Technical Appendix of Sinha and Greenleaf "The Impact of Discrete Bidding and Bidder Aggressiveness on Sellers' Strategies in Open English Auctions: Reserves and Covert Shilling."

TECHNICAL APPENDIX

All variables and functions are real valued.

Assumption 1: $f(v)/(1-F(v))$ is strictly monotonically increasing in v (see Footnote 4 for discussion).

Fact 1: Suppose $v_{(n-1)}$, the second highest order statistic of n random variables drawn from F(v), is known. Then the distribution of the highest order statistic $v_{(n)}$ is the distribution F(v) truncated to the left at $v_{(n-1)}$ and normalized (Arnold, Balakrishnan, and Nagaraja, 1992, Theorem 2.4.1, p. 23).

Definition 1: In discrete bid auction, the set of permissible bids is $\Theta = \{s_a, s_a+I, \ldots, s_a+KI\}$, where s_a is the starting auction bid, and K is an integer ≥2. Here $s_a+KI \le \bar{v}$1.

Definition and Properties of $\wp(\cdot)$

Here, we define $\wp(\cdot)$ formally, based upon the discussion of bidding aggressiveness in the text (pgs. 8-9). Let $\wp(j)|_{\text{Aggressive}}$ denote the bid probability for a relatively aggressive B_n, recalling that the "bid probability" is the probability that B_n obtains the current bid when j other bidders also remain. In a parallel manner, $\wp(j)|_{\text{Neutral}}$ and $\wp(j)|_{\text{Unaggressive}}$ describe the bid probabilities for a neutral and an unaggressive B_n. Further, note that $\wp(\cdot)$ is a function of j and some other parameter, the latter directly representing aggressiveness in any functional form appropriate for $\wp(\cdot)$.

We require that $\wp(\cdot)$ satisfy the following three properties: **(a)** For any given j, $\wp(j)|_{\text{Aggressive}} > \wp(j)|_{\text{Neutral}} > \wp(j)|_{\text{Unaggressive}}$, with the inequalities being strict; and that $\wp(j)$ is strictly increasing as aggressiveness increases, regardless of the type of aggressiveness. Note $\wp(j)|_{\text{Neutral}}=1/(j+1)$. **(b)** $(j+1)\wp(j)$ be nondecreasing in j for aggressiveness, nonincreasing in j for unaggressiveness, and constant in j for neutrality, for all j corresponding to all n. This sufficient condition ensures that for each bidder, $\wp(j) - 1/(j+1)$ is *always* either > 0, or < 0, or = 0, for all j. Also, note that different bidders are allowed to have different types of aggressiveness. **(c)** $\wp(j)$ is not a constant in j for aggressiveness, neutrality, or unaggressiveness, for all j. It follows from properties (a) and (c) that $\wp(j)$ is a monotonic function of the parameter directly representing aggressiveness, for a given j. For example, γ is that parameter in the numerical examples in the text §5. Since the proofs do not require that parameter explicitly, it does not appear as an argument in $\wp(\cdot)$.

Property (a) follows from the definition that higher aggressiveness increases B_n's bid probability, $\wp(j)$, *ceteris paribus*. The justification for property (c) has been discussed in the text (pgs. 8-9). Property (b) ensures that B_n exhibits consistent aggressiveness, and B_n does not "switch" from one type of aggressiveness to another, as j changes (discussed in text pg. 9). Next, we justify property (b) formally, by showing that violation of (b) implies that B_n does not have consistent aggressiveness, for all j, for all finite n, thus contradicting property (a).

Justification of property (b): The following is for B_n being relatively aggressive. Similar arguments can be used for the relatively unaggressive B_n.

For relatively aggressive B_n, suppose property (b) is violated. That is, we have, for some j≥j′≥2, $(j+1)\wp(j)$ is strictly decreasing. This means that a relatively aggressive B_n satisfies (b) for j=1,...,j′-1, but violates (b) for j=j′,...,n-1, where j′≥2. Or, $(j+1)\wp(j)\big|_{Aggr} > (j+2)\wp(j+1)\big|_{Aggr}$, ∀j≥j′, which implies, $\frac{\wp(j+1)}{\wp(j)}\Big|_{Aggr} < \frac{j+1}{j+2}, \forall\ j \geq j'$ 2. Let,

$\frac{\wp(j+1)}{\wp(j)}\Big|_{Aggr} = \frac{j+1}{j+2} \bullet\ \omega_j\ ,\ 0 < \omega_j < 1,\ for\ \ j = j',\ j'+1,\ldots,m,\ldots,(n-1)$ 3. Thus,

$for\ m > j', \wp(m)\big|_{Aggr} = \wp(j')\big|_{Aggr} \bullet \frac{j'+1}{j'+2}\omega_{j'} \bullet \frac{j'+2}{j'+3}\omega_{j'+1} \bullet\bullet\bullet \frac{m}{m+1}\omega_{m-1}$ 4.

Cancelling terms yields, $\wp(m)\big|_{Aggr} = \wp(j')\big|_{Aggr} \bullet\ \frac{j'+1}{m+1} \bullet\ (\omega_{j'} \bullet\bullet\bullet \omega_{m-1})$ 5.

Now, for a neutral B_n, $\wp(m)\big|_{Neut} = \wp(j')\big|_{Neut} \bullet \frac{j'+1}{j'+2} \bullet \frac{j'+2}{j'+3} \bullet\bullet\bullet \frac{m}{m+1} = \wp(j')\big|_{Neut} \bullet \frac{j'+1}{m+1}$ 6.

Although $\wp(j')\big|_{Aggr} > \wp(j')\big|_{Neut}$, by comparing the two previous expressions, we find, provided the condition $\omega_{j'} \bullet\bullet\bullet \omega_{m-1}\ le\ \frac{\wp(j')\big|_{Neut}}{\wp(j')\big|_{Aggr}}$ 7 holds, that $\wp(m)\big|_{Aggr}$ will be less than or equal to $\wp(m)\big|_{Neut}$. This violates property (a) because here we have an aggressive bidder with $\wp(m)$ less than or equal to that of a neutral bidder. This also makes B_n "switch" from one of the three aggressive categories to another. Observe that the aforementioned condition must hold for some m (for some finite n) since l.h.s. of the condition rapidly approaches zero (being the product of several ω_js, each lying strictly between 0 and 1), and the r.h.s. of the condition is a fixed quantity less than 1 (but strictly > 0) by definition of $\wp(j)$. Hence, violation of property (b) leads to contradiction of property (a).

Note that for generality our model is intended to accommodate the range of finite n without any arbitrary ceiling on n. For some small n, there may not be a violation of property (a). But if n is allowed to increase then eventually it becomes large enough so that such a "switch" will occur. This happens because as n increases the product of the ω_js becomes even smaller so that the above-mentioned condition (in the previous paragraph) holds, and the bid probability for an aggressive B_n becomes strictly less than that for a neutral B_n. While we allow different bidders to have different types of aggressiveness, note that the properties of $\wp(\cdot)$ require each bidder's behavior to belong consistently to a single type of aggressiveness across all finite n, i.e., a person who is relatively aggressive at one n, cannot become relatively neutral or unaggressive at another value of n.

It can be readily seen that the preceding approach also covers the parallel case of a relatively unaggressive B_n who violates (b) for some j′, i.e., whose $\wp(j)$ would then eventually switch to a "neutral" or "agggressive" value of $\wp(j) \geq 1/(j+1)$.

For completeness of exposition we note that bidders might exist who, as j increases, *cycle* between satisfying and violating property (b). Such bidders may or may not violate property (a), for some n. Our model is not intended to cover those bidders, who have a very complex aggressiveness behavior that could only be described by a less parsimonious model.

Definition and Properties of ξ

In understanding the impact of aggressiveness on reserve and shilling strategies, our interest is in the deviation, defined as $\chi(j) = \wp(j) - (1/(j+1))$ (text §2). When $\chi(j)$ >, {=}, [<] 0 we have relative aggressiveness, {neutrality}, and [unaggressiveness] for B_n. Note $|\chi(j)|<1$, by definition. Now consider the minimum (which is strictly >0, by definition) of the finite sequence $\{\chi(j), j=1,...,n-1\}$ (when $\chi(j)>0, \forall j$); or, the maximum (which is strictly <0) of the finite sequence $\{\chi(j), j=1,...,n-1\}$ (when $\chi(j)<0, \forall j$). Note that the sequence $\chi(j)$ cannot have values with mixed signs, by definition. Define ξ as the bound (min or max), and note that it does not depend on j. Such a bound always exists because $|\chi(j)| < 1$. Since the propositions concern directional predictions w.r.t. aggressiveness, it is sufficient, and creates simpler proofs, to use ξ as the variable representing B_n's aggressiveness. (In the numerical example we use a specific functional form for $\wp(j)$, which also gives ξ a specific functional form.)

For generality, we recognize that ξ is dependent on some parameter, representing aggressiveness in any functional form chosen for $\wp(j)$ satisfying properties (a)-(c). Moreover, ξ can be dependent on the number of bidders, n. (The dependence of ξ on these two quantities follows, as a direct implication to the properties of $\wp(j)$, based on which ξ is defined. In fact, ξ is a monotonic function of that parameter.) However, for notational simplicity, we write ξ without reference to those quantities, because we can do so without creating any confusion in the exposition. Whenever the dependence of ξ on such an aggressiveness parameter and/or, on n affects the proofs, we address those situations by recognizing different subcases and scenarios that this dependence might take. Thus, ξ >, {=}, [<] 0, stands for relative aggressiveness, {neutrality}, and [unaggressiveness], for B_n.

