Online Appendix

A Proofs of A-P Stable Propositions

The proof of Proposition 1 follows from the following two lemmas.

Lemma 1 *The strict core allocation in the advertiser-slot one-to-many matching game (Kelso Jr and Crawford 1982) is A-P stable.*

Proof of Lemma 1: In an advertiser-slot one-to-many matching game (Kelso Jr and Crawford 1982), each advertising slot is considered as an *independent agent*. The strict core condition implies:

- $\forall \langle i, k_j \rangle \in \mathcal{M}, V_i(\mathcal{M}_i) - V_i(\mathcal{M}_i \backslash \langle i, k_j \rangle) - p_{k_j} \geq V_{i0}$ and $\forall k_j, p_{k_j} \geq 0$.
- there does not exist an advertiser i, a new allocation $\mathcal{M}'$ and a price vector $\mathcal{P}'$, such that:
  - $V_i(\mathcal{M}'_i) - \sum_{\delta:\langle i,\delta \rangle \in \mathcal{M}'} p'_\delta \geq V_i(\mathcal{M}_i) - \sum_{\delta:\langle i,\delta \rangle \in \mathcal{M}} p_\delta$.
  - $\forall j \in \{j | \langle i, k_j \rangle \in \mathcal{M}'\}$, $p'_{k_j} \geq p_{k_j}$
  - Strict inequality holds for at least one condition.

Compare these conditions with the A-P stable condition in Definition 1. It is immediately clear that the strict core allocation of the advertiser-slot matching game is a subset of the A-P stable allocation. □

Lemma 2 *The strict core allocation exists.*

Proof of Lemma 2: The proof is straightforward by verifying that the $V_i(\mathcal{M})$ function satisfies the conditions MP, NFL and GS laid out in Kelso Jr and Crawford (1982). In particular, the GS condition requires that when the advertising prices rise, an advertiser does not withdraw from an advertising slot for which the price does not rise. This is true for the $V_i(\mathcal{M})$ function when the marginal benefit from an advertising slot is not decreasing as an advertiser purchases fewer other slots. □

Proof of Proposition 2: We prove the contrapositive. Consider allocations $\mathcal{M}$ which strictly Pareto dominate $\mathcal{M}'$. Thus, $\exists\, i, st. V_i(\mathcal{M}) > V_i(\mathcal{M}')$. The additive revenue function implies that $\sum_{i,k_j:\langle i,k_j\rangle\in\mathcal{M}} V_i(\langle i,k_j\rangle) > \sum_{i,k_j:\langle i,k_j\rangle\in\mathcal{M}'} V_i(\langle i,k_j\rangle)$. In other words, the total valuation is higher in allocation $\mathcal{M}$. There exists at least one advertising slot that is valued higher in allocation $\mathcal{M}$ than in allocation $\mathcal{M}'$. Call this advertising slot k_j and assume that it is assigned to advertiser i in allocation $\mathcal{M}$. It immediately follows that advertiser i and publisher j have incentives to deviate from $\mathcal{M}'$. Thus, $\mathcal{M}'$ is not A-P stable. □

B Monte Carlo Evidence of Maximum Score Estimator

To evaluate the finite sample properties of the maximum score estimator that we use in this paper, we conduct Monte Carlo experiments. Fox and Bajari (2013) and Akkus et al. (2012) both use Monte Carlo experiments to verify the performance of the Maximum Score Estimator, in one-to-one matching and one-to-many matching contexts. Our exercise here follows the same procedure and is an extension in verifying the performance of our specific many-to-many matching with transfer framework.

