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Online Appendix: "A Theoretical Analysis of the Lean Startup Method"

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Appendix A: Robustness Checks

We examine robustness checks along three fronts: (i) nonuniform distributions of $h(x|W)$, (ii) sampling multiple customers, and (iii) asymmetric responses to test product with consumer revealing their surplus with probability ξ.

A.1. Nonuniform distributions

In this section, we examine the robustness of our results to the distributional assumption regarding $h(x|W)$ by considering a shifted and scaled beta distribution with parameters (α,β). Recall that distribution $h(x|W)$ influences both the design of the test product in the first stage and the pivoting decision in the second stage.

First, for a given $\tilde{r}$, the optimal pivoting decision Λ^* for second stage problem (2) satisfies

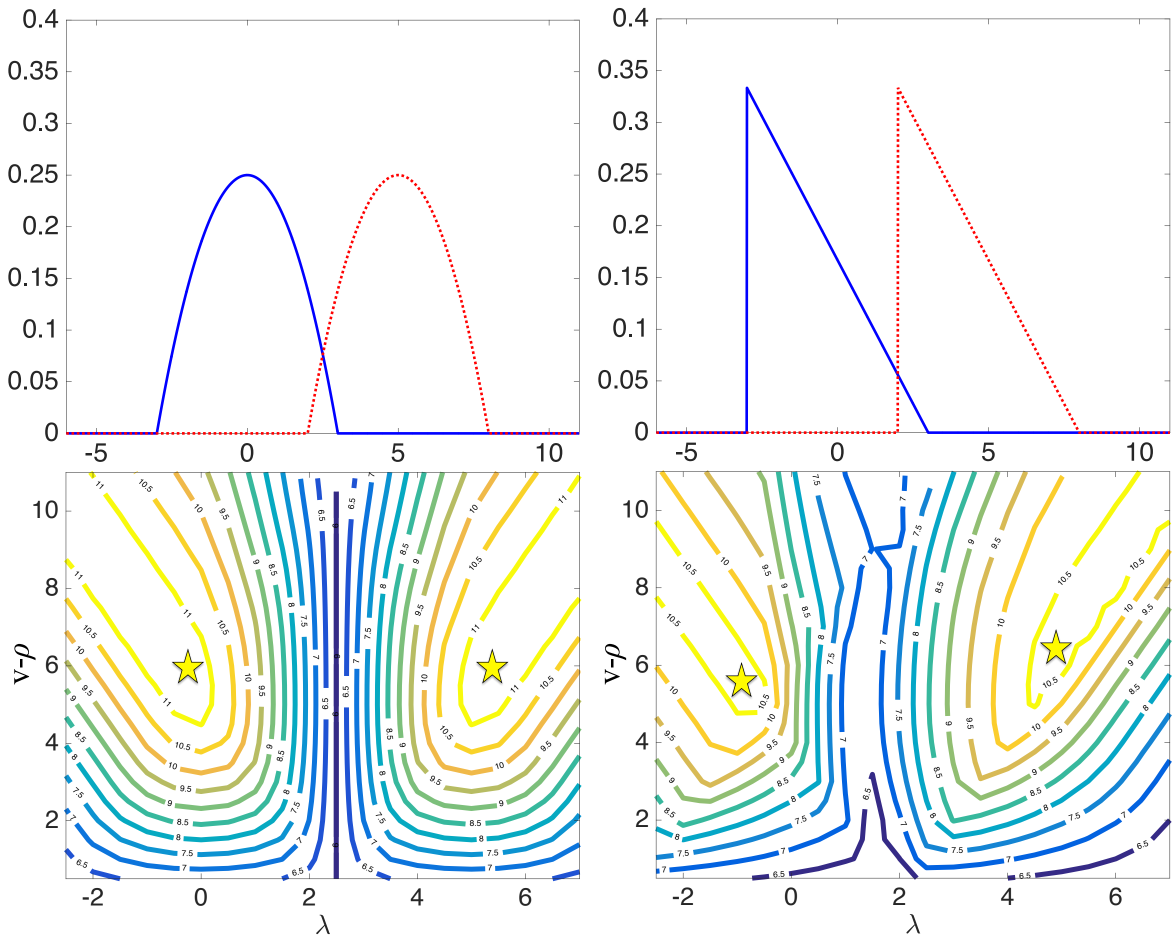
$$\Lambda^* = \arg\max_{\Lambda} \left\{ \int_{\Lambda-\frac{V-p}{t}}^{\Lambda+\frac{V-p}{t}} \left[\tilde{r}h(x|W=0) + (1-\tilde{r})h(x|W=C) \right] \right\}.$$

For general distributions $h(x|W)$, this Λ^* does not have a simple closed form, and must be computed numerically. It can be observed that the optimal final product has design $\Lambda^* \in [0,C]$. The complexity is compounded in the optimal design of the test product in the first stage, and the problem (3) is analytically intractable. Therefore the optimal test product is computed numerically.

Recall that the beta distribution with parameter $(\alpha,\beta)=(1,1)$ corresponds to a uniform distribution, which we have examined analytically in §4. Using this case as a benchmark, we will compare two cases of beta distributions, with parameters $(\alpha,\beta)=(2,2)$ and $(\alpha,\beta)=(1,2)$, plotted in the upper left and upper right panels of Figure A-1), respectively. The former reveals the impact of the shape of $h(x|W)$, while the second reveals the impact of asymmetry of $h(x|W)$. The contour plots for expected revenue $\mathbb{E}_{s_i(\lambda,v,\rho)}\pi^*(\tilde{r}(\lambda,v,\rho))$

as a function of test product $(\lambda, v - rho)$ are given in Figure A-1 for $(\alpha, \beta) = (2, 2)$ (bottom left panel) and $(\alpha, \beta) = (1, 2)$ (bottom right panel).

Figure A-1 **Nonuniform distributions of** $h(x|W)$ **and corresponding contour plots of** $\mathbb{E}_{s_i(\lambda,v,\rho)}\pi^*(\tilde{r}(\lambda, v, \rho))$ **on** $(\lambda, v - \rho)$ **plane.**



Note. Parameters: $r = 0.5$, $(C, \epsilon, t) = (5, 3, 2)$.

We observe that the contour plot is less "rectangular" (left column) or is shifted to the left to account for greater mass on the left (right column). Apart from these differences, the contour plots and the optimal locations of the test product are similar to those found in §4. Namely, the optimal test products $(\lambda, v - \rho)$ do not change much as a result of change in the distributions $h(x|W)$, the expected revenue $\mathbb{E}_{s_i}\pi^*(\tilde{r})$ is increasing–decreasing in the quality $v - \rho$ of the test product, and learning does not occur when λ is in the interior. In other words, our insights in §4 are robust to the assumptions regarding the underlying distribution $h(x|W)$.

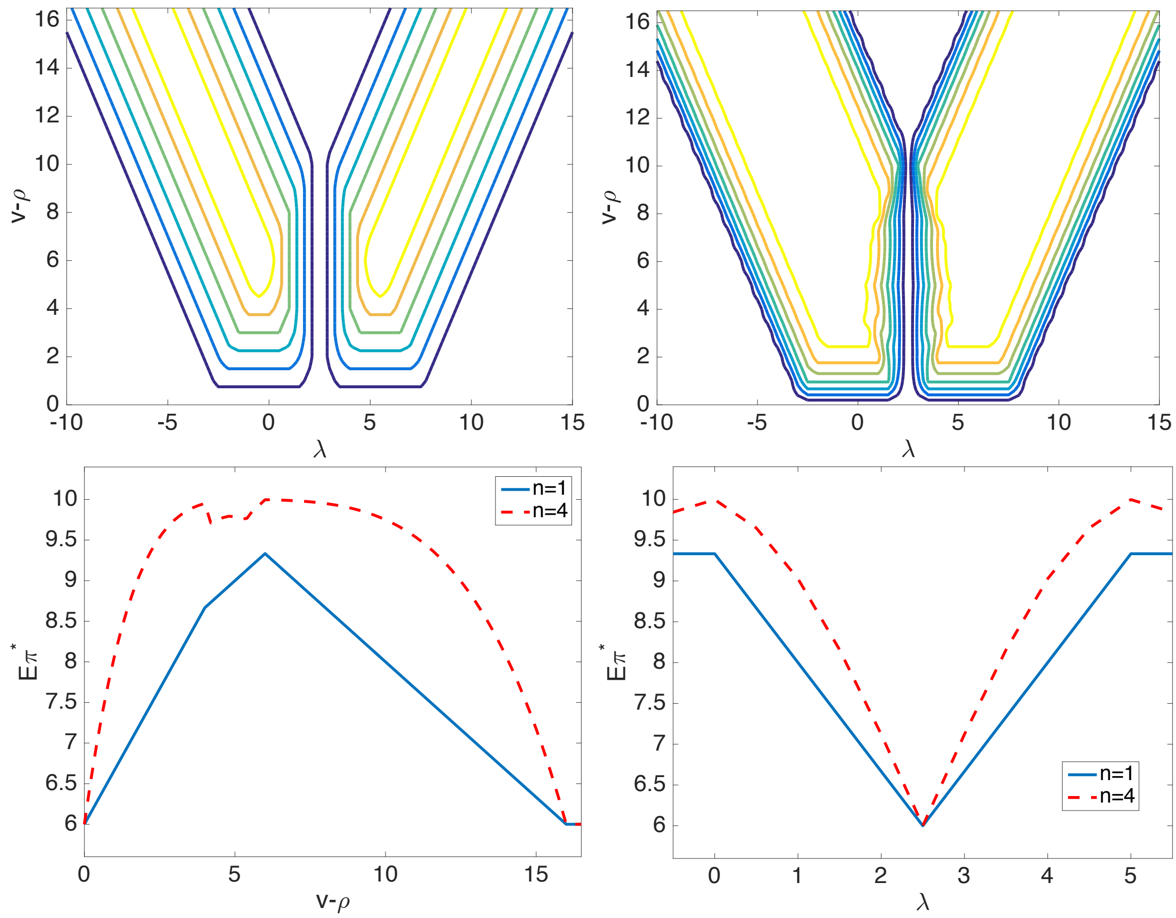
A.2. Sampling Multiple Customers

Our main model in §4 examined learning that results from sampling a single customer. In this section, we examine the impact of sampling multiple independent customers.