For completeness we observe, (i) for aggressive B_n: $\chi(j) \geq \xi, \forall j$, with $\xi>0$; (ii) for unaggressive B_n: $\chi(j) \leq \xi, \forall j$, with $\xi<0$; (iii) for neutral B_n: $\chi(j)=\xi, \forall j$, with $\xi=0$.

Next, we state all lemmas together for ease of reference when reading the proofs.

Lemma 1: $\dfrac{n\,F^{n-1}(v)}{F^{n}(v) - F^{n}(v-I)}$ 8 *strictly increases as n increases and is unbounded.*

Proof: Let, F(v) = A·F(v-I), where A>1. Then, the lemma expression reduces to $\frac{n A^{n-1}}{A^n - 1} \frac{1}{F(v-I)}$ 9.

Note $h(x)=\frac{x A^{x-1}}{A^x - 1}$ 10 is increasing in x for x>1, because $h'(x)=\frac{A^{2x-1} - A^{x-1} - x A^{x-1}\log(A)}{(A^x - 1)^2} > 0$ 11.

The numerator is non-zero for A>1, and x>1. ∴ local maximum or minimum cannot exist.

Also, $h(x) > \frac{x A^{x-1}}{A^x} = \frac{x}{A},\ for\ \ all\ \ A>1,\ x>1.\ \ As\ \ x \to \infty, \frac{x}{A} \to \infty.\ \ In\,other\ words\ , h(x) \to \infty.$ 12 Q.E.D.

Lemma 2: $\frac{n F^{n-1}(v-I)}{F^n(v) - F^n(v-I)}$ 13 *strictly decreases as n increases and has a min > 0.*

Proof: The lemma expression reduces to $\frac{n}{A^n - 1} \frac{1}{F(v-I)}$ 14. Observe, $h(x)=\frac{x}{A^x - 1}$ 15 is decreasing in x for x>1. Also, no local maximum or minimum exists. The expression is >0 everywhere; thus, the bound.

Lemma 3: $\frac{U(v - I - v_0)}{U(v - v_0) - U(v - I - v_0)}$ 16 *strictly increases as v increases.*

Proof: Numerator is strictly increasing and denominator is non-increasing, by definition of U(·).Q.E.D.

Lemma 4: $\frac{U(v - I - v_0)}{U(v - v_0) - U(v - I - v_0)}$ 17 *increases as the seller's absolute risk aversion increases.*

Proof: Consider two utility functions $U_1(v-v_0)$ and $U_2(v-v_0)$, with the absolute risk aversion of U_1 greater than that of U_2. That is, $-\frac{U_1''(v-v_0)}{U_1'(v-v_0)}$ 18≥ $-\frac{U_2''(v-v_0)}{U_2'(v-v_0)}$ 19.

This implies, (by Pratt 1964, Theorem 1), $\frac{U_1(v - I - v_0)}{U_1(v - v_0) - U_1(v - I - v_0)}$ 20≥ $\frac{U_2(v - I - v_0)}{U_2(v - v_0) - U_2(v - I - v_0)}$ 21.

The strict inequality holds if the absolute risk aversion of U_1 is strictly greater than that of U_2. Q.E.D.

Lemma 5: $\frac{1 - F(v)}{F(v) - F(v-I)}$ 22 *strictly decreases as v increases.*

Proof: By differentiating and as a consequence of Assumption 1. Q.E.D.

Lemma 6: (a) $P(v_{(n),only})$ decreases as n increases if ξ≤0, provided $n > \left[\left|\log\frac{F(s_0 - I)}{F(s_0)}\right|\right]^{-1}$ 23; and is not *necessarily monotonic if ξ>0.*

(b) $P(v_{(n),only})$ lies strictly between 0 and 1.

Proof (Lemma 6a):

From text (Section 3(i)), there are three mutually exclusive and exhaustive ways bidding can stop at s_0. Their probabilities are:

$$P_1 = \sum_{j=1}^{n-1} n\,[1 - F(s_0 + I)]\left(\frac{1}{j+1} + \xi\right)\binom{n-1}{j}[F(s_0 + I) - F(s_0)]^{j}\, F^{n-1-j}(s_0) \text{ 24,} \qquad \text{(A1a)}$$

$$P_2 = \sum_{j=1}^{n-1} n\,[1 - F(s_0)] \left(\frac{j}{j+1} - \xi \right) \binom{n-1}{j} [F(s_0) - F(s_0 - I)]^{\,j} \; F^{\,n-1-j}(s_0 - I) \text{25,} \qquad \text{(A1b)}$$

$$P_3 = \sum_{j=2}^{n} \binom{n}{j} [F(s_0 + I) - F(s_0)]^{\,j} \; F^{\,n-j}(s_0) \text{26.} \qquad \text{(A1c)}$$

Re-expressing
(A1a),

$$P_1 = n\,[1 - F(s_0 + I)] \; F^{\,n-1}(s_0 + I) \sum_{j=1}^{n-1} \left(\frac{1}{j+1} + \xi \right) \binom{n-1}{j} \left[\frac{F(s_0 + I) - F(s_0)}{F(s_0 + I)} \right]^{j} \left[\frac{F(s_0)}{F(s_0 + I)} \right]^{n-1-j} \text{27.}$$

Similarly, (A1b) and (A1c) can be expressed. Summing over the binomial terms we can write,

$$P_1 = \frac{1 - F(s_0 + I)}{F(s_0 + I) - F(s_0)} \Big[\, [F^{n}(s_0 + I) - F^{n}(s_0)] - n\,[F(s_0 + I) - F(s_0)] \; F^{n-1}(s_0) \, \Big] + n\,\xi\,[1 - F(s_0 + I)]\,[\; F^{n-1}(s_0 +$$

28 (A2a)

$$P_2 = \frac{1 - F(s_0)}{F(s_0) - F(s_0 - I)} \Big[\, n\,[F(s_0) - F(s_0 - I)]\, F^{\,n-1}(s_0) - [F^{n}(s_0) - F^{n}(s_0 - I)] \, \Big] - n\,\xi\,[1 - F(s_0)]\,[\; F^{\,n-1}(s_0) -$$

29 (A2b)

$$P_3 = [F^{n}(s_0 + I) - F^{n}(s_0)] - n\,[F(s_0 + I) - F(s_0)] \; F^{n-1}(s_0) \text{30.} \qquad \text{(A2c)}$$

Further, for algebraic convenience we write, $P_1 + P_2 = E_1 + E_2 + E_3$, where,

$$E_1 = n\,F^{n-1}(s_0)\,[F(s_0 + I) - F(s_0)] - [F^{n}(s_0 + I) - F^{n}(s_0)] \;, \text{31}$$

$$E_2 = [1 - F(s_0)] \left[\frac{F^{n}(s_0 + I) - F^{n}(s_0)}{F(s_0 + I) - F(s_0)} - \frac{F^{n}(s_0) - F^{n}(s_0 - I)}{F(s_0) - F(s_0 - I)} \right] , \text{32}$$

$$\text{and, } E_3 = n\,\xi \Big[[1 - F(s_0 + I)]\,[\; F^{n-1}(s_0 + I) - F^{n-1}(s_0) \;] - [1 - F(s_0)] \; [F^{n-1}(s_0) - F^{n-1}(s_0 - I)] \Big] . \text{33}$$

Also, simplification yields, $P_1 + P_2 + P_3 = E_2 + E_3$ (note, $E_1 < 0$).

We next derive $P(v_{(n),\text{only}} \,|\, s_0)$ for (i) $\xi=0$, (ii) $\xi<0$, and (iii) $\xi>0$.

Note, $$P(v_{(n),\,only} \,|\, s_0) = \frac{P_1 + P_2}{P_1 + P_2 + P_3} \cdot \text{34} \qquad \text{(A3)}$$

~~Case (i): B_n has Neutral Relative Aggressiveness, $\xi=0$.~~ It $\Rightarrow$ $E_3=0$. $\therefore$, $P(v_{(n),\text{ only}}) \,|_{\,\xi=0} = 1-[(-E_1)/E_2]$. It is sufficient to show $[(-E_1)/E_2]$ strictly $\uparrow$ as n $\uparrow$. Write, $(-E_1)/E_2 = (E_4/E_2) - (E_5/E_2)$, where

$$\frac{E_4}{E_2} = \left[\frac{1 - F(s_0)}{F(s_0 + I) - F(s_0)} - \frac{1 - F(s_0)}{F(s_0) - F(s_0 - I)} \bullet \frac{F^{n}(s_0) - F^{n}(s_0 - I)}{F^{n}(s_0 + I) - F^{n}(s_0)} \right]^{-1} \text{35, or}$$

$$\frac{E_4}{E_2} = \left[\frac{1 - F(s_0)}{F(s_0 + I) - F(s_0)} + \frac{1 - F(s_0)}{F(s_0) - F(s_0 - I)} \left(\frac{\left[\frac{F(s_0 - I)}{F(s_0)}\right]^n - 1}{\left[\frac{F(s_0 + I)}{F(s_0)}\right]^n - 1} \right) \right]^{-1}$$ 36, and,

$$\frac{E_5}{E_2} = \Bigg[\left(\frac{1 - F(s_0)}{[F(s_0 + I) - F(s_0)]^2} \bullet \frac{F^n(s_0 + I) - F^n(s_0)}{n F^{n-1}(s_0)} \right) - \left(\frac{1 - F(s_0)}{[F(s_0 + I) - F(s_0)][F(s_0) - F(s_0 - I)]} \bullet \frac{F^n(s_0) - F}{n F^{n-1}(} \right.$$
37.