We simulate an advertising network with 100 publishers that sells 242 advertising slots to 500 potential advertisers. We simulate one advertiser attribute and one publisher attribute that would influence the base value. We use the slot position as the slot attribute. We also simulate advertiser-publisher matching specific attributes. The distributions for each attribute are shown in Table B.1. The value function has the same specification as in our model. Consistent with our model, we add an integrated error term to the value function $e_{ik_j} = \nu_j + \kappa_k + \mu_i + \varepsilon_{ij}$. In our simulation, drawing error terms from different components has the same effect as drawing a compounded error term. Thus, we directly draw e_{ik_j} from an iid normal distribution with different levels of standard deviations σ_e. Finally, each advertiser has an outside option value that equals the median of its valuation towards all the advertising

Table B.1: Monte Carlo Results of Maximum Score Estimator

Parameters	Distribution	True value	$\sigma_e = 0.01$		$\sigma_e = 0.05$		$\sigma_e = 0.10$	
			Bias	RMSE	Bias	RMSE	Bias	RMSE
Advertiser attribute	$N(2, 0.5)$	0.5	-0.00	0.02	0.06	0.08	0.22	0.28
Publisher attribute	$N(2, 0.3)$	0.3	0.01	0.01	-0.02	0.04	-0.08	0.09
Slot attribute	1,2,3,4,5	-0.5	-0.04	0.04	0.00	0.05	0.11	0.15
Matching attribute I	$N(1, 0.5)$	0.4	0.01	0.01	0.01	0.02	0.02	0.04
Matching attribute II	$U(0, 3)$	0.6	-0.03	0.03	-0.01	0.04	-0.00	0.05

Note: $Bias = \frac{1}{n}\sum_{i=1}^{n}(\hat{\beta}_i - \beta_{true})$, $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{\beta}_i - \beta_{true})^2}$, and $n = 100$ in our simulations.

slots. To generate the market equilibrium allocation and prices, we use linear programming to solve for the social planner's problem of maximizing total advertiser value. With the implied matches under the solution, we use linear programming to generate a vector of maximum prices and a vector of minimum prices that would support the obtained assignment as an equilibrium outcome (Shapley and Shubik 1979). We use the average of the two price vectors as the equilibrium price. In our Monte Carlo experiment, we vary the standard deviation of the error term at levels of 0.01, 0.05 and 0.1 to examine the performance of the estimator at different noise levels. A large value of σ_e indicates more variations in the percentage of values that are not captured by the deterministic part. For example, $\sigma_e = 0.05$ implies that the standard deviation of the percentage of the value not captured is 5%. For each level, we generate 100 simulated markets to obtain our maximum score estimates. We report the mean bias and root mean squared error (RMSE) based on each of the 100 replications in Table B.1.

Results suggest that our maximum score estimator performs quite well when we have relatively small error components ($\sigma_e = 0.01, 0.05$), such that the variations in the model are well captured in the deterministic parts. When we have relatively larger variation uncaptured in the model, the bias for the advertiser, publisher and slot attributes increases. However, we still get very consistent estimates for the matching attributes.

C Imputation of Stochastic Components and Outside Option

We can directly compute the deterministic part of the value function based on the model estimates. However, the error components that are important for our policy analysis purposes remain unspecified. One possible method, used in Yang et al. (2009), is to simulate the stochastic terms from an assumed distribution and use the values in the policy analysis. For our application, this method may not be the optimal. First, the maximum score estimator is a semi-parametric estimation method without any specification on the distributions of the stochastic terms. It is arbitrary in terms of which distribution to choose to simulate the error terms. More importantly, given that our market consists of a large number of advertisers and publishers, the total number of inequality conditions is large. Thus, the probability that a randomly generated vector of the error terms to satisfy all of the necessary conditions is extremely low, and the procedure is very expensive in terms of computation time if we keep only the draws that satisfy the necessary conditions.

