, \theta)$$

To complete the model, we specify the prior distribution. The prior probability of the unobserved knowledge variables, ν_{jj}, is reported in equation 11 in the main text. We enforce the SOC through the prior by imposing two sufficient conditions. First, we directly impose the SOC given in equation 9a for the first stage decision (E_j^*) conditional on the optimal decision function in the second stage. We denote $1(SOC_E)$ as the indicator function taking value 1 when this SOC is met and zero otherwise and incorporate this indicator function into the prior. Second, we use sufficient conditions for the first stage SOC (i.e., for e_j^*) which places restrictions on the α, β, and γ jointly. In particular, $\alpha\gamma < 1$ and $\beta\gamma < 1$. We also note that since the knowledge variables in the reduced form regressions are significant, we expect that α and β do not equal one. Further, since the mapping to e^* provided in equation 1 becomes degenerate when α or β are one and numerically unstable as they approach one, we incorporate a low probability of approaching one into the prior. To do so we use the following joint prior.

$$\begin{aligned}
p(\alpha, \beta, \gamma) &= p(\alpha|\gamma)p(\beta|\gamma)p(\gamma) \\
p(\alpha|\gamma) &= .5\left[1(\alpha < 1)f_{Beta}(\alpha; \alpha_p, \alpha_p) + 1(\alpha > 1)f_{Beta}(\frac{\alpha\gamma - \gamma}{1-\gamma}; \alpha_p, \alpha_p)\right] \\
p(\beta|\gamma) &= .5\left[1(\beta < 1)f_{Beta}(\beta; \beta_p, \beta_p) + 1(\beta > 1)f_{Beta}(\frac{\beta\gamma - \gamma}{1-\gamma}; \beta_p, \beta_p)\right] \\
p(\gamma) &= f_{Beta}(\gamma_p, \gamma_p) \qquad (13)
\end{aligned}$$

where f_{Beta} denotes the beta distribution with the standard parameterization. For the parameters a_x, b_x we use normally distributed priors, denoting these as $f_N(x; m, v)$ where x is the variable, m is the mean, and v is the variance. For the variance parameters we use $\pi(\sigma_h^2) = \frac{1}{\sigma_h^2}$, for the correlation coefficient, η, we use a uniformly distributed variable in the interval $(-1, 1)$, and for c_x, e_x, ρ we use improper uniform priors. Hence, we have as the joint prior

$$p(\theta) \propto p(\alpha, \beta, \gamma)p(v_{jj})f_N(a_x; 0, \sigma_{a_x}^2)f_N(b_x; 0, \sigma_{b_x}^2)\sigma_e^{-2}\sigma_w^{-2}\sigma_a^{-2}\sigma_b^{-2}1(SOC_E)1(\eta \in (-1, 1)) \qquad (14)$$

In practice, we set the prior parameters $\sigma_{a_x}^2 = \sigma_{b_x}^2 = 5$ and $\alpha_p = \beta_p = \gamma_p = 5$. Also, in practice, we have rescaled the budget variable for numerical stability (specifically, we have rescaled it to a range between 0 and 2.5) and set $\gamma = 0.7$ because it was not reliably estimable with our data. We test robustness by examining values around 0.7 and verify that the qualitative results do not change. We plot non-parametric estimates of the marginal posterior densities for α and β for three values of γ in Figure 1.

We sample from the posterior by drawing θ in a single Metropolis-Hastings step conditional on the ν_{jj}. We then sample each set of ν_{jj} independently using a Metropolis-Hastings step conditional on θ.

For the Monte Carlo study in Web Appendix 3 and to initialize the endogenous budget model, we also estimate an exogenous budget model with fully observed knowledge. In this model we set the ν_j^a and ν_j^b to zero (and, of course, do not estimate the covariance matrix terms), and we do not include the probability of observing (E_j^*, E_{-j}^*) pairs in the likelihood (and, of course, do not estimate the parameters c_x, ρ, and σ_w^2).

