The Equivalence of Bundling and Advance Sales: Online Appendix

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1 Time preferences: Different discount factors

If consumers have time preferences as above, but the firm does not, then the equivalence result of the previous corollary breaks down (however, see the special case in the next subsection). In this case, we do not need to scale the costs in the bundling model by $1/\delta$ and so we keep the cost parametrization of the bundling model the same as the benchmark. If the bundle price is $(1/\delta)p^{AS}$, product 1's price is ap_1^{SS}, and product 2's price is $(1-a)p_2^{SS}$, then the demand for the bundle, the demand for only product 1, and the demand for only product 2 correspond respectively to the advance selling demand, demand for spot sales in state 1, and the demand for spot sales in state 2. However, at those prices, the firm's profit is higher in the bundling model than it is in the advance selling model since in the former model specification it receives a higher price, $(1/\delta)p^{AS}$ rather than p^{AS}, from the same number of consumers. Hence, it is not possible to obtain the exact equivalence of the advance selling model to the bundling model in this case. More generally, the equivalence breaks down if the firm discounts future at a different rate from the discount rate of consumers, since then it is not possible to obtain the equivalence of demands as well as of profits with a unique parametrization of prices. In this case we obtain the following result from the comparison of the two problems.

Proposition 1 *Suppose that in the advance selling model consumers discount future (consumption and payments at the spot selling period) by $0 < \delta_c \leq 1$ (compared to the advance selling period), and the firm discounts future (profits at the spot selling period) by $0 < \delta_f \leq 1$*

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such that $\delta_f \neq \delta_c$. Suppose that in the bundling problem the marginal cost of product 1 is ac_1/δ_f and the marginal cost of product 2 is $(1-a)c_2/\delta_f$. If the firm discounts future less than consumers, i.e., $\delta_f > \delta_c$, the region where the advance selling is profitable is smaller than the region where bundling is profitable, and the latter includes the former. The opposite is true otherwise.

Proof. The utility from buying in advance is now:

$$av_1 + (1-a)v_2 - (1/\delta_c)p^{AS},$$

Apart from this, the rest of Table 1 remains the same. But now we need to scale the costs in the bundling model by $1/\delta_f$, so the marginal cost of product 1 becomes ac_1/δ_f and the marginal cost of product 2 is $(1-a)c_2/\delta_f$. If the bundle price is $(1/\delta_c)p^{AS}$, product 1's price is ap_1^{SS}, and product 2's price is $(1-a)p_2^{SS}$, then the demand for the bundle, the demand for only product 1, and the demand for only product 2 correspond respectively to the advance selling demand, demand for spot sales in state 1, and the demand for spot sales in state 2. At those prices in the advance selling model the firm gets

$$\Pi^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}) = (1/\delta_f)(p^{AS} - ac_1 - (1-a)c_2)D^{AS} + a(p_1^{SS} - (1/\delta_f)c_1)D_1^{SS} + (1-a)(p_2^{SS} - (1/\delta_f)c_2)D_2^{SS}.$$

On the other hand, at those prices in the bundling model the firm gets

$$\begin{aligned}\Pi^{B}(ap_1^{SS}, (1-a)p_2^{SS}, (1/\delta_c)p^{AS}) = {} & ((1/\delta_c)p^{AS} - (1/\delta_f)(ac_1 + (1-a)c_2))D^{AS} \\ & + a(p_1^{SS} - (1/\delta_f)c_1)D_1^{SS} + (1-a)(p_2^{SS} - (1/\delta_f)c_2)D_2^{SS}.\end{aligned}$$

Obviously, when $\delta_f < \delta_c$, $\Pi^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}) > \Pi^{B}(ap_1^{SS}, (1-a)p_2^{SS}, (1/\delta_c)p^{AS})$, and so for parameter values where the bundling is profitable, it must be the case that advance selling is profitable. ■

This implies that if consumers are more impatient than the firm, which is more likely to be true as the firm should have easier access to capital market than individuals, for all parameter values where bundling is not profitable, advance selling is not profitable either. More interestingly, this result shows that even if one cannot obtain the exact equivalence between the bundling problem and the advance selling problem, the comparison of the two models enables us to obtain insights on the profitability of advance selling if we know whether the firm or consumers are more impatient and have a characterization of the profitability of the bundling strategy.