Our solution relies on formulating it as a mathematical programming problem. Since it is hard to separate the error components, and ε_{ij} would in general absorb the variation in ν_j and μ_i, we use a combined error term in our imputation. That is, we search for a vector of error terms $e_{ik_j} = \nu_j + \kappa_k + \mu_i + \varepsilon_{ij}$ that satisfies every necessary condition from our defined equilibrium. Specifically, we define the function to be minimized as:

$$
\begin{aligned}
& Q(e) = \sum_{ik_j} (\exp(e_{ik_j}) - 1)^2 \\
s.t. \quad & V_{ik_j} - p_{k_j} \geq V_{ik'_{j'}} - p_{k'_{j'}}, \ \forall \langle i, k_j \rangle \in \mathcal{M}, \langle i, k'_{j'} \rangle \notin \mathcal{M}, \\
& V_{ik_j} + V_{i'k'_j} \geq V_{ik'_j} + V_{i'k_j}, \ \forall \langle i, k_j \rangle \in \mathcal{M}, \langle i', k'_j \rangle \in \mathcal{M} \\
& V_{ik_j} - p_{k_j} \geq 0, \ \forall \langle i, k_j \rangle \in \mathcal{M}.
\end{aligned}
$$

That is, we minimize a square function on the error terms, to the constraint that all the three sets of pairwise inequalities used in our estimation are satisfied. The above defined objective function transforms our problem to a Quadratic Linear Programming (QLP) problem, which could be solved quickly using some standard optimization solvers (we use the Gurobi solver).[1] The solution to the above constrained minimization problem may not be unique, and we use the one that the solver returns. Most of the error terms from this imputation remain at 0, and only 3.6% of the error terms are non-zero, with a mean value of -0.004, which shows that we have captured most of the variation in the deterministic part of the model.

In terms of the outside option value, V_{i0} enters the individual rationality condition, $V_i(\langle i, k_j \rangle) - p_{k_j} \geq V_{i0}$. When the pricing policy of the platform changes, advertisers may choose not to purchase advertisements once individual rationality conditions are unsatisfied. From the equilibrium conditions, we can obtain the lower and upper bounds for V_{i0}. The lower bound will be the maximum profit in the set of advertising slots not purchased by the advertiser and the upper bound be the minimum profit in the set of advertising slots currently purchased. The two bounds turn out to be quite close to each other in our result, with the mean ratio of lower bound to upper bound being 0.97. Thus, we use the average of the two bounds as the outside option value for each advertiser. Since any value within the two bounds will satisfy the individual rationality conditions, the average of the two bounds is a valid estimate for the outside option value.

D GSP Equilibrium Under General Quality Score Assignments

Varian (2007) and Edelman et al. (2007) independently derive the equilibrium conditions of the generalized second price auction (GSP) that is used by search engines such as Google. Here, we follow the discussion in Varian (2007) to derive the Symmetric Nash Equilibrium under a general quality score assignment that is not necessarily consistent with click-through rates. In this extension we also generalize the setting by allowing each advertiser to have an outside option value.

[1] We can also use the criterion function $Q(e) = \sum_{ik_j}(e_{ik_j} - 1)^2$. However, the stochastic terms enter the constraints non-linearly, leading to a quadratic non-linear programming problem, which is far more complicated. Our problem involves a large number of variables, and we do not find a feasible solution under this criterion.

Consider the case in which n advertisers bid for appearing on s ($n \geq s$) ad slots provided by a publisher j. Denote the position effect of $k-th$ slot to be s_k, and the expected value for advertiser i conditional on click-through to be $\tilde{v}_{ij}$. In our specific context, $\tilde{v}_{ij} = EI_j \times v_i$ as in equation (2). Also denote advertiser i's base click-through rate to be m_{ij}; then the observed click-through rate for advertiser i appearing on position k is $z_{ikj} = m_{ij}s_k$. We further assume that advertiser i is assigned a quality score of q_{ij} by the platform and submits a per-click-level bid of b_{ij}. In this situation, according to the GSP rules, advertiser i's AdRank score is $b_{ij}q_{ij}$ and advertisers are ordered by this AdRank. We also assume that advertiser i has an outside option value of V_{i0}. The total equilibrium value from advertising must be greater than this V_{i0} for advertiser i to stay in competition.

