Regulating Deceptive Advertising:
False Claims and Skeptical Consumers
Online Appendix

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Proof of Lemma 5

The proof largely follows the proof of Lemma 2, and thus here we only highlight two key differences. First, in any separating equilibrium, by mimicking firm H, firm L faces a penalty $b(1-x)$ because all consumers purchase without verification. As a result, the equilibrium price is $p_H^{\text{sep}} = \min\left\{V_H,\ V_L + \min_x[c(x) + b(1-x)]\right\}$.

Second, we will show that the stated pooling equilibrium is a PBE iff b is sufficiently small (i.e., $b \leq \widehat{\kappa}$, determined in the proof below). We have $\pi_L = D_L\overline{V} - c(x) - b(1-x)D_L = [1-(1-x)F(\widetilde{s})][\overline{V} - b(1-x)] - c(x)$, where $\widetilde{s} = \phi(1-\phi)(1-x^e)(V_H - V_L)$. Thus,

$$\frac{d\pi_L}{dx} = F(\widetilde{s})[\overline{V} - 2b(1-x)] - c'(x) + b, \tag{W1}$$

which is decreasing in x for $c'' > 2b$ (assumed in the extension). Given rational expectation of x, the equilibrium deceptiveness is $x^* = x^{\text{pool}} \equiv \min\left\{\overline{x}, \widehat{\chi}\right\}$, where $\widehat{\chi} \in (0, +\infty)$ is determined by the first-order condition:

$$\widehat{G}(\widehat{\chi}) \equiv F(\phi(1-\phi)(1-\widehat{\chi})(V_H - V_L))[\overline{V} - 2b(1-\widehat{\chi})] - c'(\widehat{\chi}) + b = 0. \tag{W2}$$

Note that if $\overline{V} < b(1-\widehat{\chi})$ (which requires sufficiently large b), then no pooling equilibrium can exist because of negative profit. Thus, in the rest of this proof regarding the pooling equilibrium, we can focus on $\overline{V} \geq b(1-\widehat{\chi})$. An interior solution $\widehat{\chi} \in (0, +\infty)$ exists because $\widehat{G}(0) > 0$ ($\because \overline{V} \geq b(1-\widehat{\chi})$) and $\widehat{G}(+\infty) < 0$. We have $\widehat{G}'(\widehat{\chi}) = -\phi(1-\phi)(V_H - V_L)f(\widetilde{s})[\overline{V} -$

$2b(1-\widehat{\chi})] + 2bF(\widetilde{s}) - c''(\widehat{\chi}) = -\phi(1-\phi)(V_H - V_L)f(\widetilde{s})\overline{V} - (c''(\widehat{\chi}) - 2b) - 2b[\overline{F}(\widetilde{s}) - (1 - \widehat{\chi})\phi(1-\phi)(V_H - V_L)f(\widetilde{s})] < 0$ because $\overline{F}(\widetilde{s})$ is sufficiently large compared to $f(\cdot)$. Thus, $\widehat{\chi}$ is unique.

Firm H has no incentive to deviate to any off-equilibrium strategy under $\psi = 0$. Firm L is not willing to deviate to any off-equilibrium strategy iff $\pi_L \geq V_L$, which is the highest profit it can earn under $\psi = 0$. To complete this proof, we just need to show that π_L is decreasing in b for any interior $x^{\text{pool}} = \widehat{\chi}$. We have

$$\frac{d\pi_L}{db} = \frac{\partial \pi_L}{\partial b} + \frac{\partial \pi_L}{\partial x^e}\frac{d\widehat{\chi}}{db}, \tag{W3}$$

where

$$\frac{\partial \pi_L}{\partial b} = -(1-\widehat{\chi})[1-(1-\widehat{\chi})F(\widetilde{s})]; \tag{W4}$$

$$\frac{\partial \pi_L}{\partial x^e} = (1-\widehat{\chi})\phi(1-\phi)(V_H - V_L)f(\widetilde{s})[\overline{V} - b(1-\widehat{\chi})]; \tag{W5}$$

$$\frac{d\widehat{\chi}}{db} = -\frac{\frac{\partial \widehat{G}(\widehat{\chi})}{\partial b}}{\widehat{G}'(\widehat{\chi})} = \frac{1-2(1-\widehat{\chi})F(\widetilde{s})}{-\widehat{G}'(\widehat{\chi})}. \tag{W6}$$