2 More than two states of the world

Chen and Riordan (2013) also analyze the bundling problem of a monopolist selling more than two goods. To apply their findings of that case to the advance selling problem, we extend the model of advance selling by allowing for $N > 2$ states of nature where state i occurs with probability $a_i \in [0, 1]$ such that $\sum_i a_i = 1$ for $i = 1, .., N$. A consumer receives utility v_i if she consumes the product in state i. Let $f(\bullet, ..., \bullet)$ denote the joint probability distribution function of valuations across the states over region $[\underline{v_1}, \overline{v_1}]$ x...x $[\underline{v_N}, \overline{v_N}]$. As before, we do not make any assumption on the type of correlation between the valuations across the states. Furthermore, the marginal cost of the firm is c_i in state i. We assume that at least some consumers have valuations above the respective marginal costs, and that the monopolist's second order condition is satisfied (there exists a global maximum). Assuming the same timing as before, we analyze the profitability of selling the product before the state is realized (advance selling) at price p^{AS} or after the state realization (spot selling) at price p_i^{SS} in state i, or offering both an advance selling price and spot selling prices.

Similarly, we extend the bundling model to more than two products: suppose that the seller offers $N > 2$ products and consumers have unit demand for each product. Suppose that a consumer receives utility $a_i v_i$ from consuming product i and let $f(\bullet, ..., \bullet)$ denote the joint distribution function of the valuations $v_1, ..., v_N$ over region $[\underline{v_1}, \overline{v_1}]$x... x $[\underline{v_N}, \overline{v_N}]$. Assume also that the seller's marginal cost of product i is $a_i c_i$ and that the firm cannot prevent consumers from purchasing any two (or more) products separately. The bundling problem involves analyzing the profitability of selling the products as a bundle at price p^B or selling only the individual products separately at prices $p_1^I, .., p_N^I$, or offering both a bundle price and individual product prices (mixed bundling).

Following a similar argument as the one in Proposition 1, one can show the mathematical equivalence of these two problems.

Proposition 2 *The problem of finding the optimal pricing strategy in the advance selling model when there are $N > 2$ possible states of the world (described above) is mathematically equivalent to the problem of finding the optimal pricing strategy in the bundling problem when the firm sells $N > 2$ products (described above). Moreover, if the optimal bundle price is p^*, and the stand-alone optimal price for product i is $a_i p_i^*$, then the optimal advance selling price is p^* and the optimal spot selling price in state i is p_i^*.*

Using the previous bundling literature, this equivalence result implies that advance selling with more than two possible future states is profitable under similar conditions as the ones for two states.

Corollary 1 *Consider the advance selling problem where there are $N > 2$ possible states of the world. If the valuations across any two states are either negatively dependent, independent, or not too positively dependent, then advance selling increases profits.*

Another potential issue arising from the bundling literature is sometimes firms bundle only a subset of the products, if only for the practical purposes of having a manageable size menu. For example, with mixed bundling setting prices to 41 products results in $2^{41} - 1$ possible prices, and no firm will realistically subject its consumers to this number of choices. Chu, Leslie, and Sorensen (2011) show numerically (and confirm empirically) that, for particular distributions, bundle-size pricing (setting prices that depend only on the size of the bundle purchased) tends to closely approximate the profits from mixed bundling. This result significantly simplifies the analysis of bundling with many products.

Similarly, in the problem of advance selling, we can think of selling different types of advance tickets such that each ticket incorporates a different number of possible states. For example, a consumer can have a ticket for the basketball game at the end of the season no matter what the standings are, or she can buy a ticket that lets her attend only if the team won more than x out of the preceding 81 games.[1]

Using Chu *et. al.* (2011), our equivalence result (Proposition 2) implies the following:

Corollary 2 *Consider the problem of advance selling with $N > 2$ states of the world where the firm can offer advance purchase tickets for each of the possible combinations of states. The firm's profit under this problem is closely approximated by offering advance purchase tickets such that their prices depend only on the number of states in which they are valid.*

Bakos and Brynjolfsson (1999) consider the problem of monopoly bundling when the number of products is very large (goes to infinity). In the case where consumer valuations across states are independent and products have zero marginal cost (digital products, tickets to events where almost all costs are fixed, and so on), our equivalence result brings out the following finding.