Relabeling subscripts for advertisers such that they are indexed by the position they occupy according to AdRanks in equilibrium—i.e., $b_{1j}q_{1j} \geq b_{2j}q_{2j} \geq \cdots \geq b_{nj}q_{nj}$. Then, using the symmetric Nash equilibrium definition (no one wants to deviate to any other position) as in Varian (2007), we have the following conditions:

$$(\tilde{v}_{ij} - \frac{b_{(i+1)j}q_{(i+1)j}}{q_{ij}})s_i m_{ij} \geq (\tilde{v}_{ij} - \frac{b_{(k+1)j}q_{(k+1)j}}{q_{ij}})s_k m_{ij};$$
$$(\tilde{v}_{ij} - \frac{b_{(i+1)j}q_{(i+1)j}}{q_{ij}})s_i m_{ij} \geq V_{i0},\ \forall i, k,$$

where $\frac{b_{(k+1)j}q_{(k+1)j}}{q_{ij}}$ is the cost per click that advertiser i needs to pay for being displayed on position k—i.e., the quality score adjusted price according to the GSP rule. Rearranging the above equations as in the "one step" solution in Varian (2007) would generate a chain of inequalities that characterize the equilibrium. Specifically, starting from the one obtaining the last position s, we would have $(\tilde{v}_{sj} - \frac{b_{(s+1)j}q_{(s+1)j}}{q_{sj}})s_s m_{sj} \geq V_{s0}$ and $(\tilde{v}_s - \frac{b_{(s+1)j}q_{(s+1)j}}{q_{sj}})s_s m_{sj} \geq (\tilde{v}_s - \frac{b_{sj}q_{sj}}{q_{sj}})s_{s-1}m_{sj}$, which implies that $c_{sj} \leq (\tilde{v}_{sj} - \frac{V_{s0}}{s_s m_{sj}})s_s q_{sj}$ and $c_{(s-1)j} \geq c_{sj} + \tilde{v}_{sj}q_{sj}(s_{s-1} - s_s)$, where $c_{kj} = b_{(k+1)j}q_{(k+1)j}s_k$. Doing this recursively for the other slots up to the first one would result in:

$$
\begin{aligned}
\tilde{v}_{1j}q_{1j} &\geq \frac{c_{1j}-c_{2j}}{s_1-s_2} \geq \\
\tilde{v}_{2j}q_{2j} &\geq \frac{c_{2j}-c_{3j}}{s_2-s_3} \geq \\
&\vdots \\
\tilde{v}_{sj}q_{sj} &\geq \frac{c_{sj}}{s_s} + \frac{V_{s0}}{s_s m_{sj}}.
\end{aligned}
$$

Thus, in equilibrium the advertisers that occupy each slot are rank ordered by $\tilde{v}_{ij}q_{ij}$, the product of per click value and the assigned quality score. In addition, individual rationality conditions require that each advertiser gets a payoff higher than its outside option. Thus, we also have:

$$c_{kj} \leq (\tilde{v}_{kj} - \frac{V_{k0}}{m_{kj}s_k})s_k q_{kj}.$$

These are the inequality conditions that regulate the market equilibrium.

In equilibrium, advertiser k is displayed on position k, and will generate $s_k m_{kj}$ clicks. Thus, the total price paid by advertiser k is:

$$p_{kj} = \frac{b_{(k+1)j}q_{(k+1)j}}{q_{kj}} s_k m_{kj} = \frac{m_{kj}}{q_{kj}} c_{kj}.$$

When we add up the prices paid by each individual advertiser, we would have the total revenue of publisher j, P_j as:

$$P_j = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} c_{kj},$$

The range of P_j depends on the bounds of c_{kj} in equilibrium. Denote the lower bound and the upper bound to be $\underline{c}_{kj}$ and $\overline{c}_{kj}$, respectively, the inequality conditions imply that:

$$
\begin{aligned}
\underline{c}_{kj} &= \underline{c}_{(k+1)j} + \tilde{v}_{(k+1)j}q_{(k+1)j}(s_k - s_{k+1}),\ \forall k < s \\
\overline{c}_{kj} &= \min\left\{\overline{c}_{(k+1)j} + \tilde{v}_{kj}q_{kj}(s_k - s_{k+1}), w_{(k-1)j} - \tilde{v}_{kj}q_{kj}(s_{k-1} - s_k), w_{kj}\right\},\ \forall k < s
\end{aligned}
$$