Consider E4 / E2. The second term is strictly ↓ as n ↑. Thus, E4 / E2 strictly ↑ as n ↑.

Consider E5 / E2. The first term strictly ↑ as n ↑, by Lemma 2. The second term strictly ↓ as n ↑, by Lemma 1. Thus, E5 / E2 strictly ↓ as n ↑.

~~Cases (ii) and (iii): B_n is Relatively Unaggressive, or Aggressive, i.e., ξ≠0.~~ Now,

$$P(v_{(n),only})\big|_{\xi ne0} = \left[1 + \frac{1}{\frac{E_1 + E_2}{-E_1} + \frac{E_3}{-E_1}} \right]^{-1}$$ 38.

First, we show that E3 / (-E1) strictly ↓ as n ↑.

$$\frac{E_3}{-E_1} = \xi \left[\frac{n[1 - F(s_0 + I)][F^{n-1}(s_0 + I) - F^{n-1}(s_0)]}{-E_1} - \frac{n[1 - F(s_0)][F^{n-1}(s_0) - F^{n-1}(s_0 - I)]}{-E_1} \right]$$ 39.

Now, let F(s0+I) = A·F(s0), A>1. We re-write the first expression within bracket in E3 / (-E1) as

$$\frac{1 - F(s_0 + I)}{F(s_0)} \bullet \frac{n[A^{n-1} - 1]}{A^n - n(A - 1) - 1}$$ 40.

Using approach similar to Lemma 1 it can be checked that this expression strictly, unboundedly ↑ as n ↑. This occurs because the continuous analog of the above expression involving n, viz.,

$$\frac{x[A^{x-1} - 1]}{A^x - x(A - 1) - 1}, \textit{is greater than } \frac{x}{A}, \forall A > 1, \textit{and everywhere in } x > 1$$ 41.

The second expression within bracket in E3 / (-E1) can be re-written as E6 (after simplification)

$$E_6 = \frac{[1 - F(s_0)][F(s_0) - F(s_0 - I)]}{F(s_0 + I) - F(s_0)} \; \frac{\sum_{i=1}^{n-1} n \frac{F^{n-(i+1)}(s_0 - I)}{F^{n-(i+1)}(s_0)}}{\sum_{i=1}^{n-1} \frac{F^{n-i}(s_0 + I) - F^{n-i}(s_0)}{F^{n-(i+1)}(s_0)}}$$ 42.

It is noted $n \dfrac{F^{n-(i+1)}(s_0 - I)}{F^{n-(i+1)}(s_0)}$ 43↓ as n ↑, for $n > \left[\left| \log \dfrac{F(s_0 - I)}{F(s_0)} \right| \right]^{-1}$ 44.

Also, $\dfrac{F^{n-i}(s_0 + I) - F^{n-i}(s_0)}{F^{n-(i+1)}(s_0)}$ 45→∞ as n→∞.

Thus, as $n \to \infty$, the incremental contribution to the numerator sum in E_6 decreases, while the incremental contribution to the denominator sum in E_6 increases. Moreover, the numerator and denominator sums have identical number of terms. Hence, we conclude E_6 ↓ as n ↑, provided

$$n > \left[\left| \log \frac{F(s_0 - I)}{F(s_0)} \right| \right]^{-1} \quad 46.$$

Or, the expression in $E_3/(-E_1)$ without ξ, increases strictly without bound as n increases.
Next, we characterize $E_3/(-E_1)$ with ξ included. Since we allow ξ to be dependent on n, we study the following two mutually exclusive and exhaustive subcases to capture types of this dependence.
Subcase (a): $\xi < (>) 0$, can be nonincreasing (nondecreasing) in n. It follows that $E_3/(-E_1)$ strictly ↓ (↑) if n ↑, since $\xi < (>) 0$. Note this subcase includes all instances where ξ does not depend on n.
Subcase (b): $\xi < (>) 0$, can be strictly increasing (decreasing) in n. First, we show, $\xi \cdot$*(first expression within brackets in $E_3/(-E_1)$)* decreases (increases) with n, for $\xi < (>) 0$. From $E_3/(-E_1)$, we see that,

$$\xi \cdot \textit{(first expression within brackets in } E_3/(-E_1)\textit{)} \;<(>)\; \xi \bullet \frac{1 - F(s_0 + I)}{F(s_0)} \bullet \frac{n}{A}, \text{ for } n \geq 2 \quad 47,$$

since $\xi < (>) 0$, and $\frac{n[A^{n-1} - 1]}{A^n - n(A-1) - 1} > \frac{n}{A}$ $, \forall A > 1, \text{and } \forall\, n \geq 2$ 48 (as shown one-half page earlier).

Now, ξ's dependence on n can take one of two scenarios: (1) ξ depends on n for all n, (2) ξ depends on n for $n \leq n'$, and is constant for $n \geq n'+1$. Note that (2) occurs when $\max\{\chi(j): j=1,\ldots,n-1\}$ for $\xi<0$, (or, $\min\{\chi(j): j=1,\ldots,n-1\}$ for $\xi>0$) does *not* occcur at $j=n-1$, for some n. For scenario (1), $\xi = \wp(n-1)-(1/n)$.
Therefore, $\xi \bullet \frac{1 - F(s_0 + I)}{F(s_0)} \bullet \frac{n}{A}$ 49= $\frac{1 - F(s_0 + I)}{F(s_0 + I)} \bullet [\, n \wp(n-1) - 1]$ 50, for all $n \geq 2$. The r.h.s. is nonincreasing (nondecreasing) in n by property (b) of $\wp(j)$ for $\xi < (>) 0$. Using the inequality six lines before, this implies, that $\xi \cdot$*(first expression within brackets in $E_3/(-E_1)$)* is decreasing (increasing) in n since $\xi < (>) 0$ and we have shown already that the first expression within bracket in $E_3/(-E_1)$ strictly, unboundedly increases in n. For scenario (2), the immediate preceding gives us the result for $n \leq n'$, and for $n \geq n'+1$, the result follows from Subcase (a) above because ξ does not depend on n for such n. Thus, throughout the range of n we obtain the result that $\xi \cdot$*(first expression within brackets in $E_3/(-E_1)$)* is decreasing (increasing) in n, for $\xi < (>) 0$. Now, we focus on the second expression within brackets in $E_3/(-E_1)$ and for clarity treat the cases of unaggressiveness and aggressiveness separately.
Case (ii): B_n is Relatively Unaggressive, i.e., $\xi<0$. It can be seen that $\xi \cdot (-E_6)$ strictly decreases as n increases, because ξ and $-E_6$ are both <0, and ξ either strictly increases in n over the entire range of n (as in Subcase (b), scenario (1)) or, ξ strictly increases in n over the relevant range of n and then remains constant (as in Subcase (b), scenario (2)), and $-E_6$ strictly increases in n. Hence, we get that

$E_3/(-E_1)$ strictly ↓ as n ↑. By Case (i), we conclude $(E_1+E_2)/(-E_1)$ strictly ↓ as n ↑. This proves the monotonic relationship between $P(v_{(n),\text{ only}})$ and n, for $\xi<0$.

Case (iii): B_n is Relatively Aggressive, i.e., $\xi>0$. It can be checked that $\xi\cdot E_6$ strictly decreases as n increases, because ξ and E_6 are both >0, and both strictly decreasing in n. Thus, we get that $E_3/(-E_1)$ strictly ↑ as n ↑. By Case (i), we have $(E_1+E_2)/(-E_1)$ strictly ↓ as n ↑. This suggests that for $\xi>0$, there is *not* always a monotonic relationship between $P(v_{(n),\text{ only}})$ and n. Q.E.D.

Proof (Lemma 6b): By definition of $P(v_{(n),\text{ only}})$, it is strictly < 1, because P_3 is strictly > 0. Also, since P_1+P_2 is strictly > 0 for all n, $P(v_{(n),\text{ only}})$ is strictly > 0, for all n. Q.E.D.