The final samples use 4 million iterations after burnin thinned by 1000.

3 Identification

The identification of some of the parameters is obvious. In this appendix we briefly discuss the identification of a_x versus b_x and then focus on the distinction among three parameters, α, β, and γ.

3.1 a_x versus b_x

Although most of our knowledge variables are not type specific (i.e., good traits versus bad traits) we can separately identify a_x from b_x because they are identified by different moments. While a_x is based on the empirical relationship between the knowledge on candidate j and her tendency to go negative, b_x is based on the relationship between her tendency to go negative and the knowledge on her opponent.

3.2 α, β, and γ

One simple way to demonstrate the identification of these three parameters is via Monte Carlo experiments. To conduct this test, we simplified the model to reduce the computational burden of estimation and allow for many repetitions. We use a model with exogenous budget and no partially observed knowledge (i.e., the knowledge variables are based only on the observables), but the same prior structure. We created data by drawing from random variables for (i) the knowledge observables, x^a and x^b (uniform between 0 and 5), (ii) the mean shifters of the measurement error, x^e, (uniform between 0 and 5), (iii) the stochastic component of the measurement error, and (iv) the exogenous budget levels, E (uniform between 0 and 2.5 – i.e., the same scale as in the real data). We also include in each of x^a, x^b, and x^e an intercept, and fix the parameter values at those shown in Web Appendix Table 1. The budget spent on positive ads, e^*, is calculated using the first order condition for e as laid out in the paper. This allows us to calculate both the optimal and the observed negativity.

We create 250 datasets and run an MCMC chain for each dataset using a Metropolis-Hastings algorithm to sample from the posterior distribution. We initialize the chains at random starting points centered at the true values with a standard deviation of .05. This leads to starting points that diverge meaningfully from the true values and some chains require a large number of iterations before converging. We confirm convergence for the chains using KS statistics and use a thinned post-burnin sample for inference. We run the chains for 4,000,000 million iterations.

Web Appendix Table 1 presents the minimum of the lower bound and maximum of the upper bound of the estimated 95% credible intervals, the mean of the posterior means, the portion of estimated 95% credible intervals that cover the true parameter values, and the true parameter values used. The coverage portion suggests very good properties with most parameters falling within the 95% interval more than 95 percent of the time. The only exception, e_0 (the intercept for the measurement error) is not of interest for this study. Some small bias is apparent in the posterior means, but we note that most of these distributions are skewed and that the posterior mean may not be the best point estimate. This bias does not concern us since our primary interest is in the range of the estimates rather than a point estimate.

For this purpose, it appears our model performs well. In particular, we note that the estimates of α, β, and γ all diverge from the prior means, implying the data contains information about each of these parameters. However, importantly, we note that the γ parameter is closely related to the other structural parameters in a non-linear way, making it much more difficult to achieve convergence due to slower mixing. This increases the computational difficulty of the problem dramatically when the numbers of parameters is larger. In the real data this is the case, since we have multiple variables for positive knowledge, negative knowledge and the measurement errors. Once moving to the endogenous budget model the number of parameters increases even further. To make the problem computationally feasible, we fix γ and estimate the model for values above and below this fixed γ value. This demonstrates that our results are not dramatically affected by the choice of a specific value for γ.

In the rest of this appendix we shed some light on the moments that are involved in identifying these parameters. The discussion is not formal and is incomplete (i.e., it does not include the other moments and the other parameters), but it can provide some elementary intuition.

To simplify the discussion we focus on the model that was used in the Monte Carlo experiment in which the budget is exogenous. As pointed above, endogenizing the budget adds information (e.g., the correlation between prior knowledge and budget) and can assist in the identification of α, β, and γ.