Corollary 3 *Consider the problem of advance selling with $N > 2$ states of the world. Assume that consumer valuations are distributed independently across the states, uniformly bounded with continuous density functions and nonnegative support, and marginal costs are zero. Then, as N increases, the consumer surplus from purchasing the bundle decreases to zero, and the firm extracts nearly all consumer surplus.*

[1] The alternative results in the advance selling menu of the size of the cardinality of the power set with 82 elements (2^{82}): an advance sale ticket that is valid only if the team won 0 or 1 games, an advance sale ticket that is valid only if the team won 1 or 2 games... an advance ticket that is valid only if the team won 0, 1, or 2 games.... culminating with an advance ticket that is valid in any state.

The intuition comes from the law of large numbers. More and more possible future states result in more and more independent draws for each consumer. As long as consumer valuations are drawn from the same distribution (within the same state, not necessarily the same distribution across the states), their expected values of the advance selling ticket covering all the states become more and more homogeneous, and in the limit the monopolist simply charges that price in advance, sells to everyone in advance, and captures all consumer surplus.

3 Competition

Assume that consumer value from the dominant firm's product is v_i in state i, for $i = 1, 2$, and their value from the rival product is $\tilde{v}_1$ in state 1, and state 1 occurs with probability a. Let $f(\bullet, \bullet)$ denote the joint distribution of (v_1, v_2) over region $[\underline{v_1}, \overline{v_1}]$ x $[\underline{v_2}, \overline{v_2}]$ (as in the benchmark) and also let $\tilde{f}(\bullet, \bullet)$ denote the joint distribution of $(\tilde{v}_1, v_2)$ over region $[\underline{\tilde{v}_1}, \overline{\tilde{v}_1}]$ x $[\underline{v_2}, \overline{v_2}]$. Assume that the dominant seller's marginal cost of product 1 is c_1 and of product 2 is c_2, and the entrant's marginal cost is $\tilde{c}_1$. The timing of the events is the following:

The timing of the game is as described in the body of the text.

Similar to the monopoly model of advance selling (section 2.1), Table 3 shows consumers' purchasing options at the advance selling stage and their corresponding utilities.

Table 1: The expected utility of a consumer from different purchasing options in the advance selling model with competition.

Purchasing options	**Expected utility**
Advance purchase	$av_1 + (1-a)v_2 - p^{AS}$
Spot purchase only the dominant firm's product in state 1	$a(v_1 - p_1^{SS})$
Spot purchase only the competitor's product in state 1	$a(\tilde{v}_1 - \tilde{p}_1^{SS})$
Spot purchase only in state 2	$(1-a)(v_2 - p_2^{SS})$
No purchase	0

Let $D^{AS}, D_1^{SS}, \tilde{D}_1^{SS}, D_2^{SS}$ denote respectively the demand for advance purchasing, the demand for spot purchasing only the dominant firm's product in state 1, the demand for spot purchasing only the competitor's product in state 1, and the demand for spot purchasing only in state 2. Note that these demand functions are different from their counterparts of the benchmark (derived in appendix A) since now while deriving demands we need to account for one more option for consumers in state 1: Purchasing the product of the competitor. The dominant firm's profit is the sum of the *expected* profits from advance selling and from spot selling:

$$\Pi^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}, \tilde{p}_1^{SS}) = (p^{AS} - ac_1 - (1-a)c_2)D^{AS} + a(p_1^{SS} - c_1)D_1^{SS} + (1-a)(p_2^{SS} - c_2)D_2^{SS}. \quad (1)$$

The competitor's profit is the expected profit from spot selling only:

$$\tilde{\Pi}^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}, \tilde{p}_1^{SS}) = a(\tilde{p}_1^{SS} - \tilde{c}_1)\tilde{D}_1^{SS}. \quad (2)$$

