Online Appendix for
"The Cross Attributes Flexible Substitution Logit:
Uncovering Category Expansion and Share Impacts of Marketing Instruments"
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<u>Online Appendix 1:</u>

Proof:

$$\frac{\partial P_k}{\partial v_j}=\begin{cases}\dfrac{P_j}{\delta_n}(1+(\delta_n-1)P_{j|B_n}-\delta_n P_j) & for\ k=j,\ and\ k,j\in B_n\\ \dfrac{P_k}{\delta_n}((\delta_n-1)P_{j|B_n}-\delta_n P_j) & for\ k\neq j,\ and\ k,j\in B_n\\ -P_kP_j & for\ k\neq j,\ and\ k\in B_{m,}\ j\in B_n\end{cases}$$

$$\frac{\partial v_k}{\partial x_{ja}}=\begin{cases}\beta_a+\gamma_a & for\ k=j\\ \gamma_a & for\ k\neq j,\ and\ k,j\in B_n\\ 0 & for\ k\neq j,\ and\ k\in B_{m,}\ j\in B_n\end{cases}$$

$$\begin{aligned}\frac{\partial P_j}{\partial x_{ja}}&=\sum_{l=1}^{J}\frac{\partial P_j}{\partial v_l}\cdot\frac{\partial v_l}{\partial x_{ja}}\\ &=\frac{\partial P_j}{\partial v_j}\cdot\frac{\partial v_j}{\partial x_{ja}}+\sum_{\substack{l=1\\ l\neq j}}^{J}\frac{\partial P_j}{\partial v_l}\cdot\frac{\partial v_l}{\partial x_{ja}}\\ &=\frac{P_j}{\delta_n}\left(1+(\delta_n-1)P_{j|B_n}-\delta_nP_j\right)\left(\beta_a+\gamma_a\right)+\sum_{l\in B_n,l\neq j}\frac{P_j}{\delta_n}\left((\delta_n-1)P_{l|B_n}-\delta_nP_l\right)\cdot\gamma_a\\ &=\frac{P_j}{\delta_n}\left(\left(1+(\delta_n-1)P_{j|B_n}-\delta_nP_j\right)\left(\beta_a+\gamma_a\right)+\left((\delta_n-1)(1-P_{j|B_n})-\delta_n(P_n-P_j)\right)\gamma_a\right)\\ &=\frac{P_j}{\delta_n}\left(\left(1+(\delta_n-1)P_{j|B_n}-\delta_nP_j\right)\beta_a+\delta_n\left(1-P_n\right)\gamma_a\right)\end{aligned}$$

For $k \neq j$ *and* $k, j \in B_n$

$$
\begin{aligned}
\frac{\partial P_k}{\partial x_{ja}} &= \sum_{l=1}^{J} \frac{\partial P_k}{\partial v_l} \cdot \frac{\partial v_l}{\partial x_{ja}} \\
&= \frac{\partial P_k}{\partial v_j} \cdot \frac{\partial v_j}{\partial x_{ja}} + \frac{\partial P_k}{\partial v_k} \cdot \frac{\partial v_k}{\partial x_{ja}} + \sum_{\substack{l \in B_n \\ l \neq j,k}} \frac{\partial P_k}{\partial v_l} \cdot \frac{\partial v_l}{\partial x_{ja}} \\
&= \frac{P_k}{\delta_n}((\delta_n - 1)P_{j|B_n} - \delta_n P_j)\left(\beta_a + \gamma_a\right) + \frac{P_k}{\delta_n}(1 + (\delta_n - 1)P_{k|B_n} - \delta_n P_k) \cdot \gamma_a \\
&\quad + \sum_{\substack{l \in B_n \\ l \neq j,k}} \frac{P_k}{\delta_n}((\delta_n - 1)P_{l|B_n} - \delta_n P_l) \cdot \gamma_a \\
&= \frac{P_k}{\delta_n}\begin{pmatrix} ((\delta_n - 1)P_{j|B_n} - \delta_n P_j)\left(\beta_a + \gamma_a\right) + (1 + (\delta_n - 1)P_{k|B_n} - \delta_n P_k) \cdot \gamma_a \\ +(\delta_n - 1)(1 - P_{j|B_n} - P_{k|B_n}) \cdot \gamma_a - \delta_n(P_n - P_j - P_k) \cdot \gamma_a \end{pmatrix} \\
&= \frac{P_k}{\delta_n}\left(\left((\delta_n - 1)P_{j|B_n} - \delta_n P_j\right)\beta_a + \delta_n\left(1 - P_n\right)\gamma_a\right)
\end{aligned}
$$