Let $\tilde{r}(n)$ denote the updated posterior belief given the outcome from sampling n customers. Let us denote the sales outcome (sale or no sale) of n customers by a vector $\mathbf{s}$. If the entrepreneur increases the sample of customers, or as $n \to \infty$, we know that $\tilde{r}(n) \to 0$ or $\tilde{r}(n) \to 1$, resulting in a lower expected probability of launching the wrong product, $\mathbb{E}_{\mathbf{s}} \min\{\tilde{r}(n), 1-\tilde{r}(n)\}$. The distribution of the posterior $\tilde{r}(n)$ depends on 2^n combination of each customer's sales outcomes. Hence, it is infeasible to analytically examine the signal for a general $n > 1$, so we resort to numerical analysis.

The top panels of Figure A-2 compare the contour curves for $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(n))$ on the (λ, $v-\rho$) space for $n=1$ (upper left panel) and $n=4$ (upper right panel). We observe that the overall shapes of the contour curves remain the same, and that the optimal test product that maximizes the $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(n))$ are equivalent for both values of n.

Figure A-2 **Impact of sampling multiple customers. Top row shows the impact of sampling on contours of** $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(n))$ **as a function of test product's attribute and optimal test product on** $(\lambda, v-\rho)$ **plane (**$n=1$ **in left panel and** $n=4$ **in right panel). Bottom row examines the cross section of** $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(n))$ **evaluated at** $\lambda=0$ **with respect to** $v-\rho$ **(left panel) and evaluated at** $v-\rho=6$ **with respect to** λ **(right panel).**



Note. Parameters: $r=0.5$, $(C,\epsilon,t)=(5,3,2)$.

The bottom panels of Figure A-2 compare the cross section of the two contour curves with respect to v (lower left panel) and λ (lower right panel). The solid (blue) curves represent the case where $n = 1$, and the dotted (red) curves represent the case where $n = 4$. Examining the values of $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(4))$ more closely, we observe that it is (largely) unimodal in the vertical quality $v - \rho$ and also decreases as $\lambda \to C/2$. However, the value $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(4))$ is greater than the value of $\mathbb{E}_{\mathbf{s}}\pi^*(\tilde{r}(1))$ and appears less sensitive to the test product's attributes λ and $v - \rho$ at the optimal. Ultimately, the number of samples n influences the distribution of posterior distribution $\tilde{r}$ and improves the effectiveness of LSM. However, the key insights of our analysis regarding the optimal test product and its key features remain same as in §4.

A.3. Asymmetric consumer responses $\xi > 0$

So far, we have assumed that a sampled customer will either purchase the test product or not purchase the test product. We now assume that consumers can provide more detailed feedback (e.g., through comments or posting texts) by revealing their surplus information s_i. To incorporate the asymmetry between feedback after purchase and after no purchase, we assume that consumers reveal surplus with probability ξ only after purchase.

Thus, conditional on sales, the consumer reveals surplus information with probability ξ and fails to reveal it with probability $1 - \xi$. If the surplus is negative, she does not purchase the product and therefore does not reveal her surplus. In other words, the entrepreneur's sales effort may lead to one of three outcomes: (i) no sales (and no surplus information), (ii) sales without surplus information, and (iii) sales with surplus information. This updates the entrepreneur's learning and updates Lemma 2 as follows.

Lemma A.1 **(Posterior Belief)**. *After launching the test product (λ, v, ρ), the prior belief r is updated to the posterior belief $\tilde{r}$ as follows:*

(i) If the surplus s_i is revealed by the customer, then

$$\tilde{r}(s_i|\lambda, v, \rho) \triangleq \frac{r g(s_i|(\lambda, v, \rho), W = 0)}{r g(s_i|(\lambda, v, \rho), W = 0) + (1 - r) g(s_i|(\lambda, v, \rho), W = C)}.$$

(ii) If only the sale event $s_i > 0$ is observed, then

$$\tilde{r}(s_i > 0|\lambda, v, \rho) \triangleq \frac{r \int_0^\infty g(s_i|(\lambda, v, \rho), W = 0) ds}{r \int_0^\infty g(s_i|(\lambda, v, \rho), W = 0) ds + (1 - r) \int_0^\infty g(s_i|(\lambda, v, \rho), W = C) ds}.$$

(iii) If only the no-sale event $s_i \leq 0$ is observed, then

$$\tilde{r}(s_i \leq 0|\lambda, v, \rho) \triangleq \frac{r \int_{-\infty}^0 g(s_i|(\lambda, v, \rho), W = 0) ds}{r \int_{-\infty}^0 g(s_i|(\lambda, v, \rho), W = 0) ds + (1 - r) \int_{-\infty}^0 g(s_i|(\lambda, v, \rho), W = C) ds}.$$

The first stage problem (3) then becomes:

$$\begin{aligned}
\max_{(\lambda, v, \rho)} \; & \mathbb{E}_{s_i(\lambda, v, \rho)} \pi^*(\tilde{r}(\lambda, v, \rho)) \\
= & \int_0^{v-\rho} \left[\xi \pi^* \left(\tilde{r}(s_i|\lambda, v, \rho), \lambda, v \right) + (1 - \xi) \pi^* \left(\tilde{r}(s_i > 0|\lambda, v, \rho), \lambda, v \right) \right] \times \\
& \qquad \left[r \cdot g\left(s_i \middle| (\lambda, v, \rho), W = 0 \right) + (1 - r) \cdot g\left(s_i \middle| (\lambda, v, \rho), W = C \right) \right] ds_i \\
& + \int_{-\infty}^0 \pi^* \left(\tilde{r}(s_i < 0|\lambda, v, \rho), \lambda, v \right) \left[r \cdot g\left(s_i \middle| (\lambda, v, \rho), W = 0 \right) + (1 - r) \cdot g\left(s_i \middle| (\lambda, v, \rho), W = C \right) \right] ds_i.
\end{aligned}$$

Observe that if $\xi = 0$ the entrepreneur's problem and the learning reduces to our original problem in §4.

The following result reveals that ξ has no impact on the optimal choice of the test product.

Proposition A.1 **(Impact of ξ on Optimal Test Product)**. *The set of optimal test products* (λ^*, v^*, ρ^*) *and* $\mathbb{E}_{s_i(\lambda^*,v^*,\rho^*)}\pi^*(\tilde{r}(\lambda^*, v^*, \rho^*))$ *when* $\xi \in (0,1)$ *are equivalent to those when* $\xi = 0$ *in Proposition 2.*

In other words, when the test product is optimally designed, the level of learning does not improve from obtaining additional surplus information from consumers. When sales occur and a surplus is revealed, the surplus s_i can sometimes reveal the location of the ideal product (i.e., $\tilde{r} = 1$); however it can also sometimes lead to no updating at all (i.e., $\tilde{r} = r$), thus offsetting the benefits of additional information. When test product is chosen optimally, the benefit of additional surplus information does not manifest in additional profit.

Nevertheless, the additional surplus information does help entrepreneur's profit when he chooses test product suboptimally.

Corollary A.1 **(Effectiveness of the LSM)**. *The effectiveness of the LSM is characterized by its: (i) peak benefit; and (ii) sensitivity to implementation, which are given respectively:*

$$(i) \quad \beta(\lambda^*, v^*, \rho^*|0.5) = \frac{C}{4\epsilon},$$

$$(ii) \quad \frac{\partial \beta(\lambda, v, \rho|0.5)}{\partial (v-\rho)}\bigg|_{(\lambda, v-\rho)=(0,\epsilon t)} = \begin{cases} \frac{1}{4\epsilon t}, & \text{if } v-\rho < \epsilon t, \\ -\frac{\xi}{4\epsilon t}, & \text{if } v-\rho > \epsilon t, \end{cases},$$

$$\frac{\partial \beta(0.5)}{\partial \lambda}\bigg|_{(\lambda, v-\rho)=(0,\epsilon t)} = \begin{cases} 0, & \text{if } \lambda < 0, \\ \frac{1}{\epsilon}, & \text{if } \lambda > 0. \end{cases}$$

Specifically, having surplus information makes LSM more forgiving to overshooting on quality of test product. This is illustrated in Figure A-3. We observe that the expected profit $\mathbb{E}_{s_i(\lambda,v,\rho)}\pi^*(\tilde{r}(\lambda, v, \rho))$ when the entrepreneur overshoots the optimal quality $(v^* - \rho^*)$ increases as $\xi \to 1$.