To obtain a bundling equivalent of the advance selling model with competition, we modify the benchmark model of bundling (section 2.2) as the following: Consider a static problem of competitive bundling where a dominant seller offers two products, a competitor offers a differentiated version of product 1 and consumers have unit demand for each product. Suppose that a consumer receives utility av_1 from consuming product 1 of the dominant firm, $a\tilde{v_1}$ from consuming product 1 of the entrant, and $(1-a)v_2$ from consuming product 2. As before, let $f(\bullet, \bullet)$ denote the joint distribution function of the valuations (v_1, v_2) over region $[\underline{v_1}, \overline{v_1}]$ x $[\underline{v_2}, \overline{v_2}]$ and also let $\tilde{f}(\bullet, \bullet)$ denote the joint distribution of $(\tilde{v}_1, v_2)$ over region $[\underline{\tilde{v}_1}, \overline{\tilde{v}_1}]$ x $[\underline{v_2}, \overline{v_2}]$. Assume that the dominant seller's marginal cost of product 1 is ac_1 and of product 2 is $(1-a)c_2$, and the entrant's marginal cost is $a\tilde{c_1}$. Assume also that the dominant firm cannot prevent consumers from purchasing both products separately. The timing of the events is the following:

1. The dominant firm sets a bundle price (p^B) and individual product prices (p_1^I, p_2^I). Simultaneously, the competitor sets its price, $\tilde{p}_1^I$.

2. Consumers decide whether to purchase the bundle or individual products only. In the latter case, they decide from which firm to buy product 1.

Similar to the previous advance selling problem, we write consumer options and their corresponding utilities:

Table 2: The utility of a consumer from different purchasing options in the bundling model with competition.

Purchasing options	**Expected utility**
Bundle of products 1 and 2	$av_1 + (1-a)v_2 - p^B$
Only product 1 from the dominant firm	$av_1 - p_1^I$
Only product 1 from the competitor	$a\tilde{v}_1 - \tilde{p}_1^I$
Only product 2	$(1-a)v_2 - p_2^I$
No purchase	0

Consumers choose the option which gives them the highest utility. The dominant firm's profit in the bundling model is

$$\Pi^B(p_1^I, p_2^I, p^B, \tilde{p}_1^I) = (p^B - ac_1 - (1-a)c_2)D^B + (p_1^I - ac_1)D_1^I + (p_2^I - (1-a)c_2)D_2^I, \quad (3)$$

and the competitor's profit is

$$\tilde{\Pi}^B(p_1^I, p_2^I, p^B, \tilde{p}_1^I) = (\tilde{p}_1^I - a\tilde{c}_1)\tilde{D}_1^I, \quad (4)$$

Comparing table 3 and table 4 shows that at prices $p^B = p^{AS}$, $p_1^I = ap_1^{SS}$, $\tilde{p}_1^I = a\tilde{p}_1^{SS}$, $p_2^I = (1-a)p_2^{SS}$ consumers face exactly the same set of purchasing options and utilities in the advance selling model and in the bundling model. This implies that at these prices the demand for advance purchasing corresponds to the demand for the bundle, $D^{AS} = D^B$, the demand for spot purchasing the dominant firm's product in state 1 corresponds to the demand for only product 1 from the dominant firm, $D_1^{SS} = D_1^I$, the demand for spot purchasing the competitor's product in state 1 corresponds to the demand for only product 1 from the competitor, $\tilde{D}_1^{SS} = \tilde{D}_1^I$ and finally the demand for spot purchasing only in state 2 corresponds to the demand for only product 2, $D_1^{SS} = D_1^I$. But then at those prices the profits of both models coincide:

$$\Pi^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}, \tilde{p}_1^{SS}) = \Pi^B(ap_1^{SS}, (1-a)p_2^{SS}, p^{AS}, a\tilde{p}_1^{SS})$$
$$\tilde{\Pi}^{AS}(p_1^{SS}, p_2^{SS}, p^{AS}, \tilde{p}_1^{SS}) = \tilde{\Pi}^B(ap_1^{SS}, (1-a)p_2^{SS}, p^{AS}, (1-a)\tilde{p}_1^{SS})$$

Hence, we obtain our equivalence result

Proposition 3 *The problem of finding the optimal pricing strategy in the competitive advance selling model (described above) is mathematically equivalent to the problem of finding the optimal pricing strategy in the competitive bundling model (described above). Moreover, if the dominant firm's optimal bundle price is p^*, and optimal prices for the individual products are ap_1^* and $(1-a)p_2^*$, and the competitor's product price is $a\tilde{p}_1^*$, then in the advance selling model the dominant firm's optimal advance selling price is p^* and the optimal spot selling prices are p_1^* and p_2^*, and the competitor's product price is $\tilde{p}_1^*$.*