Proofs of Propositions

Proof of Proposition 1a-1c (showing existence of v_D^**)*

Using exposition in text, and denoting probabilities of gain and loss as P_{gain} , and P_{loss} :

$P_{gain,a} = n\,[1 - F(v)]\,F^{n-1}(v-I)$, (corresponds to sub-case (i) in case (c) preceding eq. (1) in text)

$$P_{gain,b} = \sum_{j=1}^{n-1} n\,[1 - F(v)]\left(\frac{1}{j+1} + \xi\right)\binom{n-1}{j}[F(v) - F(v-I)]^{\,j}\;F^{\,n-1-j}(v-I)$$ 51, (corresponds to sub-case (ii) in case (c))

$$P_{gain} = P_{gain,a} + P_{gain,b} = \frac{1 - F(v)}{F(v) - F(v-I)}\,[\,F^n(v) - F^n(v-I)\,] + n\,\xi\,[1 - F(v)]\,[\,F^{n-1}(v) - F^{n-1}(v-I)\,]$$ 52,(A4)

$P_{loss} = F^n(v) - F^n(v-I)$ (corresponds to case (d) preceding eq. (1) in text). (A5)

From (1) in text, *Expected Gain* $= q(v,v_0,n) = P_{gain}\,[\,U(v-v_0) - U(v-I-v_0)\,]$. (A6)

From (2) in text, *Absolute Expected Loss* $= r(v,v_0,n) = P_{loss}\;U(v-I-v_0)$. (A7)

Definition 2: Define $v_D^* \in \Theta$, as the maximum discrete bid level where,

$$q(v_D^*,v_0,n) \geq r(v_D^*,v_0,n)\ , \text{ and }\ q(v_D^*+I,v_0,n) < r(v_D^*+I,v_0,n).$$

In order to show that such a v_D^* exists it is sufficient to prove the following Claim 1:

Claim 1: The functions q and r intersect each other; q starts from above and reaches below r, w.r.t. v. Combining (i) at $v=v_0+I$, $q(\cdot) > r(\cdot)$; and (ii) at $v = v$, $q(\cdot) < r(\cdot)$, proves Claim 1.

We first consider v as a continuous variable in $[v_0+I,v)$ and then prove the proposition for $v \in \Theta$.

By Claim 1, $\exists\ v' \in [v_0+I,v)$ for which $q(v',v_0,n) = r(v',v_0,n)$. (A8)

If $v' \notin \Theta$, then subtract a $\delta>0$ ($\delta<I$) from v' which will make $v'-\delta \in \Theta$, and immediately below v', for which, (using continuity of $q(\cdot)$, $r(\cdot)$, and Claim 1) $q(v'-\delta,v_0,n) > r(v'-\delta,v_0,n)$, and $q(v'-\delta+I,v_0,n) < r(v'-\delta+I,v_0,n)$. Thus, v_D^* is as follows:

$$v_D^* = v'-\delta, \text{ if } v' \notin \Theta; \qquad \text{(A9)}$$
$$= v', \text{ if } v' \in \Theta.$$

From (A8), equating ratio of utilities of loss and gain to the inverse ratio of associated probabilities,

$$\frac{U(v'-I-v_0)}{U(v'-v_0) - U(v'-I-v_0)} = \frac{1 - F(v')}{F(v') - F(v'-I)} + \xi\,[1 - F(v')]\left[\frac{n\,F^{n-1}(v')}{F^n(v') - F^n(v'-I)} - \frac{n\,F^{n-1}(v'-I)}{F^n(v') - F^n(v'-I)}\right]$$ 53. (A10)

When $\xi=0$, to show v_D^* is unique for general $F(\cdot)$, choose v' satisfying (A10). We further note that v' in (A10) is unique because (i) l.h.s. and r.h.s. of (A10) are increasing and decreasing, respectively, as v' increases (Lemmas 4 and 6); (ii) at $v'=v_0+I$, l.h.s. < r.h.s., and at $v'=v$, l.h.s. > r.h.s.

When $\xi \neq 0$, we have (ii), and (iii) l.h.s. is increasing in v' (Lemma 3). But, the r.h.s. is not necessarily monotonic in the entire range of $v \in (v_0,v]$ for general $F(\cdot)$. Hence, there may exist multiple roots for v' satisfying (A10), in which case it is necessary to evaluate the expected utilities to determine the global maximum (also true of continuous auctions - Riley and Samuelson, 1981, footnote 8). Q.E.D.

Proof of Proposition 1a-1c (when $\xi > [<] \{=\} 0$, v_D^* increases [decreases] {unchanged} with n)
Rearranging terms from (A10), and defining,

$$Y = \frac{n\,F^{n-1}(v')}{F^{n}(v') - F^{n}(v'-I)} \ , and\ Z = \frac{n\,F^{n-1}(v'-I)}{F^{n}(v') - F^{n}(v'-I)}$$

[54]; we obtain

$$\left[\frac{1-F(v')}{F(v')-F(v'-I)} + \xi\ [1-F(v')]\ \ [\ Y\ -\ Z\ ] \right] - \frac{U(v'-I-v_0)}{U(v'-v_0)\ -\ U(v'-I-v_0)} = 0 \qquad \text{(A11)}$$

[55].

Observe that Y unboundedly strictly ↑ as n ↑, by Lemma 1; and Z strictly ↓ as n ↑, by Lemma 2. To study the behavior of $\xi \cdot [Y-Z]$ we look at the following subcases.

Subcase (a): $\xi > (<) 0$, and can be nondecreasing (nonincreasing) in n. From the immediate preceding line, [Y-Z] unboundedly strictly ↑ as n ↑. Thus, if $\xi > (<) 0$, then $\xi[1-F(v')][Y-Z]$ is unboundedly strictly ↑ (↓) as n ↑. Also note, the fraction within bracket in (A11) does not depend upon n.

Subcase (b): $\xi > (<) 0$, and can be strictly decreasing (strictly increasing) in n. First, focus on $\xi \cdot Y$. From Proof of Lemma 1 note that, $Y > \frac{n}{F(v')}$ [56], ∀ $n \geq 2$. Or, $\xi \cdot Y > (<)\ \xi \cdot (n/F(v'))$, for $\xi > (<) 0$. Now, as explained earlier, ξ's dependence on n can take one of two scenarios: (1) ξ depends on n for all n, (2) ξ depends on n for $n \leq n'$, and is constant for $n \geq n'+1$. For scenario (1), $\xi = \wp(n-1)-(1/n)$. Therefore, $\xi \cdot (n/F(v')) = (1/F(v')) \cdot [n\wp(n-1) - 1]$. The r.h.s. is non-decreasing (non-increasing) in n (property (b) of $\wp(j)$ for aggressiveness (unaggressiveness)). Using the inequality involving $\xi \cdot Y$ four lines before, this implies that $\xi \cdot Y$ is unboundedly increasing (decreasing) in n since $\xi > 0$ (<0) and we have shown in Lemma 1 that Y strictly, unboundedly increases in n. For scenario (2), the immediate preceding gives us the result for $n \leq n'$, and for $n \geq n'+1$, the result follows from Subcase (a) above because ξ does not depend on n for such n. Thus, throughout the range of n we obtain the result that $\xi \cdot Y$ is strictly increasing (decreasing) in n for $\xi > (<) 0$.
Now we focus on $\xi \cdot Z$. For $\xi > 0$ and strictly decreasing (throughout the range of n as in scenario (1), or, in the relevant range of n as in scenario (2)), $\xi \cdot Z > 0$, and monotonically strictly decreases with n, by Lemma 2, and hence $(\xi \cdot Y - \xi \cdot Z)$ strictly unboundedly increases as n increases. Similarly, for $\xi < 0$ and strictly increasing, $\xi \cdot Z < 0$, and strictly increases with n, and hence $(\xi \cdot Y - \xi \cdot Z)$ unboundedly strictly decreases as n increases.

Finally, we study the impact of n on the functions $q(\cdot)$ and $r(\cdot)$, as defined in (A6) and (A7).
Let $\xi > 0$. It is sufficient to show v' increases as n increases.
At v', $q(v', v_0, n) - r(v', v_0, n) = 0$. Suppose n increases. As shown in the preceding, the term within bracket in (A11) increases unboundedly, implies the difference in (A11) is greater than zero (irrespective of whether the expression within bracket in (A11) is decreasing or increasing with

$v \in [v'-I, v'+I]$). In order to make the difference equal to zero, v' must increase to $v'_{new} > v'$, since the third term in (A11) strictly ↑ as v' ↑ (Lemma 3) and does not depend upon n. Thus, $v_D^*{}_{new} \big|_{\xi>0} \geq v_D^*$, where $v_D^*{}_{new} \,|_{\,\xi>0}$ $= v_D^*$, if $v'-\delta+I > v'_{new} > v'$;

$\geq v_D^*+I$, if $v'_{new} \geq v'-\delta+I$.

If $\xi<0$, it can be similarly proved that $v_D^*{}_{new} \big|_{\xi<0} \leq v_D^*$.

If $\xi=0$, observe in (A11) the 2nd term is zero, and the other terms not dependent on n. Q.E.D.

Proof-Proposition 1a-1b (if $\xi >$ (<) 0, v_D^* may attain any discrete bid below v (above v_0), if n is large)

We have already established above that v_D^* changes monotonically with n for a given sign of ξ.

Let $\xi>0$. By definition, v_D^* must satisfy $q(v_D^*,v_0,n) \geq r(v_D^*,v_0,n)$. At $v=v$, $q(v,v_0,n)=0$, and $r(v,v_0,n)>0$. Thus, $q(\cdot)$ and $r(\cdot)$ must be equal at some v' which is strictly less than v . This in turn implies (from A9) that least upper bound (l.u.b.) of v_D^* will be strictly less than v . We now establish that the l.u.b. is indeed the highest discrete value just below v . Consider v' in (A10). For a given n, $\exists\, v'$ that satisfies (A10). Since (i) the l.h.s. of (A10) is strictly >0, strictly increases as v' increases (Lemma 3), and is finite; (ii) the r.h.s. of (A10) is strictly > 0 for $v'<v$; and (iii) the r.h.s. of (A10) goes to infinity as n goes to infinity (since $\xi>0$ - see Proof after (A11)), one can find a $v''(=v'+I)$ that satisfies (A10) by increasing n sufficiently to some $n''>>n$. Thus, as n ↑, the maximum value of v'' that satisfies (A10) is just below v . By definition in (A9) l.u.b. of v_D^* must be the highest discrete value below v .