As illustrated in the text, the identification of α and β, rests on the impact of prior knowledge on negativity. Specifically, unless the good traits of a candidate affect her tendency to go negative, α is equal to one. Accordingly, unless this tendency is also a function of the bad traits of her rival, β is equal to one. Furthermore, the relationship between the prior knowledge and the budget on the one hand and negativity on the other hand directly identify these parameters. For example, a negative (positive) correlation between the prior knowledge of a candidate (her rival) and her tendency to go negative implies that alpha is smaller than one (beta is bigger than one). Moreover, the exact values of alpha and beta depend on the magnitude of these correlations as well as on the magnitude of the correlation between the budget and negativity. Specifically, the stronger these correlations the further are alpha and beta from one. Note: the relationship between prior knowledge and negativity identify both the a_x and b_x vectors and the values of α and β. This is enabled by the non-linear nature of the model, the exclusion restrictions that a_x cannot contain (all of) the variables in b_x, and by the correlation between the budget and negativity. Finally, the first order condition in equation 1 illustrates that α and β are not only involved in the effect of prior knowledge and budget on negativity but also in the mean value of negativity. The higher the ratio β/α the more negative are campaigns on average. This brings us to the identification of γ. As illustrated by equation 1 γ is identified by the same moments as α and β. When γ is close to zero the marginal effectiveness of advertising diminishes quickly and the candidates tend to diversify between positive and negative ads. In such a case we would rarely observe candidates who concentrate their effort around one type of ads (positive or negative). The opposite holds when γ is closer to one. Furthermore, it is immediate from equation 1 that the effects of both prior knowledge and budget on the tendency to go negative also depend on γ – the higher the γ the more responsive is negativity to these variables. Again, non-linearity assists in separating this responsiveness from that of α and β. To summarize: α, β, and γ are identified by the same moments – the average tendency to go negative, the concentration of negativity, and the relationship between prior knowledge and budget on the one hand and negativity on the other. These parameters are separately identified because their role in determining these moments is somewhat different – e.g., while α is closely related to the effect of good traits and β to the effect of bad traits, γ is symmetric with respect to these two types of traits. Because they are identified by the same moments, the identification is based on the structure of the model. Furthermore, identification via the structure often leads to estimability problems as the number of parameters increases and we are indeed facing such an issue in the estimation of γ.[1]

4 The direct effect of competitor's negativity

Under the model's assumptions competitor's negativity affects the focal candidate's negativity only indirectly – through the choice of budgets in the first stage of the game. This appendix examines whether we can indeed rule out the hypothesis that there is a direct effect. We show that, while we cannot rule out this hypothesis: (1) the explanatory power of the direct effect is small and possibly

[1] In the full model there are additional parameters that are identified via the moments discussed above (e.g., the variance of the negativity measurement error). This makes the identification even weaker, but as illustrated by the Monte Carlo experiments, the model is still identified.

insignificant, and in any case (2) including the direct effect does not change our theoretical insights. We conclude this appendix by explaining our decision to exclude the direct effect from the model.

We conduct the empirical analysis using the non-structural model described in the paper. Since this analysis is based on the relationship between the negativity of both candidates within a district, we exclude the 12 districts in which only one candidate aired ads. We start our examination using the flawed assumption that the rival's negativity is exogenous. Such an analysis will provide us with an upper limit for the power and impact of the direct effect. As demonstrated later, the estimated role of the direct effect is, of course, much lower when the endogeneity problem is dealt with.

Web Appendix Table 2 gives a preliminary sense of the explanatory power of the direct effect. It presents the estimates of four different models. We start by comparing the first three models. Model I is the base model that exclude both the variables presented by our theory (i.e., knowledge and budget) and the direct effect of negativity. Model II excludes the direct effect but includes knowledge and budget, while Model III is the opposite (i.e., included direct effect, but exclude knowledge and budget). The results demonstrate that the explanatory power of the direct effect is smaller than the power of knowledge and budget. Specifically, the improvement in the likelihood (and AIC and BIC) between models II and I is much larger than the improvement between models III and I (for example, while the likelihood of model II is -287.62 the one for model III is -318.36).