For $k \neq j$ *and* $k \in B_m, j \in B_n$

$$
\begin{aligned}
\frac{\partial P_k}{\partial x_{ja}} &= \sum_{l=1}^{J} \frac{\partial P_k}{\partial v_l} \cdot \frac{\partial v_l}{\partial x_{ja}} \\
&= \frac{\partial P_k}{\partial v_j} \cdot \frac{\partial v_j}{\partial x_{ja}} + \sum_{\substack{l \in B_n \\ l \neq j}} \frac{\partial P_k}{\partial v_l} \cdot \frac{\partial v_l}{\partial x_{ja}} \\
&= -P_k P_j\left(\beta_a + \gamma_a\right) - \sum_{\substack{l \in B_n \\ l \neq j}} P_k P_l \gamma_a \\
&= -P_k\left(P_j \beta_a + P_j \gamma_a + (P_n - P_j)\gamma_a\right) \\
&= -P_k\left(P_j \beta_a + P_n \gamma_a\right)
\end{aligned}
$$

$$
\frac{-\partial P_k / \partial x_{ja}}{\partial P_j / \partial x_{ja}} = \begin{cases} \dfrac{-P_k\left(\left((\delta_n - 1)P_{j|B_n} - \delta_n P_j\right)\beta_a + \left(1 - P_n\right)\delta_n \gamma_a\right)}{P_j\left(\left(1 + (\delta_n - 1)P_{j|B_n} - \delta_n P_j\right)\beta_a + \left(1 - P_n\right)\delta_n \gamma_a\right)} & \\ \qquad\qquad\qquad\qquad \textit{for } k \neq j, \textit{ and } k, j \in B_n & \\ \dfrac{\delta_n P_k\left(P_j \beta_a + P_n \gamma_a\right)}{P_j\left(\left(1 + (\delta_n - 1)P_{j|B_n} - \delta_n P_j\right)\beta_a + \left(1 - P_n\right)\delta_n \gamma_a\right)} & \\ \qquad\qquad\qquad\qquad \textit{for } k \neq j, \textit{ and } k \in B_{m,}\ j \in B_n & \end{cases}
$$

Online Appendix 2: Simulation Experiment

The purpose of the simulation experiment is to assess the extent to which a standard RUM model with parameter heterogeneity can capture non-IPS patterns. Given the purpose of the experiment, we assume the source of growth substantially varies. In a market in which we assume competing brands use three marketing instruments, the first and third marketing instruments draw most of the demand (98.4% and 74.7% respectively) away from competing alternatives, whereas the second marketing instrument draws mostly from the outside option (81.5%). The experiment will show that the standard RUM models systematically underestimate these differences.

We assume there are four brands in the market and a no purchase option. The data generating process in based on the proposed CASFL model wherein we assume that the four brands are in one nest and the no-purchase option is in another. We generate data for 300 households, each of whom makes 50 purchase decisions. Individual consumers' preferences and responsiveness to instruments are assumed to follow a normal distribution, i.e. $\theta_i \sim N(\theta, \Sigma)$. The true value of θ and the sample mean of simulated values of θ_i are given in Table A1. The values of Σ and the simulated variance-covariance of β_i are provided in Online Appendix 3 (Tables A3-A4).