Appendix B: Probability distribution of surplus s_i and posterior belief $\tilde{r}$, and $\mathbb{E}_{s_i}\min\{\tilde{r}, 1-\tilde{r}\}$ as a function of test product's attributes (λ, v, ρ).

We consider five ranges of $\lambda \in (-\infty, C/2]$: (1) $\lambda < -\epsilon$, (2) $\lambda \in [-\epsilon, C/2 - \epsilon]$, (3) $\lambda \in [C/2 - \epsilon, 0]$, (4) $\lambda \in [0, C - \epsilon]$, and (5) $\lambda \in [C - \epsilon, C/2]$. (The other five ranges in $\lambda \in [C/2, \infty)$ can be found by interchanging λ and $C - \lambda$ and interchanging $W = 0$ and $W = C$.) These five ranges of λ correspond to five different settings of surplus density functions $g(s_i|W = 0)$ and $g(s_i|W = C)$, as illustrated in Figure B-1.

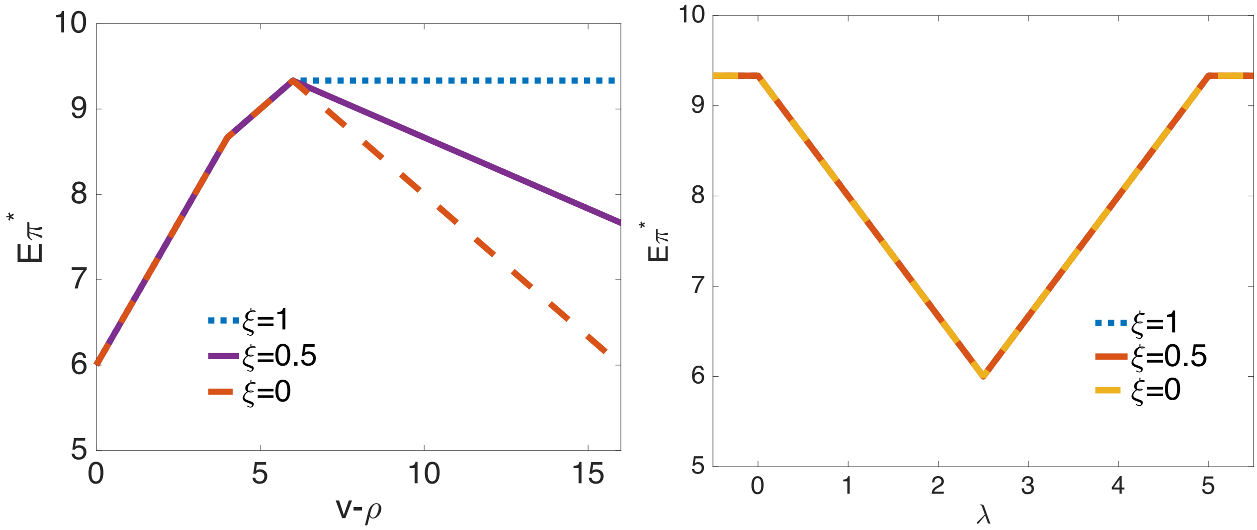
Recall that consumers reveal their surplus information s_i (with probability ξ) only if the test product results in a sale, i.e., $s_i > 0$. To account for this asymmetry, for each range of λ we must also consider five different ranges of vertical attribute $v - \rho$ to account for five ways in which surplus can be less than 0:

For case (1), we have the following 5 ranges of $(v - \rho)$: (A) $(v - \rho) > (C - \lambda + \epsilon)t$, (B) $(v - \rho) \in [(-\lambda + \epsilon)t, (C - \lambda + \epsilon)t]$, (C) $(v - \rho) \in [(C - \lambda - \epsilon)t, (-\lambda + \epsilon)t]$, (D) $(v - \rho) \in [(-\epsilon - \lambda)t, (C - \epsilon - \lambda)t]$, and (E) $(v - \rho) \in (0, (-\epsilon - \lambda)t]$.

For case (2), we have the following ranges of $(v-\rho)$: (A) $(v-\rho) > (C-\lambda+\epsilon)t$, (B) $(v-\rho) \in [(-\lambda+\epsilon)t, (C-\lambda+\epsilon)t]$, (C) $(v-\rho) \in [(C-\lambda-\epsilon)t, (-\lambda+\epsilon)t]$, (D) $(v-\rho) \in [(\epsilon+\lambda)t, (C-\epsilon-\lambda)t]$, and (E) $(v-\rho) \in (0, (\epsilon+\lambda)t]$.

For case (3), we have the following ranges of $(v-\rho)$: (A) $(v-\rho) > (C-\lambda+\epsilon)t$, (B) $(v-\rho) \in [(-\lambda+\epsilon)t, (C-\lambda+\epsilon)t]$, (C) $(v-\rho) \in [(\lambda+\epsilon)t, (-\lambda+\epsilon)t]$, (D) $(v-\rho) \in [(C-\epsilon-\lambda)t, (\epsilon+\lambda)t]$, and (E) $(v-\rho) \in (0, (C-\epsilon-\lambda)t]$.

Figure A-3 **Sensitivity of $\mathbb{E}_{s_i(\lambda,v,\rho)}\pi^*(\tilde{r}(\lambda,v,\rho))$ with respect to quality $v-\rho$ (left panel) and with respect to λ (right panel). Left panel shows that increasing the chance of receiving surplus information ξ does not improve the optimum but it mitigates the cost of overshooting on quality. Right panel shows that no learning occurs when $\lambda = C/2$.**



Note. Parameters: $r = 0.5$, $(C,\epsilon,t) = (5,3,2)$.

For case (4), we have the following ranges of $(v-\rho)$: (A) $(v-\rho) > (C-\lambda+\epsilon)t$, (B) $(v-\rho) \in [(\lambda+\epsilon)t, (C-\lambda+\epsilon)t]$, (C) $(v-\rho) \in [(-\lambda+\epsilon)t, (\lambda+\epsilon)t]$, (D) $(v-\rho) \in [(C-\epsilon-\lambda)t, (-\lambda+\epsilon)t]$, and (E) $(v-\rho) \in (0, (C-\epsilon-\lambda)t]$.

For case (5), we have the following ranges of $(v-\rho)$: (A) $(v-\rho) > (C-\lambda+\epsilon)t$, (B) $(v-\rho) \in [(\lambda+\epsilon)t, (C-\lambda+\epsilon)t]$, (C) $(v-\rho) \in [(-\lambda+\epsilon)t, (\lambda+\epsilon)t]$, (D) $(v-\rho) \in [(\lambda-C+\epsilon)t, (-\lambda+\epsilon)t]$, and (E) $(v-\rho) \in (0, (\lambda-C+\epsilon)t]$.

This leads to 50 discrete regions in $(\lambda, v-\rho)$-plane as illustrated Figure B-2. a test product with attributes (λ, v, ρ) in a region leads to the same closed form expressions for the probability distribution of s_i and updated posterior belief $\tilde{r}$, and $\mathbb{E}_{s_i}\min\{\tilde{r}, 1-\tilde{r}\}$. We provide them next for the 25 regions that are labeled. (The other 5 cases are given after changes of values $\lambda \leftrightarrow C-\lambda$ and variables $W=0 \rightarrow W=C$ due to symmetry.)

$\underline{(\lambda, v-\rho) \in (1A)}$

$P(s_i < 0) = 0;$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 1n2) = (1-r)\dfrac{C}{2\epsilon}, \quad \tilde{r} = 0,$

$P(s_i > 0, 2n3) = 1 - \dfrac{C}{2\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\dfrac{C}{2\epsilon}, \quad \tilde{r} = 1;$

