Online Technical Appendix: Estimation Details

Following Netzer, Lattin and Srinivasan (2005), the model parameters to be estimated can be divided into three parts: (1) the fixed effects governing the evaluation, incidence, and latent experience components of the model which do not vary across individuals or products, (2) individual-specific deviations from the fixed effects governing evaluation and incidence behavior, and (3) product-specific deviations from the fixed effects in the evaluation model and latent experience component.

We denote by ψ the set of fixed effects from the evaluation model ($\bar{\gamma}$ and η), the incidence model ($\bar{\beta}$, δ_1 and δ_2), and latent experience model (θ). We let ζ_i be a vector of the individual-specific deviations from the fixed effects $\bar{\gamma}$ and $\bar{\beta}$. As noted in equation (11), $\zeta_i \sim MVN(0,\Sigma)$. We let ξ_i denote the individual-specific deviations from η, where $\xi_i \sim MVN(0,T)$.

The MCMC procedure draws parameters from the conditional distribution of the model parameters:

$$
\begin{gathered}
\zeta_i | \mathbf{Y_i}, \mathbf{X_i}, \xi_i, \psi, \{\kappa\}, \{\tau\}, \Sigma \\
\xi_i | \mathbf{Y_i}, \mathbf{X_i}, \zeta_i, \psi, \{\kappa\}, \{\tau\}, T \\
\{\kappa\} | \mathbf{Y}, \mathbf{X}, \{\zeta_i\}, \{\xi_i\}, \psi, \{\tau\}, \sigma_\kappa \\
\{\tau\} | \mathbf{Y}, \mathbf{X}, \{\zeta_i\}, \{\xi_i\}, \psi, \{\kappa\}, \sigma_\tau \\
\Sigma \,|\, \{\zeta_i\} \\
T \,|\, \{\xi_i\} \\
\sigma_\kappa^2 | \{\kappa\} \\
\sigma_\tau^2 | \{\tau\} \\
\psi | \mathbf{Y}, \mathbf{X}, \{\zeta_i\}, \{\xi_i\}, \{\kappa\}, \{\tau\}
\end{gathered}
$$

where $\mathbf{Y_i}$ the set of observations from *i* (incidence decisions $\mathbf{z_i}$ and evaluation decisions $\mathbf{y_i}$) and $\mathbf{X_i}$ denotes the state of the ratings environment at the time of the observations.

We next describe how we sample the individual-specific random effects (ζ_i and ξ_i), product-specific random effects ({κ} and {τ}), covariance matrices governing the random effects (Σ, T, σ_κ, and σ_τ), and fixed effects (ψ).

(1) **Generating individual-specific random effects for individual *i***

$$f(\zeta_i|\mathbf{Y_i},\mathbf{X_i},\xi_i,\psi,\{\kappa\},\{\tau\},\Sigma)$$

$$\propto \pi(\zeta_i|\Sigma)L(\mathbf{Y_i}|\zeta_i)$$

$$\propto|\Sigma|^{-.5}\exp[.5\zeta_i'\Sigma^{-1}\zeta_i]L(\mathbf{Y_i}|\zeta_i) \tag{A1}$$

where $\pi(\zeta_i|\Sigma)$ is the prior distribution of ζ_i and $L(\mathbf{Y_i}|\zeta_i)$ is the individual-specific likelihood given by equation (10). Since (A1) does not have a closed form, we use a random walk Metropolis-Hastings algorithm with a Gaussian jumping distribution to draw from the conditional distribution of ζ_i. Letting $\zeta_i^{(t)}$ denote the value of the vector on draw t, the probability that of accepting draw t+1 is given by:

$$\Pr(\text{accept}) = \min\left(\frac{\exp[-.5\left(\zeta_i^{(t+1)}\right)'\Sigma^{-1}\zeta_i^{(t+1)}]L(\mathbf{Y}_i \mid \zeta_i^{(t+1)})}{\exp[-.5\left(\zeta_i^{(t)}\right)'\Sigma^{-1}\zeta_i^{(t)}]L(\mathbf{Y}_i \mid \zeta_i^{(t)})},1\right) \tag{A2}$$

The procedure to draw ξ_i is similar:

$$f(\xi_i|\mathbf{Y_i},\mathbf{X_i},\zeta_i,\psi,\{\kappa\},\{\tau\},\mathrm{T})$$

$$\propto \pi(\xi_i|\mathrm{T})L(\mathbf{Y_i}|\xi_i)$$

$$\propto|\mathrm{T}|^{-.5}\exp[.5\xi_i'\mathrm{T}^{-1}\xi_i]L(\mathbf{Y_i}|\xi_i) \tag{A3}$$

A random walk Metropolis-Hastings algorithm is used to draw from the conditional distribution of ξ_i, where the probability of accepting a new draw is:

$$\Pr(\text{accept}) = \min\left(\frac{\exp[-.5\left(\xi_i^{(t+1)}\right)' \mathrm{T}^{-1} \zeta_i^{(t+1)}] L(\mathbf{Y}_i \mid \xi_i^{(t+1)})}{\exp[-.5\left(\xi_i^{(t)}\right)' \mathrm{T}^{-1} \zeta_i^{(t)}] L(\mathbf{Y}_i \mid \xi_i^{(t)})}, 1 \right) \quad \text{(A4)}$$