In addition to the RC CAFSL model, we estimate two other models on the simulated data: a RC logit model and a RC GNL model.[1] Parameter estimates for the three models are shown in Table A1. Estimates for Σ are provided in Appendix 3 Table A5-A7. As might be expected, the RC CAFSL model outperforms the other models in terms of model fit. The real aim of the experiment, however, is not to compare the models based on fit but rather to better understand how well the benchmark models recover substitution patterns when the origins of demand vary across marketing instruments. We now turn to comparing the aggregate substitution patterns found by each model across the marketing instruments.

In Table A2, we present the own elasticities and the substitution matrix for the three marketing instruments of Brand 1. We note all three models do well in recovering elasticities of the three marketing instruments. Only the CAFSL model, however, is able to recover the substitution patterns found in the simulated data. For example, the CASFL implies the proportion of demand drawn from the outside option is 1.5%, 81.8% and 36.8% respectively across the marketing instruments,

[1] To better explore the flexibility offered by the RC GNL model, we considered two kinds of nesting structures. In the first structure we allow each alternative (including no-purchase) to belong to each of two nests with some probability. In the second structure, we set up three nests that are respectively, the nest for all brands, the nest for all alternatives including no-purchase, and the nest for the no-purchase option only. In this structure any brand can belong to each of the first two nests with a certain probability and the no-purchase option can belong to each of the latter two nests with a certain probability. The RC GNL with two nests outperforms the model with three nests in fit, and therefore we used this model to make our comparisons.

which accords well with the proportions actually found in the data of 1.6%, 81.5% and 35.3% respectively.

The substitution patterns predicted by the RC logit model and the RC GNL model do not accord well with the data. While adding heterogeneity to the logit and GNL models via the RC specification allows some differences to exist across the marketing instruments, these differences are small and do not necessarily reflect the true substitution patterns. The RC logit model implies the proportion of demand drawn from the outside option is 48.2%, 44.2% and 43.4% respectively across the marketing instruments. The RC GNL model implies the proportion of demand drawn from the outside option is 29.5%, 29.6% and 31.8% respectively[2].

Table 1: Simulation Study – Parameter Value and Estimates for RC Models

	True Model		RC Logit		RC GNL		RC CAFSL	
	True Value	Sample Mean of θ_i	Mean	95% Interval	Mean	95% Interval	Mean	95% Interval
Brand Intercept								
α_1	-0.9	-0.797	-1.577	(-1.756, -1.398)	-0.268	(-0.361, -0.175)	-0.880	(-1.081, -0.700)
α_2	-0.7	-0.756	-1.524	(-1.706, -1.345)	-0.245	(-0.338, -0.154)	-0.841	(-1.034, -0.661)
α_3	-0.75	-0.750	-1.467	(-1.637, -1.304)	-0.238	(-0.330, -0.150)	-0.797	(-0.985, -0.624)
α_4	-1.0	-0.964	-1.752	(-1.929, -1.581)	-0.337	(-0.432, -0.243)	-1.028	(-1.228, -0.846)
Marketing Instruments								
β_1	-3.0	-2.935	-3.729	(-3.919, -3.544)	-1.456	(-1.615, -1.302)	-3.029	(-3.279, -2.808)
β_2	0.25	0.222	0.344	(0.235, 0.453)	0.121	(0.080, 0.164)	0.206	(0.131, 0.281)
β_3	1.35	1.382	1.801	(1.637, 1.967)	0.688	(0.597, 0.785)	1.412	(1.263, 1.564)

[2] We also estimate on the simulated data a RC CAFSL model that includes a parameter for the summation of instrument 3 (whereas the true value of γ_3 is zero). As expected, the misspecified CAFSL model provides a non-significant estimate of γ_3, indicating that the proposed model can identify the absence of spill-over effects.