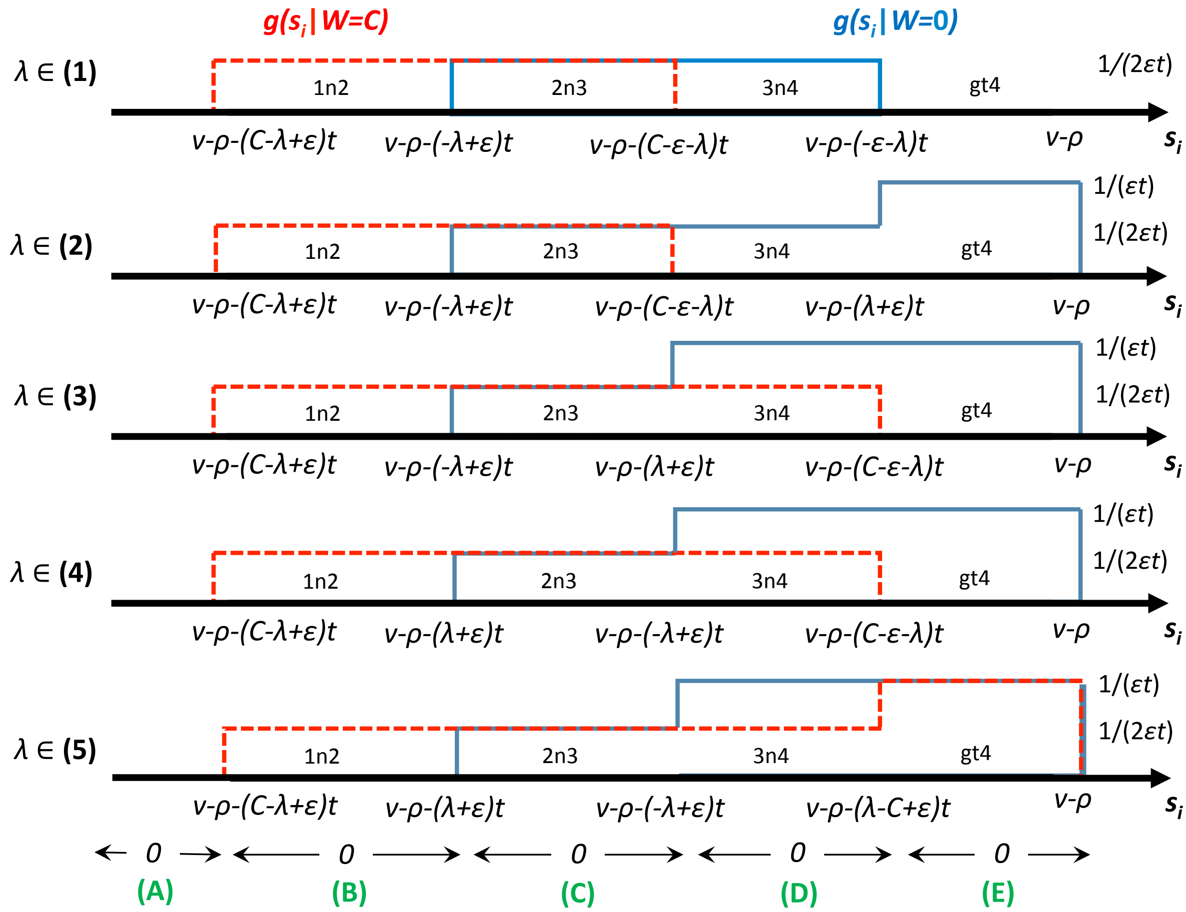
(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1, \quad \tilde{r} = r.$

$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1 - \dfrac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)\min\{r, 1-r\}.$

$\underline{(\lambda, v-\rho) \in (1B)}$

Figure B-1 Illustration of two different probability density functions $g(s_i|W=0)$ and $g(s_i|W=C)$ for 5 different scenarios based on position of λ. (The other 5 cases are given after changes of values $\lambda \leftrightarrow C-\lambda$ and variables $W=0 \to W=C$ due to symmetry.)



$$P(s_i<0)=(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right),\quad \tilde r=0;$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,1n2)=(1-r)\left(\frac{v-\rho}{2\epsilon t}-\frac{1}{2}+\frac{\lambda}{2\epsilon}\right),\quad \tilde r=0,$$

$$P(s_i>0,2n3)=1-\frac{C}{2\epsilon},\quad \tilde r=r,$$

$$P(s_i>0,3n4)=r\frac{C}{2\epsilon},\quad \tilde r=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

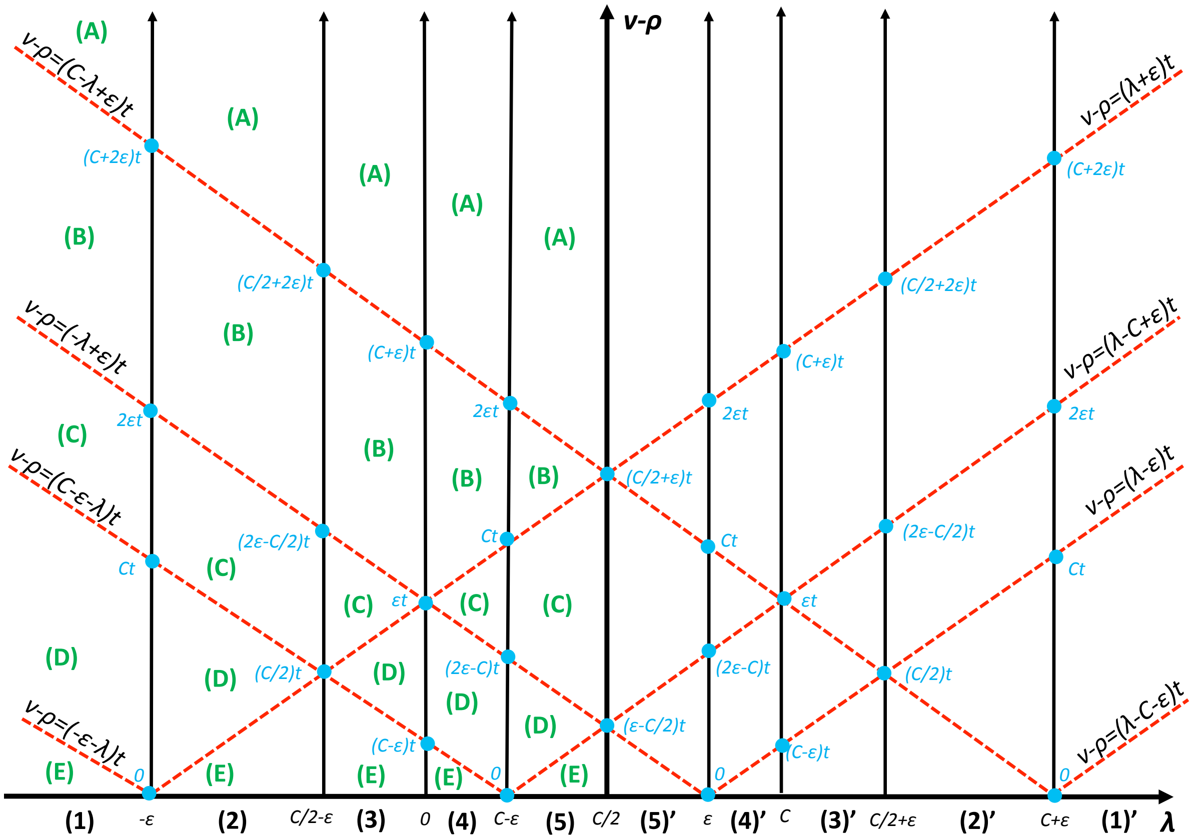
$$P(s_i>0)=1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right),\quad \tilde r=\frac{r}{1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right)}.$$

$$\mathbb{E}\min\{\tilde r,\ 1-\tilde r\}=\xi\left(1-\frac{C}{2\epsilon}\right)\min\{r,1-r\}+(1-\xi)\min\left\{r,\ 1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right)-r\right\}.$$

$\underline{(\lambda, v-\rho)\in(1C)}$

$$P(s_i<0)=r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)$$

Figure B-2 50 discrete regions. We analyze 25 regions that are labeled.



$$= \left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\frac{C}{2\epsilon}, \quad \tilde{r}=\frac{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,2n3)=\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}, \quad \tilde{r}=r,$$

$$P(s_i>0,3n4)=r\frac{C}{2\epsilon}, \quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]$$

$$\tilde{r}=\frac{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right]}{1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\,1-\tilde{r}\}=\xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right),(1-r)\frac{C}{2\epsilon}\right\}+\left(\frac{v-\rho}{2\epsilon t}-\frac{C-\epsilon-t}{2\epsilon}\right)\min\{r,1-r\}\right]$$

$$+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right.$$

$$\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].$$

$\underline{(\lambda, v-\rho)\in(1D)}$

$P(s_i < 0) = r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r), \quad \tilde{r} = \frac{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right)}{\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)};$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 3n4) = r\left(\frac{v-\rho}{2\epsilon t} + \frac{-\epsilon-\lambda}{2\epsilon}\right), \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - \left[r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\right], \quad \tilde{r} = 1.$

$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi \min\left\{r\left(-\frac{v-\rho}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right), 1-r\right\} + (1-\xi)\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right),\, (1-r)\right\}.$

$\underline{(\lambda, v-\rho) \in (1E)}$

$P(s_i < 0) = 1, \quad \tilde{r} = r.$

$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\{r, 1-r\}.$

$\underline{(\lambda, v-\rho) \in (2A)}$

$P(s_i < 0) = 0;$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 1n2) = (1-r)\frac{C}{2\epsilon}, \quad \tilde{r} = 0,$

$P(s_i > 0, 2n3) = r\left(1 - \frac{C}{2\epsilon}\right) + (1-r)\left(1 - \frac{C}{2\epsilon}\right) = 1 - \frac{C}{2\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\frac{C - 2\lambda - 2\epsilon}{2\epsilon}, \quad \tilde{r} = 1,$

$P(s_i > 0, gt4) = r\frac{\lambda+\epsilon}{\epsilon}, \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1, \quad \tilde{r} = r.$

$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1 - \frac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)\min\{r, 1-r\}.$

$\underline{(\lambda, v-\rho) \in (2B)}$

$P(s_i < 0) = (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = 0;$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 1n2) = (1-r)\left(\frac{v-\rho}{2\epsilon t} - \frac{(-\lambda+\epsilon)}{2\epsilon}\right), \quad \tilde{r} = 0,$