4 Consumers with risk preferences

In the advance selling model we assume that consumers are risk-neutral. In particular, this implies that a consumer's utility from consuming a unit of a good bringing utility v at a price p is $U(w+v-p) = u(w)+v-p$ where w refers to the consumer's endowment. In most microeconomic and marketing problems, the endowment is assumed to be fixed and thus the $u(w)$ term generally drops out of the calculations. In this section, we show how an advance selling problem can be approximated by a bundling problem even when consumers are not risk-neutral.

The only difference from the setup of Section 2 is that in both the advance selling model and the bundling model we assume that consumers have a utility function of $U(\bullet)$, which incorporates consumer risk preferences. In particular, a concave U implies risk-aversion, a convex U implies risk-lovingness, and a linear U degenerates to our benchmark models. We again assume that in both models there is a unique solution to the seller's optimal pricing problem (the second order conditions are satisfied). Keep the notation as before. Denote consumer i's current wealth level by w_i. Similar to Tables 1 and 2, Tables 5 and 6 show the purchasing options of a consumer in each model.

Table 3: The utility of a consumer from different purchasing options in the advance selling model.

Purchasing options	**Expected utility**
Advance purchase	$aU(w_i+v_1-p^{AS})+(1-a)U(w_i+v_2-p^{AS})$
Spot purchase only in state 1	$aU(w_i+v_1-p_1^{SS})+(1-a)U(w_i)$
Spot purchase only in state 2	$aU(w_i)+(1-a)U(w_i+v_2-p_2^{SS})$
No purchase	$U(w_i)$

Table 4: The utility of a consumer from different purchasing options in the bundling model.

Purchasing options	**Utility**
Bundle of products 1 and 2	$U(w_i+av_1+(1-a)v_2-p^B)$
Only product 1	$U(w_i+av_1-p_1^I)$
Only product 2	$U(w_i+(1-a)v_2-p_2^I)$
No purchase	$U(w_i)$

Each consumer is going to pick the purchasing option which gives her the maximum (expected) utility. Using Taylor Expansion, we transform Tables 5 and 6 into Table 7 and 8, respectively.

Comparing Tables 7 and 8 shows that if the price for the bundle is p^{AS} and the prices for product 1 and product 2 are, respectively, ap_1^{SS} and $(1-a)p_2^{SS}$, the purchasing options of the bundling model would differ from the purchasing options of the advance selling model for Taylor expansion of orders two and more. And so how close the bundling model approximates the advance selling model depends on the magnitude of the higher order derivatives of the utility function.

Table 5: The (expected) utility of a consumer from different options in the advance selling model (Taylor Expansion).

Purchasing options	**Expected Utility**
Advance purchasing	$U(w_i) + U'(w_i)(av_1 + (1-a)v_2 - p^{AS}) + U''(w_i)(a(v_1 - p^{AS})^2 + (1-a)(v_2 - p^{AS})^2)/2 + ...$
only in spot 1	$U(w_i) + aU'(w_i)(v_1 - p_1^{SS}) + aU''(w_i)(v_1 - p_1^{SS})^2/2 + ...$
only in spot 2	$U(w_i) + (1-a)U'(w_i)(v_2 - p_2^{SS}) + (1-a)U''(w_i)(v_2 - p_2^{SS})^2/2 + ...$
No purchase	$U(w_i)$

Table 6: The utility of a consumer from different options in the bundling model (Taylor Expansion).

Purchasing options	**Utility**
Bundle	$U(w_i) + U'(w_i)(av_1 + (1-a)v_2 - p^B) + U''(w_i)(av_1 + (1-a)v_2 - p^B)^2/2...$
Product 1 only	$U(w_i) + U'(w_i)(av_1 - ap_1^I) + U''(w_i)(av_1 - ap_1^I)^2/2 + ...$
Product 2 only	$U(w_i) + U'(w_i)((1-a)v_2 - (1-a)p_2^I) + U''(w_i)((1-a)v_2 - (1-a)p_2^I)^2/2 + ...$
No purchase	$U(w_i)$