Summation of Instruments								
γ_1	0.35	0.378	-	-	-	-	0.400	(0.298, 0.502)
γ_2	0.15	0.126	-	-	-	-	0.103	(0.031, 0.171)
γ_3	-	-	-	-	-	-	-	-
Inclusive Value								
δ_1	0.5	-	-	-	0.232	(0.206, 0.261)	0.523	(0.487, 0.568)
δ_2	-	-	-	-	0.352	(0.284, 0.425)	-	-
Probability of Membership in Nest 1								
τ_1	-	-	-	-	0.802	(0.764, 0.840)	-	-
τ_2	-	-	-	-	0.792	(0.753, 0.832)	-	-
τ_3	-	-	-	-	0.809	(0.765, 0.856)	-	-
τ_4	-	-	-	-	0.786	(0.748, 0.823)	-	-
τ_5	-	-	-	-	0.228	(0.171, 0.281)	-	-
Log - ML*			-12244		-11422		-10641	

*Log Marginal Likelihood.

Table 2: Substitution Matrices for Simulation Study

	Simulated Data			RC Logit Model			RC Generalized Nested Logit Model			RC CAFSL Model		
	I1	*I2*	*I3*	*I1*	*I2*	*I3*	*I1*	*I2*	*I3*	*I1*	*I2*	*I3*
B1	-	-	-	-	-	-	-	-	-	-	-	-
B2	31.7	1.8	21.6	17.2	16.7	19.6	22.9	22.2	23.4	32.1	6.9	21.6
B3	34.1	4	22.3	17.6	20.6	19.3	24.1	24.9	23.5	34.4	3.1	21.7
B4	32.6	12.7	20.9	17.1	18.5	17.8	23.4	23.3	21.3	32.1	8.2	19.9
No-purch.	1.6	81.5	35.3	48.2	44.2	43.4	29.5	29.6	31.8	1.5	81.8	36.8
Total	100	100	100	100	100	100	100	100	100	100	100	100
Elasticities	-0.828	0.129	0.812	-0.832	0.117	0.822	-0.837	0.110	0.801	-0.832	0.112	0.812

For each model, cell entries in each column indicate the percentage of sales increase of the brand B1 due to a 1% increase in its marketing instrument (e.g. I1) that is drawn from the alternative indicated in the row. For example, the logit model predicts that if B1 decreases its I1 by 1%, 31.7% of its incremental sales will come from the brand B2.

Online Appendix 3: Variance-Covariance Matrices for Simulation and Empirical Studies

Table 3: Simulation Study (Σ)– True Values

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_1	γ_2
α_1	1.0	0.1	0.1	0.1	-0.1	0.1	-0.1	-0.1	0.1
α_2	-0.1	1.0	0.1	-0.1	-0.1	-0.1	0.1	0.1	-0.1
α_3	-0.1	-0.1	1.0	0.1	-0.1	-0.1	-0.1	-0.1	0.1
α_4	-0.1	-0.1	-0.1	1.0	0.1	-0.1	-0.1	0.1	-0.1
β_1	-0.1	-0.1	0.1	-0.1	1.0	0.1	0.1	0.1	-0.1
β_2	-0.1	-0.1	0.1	0.1	0.1	0.25	-0.1	-0.1	0.1
β_3	-0.1	0.1	-0.1	-0.1	0.1	0.1	1.0	0.1	-0.1
γ_1	-0.1	-0.1	-0.1	-0.1	0.1	-0.1	0.1	0.5	0.1
γ_2	-01	-0.1	0.1	-0.1	-0.1	0.1	0.1	-0.1	0.25

Table 4: Simulation Study (Σ)– Simulated Sample

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_1	γ_2
α_1	1.079	-0.15	-0.136	-0.103	-0.042	-0.091	-0.074	-0.105	-0.095
α_2	-0.15	1.032	-0.041	-0.094	-0.207	-0.075	0.107	-0.117	-0.059
α_3	-0.136	-0.041	0.953	-0.15	0.246	0.047	-0.165	-0.084	0.052
α_4	-0.103	-0.094	-0.15	0.951	-0.101	0.123	-0.041	-0.098	-0.072
β_1	-0.042	-0.207	0.246	-0.101	0.972	0.132	0.009	0.021	-0.064
β_2	-0.091	-0.075	0.047	0.123	0.132	0.24	0.083	-0.124	0.079
β_3	-0.074	0.107	-0.165	-0.041	0.009	0.083	0.98	0.103	0.079
γ_1	-0.105	-0.117	-0.084	-0.098	0.021	-0.124	0.103	0.546	-0.123
γ_2	-0.095	-0.059	0.052	-0.072	-0.064	0.079	0.079	-0.123	0.217