$P(s_i > 0, 2n3) = 1 - \frac{C}{2\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\frac{C - 2\lambda - 2\epsilon}{2\epsilon}, \quad \tilde{r} = 1,$

$P(s_i > 0, gt4) = r\frac{\lambda+\epsilon}{\epsilon}, \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = \frac{r}{1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)}.$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1-\frac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)\min\left\{r, 1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)-r\right\}.$$

$(\lambda, v-\rho) \in (2C)$

$$P(s_i<0) = r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)$$

$$=\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\frac{C}{2\epsilon},\quad \tilde{r}=\frac{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0, 2n3) = \frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon},\quad \tilde{r}=r,$$

$$P(s_i>0, 3n4) = r\left(\frac{C-2\lambda-2\epsilon}{2\epsilon}\right),\quad \tilde{r}=1,$$

$$P(s_i>0, gt4) = r\left(\frac{\lambda+\epsilon}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0) = 1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right],$$

$$\tilde{r}=\frac{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right]}{1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}$$

$$=\xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right), (1-r)\left(-\frac{v-\rho}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\left(\frac{v-\rho}{2\epsilon t}-\frac{C-\epsilon-\lambda}{2\epsilon}\right)\min\{r,1-r\}\right]$$

$$+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right.$$

$$\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].$$

$(\lambda, v-\rho) \in (2D)$

$$P(s_i<0) = r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r),\quad \tilde{r}=\frac{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0, 3n4) = r\left(\frac{v-\rho}{2\epsilon t}-\frac{\epsilon+\lambda}{2\epsilon}\right),\quad \tilde{r}=1,$$

$$P(s_i>0, gt4) = r\left(\frac{\epsilon+\lambda}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0) = 1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\right],\quad \tilde{r}=1.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right), 1-r\right\}.$$

$(\lambda, v-\rho) \in (2E)$

$$P(s_i<0) = r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r),\quad \tilde{r}=\frac{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)}{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$P(s_i > 0, gt4) = r\left(\frac{v-\rho}{\epsilon t}\right), \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - \left[r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\right], \quad \tilde{r} = 1.$

$\mathbb{E}\min\{\tilde{r}, 1-\tilde{r}\} = \xi\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right), 1-r\right\} + (1-\xi)\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right), (1-r)\right\}.$

<u>$(\lambda, v-\rho) \in (3A)$</u>

$P(s_i < 0) = 0;$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 1n2) = (1-r)\frac{C}{2\epsilon}, \quad \tilde{r} = 0,$

$P(s_i > 0, 2n3) = r\left(-\frac{\lambda}{\epsilon}\right) + (1-r)\left(-\frac{\lambda}{\epsilon}\right) = -\frac{\lambda}{\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right), \quad \tilde{r} = \frac{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right)},$

$P(s_i > 0, gt4) = r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right), \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1, \quad \tilde{r} = r.$

$\mathbb{E}\min\{\tilde{r}, 1-\tilde{r}\} = \xi\left[\left(-\frac{\lambda}{\epsilon}\right)\min\{r, 1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r, 1-r\}\right] + (1-\xi)\min\{r, 1-r\}.$

<u>$(\lambda, v-\rho) \in (3B)$</u>

$P(s_i < 0) = (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = 0;$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 1n2) = (1-r)\left(\frac{v-\rho}{2\epsilon t} - \frac{(-\lambda+\epsilon)}{2\epsilon}\right), \quad \tilde{r} = 0,$

$P(s_i > 0, 2n3) = -\frac{\lambda}{\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right), \quad \tilde{r} = \frac{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right)},$

$P(s_i > 0, gt4) = r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right), \quad \tilde{r} = 1;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - \left[(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right], \quad \tilde{r} = \frac{r}{1 - \left[(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]}.$

$\mathbb{E}\min\{\tilde{r}, 1-\tilde{r}\} = \xi\left[\left(-\frac{\lambda}{\epsilon}\right)\min\{r, 1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r, 1-r\}\right]$

$+(1-\xi)\min\left\{r, 1 - \left[(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right] - r\right\}.$

<u>$(\lambda, v-\rho) \in (3C)$</u>

$P(s_i < 0) = r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)$

$$= \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\frac{C}{2\epsilon}, \quad \tilde{r} = \frac{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$$P(s_i > 0, 2n3) = \frac{v-\rho}{2\epsilon t} - \frac{(\epsilon+\lambda)}{2\epsilon}, \quad \tilde{r} = r,$$

$$P(s_i > 0, 3n4) = r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right), \quad \tilde{r} = \frac{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\lambda+2\epsilon}{\epsilon}\right) + (1-r)\left(\frac{-C+2\lambda+2\epsilon}{2\epsilon}\right)},$$

$$P(s_i > 0, gt4) = r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right), \quad \tilde{r} = 1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$$P(s_i > 0) = 1 - \left[r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right],$$

$$\tilde{r} = \frac{r\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right]}{1 - \left[r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right), (1-r)\left(-\frac{v-\rho}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.$$
$$\left. + \left(\frac{v-\rho}{2\epsilon t} - \frac{\lambda+\epsilon}{2\epsilon}\right)\min\{r, 1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r, 1-r\}\right]$$
$$+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right),\, (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\} + \right.$$
$$\left.\min\left\{r\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right],\, (1-r)\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].$$

$\underline{(\lambda, v-\rho) \in (3D)}$

$$P(s_i < 0) = r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = \frac{r\left(1 - \frac{(v-\rho)}{\epsilon t}\right)}{r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$$P(s_i > 0, 3n4) = r\left(\frac{v-\rho}{\epsilon t} - \frac{(C-\epsilon-\lambda)}{\epsilon}\right) + (1-r)\left(\frac{v-\rho}{2\epsilon t} - \frac{(C-\epsilon-\lambda)}{2\epsilon}\right),$$

$$\tilde{r} = \frac{r\left(\frac{v-\rho}{\epsilon t} - \frac{(C-\epsilon-\lambda)}{\epsilon}\right)}{r\left(\frac{v-\rho}{\epsilon t} - \frac{(C-\epsilon-\lambda)}{\epsilon}\right) + (1-r)\left(\frac{v-\rho}{2\epsilon t} - \frac{(C-\epsilon-\lambda)}{2\epsilon}\right)},$$

$$P(s_i > 0, gt4) = r\left(\frac{C-\epsilon-\lambda}{\epsilon}\right), \quad \tilde{r} = 1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$$P(s_i > 0) = 1 - \left[r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right],$$

$$\tilde{r} = \frac{r\left[1 - \left(1 - \frac{(v-\rho)}{\epsilon t}\right)\right]}{1 - \left[r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}$$
$$= \xi\left[\min\left\{r\left(1 - \frac{v-\rho}{\epsilon t}\right), (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\} + \left(\frac{v-\rho}{2\epsilon t} - \frac{(C-\epsilon-\lambda)}{2\epsilon}\right)\min\{2r, 1-r\}\right]$$
$$+(1-\xi)\left[\min\left\{r\left(1 - \frac{(v-\rho)}{\epsilon t}\right),\, (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\} + \right.$$

$$\min\left\{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\Bigg].$$

$\underline{(\lambda, v-\rho)\in(3E)}$

$$P(s_i<0)=r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r),\quad \tilde{r}=\frac{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)}{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,gt4)=r\left(\frac{v-\rho}{\epsilon t}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-\left[r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\right],\quad \tilde{r}=1.$$

$$\mathbb{E}\min\{\tilde{r},\,1-\tilde{r}\}=\xi\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),1-r\right\}+(1-\xi)\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right),\,(1-r)\right\}.$$

$\underline{(\lambda, v-\rho)\in(4A)}$

$$P(s_i<0)=0;$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,1n2)=(1-r)\left(\frac{C-2\lambda}{2\epsilon}\right),\quad \tilde{r}=0,$$

$$P(s_i>0,2n3)=\frac{\lambda}{\epsilon},\quad \tilde{r}=r,$$

$$P(s_i>0,3n4)=r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right),\quad \tilde{r}=\frac{r\left(\frac{-C+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right)},$$

$$P(s_i>0,gt4)=r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1,\quad \tilde{r}=r.$$

$$\mathbb{E}\min\{\tilde{r},\,1-\tilde{r}\}=\xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right]+(1-\xi)\min\{r,1-r\}.$$

$\underline{(\lambda, v-\rho)\in(4B)}$

$$P(s_i<0)=(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right),\quad \tilde{r}=0;$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,1n2)=(1-r)\left(\frac{v-\rho}{2\epsilon t}-\frac{(\lambda+\epsilon)}{2\epsilon}\right),\quad \tilde{r}=0,$$

$$P(s_i>0,2n3)=\frac{\lambda}{\epsilon},\quad \tilde{r}=r,$$

$$P(s_i>0,3n4)=r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right),\quad \tilde{r}=\frac{r\left(\frac{-C+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right)},$$

$$P(s_i>0,gt4)=r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\quad \tilde{r}=\frac{r}{1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)}.$$

$$\mathbb{E}\min\{\tilde{r},\,1-\tilde{r}\}=\xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right]$$
$$+(1-\xi)\min\left\{r,1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)-r\right\}.$$