When $\xi<0$, it can be similarly proved that v_D^* can attain any discrete bid level above v_0. Q.E.D.

Proof of Proposition 1d ($v_D^* > v_0$)

As already shown, at $v=v_0+I$, $q(\cdot) > r(\cdot)$. That is, $v'>v_0+I$. Or, $v_D^* \geq v_0+I > v_0$. Note that if I is very large compared to $(v - v_0)$ we can have $v_D^*=v_0$. By Def. 1, such large I is disallowed. Q.E.D.

Proof of Proposition 1d (v_D^* increases with v_0)

We shall first appeal to v as a continuous variable and then establish the result for discrete reserve. It can be checked from (A6) and (A7) by differentiating w.r.t. v_0, that $q(v,v_0,n)$ and $r(v,v_0,n)$ are strictly monotonically increasing and decreasing, respectively, as v_0 increases. Let v_0 be increased to v_0+x $(0<x<I)$. Hold v' as before. Then $q(v',v_0+x,n) \geq q(v',v_0,n)$, and $r(v',v_0+x,n) < r(v',v_0,n)$. From (A8) we get, $q(v',v_0,n) = r(v',v_0,n)$. Thus, $q(v',v_0+x,n) > r(v',v_0+x,n)$. But we know from (A6) & (A7), at $v=v$, $q(v,v_0+x,n) < r(v,v_0+x,n)$. Hence, by Claim 1, q and r must intersect at a point $v'_{new}>v'$. Thus, the new reserve $v_D^*{}_{new} \geq v_D^*$, where,

$v_D^*{}_{new} = v_D^*$, if $v' < v'_{new} < v'-\delta+I$;

$\geq v_D^*+I$, if $v'_{new} \geq v'-\delta+I$. Q.E.D.

Proof of Proposition 1d (v_D^* ↑ as ξ ↑, for a given n)

Let $\xi>0$. Suppose ξ ↑ for a given n (please see "Definition and Properties of ξ" towards the beginning of the Appendix to check that ξ is also a function of an aggressiveness parameter, besides n). Then the r.h.s. in (A10) $\uparrow$. The l.h.s. needs to increase to maintain equality and the l.h.s. strictly increases in v' (Lemma 3). Therefore, v' increases, and so $v_{D\,new}^* \geq v_D^*$ (by definition of v_D^*, and applying Lemma 5 to first part of r.h.s. in A10). Note that, since v_D^* can increase only in discrete increments of I, the strict inequality must hold for sufficiently large n. This happens because, for $\xi>0$, the r.h.s. of (A10) strictly $\uparrow$ as n $\uparrow$, no matter whether and how ξ depends on n, as shown in discussion immediately after (A11).

Similarly, for $\xi<0$, it can be shown that v_D^* still $\uparrow$ as ξ $\uparrow$. Q.E.D.

Proof of Proposition 1d: (v_D^* approaches v_C^* as $I\to 0$, $\forall\,\xi$)

v_D^* satisfies (by Definition 2): $q(v_D^*,v_0,n) \geq r(v_D^*,v_0,n)$. (A12)

From definition of v_D^* in (A9), as $I\to 0$, $v_D^*\to v'$, since $\delta<I$. Thus, in the limit as $I\to 0$, v_D^* satisfies the equality in (A12). In (A12), dividing both sides by I, then taking limit $I\to 0$, using (A10) and noting that the second term on the r.h.s of (A10) becomes zero, and equating we get

$$\lim I \to 0 \;\; \frac{U(v_D^* - I - v_0)}{\dfrac{U(v_D^* - v_0) - U(v_D^* - I - v_o)}{I}} = \lim I \to 0 \;\; \frac{1 - F(v_D^*)}{\dfrac{F(v_D^*) - F(v_D^* - I)}{I}} \tag{A13}$$

[57].

Since $U(\cdot)$ and $F(\cdot)$ are both differentiable everywhere, derivatives exist at (v_D^*-v_0) for $U(\cdot)$, and at v_D^* for $F(\cdot)$. Since the functions are differentiable at these points, the left and the right hand limit of the functions at these points exist and are equal. Using the left-hand limit on (A13) and inverting the terms gives $\dfrac{U'(v_D^* - v_0)}{U(v_D^* - v_0)} = \dfrac{f(v_D^*)}{1 - F(v_D^*)}$[58]. This condition for optimal reserve is identical to that for optimal reserve in continuous bid auctions (Matthews, 1980, eq. 1 - using utility for zero profit is 0). Q.E.D.

Proof of Proposition 1d (when $\xi\leq 0$, v_D^* decreases as absolute risk aversion increases)

It is sufficient to show v' decreases as absolute risk aversion increases. At v', $q(v',v_0,n) - r(v',v_0,n) = 0$. When $\xi=0$, (A10) gives

$$\frac{1-F(v')}{F(v')-F(v'-I)} - \frac{U(v'-I-v_0)}{U(v'-v_0) - U(v'-I-v_0)} = 0 \tag{A14}$$

[59].

As absolute risk aversion increases, by Lemma 4, the second term increases, implying the difference decreases from zero. Call this difference (with increased absolute risk aversion) as W, W<0. Suppose v' increases. Then, by Lemma 3, the second term in (A14) increases; from Lemma 5, the first term decreases $\Rightarrow$ W decreases further below zero. Thus, W=0 cannot be achieved. However, if v' decreases, the first term increases (Lemma 5), the second term decreases (Lemma 3) and $\exists\; v'_{new}<v'$, for which W=0. Thus, new reserve $v_{D\,new,\xi=0}^* \leq v_D^*$, where $v_{D\,new,\xi=0}^*$ is

$$\begin{array}{ll} v_{D\,new,\xi=0}^* & = v_D^*, \text{ if } v'-\delta \leq v'_{new} < v'; \\ & \leq v_D^*-I, \text{ if } v'_{new} < v'-\delta. \end{array}$$

Q.E.D.

Now decrease ξ, i.e., $\xi<0$. As shown in end of Proof of Proposition 1a-1c, $v_{D\,new,\xi<0}^* \leq v_{D\,new,\xi=0}^*$, where strict inequality holds when n is large. So with relative unaggressiveness, v_D^* decreases as absolute risk aversion increases.

On the other hand, let ξ increase from zero, i.e., $\xi>0$. As shown in end of Proof of Proposition 1a-1c $v_D^*{}_{new,\xi>0} \geq v_D^*{}_{new,\xi=0}$, with strict inequality holding when n is large. Hence, with aggressiveness, it is not necessary that v_D^* change monotonically as absolute risk aversion increases. Q.E.D.

Proof of Proposition 2a: ($E[U[(\pi_S)]$ at v_D^*, increases as ξ decreases)

First, we formulate the expression for expected utility in a discrete auction with reserve. To our knowledge this is not available in the extant literature (Rothkopf and Harstad (1994b) formulated for discrete auction without reserve.) In a discrete auction with reserve, there are three mutually exclusive and exhaustive outcomes, as follows:

(I) The item does not sell. This happens iff $v_{(n)} < v^*_D$, with probability $P_I = F^n(v^*_D)$. This yields $\pi_S = 0$.

(II) The item sells at v^*_D. This happens in three mutually exclusive and exhaustive ways:

(a) Only B_n's value is $\geq v_D^*$. With "correct foot" bidding for the reserve, the item sells at v_D^*. The probability of this event is: $P_{II,a} = \binom{n}{1} [1 - F(v_D^*)]\, F^{n-1}(v_D^*)$ 60.

(b) j ($j\geq 2$) bidders have $v_D^* \leq v_{(i)} < v_D^*+I$, but none has $v_{(i)} \geq v_D^*+I$. The associated probability is:

$$P_{II,b} = \sum_{j=2}^{n} \binom{n}{j} [F(v_D^*+I) - F(v_D^*)]^{j}\, F^{n-j}(v_D^*) \quad 61.$$

(c) Only B_n's value is $\geq v_D^*+I$, and at least one of the (n-1) remaining bidders has $v_D^* \leq v_{(i)} < v_D^*+I$, and B_n has bid at v_D^*. The corresponding probability is:

$$P_{II,c} = \sum_{j=1}^{n-1} n\,[1 - F(v_D^*+I)]\,\left(\frac{1}{j+1} + \xi\right) \binom{n-1}{j} [F(v_D^*+I) - F(v_D^*)]^{j}\, F^{n-1-j}(v_D^*) \quad 62.$$

(III) The item sells for above the reserve, due to competitive bidding. This happens in three ways for each

bid level $v_D^*+kI \leq v$, for k=1,2,...,M; where M denotes the number of possible bid levels above v_D^*.

(a) Only B_n's value is $\geq v_D^*+(k+1)I$, and at least one of the (n-1) remaining bidders has $v_D^*+kI \leq v_{(i)} < v_D^*+(k+1)I$, and B_n has bid at v_D^*+kI. The associated probability is:

$$P_{III,a} = \sum_{j=1}^{n-1} n\,[1 - F(v_D^*+(k+1)I)]\,\left(\frac{1}{j+1} + \xi\right) \binom{n-1}{j} [F(v_D^*+(k+1)I) - F(v_D^*+kI)]^{j}\, F^{n-1-j}(v_D^*+kI)$$

63.