This is indeed a preliminary and a descriptive result. A more interesting analysis is introduced in Model IV that includes the direct effect as well as knowledge and budget. It turns out that the coefficient of the direct effect is positive and significant. Ignoring for the moment the endogeneity problem, this suggests that we cannot rule out the direct effect. However, another interesting finding in Model IV is that the substantial findings from our theoretical variables (knowledge and budget) are essentially the same; the signs are the same and the significance levels are similar. This means that ignoring the direct effect does not seem to affect the theoretical results concerning knowledge and budget.

As noted above, all of these results are based on the assumption that the rival's negativity is exogenous. However, such an assumption is clearly flawed. In order to resolve the endogeneity of the rival's negativity and get a consistent estimate of the direct effect one needs an instrumental variable. The rival's budget is a good possibility for two reasons. First, the budget is a highly significant variable on negativity. Second, it is possible to show that when the theoretical model is enriched to account for the direct effect, the opponent's budget doesn't have a direct effect on the focal candidate's negativity. In other words, in the second stage of the game, the rival's budget can serve as a good instrument for his negativity. However, we cannot use the rival's budget as an instrument since (while exogenous in the second stage of the game) it is endogenous in the first stage of the game. Thus, we need to instrument the rival's budget as well. The two possible instruments for this task are (i) rival's wealth and (ii) his collection of contribution in previous races.

Following this logic Web Appendix Table 2 offers preliminary evidence on the role of the rival's negativity when its endogeneity is accounted for in the column for Model V. We re-run Model IV with one change – we replaced the rival's negativity with these two instruments. Notice that we do not have a reason to expect that these two variables have a direct effect on the focal candidate's negativity (i.e., exclusion restriction). Thus, any evidence that they do influence negativity (when rival's negativity is excluded from the estimation) can support the direct effect of the rival's negativity. We find that only one of these variables has the expected sign and even it is only marginally significant. Specifically, the coefficient of "past contribution" is 0.011 with a standard error of 0.006. The takeaway from this result is that either rival's negativity does not have a direct effect when its endogeneity is accounted for or our instruments are too weak to capture this effect. Note that even when we approach this test in a more formal way, estimating a simultaneous linear system of equations for negativity, budget, opponent negativity, and opponent budget with the added instruments as described above, we still get the same result – the direct effect of rival's negativity is not statistically different from zero. [2]

While the evidence in this appendix is not strong enough to make a compelling argument against the direct impact of rival's negativity, it certainly demonstrates that such direct effect cannot be supported convincingly with our data. It is quite possible that moving from the static setting employed here to a dynamic setting would enable a more reliable examination of the direct effect of rival's negativity, and that in such a case the direct effect would be supported by the data. However, since (i) the instruments in the static setting are not strong enough to support the role of rival's negativity and (ii) it does not seem that the results of our study are sensitive to the inclusion of rival's negativity (even when its endogeneity is ignored), we have decided to exclude it from our model.

[2] We also searched for alternative instruments that could be argued as influencing negativity directly (i.e., not through budget). Our idea was to collect variables that might suggest a tendency toward negative or positive campaign tones due to personality or personal experiences. We tested biographical data on whether the candidate was female, of a different ethnicity/race, a TV or radio personality, or previously in the military. Unfortunately, none of these variables were significantly related to negativity and as a result could not be used as effective instruments.

Web Appendix Table 1: Monte Carlo Results

Parameter	95% Credible Interval	Posterior Mean	Coverage Proportion	True Value of Parameter
α	0.448, 0.949	0.793	0.980	0.85
β	1.030, 1.515	1.185	0.996	1.15
γ	0.398, 0.910	0.680	1.00	0.70
a_0	-0.023, 0.607	0.083	1.00	0.05
a_1	0.079, 3.514	0.520	0.964	0.50
b_0	-0.019, 1.688	0.166	1.00	0.10
b_1	0.133, 7.016	0.668	0.956	0.40
e_0	-0.141,0.083	-0.044	0.880	0.00
e_1	-0.131, -0.066	-0.101	0.948	-0.10
σ_e	0.008, 0.012	0.012	0.976	0.01