Table 5: Simulation Study (Σ) – RC Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3
α_1	1.504	-0.478	-0.433	-0.445	-0.2	-0.298	0.009
α_2	-0.478	1.57	-0.206	-0.208	-0.415	-0.142	0.386
α_3	-0.433	-0.206	1.256	-0.303	0.305	0.102	-0.135
α_4	-0.445	-0.208	-0.303	1.331	-0.391	0.014	0.025
β_1	-0.2	-0.415	0.305	-0.391	1.544	0.023	0.198
β_2	-0.298	-0.142	0.102	0.014	0.023	0.352	0.22
β_3	0.009	0.386	-0.135	0.025	0.198	0.22	1.272

Table 6: Simulation Study (Σ) – RC Generalized Nested Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3
α_1	0.233	-0.045	-0.031	-0.038	-0.028	-0.037	-0.005
α_2	-0.045	0.235	-0.007	-0.009	-0.055	-0.015	0.047
α_3	-0.031	-0.007	0.21	-0.017	0.044	0.013	-0.024
α_4	-0.038	-0.009	-0.017	0.213	-0.046	0.006	-0.008
β_1	-0.028	-0.055	0.044	-0.046	0.237	0.016	0.023
β_2	-0.037	-0.015	0.013	0.006	0.016	0.048	0.023
β_3	-0.005	0.047	-0.024	-0.008	0.023	0.023	0.206

Table 7: Simulation Study (Σ) – RC CAFSL Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_1	γ_2
α_1	1.185	-0.222	-0.123	-0.169	-0.013	-0.13	-0.07	-0.111	-0.037
α_2	-0.222	1.129	-0.076	-0.064	-0.22	-0.003	0.176	-0.057	-0.067
α_3	-0.123	-0.076	0.99	-0.087	0.265	0.097	-0.154	-0.064	0.044
α_4	-0.169	-0.064	-0.087	1.067	-0.117	0.08	-0.076	-0.082	-0.032
β_1	-0.013	-0.22	0.265	-0.117	1.116	0.13	0.037	0.022	-0.005
β_2	-0.13	-0.003	0.097	0.08	0.13	0.26	0.08	-0.174	0.111
β_3	-0.07	0.176	-0.154	-0.076	0.037	0.08	1.064	0.119	0.116
γ_1	-0.111	-0.057	-0.064	-0.082	0.022	-0.174	0.119	0.532	-0.11
γ_2	-0.037	-0.067	0.044	-0.032	-0.005	0.111	0.116	-0.11	0.19

Table 8: Yogurt Data (Σ) – RC Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_{temp}
α_1	6.179	3.531	5.192	6.789	-3.764	-0.466	-0.069	-0.039
α_2	3.531	4.65	4.36	5.186	-2.464	-0.328	-0.152	-0.035
α_3	5.192	4.36	8.362	6.804	-4.483	-0.524	-0.05	-0.045
α_4	6.789	5.186	6.804	9.764	-6.543	-0.533	-0.125	-0.038
β_1	-3.764	-2.464	-4.483	-6.543	7.075	0.244	0.055	-0.001
β_2	-0.466	-0.328	-0.524	-0.533	0.244	0.066	0.005	0.004
β_3	-0.069	-0.152	-0.05	-0.125	0.055	0.005	0.032	<0.001
β_{temp}	-0.039	-0.035	-0.045	-0.038	-0.001	0.004	<0.001	0.001