(b) Only B_n's value is $\geq v_D^*+kI$, and at least one of the (n-1) remaining bidders has $v_D^*+(k-1)I \leq v_{(i)} <$

v_D^*+kI, and B_n does not have bid at v_D^*+(k-1)I, and so must bid v_D^*+kI. The probability is:

$$P_{III,b} = \sum_{j=1}^{n-1} n\,[1 - F(v_D^* + k\,I)\,]\,(\frac{j}{j+1} - \xi\,)\binom{n-1}{j}[F(\,v_D^* + k\,I\,) - F(\,V_D^* + (k-1)\,I\,)\,]^{\,j}\;F^{n-1-j}(v_D^* + (k-1)\,I\,)$$

64.

(c) Two or more bidders have v_D^*+kI ≤ $v_{(i)}$ < v_D^*+(k+1)I, but none has $v_{(i)}$≥v_D^*+(k+1)I. The associated probability is: $P_{III,c} = \sum_{j=2}^{n}\binom{n}{j}[F(\,v_D^* + (k+1)\,I\,) - F(\,v_D^* + k\,I\,)\,]^{\,j}\;F^{\,n-j}\,(v_D^* + k\,I\,)$ 65.

Therefore, expected utility of seller, adding (I) through (III), is:

$$E\,[U(\pi_S\,)] = U(\,v_0 - v_0\,)\bullet P_I + U(\,v_D^* - v_0\,)\bullet[\,P_{II,a} + P_{II,b} + P_{II,c}\,] + \sum_{k=1}^{M} U(\,v_D^* + kI\; - \;v_0\,)\bullet[\,P_{III,a} + P_{III,b} + P_{III,c}\,]$$

66, (A15)

where M is such that v_D^*+MI ≤v , and v - (v_D^*+MI) < I.

Note that relative aggressiveness ξ affects only $P_{II,c}$, $P_{III,a}$, and $P_{III,b}$. Hence, to check how E[U(πS)] changes w.r.t. ξ we need focus only on these three probabilities. Thus, we define

$$E\,[U(\pi_S\,)]\,|_{restricted} = U(\,v_D^* - v_0\,)\bullet[\;P_{II,c}\,] + \sum_{k=1}^{M} U(\,v_D^* + k\,I\; - \;v_0\,)\bullet[\,P_{III,a} + P_{III,b}\,]$$ 67.

From Proof of Lemma 6a note that the probability P_1 (see A1a) is similar to each of the probabilities $P_{II,c}$ (using v_D^* for s_0) and $P_{III,a}$ (using v_D^*+kI for s_0) in here; while the probability P_2 (A1b) is similar to probability $P_{III,b}$ in here. From Proof of Lemma 6a further observe that both P_1 (A2a), and P_2 (A2b) can be written as sum of two parts, in which only one part depends on ξ. So we can rewrite,

$$E\,[U(\pi_S\,)]\,|_{restricted} = U(\,v_D^* - v_0\,)\bullet[\;\zeta_{\,II,c}(.) + \eta_{II,c}(\,\xi\,)] + \sum_{k=1}^{M} U(\,v_D^* + k\,I\; - \;v_0\,)\bullet[\,\zeta_{\,III,a}(.) + \eta_{III,a}(\,\xi\,)\; + \;\zeta_{\,III,b}(.) + \eta_{III,b}(\,\xi\,)\,]$$

68,

where ζ(·) + η(·) represents the sum of two parts for each probability, respectively, with only η(·) dependent on ξ. As discussed in "Definition and Properties of ξ" towards the beginning of the Appendix, ξ must depend on some parameter other than n. Thus, to check how E[U(πS)]$|_{\text{restricted}}$ varies w.r.t ξ, we hold n fixed, making ξ only a function of the aforementioned parameter. Then, taking derivative w.r.t. that parameter, indicates in which direction E[U(πS)]$|_{\text{restricted}}$ changes with aggressiveness, as does taking derivative w.r.t. to ξ, because ξ is a monotonic function of the parameter. Also, note that we could write the parameter as an inverse function of ξ (the inverse function is well-defined because the direct function is one-to-one) and accomplish the same objective. To preserve generality and to avoid introducing an additional parameter, we derive the following by using ξ and need not refer to the other parameter.

Now, differentiating w.r.t. ξ gives (noting that the terms in $E[U(\pi_S)]$ which are not dependent on ξ will not enter the expression below),

$$\frac{\partial E\,[U(\pi_S)]}{\partial\,\xi} = U(v_D^* - v_0)\frac{\partial\,\eta_{II,c}(\xi)}{\partial\xi} + \sum_{k=1}^{M} U(v_D^* + k\,I\ -\ v_0)\left[\frac{\partial\,\eta_{III,a}(\xi)}{\partial\xi} + \frac{\partial\,\eta_{III,b}(\xi)}{\partial\xi}\right] \text{69.}$$

We borrow from the simplified expressions for P_1, and P_2 in (A2a-A2b) and on manipulation get:

$$\frac{\partial E\,[U(\pi_S)]}{\partial\,\xi} = U(v_D^* - v_0)\ \left\{n\,[\,1 - F(v_D^* + I)\,]\,[\,F^{n-1}(v_D^* + I)\ -\ F^{n-1}(v_D^*)]\right\} \text{70}$$

$$+ \sum_{k=1}^{M} U(v_D^* + k\,I\ -\ v_0)\bullet\ n\,\{\ \ [\,1 - F(v_D^* + (k+1)\,I\ ]\,[\,F^{n-1}(v_D^* + (k+1)\,I)\ -\ F^{n-1}(v_D^* + k\,I)\,] \text{71}$$

$$- [1 - F(v_D^* + k\,I)\,]\,[\,F^{n-1}(v_D^* + k\,I)\ -\ F^{n-1}(v_D^* + (k-1)\,I\ ]\quad \} \text{72,}$$

$$= \sum_{k=0}^{M} U(v_D^* + k\,I\ -\ v_0)\bullet\ n\,\left\{[\,1 - F(v_D^* + (k+1)\,I\ ]\,[\,F^{n-1}(v_D^* + (k+1)\,I)\ -\ F^{n-1}(v_D^* + k\,I)\,]\right\} \text{73}$$

$$+ \sum_{k=1}^{M} U(v_D^* + k\,I\ -\ v_0)\bullet\ n\left\{[\,1 - F(v_D^* + k\,I\ ]\,[\,F^{n-1}(v_D^* + k\,I)\ -\ F^{n-1}(v_D^* + (k-1)\,I)\,]\right\} \text{74,}$$

which, after simplification yields,

$$= n\sum_{k=1}^{M}\left\{U(v_D^* + (k-1)\,I\ -\ v_0) - U(v_D^* + k\,I\ -\ v_0)\right\}\left\{[\,1 - F(v_D^* + k\,I\ ]\,[\,F^{n-1}(v_D^* + k\,I)\ -\ F^{n-1}(v_D^* + (k-1)\,I)\,]\right\}$$

$$\text{75}\qquad + n\,U(v_D^* + (M-1)\,I\ -\ v_0)\bullet[\,1 - F(v_D^* + M\ I\ )\,]\,[\,F^{n-1}(v_D^* + M\ I\ )\ -\ F^{n-1}(v_D^* + (M-1)\,I)] \text{76,}$$

< 0.

The first line of the expression is strictly < 0, always, since U(·) is a strictly increasing function. The second line is zero because $F(v_D^* + M\,I) = 1$, by setting w.l.g. $v_D^* + M\,I = v$. This is possible because the starting point for bid levels is arbitrary, and also, beyond $v_D^* + M\,I$ there is no relevant support of the valuation distribution. Q.E.D.

Proof of Proposition 2b: (existence of auctions with discrete bid $E[U(\pi_S)]$ > continuous bid $E[U(\pi_S)]$)

In continuous auction, relative aggressiveness does not affect the outcomes and hence, the seller's expected utility remains unaffected. In discrete auctions, $E[U(\pi_S)]$ strictly ↑ as ξ ↓. Thus, as ξ ↓, the difference, $E[U(\pi_S)]_{\text{discrete}} - E[U(\pi_S)]_{\text{continuous}}$, strictly ↑, and can be > 0, depending upon ξ, F(·), v_0, n, and I. This increases the set of auctions where the seller is better off with discrete than continuous bidding. We present an example of this in Section 5. Q.E.D.

Proofs of Proposition 3a-c:

The proofs examine a more general shilling strategy where shill bids can be placed one or more times in succession. First the general case is stated, and then the results are proved.

General Case of Shilling Strategies: Define a shill "run" as one or more shill bids placed consecutively by shill(s) without waiting for a response from bona fide bidders, where runs of two or more bids can be placed as in Footnote 9. Two shill runs must be separated by a bona fide bid. Denote a sequence of shill runs by s_1, s_2, ..., where, the discrete bid level at which the *i*th shill run ends is denoted s_i.

Thus, $s_i \geq s_{i-1}+2I$, i=1,2,...

A seller's planned shilling actions, if the seller decides to shill and all shill runs succeed, are completely defined by the values s_1, s_2, ... , s_t, where s_t is the final shill bid in the terminal shill run.