(2) **Generating product-specific random effect for product *j***

$$f(\{\kappa\}|\mathbf{Y},\mathbf{X},\{\zeta_i\},\{\xi_i\},\psi,\{\tau\},\sigma_\kappa)$$

$$\propto \pi(\{\kappa\}|\sigma_\kappa)\, L(\mathbf{Y}|\{\kappa\})$$

$$\propto \Pi\left(\sigma_\kappa^{-1}\exp[-\kappa_j^2/2\sigma_\kappa^2]\right) L(\mathbf{Y}|\{\kappa\}) \quad \text{(A5)}$$

where $L(\mathbf{Y}|\{\kappa\})$ denotes the likelihood of the data, derived by taking the product of the individual-specific likelihood in equation (10) across individuals. We use a random walk Metropolis-Hastings algorithm to draw from the conditional distribution of $\{\kappa\}$. The probability of accepting $\{\kappa\}^{(t+1)}$ is:

$$\Pr(\text{accept}) = \min\left(\frac{\exp\left(-\frac{1}{2}\sum_j \left(\frac{\kappa_j^{(t+1)}}{\sigma_\kappa}\right)^2\right) L(\mathbf{Y} \mid \{\kappa\}^{(t+1)})}{\exp\left(-\frac{1}{2}\sum_j \left(\frac{\kappa_j^{(t)}}{\sigma_\kappa}\right)^2\right) L(\mathbf{Y} \mid \{\kappa\}^{(t)})}, 1 \right) \quad \text{(A6)}$$

Drawing from the conditional distribution of $\{\tau\}$ follows a similar procedure:

$$f(\{\tau\}|\mathbf{Y},\mathbf{X},\{\zeta_i\},\{\xi_i\},\psi,\{\kappa\},\sigma_\tau)$$

$$\propto \pi(\{\tau\}|\sigma_\tau)\, L(\mathbf{Y}|\{\tau\})$$

$$\propto \Pi\left(\sigma_\tau^{-1}\exp[-\tau_j^2/2\sigma_\tau^2]\right) L(\mathbf{Y}|\{\tau\}) \quad \text{(A7)}$$

Using the random walk Metropolis-Hastings algorithm, the probability of accepting draw t+1 is:

$$\Pr(\text{accept}) = \min\left(\frac{\exp\left(-\frac{1}{2}\sum_j \left(\frac{\tau_j^{(t+1)}}{\sigma_\tau} \right)^2 \right) L(\mathbf{Y} \mid \{\tau\}^{(t+1)})}{\exp\left(-\frac{1}{2}\sum_j \left(\frac{\tau_j^{(t)}}{\sigma_\tau} \right)^2 \right) L(\mathbf{Y} \mid \{\tau\}^{(t)})}, 1 \right) \quad \text{(A8)}$$

For efficiency, we update the parameters {κ} and {τ} in blocks of multiple products (Chib and Greenberg 1995).

(3) **Updating Covariance Matrices**

Conditional on the values of the individual-specific random effects ($\{\zeta_i\}$ and $\{\xi_i\}$) and the product-specific random effects ({κ} and {τ}), the corresponding covariance matrices can be sampled directly. We begin by drawing values of Σ conditional on the individual-specific random effects $\{\zeta_i\}$. Following Netzer, Lattin and Srinivasan (2005), the conditional distribution of Σ is given by:

$$\Sigma \mid \{\zeta_i\} \sim IW_p\left(f_0 + N, G_0^{-1} + \sum_{i=1}^{N} \zeta_i ' \zeta_i \right) \quad \text{(A9)}$$

where IW_p denotes an inverse-Wishart distribution, p is the length of the vector ζ_i (in our case, p=6), and N is the number of observations. To assume a diffuse prior, we assume that $f_0=p+5$ and G_0^{-1} is an identity matrix of size p.

In the same fashion, the covariance matrix T is drawn from the conditional distribution $T \mid \{\xi_i\}$, the variance σ_κ^2 is drawn from the distribution $\sigma_\kappa^2 \mid \{\kappa\}$, and σ_τ^2 is drawn from $\sigma_\tau^2 \mid \{\tau\}$.

(4) **Updating Fixed Effects**

The vector of fixed effects ψ is drawn in a similar fashion to the individual-specific and product-specific random effects. The conditional distribution of ψ is given by:

$$f(\psi | \mathbf{Y},\mathbf{X},\{\zeta_i\},\{\xi_i\},\{\kappa\},\{\tau\})$$

$$\propto \pi(\psi)\, L(\mathbf{Y}|\psi)$$

$$\propto |V|^{-.5}\exp[.5\psi' V^{-1}\psi] L(\mathbf{Y}|\psi) \qquad \text{(A10)}$$

where π(ψ) is the prior distribution of ψ. We assume a diffuse normal prior by setting the mean equal to a vector of zeros of length q, where q is the length of ψ, and covariance matrix $V=5I_q$. As the conditional distribution given in (A10) does not have a closed form, we use a random walk Metropolis-Hastings algorithm to draw from the conditional distribution. The probability of accepting draw t+1 is given by:

$$\Pr(\text{accept}) = \min\left(\frac{\exp[-.5\psi^{(t+1)}{}' V^{-1}\psi^{(t+1)}] L(\mathbf{Y} \mid \psi^{(t+1)})}{\exp[-.5\psi^{(t)}{}' V^{-1}\psi^{(t)}] L(\mathbf{Y} \mid \psi^{(t)})}, 1 \right) \qquad \text{(A11)}$$

Additional References

Chib, Siddhartha and Edward Greenberg (1995), "Understanding the Metropolis-Hastings Algorithm," *The American Statistician*, 49 (4), 327-335.

Netzer, Oded, James M. Lattin and V. Srinivasan (2008), "A Hidden Markov Model of Customer Relationship Dynamics," *Marketing* Science, 27 (2), 185-204.