$(\lambda, v-\rho)\in(4C)$

$$P(s_i<0)=r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right),$$
$$\tilde{r}=\frac{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,2n3)=\frac{v-\rho}{2\epsilon t}-\frac{(-\lambda+\epsilon)}{2\epsilon},\quad \tilde{r}=r,$$
$$P(s_i>0,3n4)=r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right),\quad \tilde{r}=\frac{r\left(\frac{-C+2\epsilon}{\epsilon}\right)}{r\left(\frac{-C+2\epsilon}{\epsilon}\right)+(1-r)\left(\frac{-C+2\epsilon}{2\epsilon}\right)},$$
$$P(s_i>0,gt4)=r\left(\frac{C-\lambda-\epsilon}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right],$$
$$\tilde{r}=\frac{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)\right]}{1-\left[r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\,1-\tilde{r}\}=\xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{\lambda+\epsilon}{2\epsilon}\right),(1-r)\left(-\frac{v-\rho}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.$$
$$\left.+\left(\frac{v-\rho}{2\epsilon t}-\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\min\{r,1-r\}+\left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right]$$
$$+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right.$$
$$\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].$$

$(\lambda, v-\rho)\in(4D)$

$$P(s_i<0)=r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right),\quad \tilde{r}=\frac{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)}{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,3n4)=r\left(\frac{v-\rho}{\epsilon t}-\frac{(C-\epsilon-\lambda)}{\epsilon}\right)+(1-r)\left(\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}\right),$$
$$\tilde{r}=\frac{r\left(\frac{v-\rho}{\epsilon t}-\frac{(C-\epsilon-\lambda)}{\epsilon}\right)}{r\left(\frac{v-\rho}{\epsilon t}-\frac{(C-\epsilon-\lambda)}{\epsilon}\right)+(1-r)\left(\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}\right)},$$
$$P(s_i>0,gt4)=r\left(\frac{C-\epsilon-\lambda}{\epsilon}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-\left[r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right],$$

$$\tilde{r} = \frac{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right]}{1-\left[r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]}.$$

$$\begin{aligned}&\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}\\ &=\xi\left[\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\left(\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}\right)\min\{2r,1-r\}\right]\\ &+(1-\xi)\left[\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\right.\\ &\left.\min\left\{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

<u>$(\lambda, v-\rho)\in(4E)$</u>

$$P(s_i<0)=r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r),\quad \tilde{r}=\frac{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)}{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)};$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,gt4)=r\left(\frac{v-\rho}{\epsilon t}\right),\quad \tilde{r}=1;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$$P(s_i>0)=1-\left[r\left(1-\frac{(v-\rho)}{\epsilon t}\right)+(1-r)\right],\quad \tilde{r}=1.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),1-r\right\}.$$

<u>$(\lambda, v-\rho)\in(5A)$</u>

$P(s_i<0)=0;$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,1n2)=(1-r)\left(\frac{C-2\lambda}{2\epsilon}\right),\quad \tilde{r}=0,$$

$$P(s_i>0,2n3)=\frac{\lambda}{\epsilon},\quad \tilde{r}=r,$$

$$P(s_i>0,3n4)=r\left(\frac{C-2\lambda}{\epsilon}\right)+(1-r)\left(\frac{C-2\lambda}{2\epsilon}\right),\quad \tilde{r}=\frac{r\left(\frac{C-2\lambda}{\epsilon}\right)}{r\left(\frac{C-2\lambda}{\epsilon}\right)+(1-r)\left(\frac{C-2\lambda}{2\epsilon}\right)},$$

$$P(s_i>0,gt4)=\frac{\lambda-C+\epsilon}{\epsilon},\quad \tilde{r}=r;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i>0$,

$P(s_i>0)=1,\quad \tilde{r}=r.$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=\xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r,1-r\}+\left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r,1-r\}\right]$$
$$+(1-\xi)\min\{r,1-r\}.$$

<u>$(\lambda, v-\rho)\in(5B)$</u>

$$P(s_i<0)=(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right),\quad \tilde{r}=0;$$

(i) with probability ξ, surplus information is given if $s_i>0$,

$$P(s_i>0,1n2)=(1-r)\left(\frac{v-\rho}{2\epsilon t}-\frac{(\lambda+\epsilon)}{2\epsilon}\right),\quad \tilde{r}=0,$$

$P(s_i > 0, 2n3) = \dfrac{\lambda}{\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\left(\dfrac{C-2\lambda}{\epsilon}\right) + (1-r)\left(\dfrac{C-2\lambda}{2\epsilon}\right), \quad \tilde{r} = \dfrac{r\left(\frac{C-2\lambda}{\epsilon}\right)}{r\left(\frac{C-2\lambda}{\epsilon}\right) + (1-r)\left(\frac{C-2\lambda}{2\epsilon}\right)},$

$P(s_i > 0, gt4) = \dfrac{\lambda - C + \epsilon}{\epsilon}, \quad \tilde{r} = r;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - (1-r)\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = \dfrac{r}{1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)}.$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r, 1-r\} + \left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r, 1-r\} + \left(\frac{\lambda - C+\epsilon}{\epsilon}\right)\min\{r, 1-r\}\right]$$
$$+(1-\xi)\min\left\{r, 1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right) - r\right\}.$$

<u>$(\lambda, v-\rho) \in (5C)$</u>

$P(s_i < 0) = r\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{(\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{(C-\lambda+\epsilon)}{2\epsilon}\right),$

$$\tilde{r} = \frac{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right)}{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)};$$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 2n3) = \dfrac{v-\rho}{2\epsilon t} - \dfrac{(-\lambda+\epsilon)}{2\epsilon}, \quad \tilde{r} = r,$

$P(s_i > 0, 3n4) = r\left(\dfrac{C-2\lambda}{\epsilon}\right) + (1-r)\left(\dfrac{C-2\lambda}{2\epsilon}\right), \quad \tilde{r} = \dfrac{r\left(\frac{C-2\lambda}{\epsilon}\right)}{r\left(\frac{C-2\lambda}{\epsilon}\right) + (1-r)\left(\frac{C-2\lambda}{2\epsilon}\right)},$

$P(s_i > 0, gt4) = \dfrac{\lambda - C + \epsilon}{\epsilon}, \quad \tilde{r} = r;$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$P(s_i > 0) = 1 - \left[r\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{(\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right],$

$$\tilde{r} = \frac{r\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right)\right]}{1 - \left[r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t} + \frac{\lambda+\epsilon}{2\epsilon}\right), (1-r)\left(-\frac{v-\rho}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.$$
$$\left.+\left(\frac{v-\rho}{2\epsilon t} - \frac{(-\lambda+\epsilon)}{2\epsilon}\right)\min\{r, 1-r\} + \left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r, 1-r\} + \left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r, 1-r\}\right]$$
$$+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right),\, (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\} + \right.$$
$$\left.\min\left\{r\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(\lambda+\epsilon)}{2\epsilon}\right)\right],\, (1-r)\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].$$

<u>$(\lambda, v-\rho) \in (5D)$</u>

$P(s_i < 0) = r\left(1 - \dfrac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\dfrac{(v-\rho)}{2\epsilon t} + \dfrac{C-\lambda+\epsilon}{2\epsilon}\right), \quad \tilde{r} = \dfrac{r\left(1-\frac{(v-\rho)}{\epsilon t}\right)}{r\left(1-\frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)};$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$P(s_i > 0, 3n4) = r\left(\dfrac{v-\rho}{\epsilon t} - \dfrac{(\lambda - C+\epsilon)}{\epsilon}\right) + (1-r)\left(\dfrac{v-\rho}{2\epsilon t} - \dfrac{(\lambda-C+\epsilon)}{2\epsilon}\right),$

$$\tilde{r} = \frac{r\left(\frac{v-\rho}{\epsilon t} - \frac{(\lambda - C+\epsilon)}{\epsilon}\right)}{r\left(\frac{v-\rho}{\epsilon t} - \frac{(\lambda - C+\epsilon)}{\epsilon}\right) + (1-r)\left(\frac{v-\rho}{2\epsilon t} - \frac{(\lambda - C+\epsilon)}{2\epsilon}\right)},$$

$$P(s_i > 0, gt4) = \frac{\lambda - C+\epsilon}{\epsilon}, \quad \tilde{r} = r;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

$$P(s_i > 0) = 1 - \left[r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right],$$

$$\tilde{r} = \frac{r\left[1 - \left(1 - \frac{(v-\rho)}{\epsilon t}\right)\right]}{1 - \left[r\left(1 - \frac{(v-\rho)}{\epsilon t}\right) + (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]}.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\min\left\{r\left(1 - \frac{v-\rho}{\epsilon t}\right), (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.$$
$$\left. + \left(\frac{v-\rho}{2\epsilon t} - \frac{(\lambda - C+\epsilon)}{2\epsilon}\right)\min\{2r, 1-r\} + \left(\frac{\lambda - C+\epsilon}{\epsilon}\right)\min\{r, 1-r\}\right]$$
$$+(1-\xi)\left[\min\left\{r\left(1 - \frac{(v-\rho)}{\epsilon t}\right),\, (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\} + \right.$$
$$\left.\min\left\{r\left[1 - \left(1 - \frac{(v-\rho)}{\epsilon t}\right)\right],\, (1-r)\left[1 - \left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\right].$$