The seller's expected utility from a given shilling strategy: $E[U(\pi_S) \mid s_0]$ from a series of shill runs ending at s_t is the product of $U(s_t+I-v_0)$ and the probabilities that all shill runs up to and including s_t succeed. Or,

$$E(U(\pi_{s_1,\dots,s_t} \mid s_0)) = U(s_t + I - v_0) \bullet P(v_{(n),only} \mid s_0) \bullet P(v_{(n)} \geq s_1 + I \mid s_0, v_{(n),only}) \bullet \prod_{j=2}^{j=t} P(v_{(n)} \geq s_j + I \mid v_{(n)} \geq s_{j-1} + I)$$

77 (A16)

Denote $\rho = P(v_{(n)} \geq s_1 + I \mid \{S_0 = s_0, v_{(n),only}\})$.

To compute $E[U(\pi_S) \mid s_0]$, first enumerate ρ. Then enumerate the probability that all subsequent shills succeed. $P(v_{(n),only})$ has been defined in (A3). We further show that the decisions where to begin and end each shill run, and the length of each run do not affect $E[U(\pi_S) \mid s_0]$, provided the last shill run ends in s_t^*. Combining all these gives the result.

Proof of Proposition 3a: (existence of s_t^*)

By Fact 1, knowing $v_{(n-1)}$ fully characterizes distribution of $v_{(n)}$. From discussion in text (Section 3(ii)) it follows that the event $\{S_0 = s_0$ and $v_{(n),only}\}$ provides the bound $[s_0-I, s_0+I)$ for $v_{(n-1)}$. That is, either (i) $s_0 - I \leq v_{(n-1)} < s_0$, or, (ii) $s_0 \leq v_{(n-1)} < s_0 + I$. If (i), then we know $v_{(n)} \geq s_0$, using $\{S_0 = s_0$ and $v_{(n),only}\}$, and Fact 1. If (ii), then we know $v_{(n)} > v_{(n-1)}$, using Fact 1. To enumerate the probability ρ we condition on (i) and (ii). Thus, we get, applying theorem of total probability,

$$\rho = [\, P(v_{(n)} \geq s_1 + I \mid \{s_0 \text{ and } v_{(n),only}\} \text{ and } s_0 - I \leq v_{(n-1)} < s_0) \cdot P(s_0 - I \leq v_{(n-1)} < s_0 \mid \{s_0 \text{ and } v_{(n),only}\})$$
$$+ P(v_{(n)} \geq s_1 + I \mid \{s_0 \text{ and } v_{(n),only}\} \text{ and } s_0 \leq v_{(n-1)} < s_0 + I) \cdot P(s_0 \leq v_{(n-1)} < s_0 + I \mid \{s_0 \text{ and } v_{(n),only}\})\,].$$

Using the characterization that $v_{(n-1)} \in [s_0-I, s_0+I)$, as obtained from $\{S_0 = s_0$ and $v_{(n),only}\}$,

$$\rho = [\, P(v_{(n)} \geq s_1 + I \mid \{s_0 \text{ and } v_{(n),only}\} \text{ and } s_0 - I \leq v_{(n-1)} < s_0) \cdot P(s_0 - I \leq v_{(n-1)} < s_0 \mid s_0 - I \leq v_{(n-1)} < s_0 + I)$$
$$+ P(v_{(n)} \geq s_1 + I \mid \{s_0 \text{ and } v_{(n),only}\} \text{ and } s_0 \leq v_{(n-1)} < s_0 + I) \cdot P(s_0 \leq v_{(n-1)} < s_0 + I \mid s_0 - I \leq v_{(n-1)} < s_0 + I)\,].$$

Using Fact 1 and writing 1st line of ρ to the left of the \`+' sign and 2nd line to the right in (A17), we get,

ρ

=

$$\frac{1-F(s_1+I)}{1-F(s_0)}\,\frac{G(s_0)-G(s_0-I)}{G(s_0+I)-G(s_0-I)}+\left[\int_{s_0}^{s_0+I}\frac{1-F(s_1+I)}{1-F(v_{(n-1)})}\,\frac{g(v_{(n-1)})}{G(s_0+I)-G(s_0)}\,d\,v_{(n-1)}\right]\frac{G(s_0+I)-G(s_0)}{G(s_0+I)-G(s_0-I)}$$

[78] (A17)

where G(·) and g(·) are the cdf and pdf of the unconditional distribution of $v_{(n-1)}$.

We next rewrite (A17), for ease of exposition in what follows, as

$\rho=[1 - F(s_1+I)]\ H(s_0)$ (A18)

where, $H(s_0)=\dfrac{1}{G(s_0+I)-G(s_0-I)}\left[\dfrac{G(s_0)-G(s_0-I)}{1-F(s_0)}+\displaystyle\int_{s_0}^{s_0+I}\frac{g(v_{(n-1)})}{1-F(v_{(n-1)})}\,d\,v_{(n-1)}\right]$[79].

On integration, we get,

$$H(s_0)=\frac{1}{G(s_0+I)-G(s_0-I)}\left[\frac{G(s_0)-G(s_0-I)}{1-F(s_0)}+n\left\{F^{\,n-1}(s_0+I)-F^{\,n-1}(s_0)\right\}\right]$$ [80]. (A19)

If the shill run s_1 succeeds, the probability that s_2 succeeds, where $s_2 \geq s_1+2I$, is

$P(v_{(n)} \geq s_2+I \mid v_{(n)} \geq s_1+I) = [1 - F(s_2+I)] / [1 - F(s_1+I)]$, (A20)

noting, if first shill run succeeds then remaining bidder has $v_{(n)}$ with probability one, by text Sec. 3(iii).

The expected utility of shilling with terminal shill s_t^* does not depend on the intermediate shills. Let $J(s_t)$ denote $E[U(\Pi_S) \mid s_0]$ from a set of shilling runs where final run ends in s_t. *Ex-ante*, at s_0, the expected utility from shilling to s_1 and then to s_2 is, by (A18) & (A20) and using conditional probability,

$J(s_2) = P(v_{(n),only})\ \{ [1 - F(s_1+I)]\ H(s_0) \}\ \{ [1 - F(s_2+I)]\ /\ [1 - F(s_1+I)] \}\ U(s_2+I-v_0)$

$= P(v_{(n),only})\ \{ [1 - F(s_2+I)]\ H(s_0) \}\ U(s_2+I-v_0)$, by cancelling terms.

Now suppose that an optimal terminal shill bid s_t^* exists. It can be seen by repeating the above algebra for several shill runs and cancelling terms, that $J(s_t^*)$ from shilling $s_1, s_2, \ldots, s_t^*$ is $P(v_{(n),only})\ [1 - F(s_t^*+I)]\ H(s_0)\ U(s_t^*+I-v_0)$. The intermediate shills do not affect the expected utility, so the seller can choose any sequence of shill runs to arrive at s_t^*.

The seller's objective. At s_0, the seller wants to find the bid level s_t^* (allowing property to sell for s_t^*+I), which maximizes $J(s_t) = P(v_{(n),only})\ [1 - F(s_t+I)]\ H(s_0)\ U(s_t+I-v_0)$ (A21)

subject to the constraint $\{ P(v_{(n),only})\ [1 - F(s_t^*+I)]\ H(s_0)\ U(s_t^*+I-v_0) \} > U(s_0-v_0)$. (A22)

s_t^* does not depend on s_0. In expression (A21) for the objective function, s_0 appears *only* in $P(v_{(n),only})$ and $H(s_0)$ - both of which are multiples of the expressions involving the optimizing variable s_t. Further, $P(v_{(n),only})$ and $H(s_0)$ do not depend upon s_t, observing (A1)-(A2), and (A21) and (A19). Maximizing $[1 - F(s_t+I)]\ U(s_t+I-v_0)$ w.r.t. s_t yields s_t^*, which by observation does not depend upon s_0. But, *whether* to shill is determined by (A22), which depends upon s_0.

Solving for s_t^*. We consider the objective function J(·) in (A21) as a function of a continuous variable y, and then establish the result for $s_t^* \in \Theta$.

Call $J(y\text{-}I) = P(v_{(n),only})\,[1 - F(y)]\,H(s_0)\,U(y\text{-}v_0)$. Note that at $y=s_t+I$, (A21) follows. We find $y=y^*$ which maximizes $J(y\text{-}I)$ such that $J(y^*\text{-}I)>U(s_0\text{-}v_0)$, where y^* is the continuous equivalent of s_t^*+I, not that of s_t^*. Simplifying the f.o.c. such a y^*, if exists, satisfies

$$\frac{U'(y^* - v_0)}{U(y^* - v_0)} = \frac{f(y^*)}{1 - F(y^*)} \tag{A23}$$

[81].

Assumption 1 and definition of F(v) and U(·) assures y^* in (A23) is unique.

If $y^* \notin \Theta$, then search for s_t^* on the discrete bid levels immediately below and above y^*. In doing so, we can find, by continuity and strict concavity of J(·), a $\delta>0$ ($\delta<I$), which will make $y^*\text{-}\delta \in \Theta$. Note $y^*\text{-}\delta$ is the discrete bid level immediately below y^*. To construct s_t^*, we make use of the fact the seller wants the *bidder* to have the bid which maximizes $E[U(\pi_S) \mid s_0]$, so the seller would shill up to one interval lower. Thus, subject to satisfying (A22), we get:

$$\begin{aligned} s_t^* &= y^*\text{-}I, \text{ if } y^* \in \Theta; \\ &= y^*\text{-}\delta, \text{ if } y^* \notin \Theta \text{ and } J(y^*\text{-}\delta) \geq J(y^*\text{-}\delta\text{-}I); \\ &= y^*\text{-}\delta\text{-}I, \text{ if } y^* \notin \Theta \text{ and } J(y^*\text{-}\delta) < J(y^*\text{-}\delta\text{-}I). \end{aligned} \tag{A24}$$

Q.E.D.