Web Appendix Table 2: Opponent Negativity Models

Variable	Model I Controls Only	Model II Theory-based Model	Model III Opponent Negativity	Model IV Full Model	Model V Opponent Instrumented
Positive Knowledge					
Media Coverage - Self		-0.02 (-0.81)		-0.02 (-0.75)	-0.02 (-0.83)
Prior Exposure - Self		0.01 (0.19)		0.01 (0.12)	0.00 (0.05)
Iraq Vote Against Party		0.17 (2.53)*		0.21 (2.84)*	0.16 (2.47)*
Negative Knowledge					
Opponent Is Incumbent		0.13 (2.03)*		0.15 (2.56)*	0.08 (1.34)
Media Coverage - Opponent		0.07 (2.13)*		0.06 (1.92)*	0.07 (2.23)*
Prior Exposure - Opponent		0.03 (0.86)		0.03 (0.68)	0.02 (0.55)
Budget					
Log(Total Ad Spend)		0.18 (7.64)*		0.13 (5.49)*	0.18 (8.23)*
Opponent Negativity Variables					
Opponent Negativity			0.57 (7.53)*	0.32 (4.24)	
Opponent Past Contributions					0.01 (1.81)
Opponent Wealth					-0.03 (-0.45)
Controls					
Incumbent	-0.20 (-3.22)*	-0.17 (-2.66)*	-0.21 (-4.14)*	-0.19 (-3.32)*	-0.17 (-2.85)*
Frontrunner	-0.13 (-2.27)*	-0.12 (-1.82)	-0.23 (-4.93)*	-0.13 (-2.35)*	-0.12 (-1.95)*
Partisanship	0.02 (0.33)	0.00 (0.01)	0.01 (0.22)	0.01 (0.16)	0.00 (0.06)
Party = Republican	-0.00 (-0.05)	0.04 (0.54)	-0.01 (-0.14)	0.03 (0.42)	0.04 (0.53)
Year = 2002	-0.39 (-5.11)*	-0.39 (-5.30)*	-0.30 (-4.16)*	-0.35 (-5.11)*	-0.38 (-5.46)*
Year = 2004	-0.15 (-1.85)	-0.16 (-2.32)*	-0.14 (-1.97)	-0.17 (-2.24)*	-0.17 (-2.40)*
Republican in 2002	0.16 (1.36)	0.16 (1.68)	0.22 (2.11)*	0.20 (2.13)*	0.16 (1.63)
Republican in 2004	0.13 (1.12)	0.09 (0.93)	0.19 (1.72)	0.14 (1.32)	0.10 (0.95)
Demographic Factor - Educated & Income	0.02 (0.80)	-0.06 (-2.56)*	0.02 (0.65)	-0.04 (-1.77)	-0.06 (-2.66)*
Demographic Factor - Immigrant	0.01 (0.44)	0.00 (-0.14)	0.00 (0.19)	0.00 (0.04)	0.00 (0.02)
Demographic Factor - Poor Non-white	-0.01 (-0.57)	0.03 (-1.35)	-0.01 (-0.55)	0.02 (0.99)	0.02 (1.19)
Demographics - Violent Crime	-1.52 (-1.27)	-0.62 (-0.70)	-0.83 (-0.85)	-0.61 (-0.75)	-0.50 (-0.58)
Budget Residual		-0.09 (-3.60)*		-0.07 (-2.85)*	-0.09 (-3.37)*
Scale / Random Effect Scale	0.44 / 0.005	0.37/ 0.004	0.40 / 0.004	0.37 / 0.004	0.37 / 0.004
Log Likelihood / AIC / BIC	-411.4 / 734.6 / 797.2	-287.6 / 621.2 / 717.2	-318.4 / 669.2 / 736.0	-279.3 / 606.6 / 706.8	-279.3 / 606.6 / 706.8

Figure 1: marginal posterior densities for α and β