Therefore,

$$\begin{aligned}
\frac{d\pi_L}{db} &= -(1-\widehat{\chi})[1-(1-\widehat{\chi})F(\widetilde{s})] + \frac{(1-\widehat{\chi})\phi(1-\phi)(V_H-V_L)f(\widetilde{s})[\overline{V}-b(1-\widehat{\chi})][1-2(1-\widehat{\chi})F(\widetilde{s})]}{\phi(1-\phi)(V_H-V_L)f(\widetilde{s})[\overline{V}-2b(1-\widehat{\chi})]-2bF(\widetilde{s})+c''(\widehat{\chi})} \\
&= -\frac{(1-\widehat{\chi})[c''(\widehat{\chi})-2bF(\widetilde{s})-b(1-\widehat{\chi})\phi(1-\phi)(V_H-V_L)f(\widetilde{s})]}{\phi(1-\phi)(V_H-V_L)f(\widetilde{s})[\overline{V}-2b(1-\widehat{\chi})]-2bF(\widetilde{s})+c''(\widehat{\chi})} \\
&\qquad + \frac{(1-\widehat{\chi})^2F(\widetilde{s})[c''(\widehat{\chi})-2bF(\widetilde{s})-\overline{V}\phi(1-\phi)(V_H-V_L)f(\widetilde{s})]}{\phi(1-\phi)(V_H-V_L)f(\widetilde{s})[\overline{V}-2b(1-\widehat{\chi})]-2bF(\widetilde{s})+c''(\widehat{\chi})} \\
&\leq -\frac{(1-\widehat{\chi})[1-(1-\widehat{\chi})F(\widetilde{s})][c''(\widehat{\chi})-2bF(\widetilde{s})-b(1-\widehat{\chi})\phi(1-\phi)(V_H-V_L)f(\widetilde{s})]}{\phi(1-\phi)(V_H-V_L)f(\widetilde{s})[\overline{V}-2b(1-\widehat{\chi})]-2bF(\widetilde{s})+c''(\widehat{\chi})} \\
&= -\frac{(1-\widehat{\chi})[1-(1-\widehat{\chi})F(\widetilde{s})]\big\{(c''(\widehat{\chi})-2b)+b[2\overline{F}(\widetilde{s})-(1-\widehat{\chi})\phi(1-\phi)(V_H-V_L)f(\widetilde{s})]\big\}}{-\widehat{G}'(\widehat{\chi})} < 0.
\end{aligned} \tag{W7}$$

Therefore, the stated pooling equilibrium exists iff $b \leq \widehat{\kappa}$, where the threshold $\widehat{\kappa}$ is the value of b such that $\pi_L = [1-(1-x^{\text{pool}})F(\widetilde{s})][\overline{V} - b(1-x^{\text{pool}})] - c(x^{\text{pool}}) = V_L$.

Proof of Lemma 6

This lemma is implied by the proof of Lemma 5. The boundary solution $x^{\text{pool}} = \overline{x}$ is invariant in b. The interior solution $x^{\text{pool}} = \widehat{\chi}$ is increasing in b iff $M(b) \equiv 1 - 2(1-\widehat{\chi})F(\widetilde{s}) > 0$ (see Equation (W6)). Note that M is affected by b only through $\widehat{\chi}$ and it is strictly increasing in $\widehat{\chi}$. As a result, for any b, if $M(b) > 0$, then for any $b' > b$ we have $M(b') > M(b) > 0$. Similarly, for any b, if $M(b) \leqq 0$, then for any $b' > b$ we have $M(b') \leqq M(b) \leqq 0$. Therefore, the sign of M is the same as the sign of $M(0)$.

Next, we prove that $M(0) > 0$ iff $\overline{V} > \widetilde{V}$, where $\widetilde{V}$ is determined below. If $F(\phi(1-\phi)(V_H - V_L)) < 1/2$, then $M = 1 - 2(1-\widehat{\chi})F(\phi(1-\phi)(1-\widehat{\chi})(V_H - V_L)) > 0$ for any $\overline{V}$, and thus we can set the threshold $\widetilde{V} = 0$. Otherwise, M can be negative. Because $\partial\widehat{G}(\widehat{\chi})/\partial\overline{V} = F(\widetilde{s}) > 0$, we have $d\widehat{\chi}/d\overline{V} = -\left.\frac{\partial\widehat{G}(\widehat{\chi})}{\partial\overline{V}}\middle/\widehat{G}'(\widehat{\chi})\right. > 0$. Because $F(\phi(1-\phi)(V_H - V_L)) \geq 1/2$, there exists $\widehat{\chi} = \widehat{\chi}'$ such that $M(0) = 0$. Because $\widehat{\chi}$ is increasing in $\overline{V}$ and is not converging, there exists $\overline{V} = \widetilde{V}$ such that $\widehat{\chi} = \widehat{\chi}'$ (which makes $M(0) = 0$). Therefore, $M(0) > 0$ iff $\overline{V} > \widetilde{V}$, where $\widetilde{V}$ is determined by $M(0) = 0$. Similarly, it can be shown that $M(0) \leqq 0$ iff $\overline{V} \leqq \widetilde{V}$. Thus, x^{pool} is increasing in b iff $\overline{V} > \widetilde{V}$.

Proof of Proposition 6

Following the proof of Proposition 2 (see part (i)), in any pooling equilibrium, for $x^{\text{pool}} = \overline{x}$, $dCS/db = 0$; for $x^{\text{pool}} = \widehat{\chi}$, dCS/db has the opposite sign to $d\widehat{\chi}/db$. Therefore, from Equation (W6), the consumer surplus CS is decreasing in b iff $\overline{V} > \widetilde{V}$.

Now we investigate the equilibrium shift from the pooling equilibrium to the separating equilibrium at $b = \widehat{\kappa}$. For any $b \leq \widehat{\kappa}$, we have $\pi_L \geq V_L \iff [1-(1-x^{\text{pool}})F(\widetilde{s})]\overline{V} \geq V_L + c(x^{\text{pool}}) + b(1-x^{\text{pool}})[1-(1-x^{\text{pool}})F(\widetilde{s})]$, which implies $\overline{V} > V_L + c(x^{\text{pool}}) + b(1-x^{\text{pool}}) \geq$

$V_L + \min_x[c(x) + b(1-x)] \geq p_H^{\text{sep}}$, and so $p_H^{\text{sep}} < \overline{V}$. Part (ii) of the proof of Proposition 2 can be used to show that at $b = \widehat{\kappa}$, $CS^{\text{pool}} < CS^{\text{sep}}$.

In the separating equilibrium, because $p_H^{\text{sep}} = \min\big\{V_H,\ V_L + \min_x[c(x) + b(1-x)]\big\}$ is (weakly) increasing in b, consumer surplus is (weakly) decreasing in b.