<u>$(\lambda, v-\rho) \in (5E)$</u>

$$P(s_i < 0) = 1 - \frac{(v-\rho)}{\epsilon t}, \quad \tilde{r} = r;$$

(i) with probability ξ, surplus information is given if $s_i > 0$,

$$P(s_i > 0, gt4) = \frac{v-\rho}{\epsilon t}, \quad \tilde{r} = r;$$

(ii) with probability $1-\xi$, surplus information is NOT given if $s_i > 0$,

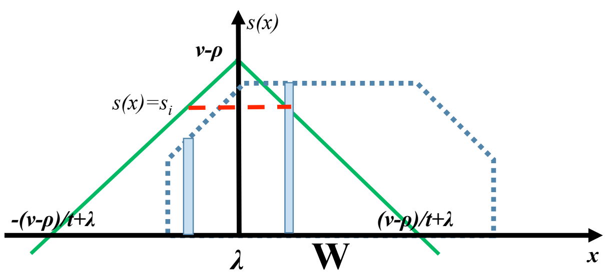
$$P(s_i > 0) = \frac{v-\rho}{\epsilon t}, \quad \tilde{r} = r.$$

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\{r, 1-r\}.$$

Appendix C: Proofs

Proof of Lemma 1. The probability that a randomly selected customer i has surplus $[s_i, s_i+\delta]$ for some small δ is given in Figure C-1.

Figure C-1 Illustration of probability that a randomly chosen consumer has surplus $[s_i, s_i+\delta]$ for some small δ.



$$P(u_i \in (s_i, s_i+\delta)|(\lambda,\, v,\, \rho),\, W) = \int_{\frac{-v+\rho+s_i}{t}+\lambda}^{\frac{-v+\rho+s_i+\delta}{t}+\lambda} h(x|W)dx + \int_{\frac{v-\rho-s_i-\delta}{t}+\lambda}^{\frac{v-\rho-s_i}{t}+\lambda} h(x|W)dx$$

$$= h\left(\frac{-v+\rho+s_i}{t}+\lambda|W\right)\cdot\frac{\delta}{t}+h\left(\frac{v-\rho-s_i}{t}+\lambda|W\right)\cdot\frac{\delta}{t}$$
$$= \frac{1}{t}\left[h\left(\frac{-v+s_i}{t}+\lambda|W\right)+h\left(\frac{v-s_i}{t}+\lambda|W\right)\right]\cdot\delta\equiv g(s_i|(\lambda,v,\rho),W)\cdot\delta. \quad \square$$

Proof of Lemma A.1 (Lemma 2 is a special case). (i) Surplus setting: Recall that $P(\lambda, u_i\in[s_i,s_i+\delta]|W)=g(\lambda,s_i|W)\cdot\delta$. So,

$$\begin{aligned}P(W=0|\lambda,u_i=s_i) &\approxeq P(W=0|\lambda,u_i\in[s_i,s_i+\delta])\\ &= \frac{P(\lambda,u_i\in[s_i,s_i+\delta]|W=0)P(W=0)}{P(\lambda,u_i\in[s_i,s_i+\delta]|W=0)P(W=0)+P(\lambda,u_i\in[s_i,s_i+\delta]|W=C)P(W=C)}\\ &= \frac{g(\lambda,s_i|W=0)\cdot\delta\cdot P(W=0)}{g(\lambda,s_i|W=0)\cdot\delta\cdot P(W=0)+g(\lambda,s_i|W=C)\cdot\delta\cdot P(W=C)}\\ &= \frac{g(\lambda,s_i|W=0)P(W=0)}{g(\lambda,s_i|W=0)P(W=0)+g(\lambda,s_i|W=C)P(W=C)}.\end{aligned}$$

(ii) sales and no surplus information:

$$P(W=0|1(s_i>0)) = \frac{P(1(s_i>0)|W=0)P(W=0)}{P(1(s_i>0)|W=0)P(W=0)+P(1(s_i>0)|W=C)P(W=C)};$$

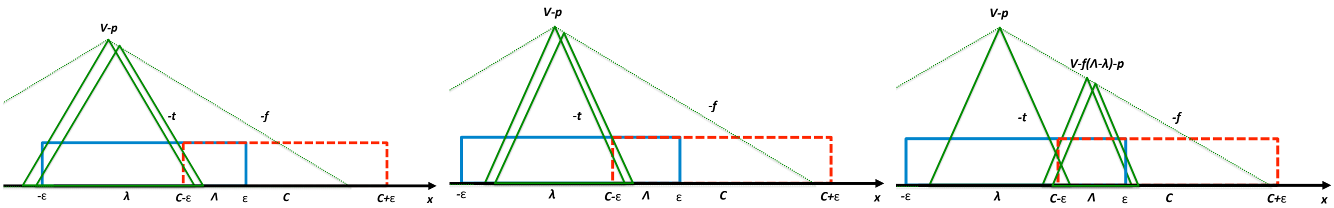
(iii) No sales and no surplus information:

$$P(W=0|1(s_i<0)) = \frac{P(1(s_i<0)|W=0)P(W=0)}{P(1(s_i<0)|W=0)P(W=0)+P(1(s_i<0)|W=C)P(W=C)}. \quad \square$$

Proof of Proposition 3 (Proposition 1 is a special case). (i) We shall examine the case when $f<t$.

To avoid redundancy, we will show the result for the setting (a) $\lambda\in\left(C-\epsilon-\frac{V-p}{t},\ \epsilon-\frac{V-p}{t}\right)$, and examine the derivative w.r.t. Λ in the following three cases. Setting (c) corresponding to $\lambda\in\left(C-\epsilon+\frac{V-p}{t},\ \epsilon+\frac{V-p}{t}\right)$

Figure C-2 Three cases: (1) – left panel ; (2) – middle panel; (3) – right panel.



can be analyzed in a similar manner. Also, setting (b) follows from combining Case (3) for setting (a) and setting (b).

Case (1). In this case $\lambda-\frac{V-p}{\epsilon}<-\epsilon$ (represented in left panel of Figure C-2). Without any pivoting, the final product's location is λ, and with incremental pivoting the final product's location is $\Lambda>\lambda$. Then, the expected revenues are respectively

$$\begin{aligned}\pi_\lambda &= \left(\frac{V-p}{t}+\lambda-(-\epsilon)\right)\cdot\tilde{r}+\left(\frac{V-p}{t}+\lambda-(C-\epsilon)\right)\cdot(1-\tilde{r});\\ \pi_\Lambda &= \left(\frac{V-p-f(\Lambda-\lambda)}{t}+\Lambda-(-\epsilon)\right)\cdot\tilde{r}+\left(\frac{V-p-f(\Lambda-\lambda)}{t}+\Lambda-(C-\epsilon)\right)\cdot(1-\tilde{r}).\end{aligned}$$

Then, we have

$$\begin{aligned}\pi_\Lambda - \pi_\lambda &= \left(-\frac{f(\Lambda-\lambda)}{t} + \Lambda - \lambda\right)\cdot\tilde{r} + \left(-\frac{f(\Lambda-\lambda)}{t} + \Lambda - \lambda\right)\cdot(1-\tilde{r})\\ &= \left(1-\frac{f}{t}\right)(\Lambda-\lambda)\tilde{r} + \left(1-\frac{f}{t}\right)(\Lambda-\lambda)(1-\tilde{r}) = \left(1-\frac{f}{t}\right)(\Lambda-\lambda) > 0.\end{aligned}$$

Hence, in this case, the pivot should happen until it borders Case (2), i.e., until $\Lambda^* - \frac{V-p-f(\Lambda^*-\lambda)}{t} = -\epsilon$.

Case (2). This case is represented by the middle panel of Figure C-2. Without any pivoting, the final product's location is λ, and with incremental pivoting the final product's location is $\Lambda > \lambda$. Then, the expected revenues are respectively

$$\begin{aligned}\pi_\lambda &= \left(\frac{V-p}{t} + \lambda - \left(-\frac{V-p}{t} + \lambda\right)\right)\cdot\tilde{r} + \left(\frac{V-p}{t} + \lambda - (C-\epsilon)\right)\cdot(1-\tilde{r});\\ \pi_\Lambda &= \left(\frac{V-p-f(\Lambda-\lambda)}{t} + \Lambda - \left(-\frac{V-p-f(\Lambda-\lambda)}{t} + \Lambda\right)\right)\cdot\tilde{r}\\ &\quad + \left(\frac{V-p-f(\Lambda-\lambda)}{t} + \Lambda - (C-\epsilon)\right)\cdot(1-\tilde{r}).\end{aligned}$$

Then, we have

$$\begin{aligned}\pi_\Lambda - \pi_\lambda &= -\frac{2f(\Lambda-\lambda)}{t}\cdot\tilde{r} + \left(-\frac{f(\Lambda-\lambda)}{t} + \Lambda - \lambda\right)\cdot(1-\tilde{r})\\ &= -\frac{2f}{t}(\Lambda-\lambda)\tilde{r} + \left(1-\frac{f}{t}\right)(\Lambda-\lambda)(1-\tilde{r}) = \left(-\frac{f}{t}2\tilde{r} + \left(1-\frac{f}{t}\right)(1-\tilde{r})\right)(\Lambda-\lambda).\end{aligned}$$

Hence, in this case, pivoting is beneficial if and only if

$$\frac{f}{t}2\tilde{r} + \left(1-\frac{f}{t}\right)(1-\tilde{r}) > 0 \iff \frac{f}{t} < \frac{1-\tilde{r}}{1+\tilde{r}}.$$

Thus, in this case, it is optimal not to pivot if $\frac{f}{t} > \frac{1-\tilde{r}}{1+\tilde{r}}$.