and,

$$\underline{c}_{sj} = \tilde{v}_{(s+1)j} s_s q_{(s+1)j} - V_{(s+1)0} \frac{q_{(s+1)j}}{m_{(s+1)j}}$$

$$\bar{c}_{sj} = \min\left\{w_{(s-1)j} - \tilde{v}_{sj} q_{sj}(s_{s-1} - s_s), w_{sj}\right\},$$

where $w_{kj} = \tilde{v}_{kj} s_k q_{kj} - V_{k0} \frac{q_{kj}}{m_{kj}}$.

The range of P_j is therefore within the following two bounds:

$$P_{j,\min} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \underline{c}_{ik} \tag{D.1}$$

$$P_{j,\max} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \bar{c}_{ik} \tag{D.2}$$

Finally, as a special case, when the outside option value of each advertiser is $V_{i0} = 0$, $\underline{c}_{ik}$ and $\bar{c}_{ik}$ have simple recursive expressions which results in,

$$P_{j,\min} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \underline{c}_{ik} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \sum_{l=k}^{s} (s_l - s_{l+1}) q_{(l+1)j} \tilde{v}_{(l+1)j}$$

$$P_{j,\max} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \bar{c}_{ik} = \sum_{k=1}^{s} \frac{m_{kj}}{q_{kj}} \sum_{l=k}^{s} (s_l - s_{l+1}) q_{lj} \tilde{v}_{lj}.$$

E Algorithm for Pricing Policy Analysis

We use the following algorithm in simulating the pricing scheme counterfactual.

1. Calculate the advertisers' revenue function for each advertiser-publisher-slot allocation, V_{ik_j}, and calculate the matching score for each advertiser-publisher pair, m_{ij}.

2. Specify 20 levels of λ_{Tech}, from 0 to 1, at the increment of 0.05.

3. For each λ_{Tech}:

   (a) Simulate a vector of q_{ij}, based on m_{ij} and λ_{Tech}, from the distribution of: $\ln(q_{ij}) \sim$

$N\Big(\mu + \lambda_{Tech}(\ln(m_{ij}) - \mu_m), (1-\lambda_{Tech}^2)\sigma_m^2\Big)$, where μ_m and σ_m^2 are the mean and variance of the calculated m_{ij};

(b) For each publisher j:

i. Put advertiser i in competition for a position on j if $\max_k\{V_{ik_j} - V_{i0} > 0\}$. The set of competing advertisers is denoted by N_j;

ii. Rank order the N_j participating advertisers in descending order based on the values of $v_i q_{ij}$. Propose allocations of $s_j + 1$ advertisers (out of N_j) to $s_j + 1$ slots, where the last advertiser occupy an outside slot that generates 0 clicks, but its value will influence the equilibrium prices. For each allocation, verify whether it is feasible in the sense that the price range is non-empty, i.e., whether $\bar{c}_{kj} \geq \underline{c}_{kj}$ holds for every $k = 1, \cdots, s$. Pick up the allocation with the highest total advertiser surplus from all the feasible allocations.

iii. Compute $P_{min,j}$ and $P_{max,j}$ for publisher j, according to the equilibrium conditions as specified in equations (D.1) and (D.2).

(c) Sum up and get $P_{min} = \sum_{j=1}^{J} P_{min,j}$, and $P_{max} = \sum_{j=1}^{J} P_{max,j}$. This gives the bounds of the total publishers' revenue under equilibrium.

(d) Based on the equilibrium allocation, sum up the allocated V_{ik_j} to get the total advertisers' revenue.

(e) Repeat the above (a)-(d) steps 100 times, and compute the mean of the total publishers' revenue and total advertisers' revenue.

4. Draw the results from each λ_{Tech} on the graph.

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