Remark 1: Note that $v_C^* = s_t^*+I$, provided $v_C^* \in \Theta$. This follows by observing the equivalence between (A23) and equation (1) of Matthews (1980).

Proof of Proposition 3b (s_t^* does not depend on s_0)

As shown above.

Proof of Proposition 3b (s_t^* does not depend on ξ)

As can be seen by observing (A23) and (A24).

Proof of Proposition 3b (s_t^* does not depend on n)

Follows from definition of s_t^* in (A24) and because y^* does not depend upon n from (A23).

Proof of Proposition 3b (s_t^* increases with v_0)

We first show $y^* \uparrow$ with v_0, using Assumption 1, $U'>0$, and $U''\leq 0$. After some simplification we get

$$\frac{\partial y^*}{\partial v_0} = \frac{f(y^*)U'(y^* - v_0) - [1 - F(y^*)]\,U''(y^* - v_0)}{\dfrac{U(y^* - v_0)}{1 - F(y^*)}\left\{ f'(y^*)[1 - F(y^*)] + 2f^2(y^*) \right\} - [1 - F(y^*)]\,U''(y^* - v_0)} > 0$$

[82]. Increase v_0 to v_0+x.

Call y^* at v_0+x, as y^*_{new} which is strictly greater than y^*. Thus, $s_{t,new}^* \geq s_t^*$. Q.E.D.

Proof of Proposition 3b (s_t^* decreases as absolute risk aversion increases)

By definition of absolute risk aversion in proof of Lemma 4, and using Theorem (1) Pratt (1964),

$$-\frac{U_1'(y^* - v_0)}{U_1(y^* - v_0)} \leq -\frac{U_2'(y^* - v_0)}{U_2(y^* - v_0)} \tag{A25}$$

[83].

By Assumption 1, the r.h.s. of (A23) is strictly $\uparrow$ in y^*. We show that the l.h.s. is $\downarrow$ in y^*.

$$\frac{\partial \frac{U'(y^*-v_0)}{U(y^*-v_0)}}{\partial y^*} = -\frac{U'(y^*-v_0)}{U(y^*-v_0)}\left[\frac{U'(y^*-v_0)}{U(y^*-v_0)} - \frac{U''(y^*-v_0)}{U'(y^*-v_0)}\right] < 0$$ 84, _ $U>0$, $U'>0$, and $U''\leq 0$, $\forall\, y^*\geq v_0+I$.

Now as absolute risk aversion increases, from (A25) we know that the l.h.s. of (A23) decreases. Therefore, $\frac{U'(y^*-v_0)}{U(y^*-v_0)} < \frac{f(y^*)}{1-F(y^*)}$ 85. Given the r.h.s. is increasing and the l.h.s. is decreasing as y^* increases, we can achieve equality only if y^* decreases. Call the new value as y^*_{new} which is strictly less than y^*. Hence, $s_{t,new}^* \leq s_t^*$. Q.E.D.

Proof of Proposition 3c: (comparing s_t^*+I with v_D^*)

Let $\xi=0$. Keep v' and y^* as continuous analog of v_D^*, and s_t^*, respectively. Without loss of generality, let $\{v',y^*\}\in\Theta$. For reserve auction, using the continuous analog of (A10) means taking the limit of $I\rightarrow 0$, which gives the condition $\frac{U'(v'-v_0)}{U(v'-v_0)} = \frac{f(v')}{1-F(v')}$ 86 for optimal reserve. This is exactly same as (A23) for optimal shill level. Now using (A9) and (A24) we conclude $v_D^* = s_t^*+I$.

Now let $\xi > (<)\ 0$. We know $v_D^* \uparrow (\downarrow)$ as $n\uparrow$ (Propositions 1a-1b), and s_t^* does not depend on n (Proposition 3b). Hence, if $\xi > 0$, then for some n, $v_D^* > s_t^*+I$, provided $s_t^*+2I < v$; if $\xi < 0$, then for some n, $v_D^* < s_t^*+I$, provided $s_t^*-I \geq v_0$. Q.E.D.

Proof of Proposition 4: (existence of auctions where seller accepts s_0 rather than shilling to s_t^*)

Focus is on $\xi\leq 0$. (i) From (A22) it follows whether or not to shill depends on s_0. (ii) Also note that $P(v_{(n),only})$ and $H(s_0)$ both depend on n. Hence, (A22) implies that the decision whether to shill depends on n. (iii) We show that $s_0<s_t^*$ does not imply seller should always shill. Assume this is *not* true. Then, condition (A22) holds for all $s_0<s_t^*$.

Or, $\{ P(v_{(n),only})\, [1 - F(s_t^*+I)]\, H(s_0)\, U(s_t^*+I-v_0) \} > U(s_0-v_0)$ for all n. (A26)

Note, $U(s_t^*+I-v_0) > U(s_0-v_0)$ always, and that, it can be checked $0 < \{ [1 - F(s_t^*+I)] / [1-F(s_0)] \} < [1 - F(s_t^*+I)]\, H(s_0) < 1$. On the other hand, $P(v_{(n),only}) < 1$ (Lemma 6b), decreases as n increases (Lemma 6a, here $\xi \leq 0$). As n increases, l.h.s. in (A26) can decrease beyond some sufficiently large n, depending on ξ, v_0, $F(v)$, s_0 and I. But the r.h.s. is a positive constant w.r.t. n. $\therefore\ \exists$ n, depending on ξ, v_0, $F(v)$, s_0 and I, for which (A26) is violated, contradicting our assumption. Thus, $\exists$ auctions in which seller is better off not shilling up to s_t^* and instead accepting s_0. Q.E.D.

Proof of Proposition 5a: (seller as well or better off with shilling compared to reserve)

Note that the information demand on the seller regarding ξ is the same in both optimal shilling and reserve. Compared to using a reserve, the shilling seller acts upon more information, i.e., knowledge of s_0 (which he cannot act upon in reserve auctions). This allows the shilling seller (*ex-ante*) to enumerate each bidders' valuation with decreased variance compared to F(v) used in reserve auction. This happens because $\{S_0=s_0\}$ refines bidders' valuations to $\{ s_0 \leq v_{(n)} \leq v \}$, $\{ s_0-I \leq v_{(n-1)} < s_0+I$

$\}$, and $\{\underline{v} \leq \{v_{(1)}, \ldots, v_{(n-2)}\} < s_0+I\}$. Further, *ex-ante*, the set of possible decisions (accept or reject bids) the seller can make is identical in both auctions. Also note that the bidders' *ex-ante* set of decisions remains unchanged from reserve to shilling (bidders don't change their bidding strategy during auction, as stated in text Section 3). Hence, the seller can never be worse off by using optimal shilling compared to using optimal reserve (appealing to Blackwell's Theorem on Comparison of Experiments, 1951).

We now show that auctions exist where, *ex-ante*, the seller is strictly better off with optimal shilling than with optimal reserve. By Blackwell's Theorem, $E[U(\pi_S)]$ is strictly higher in the shilling auction when:

(i) $s_t^*+I \neq v_D^*$, and, (ii) if $s_t^*+I=v_D^*$, and it is optimal for the seller to not shill for at least one $s_0 < s_t^*$ (by Proposition 4 such auctions exist).

In (i), shilling and reserve strategies yield different optimal lowest acceptable bids and hence different expected utilities for the seller. In (ii), optimal shilling is strictly dominating for the seller because the seller accepts at least one s_0 with shilling (based on more information) that he rejects with reserve, and he accepts that s_0 because the seller's expected utility is greater from accepting s_0 rather than shilling up to s_t^*. Now, the expected utility for the latter is the same as that from setting $v_D = v_D^*$ in the optimal reserve scenario, since $v_D^*=s_t^*+I$. Thereby, the seller's expected utility is strictly higher with optimal shilling.

If $s_t^*+I=v_D^*$ and the seller shills for all $s_0<s_t^*$, then seller is as well off as in reserve auction.

Proof of Proposition 5b: (bidder better off, thus, Pareto dominance of shilling over reserve)

(i) When $\xi>0$, consider shilling auctions with $s_t^*+I<v_D^*$ (per Proposition 3c). The probability the item sells is strictly greater with shilling than with reserve. Also, *ex ante* each bona fide bidder's expected surplus is greater in such shilling auction, because (i) with probability > 0, property would sell at some bid $< v_D^*$ in shilling auction, and (ii) P(any bidder's value $\geq s_t^*+I$) > P(bidder's value $\geq v_D^*$), since $s_t^*+I < v_D^*$.

(ii) When $\xi\leq 0$, consider shilling auction with $s_t^*+I=v_D^*$. Using Proposition 4 and the arguments we have just stated in the proof of Proposition 5a, we conclude that auctions exist in which bidders are strictly better off (unless seller shills for all $s_0 \leq s_t^*$, in which case the bidders are as well off as in reserve auction.)

By Proposition 5a, the seller is at least as well off with shilling. Pareto dominance follows. Q.E.D.

REFERENCES

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