Table 9: Yogurt Data (Σ) – RC Generalized Nested Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_{temp}
α_1	2.302	1.194	2.172	2.299	-0.13	-0.378	0.039	-0.024
α_2	1.194	1.936	1.977	2.021	0.41	-0.371	-0.005	-0.024
α_3	2.172	1.977	4.27	4.011	-1.055	-0.666	0.027	-0.03
α_4	2.299	2.021	4.011	4.091	-1.012	-0.655	0.033	-0.031
β_1	-0.13	0.41	-1.055	-1.012	2.358	0.05	-0.033	-0.011
β_2	-0.378	-0.371	-0.666	-0.655	0.05	0.136	-0.004	0.006
β_3	0.039	-0.005	0.027	0.033	-0.033	-0.004	0.01	<0.001
β_{temp}	-0.024	-0.024	-0.03	-0.031	-0.011	0.006	<0.001	0.001

Table 10: Yogurt Data (Σ)– RC CAFSL Logit

	α_1	α_2	α_3	α_4	β_1	β_2	γ_1	γ_3	β_{temp}
α_1	0.844	0.098	0.061	0.135	1.111	0.046	0.025	-0.367	-0.001
α_2	0.098	1.391	0.301	0.021	2.267	0.072	0.033	-0.761	0.001
α_3	0.061	0.301	1.325	-0.217	0.999	-0.037	0.068	-0.279	-0.005
α_4	0.135	0.021	-0.217	0.373	-0.092	-0.025	-0.03	0.125	-0.003
β_1	1.111	2.267	0.999	-0.092	8.553	0.093	0.14	-2.385	-0.003
β_2	0.046	0.072	-0.037	-0.025	0.093	0.073	0.014	-0.168	0.004
γ_1	0.025	0.033	0.068	-0.03	0.14	0.014	0.032	-0.09	<0.001
γ_3	-0.367	-0.761	-0.279	0.125	-2.385	-0.168	-0.09	1.167	-0.012
β_{temp}	-0.001	0.001	-0.005	-0.003	-0.003	0.004	<0.001	-0.012	0.001

Table 11: Statins Data (Σ) – RC Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3
α_1	1.265	0.931	0.94	0.608	0.027	-0.031	0.69
α_2	0.931	2.036	1.38	0.843	0.038	-0.198	1.201
α_3	0.94	1.38	3.241	1.001	0.027	-0.052	1.985
α_4	0.608	0.843	1.001	2.664	0.118	0.17	0.234
β_1	0.027	0.038	0.027	0.118	0.021	0.008	0.002
β_2	-0.031	-0.198	-0.052	0.17	0.008	0.088	-0.152
β_3	0.69	1.201	1.985	0.234	0.002	-0.152	1.862

Table 12: Statins Data (Σ) – RC Generalized Nested Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3
α_1	0.493	0.387	0.278	0.251	-0.005	<0.001	-0.064
α_2	0.387	0.593	0.359	0.3	<0.001	-0.009	0.022
α_3	0.278	0.359	0.634	0.254	0.005	-0.017	0.17
α_4	0.251	0.3	0.254	0.616	0.018	0.043	-0.053
β_1	-0.005	<0.001	0.005	0.018	0.009	0.001	0.003
β_2	<0.001	-0.009	-0.017	0.043	0.001	0.03	-0.013
β_3	-0.064	0.022	0.17	-0.053	0.003	-0.013	0.264

Table 13: Statins Data (Σ) – RC CAFSL Logit

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	γ_2	γ_3
α_1	1.037	0.697	0.658	0.56	0.013	-0.132	0.31	0.462	0.606
α_2	0.697	1.413	0.82	0.663	0.019	-0.24	0.376	0.538	0.832
α_3	0.658	0.82	2.284	0.773	0.038	-0.2	0.38	0.496	1.376
α_4	0.56	0.663	0.773	2.466	0.077	0.079	1.308	0.267	1.43
β_1	0.013	0.019	0.038	0.077	0.017	0.001	0.039	0.003	0.048
β_2	-0.132	-0.24	-0.2	0.079	0.001	0.105	0.047	-0.101	-0.114
β_3	0.31	0.376	0.38	1.308	0.039	0.047	0.9	0.185	0.715
γ_2	0.462	0.538	0.496	0.267	0.003	-0.101	0.185	0.399	0.345
γ_3	0.606	0.832	1.376	1.43	0.048	-0.114	0.715	0.345	1.722