If $\frac{f}{t} < \frac{1-\tilde{r}}{1+\tilde{r}}$, then pivoting should happen until it borders Case (3), i.e., until $\Lambda^* + \frac{V-p-f(\Lambda^*-\lambda)}{t} = \epsilon$.

Case (3). This case is represented by the right panel of Figure C-2. Suppose that Λ_0 represents the position that first enters this case. We examine whether incrementally pivoting to $\Lambda_1 > \Lambda_0$ is beneficial or not by examining the derivative.

$$\begin{aligned}\pi_{\Lambda_0} &= \left(C-\epsilon-\frac{V-p-f(\Lambda_0-\lambda)}{t} + \Lambda_0\right)\cdot\tilde{r} + [\epsilon-(C-\epsilon)] + \left(\frac{V-p-f(\Lambda_0-\lambda)}{t} + \Lambda_0\right)\cdot(1-\tilde{r});\\ \pi_{\Lambda_1} &= \left(C-\epsilon-\frac{V-p-f(\Lambda_1-\lambda)}{t} + \Lambda_1\right)\cdot\tilde{r} + [\epsilon-(C-\epsilon)] + \left(\frac{V-p-f(\Lambda_1-\lambda)}{t} + \Lambda_1\right)\cdot(1-\tilde{r}).\end{aligned}$$

Then, we have

$$\begin{aligned}\pi_{\Lambda_1} - \pi_{\Lambda_0} &= \left(-\frac{f\Lambda_1}{t} - \Lambda_1 - \left(-\frac{f\Lambda_0}{t} - \Lambda_0\right)\right)\cdot\tilde{r} + \left(-\frac{f\Lambda_1}{t} + \Lambda_1 - \left(-\frac{f\Lambda_0}{t} + \Lambda_0\right)\right)\cdot(1-\tilde{r})\\ &= -\left(1+\frac{f}{t}\right)(\Lambda_1-\Lambda_0)\tilde{r} + \left(1-\frac{f}{t}\right)(\Lambda_1-\Lambda_0)(1-\tilde{r})\\ &= \left[-\left(1+\frac{f}{t}\right)\tilde{r} + \left(1-\frac{f}{t}\right)(1-\tilde{r})\right](\Lambda_1-\Lambda_0) = \left[1-2\tilde{r}-\frac{f}{t}\right](\Lambda_1-\Lambda_0).\end{aligned}$$

Hence, in this case, pivoting is beneficial if and only if

$$1-2\tilde{r}-\frac{f}{t} > 0 \iff \frac{f}{t} < 1-2\tilde{r} \iff \tilde{r} < \frac{1}{2}\left(1-\frac{f}{t}\right).$$

Hence, pivoting should continue until it leaves this region, i.e., $\Lambda^* - \frac{V-p-f(\Lambda^*-\lambda)}{t} = C-\epsilon$.

(ii) If $f > t$, the coverage area does not shift as a result of pivoting. As such, in this case, pivoting is never optimal. □

Proof of Lemma 3. Let $s_i \equiv s_i(\lambda, v, \rho)$ and $\tilde{r} \equiv \tilde{r}(\lambda, v, \rho | r)$. Taking the expectation of (5), we have

$$\begin{aligned}\max_{(\lambda,v,\rho)} \mathbb{E}_{s_i}\pi^*(\tilde{r}) &= \max_{(\lambda,v,\rho)} \mathbb{E}_{s_i} p\left[\frac{V-p}{\epsilon t} - \left(\frac{V-p}{\epsilon t} - \left(1-\frac{C}{2\epsilon}\right)\right)\min\{\tilde{r}, 1-\tilde{r}\}\right] \\ &= \max_{(\lambda,v,\rho)} p\left[\frac{V-p}{\epsilon t} - \left(\frac{V-p}{\epsilon t} - \left(1-\frac{C}{2\epsilon}\right)\right)\mathbb{E}_{s_i}\min\{\tilde{r}, 1-\tilde{r}\}\right] \\ &= p\frac{V-p}{\epsilon t} - p\left(\frac{V-p}{\epsilon t} - \left(1-\frac{C}{2\epsilon}\right)\right)\min_{(\lambda,v,\rho)} \mathbb{E}_{s_i}\min\{\tilde{r}, 1-\tilde{r}\}. \quad \square\end{aligned}$$

Proof of Proposition A.1 (Proposition 2 is a special case). We solve the problem of finding test product that minimizes $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}$, whose expressions for the 25 regions of $(\lambda, v-\rho)$ are derived in Appendix B. Observe that within each region, the expression is *linear* in λ and $(v-\rho)$, i.e., it is a plane. This implies that the optimal test product that maximizes learning $(\lambda^*, (v-\rho)^*)$ occurs in the boundaries between the regions. To find out which boundaries, we examine the first derivatives with respect to λ and $(v-\rho)$ of each region.

(1A), (2A): We have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \left(1-\xi\frac{C}{2\epsilon}\right)\min\{r, 1-r\}$, so $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(1B), (2B): We have

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1-\frac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)\min\left\{r,\, (1-r)\left(1+\frac{v-\rho}{2\epsilon t} - \frac{1}{2} - \frac{C-\lambda}{2\epsilon}\right)\right\}.$$

Let $1-A \equiv 1$, and $1-B \equiv \left(1+\frac{v-\rho}{2\epsilon t} - \frac{1}{2} - \frac{C-\lambda}{2\epsilon}\right)$.

(a) If $r \le \frac{1-B}{1+1-B}$, we have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1-\frac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)r$, so $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(b) If $r > \frac{1-B}{1+1-B}$, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left(1-\frac{C}{2\epsilon}\right)\min\{r, 1-r\} + (1-\xi)(1-r)\left(1+\frac{v-\rho}{2\epsilon t} - \frac{1}{2} - \frac{C-\lambda}{2\epsilon}\right)$, so $\frac{\partial}{\partial(v-\rho)} = \frac{(1-\xi)(1-r)}{2\epsilon t} > 0$, and $\frac{\partial}{\partial\lambda} = \frac{(1-\xi)(1-r)}{2\epsilon} > 0$.

(1C), (2C): We have

$$\begin{aligned}\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi&\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right), (1-r)\frac{C}{2\epsilon}\right\} + \left(\frac{v-\rho}{2\epsilon t} - \frac{C-\epsilon-t}{2\epsilon}\right)\min\{r, 1-r\},\right] \\ &+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right),\, (1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\} + \right. \\ &\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right],\, (1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Let $A \equiv r\left(-\frac{v-\rho}{2\epsilon t} + \frac{-\lambda+\epsilon}{2\epsilon}\right)$, $B \equiv \left(-\frac{v-\rho}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)$. We have $A < B$, and thus we have three cases. We have four ranges of priors.

(a) If $r \le \frac{1-B}{1-A+1-B}$, we have $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(b) If $\frac{1-B}{1-A+1-B} < r \le \frac{1}{2}$, then we have $\frac{\partial}{\partial(v-\rho)} > 0$ and $\frac{\partial}{\partial\lambda} > 0$.

(c) If $\frac{1}{2} < r \le \frac{B}{A+B}$, then we have $\frac{\partial}{\partial(v-\rho)} < 0$, and $\frac{\partial}{\partial\lambda} < 0$.

(d) If $r > \frac{B}{A+B}$, we have $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(1D): We have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\left\{r\left(-\frac{v-\rho}{2\epsilon t} + \frac{-\lambda+\epsilon}{2\epsilon}\right),\ (1-r)\right\}$. Let $A \equiv \left(-\frac{v-\rho}{2\epsilon t} + \frac{-\lambda+\epsilon}{2\epsilon}\right)$, and $B \equiv 1$, so $A < B$.

(a) If $r \le \frac{1}{A+1}$, we have $\frac{\partial}{\partial(v-\rho)} < 0$, and $\frac{\partial}{\partial\lambda} < 0$.

(b) If $r > \frac{1}{A+1}$, then $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(1E), (5E): We have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\{r, 1-r\}$. Thus, $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(2E), (3E), (4E): We have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),\ (1-r)\right\}$. Let $A \equiv \left(1-\frac{v-\rho}{\epsilon t}\right)$, and $B \equiv 1$, so $A < B$.

(a) If $r \le \frac{1}{A+1}$, we have $\frac{\partial}{\partial(v-\rho)} < 0$, and $\frac{\partial}{\partial\lambda} = 0$.

(b) If $r > \frac{1}{A+1}$, we have $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(3A): We have

$$\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\left(-\frac{\lambda}{\epsilon}\right)\min\{r,1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r,1-r\}\right] + (1-\xi)\min\{r,1-r\}.$$

So we have,

(a) if $r < \frac{1}{3}$, then $\frac{\partial}{\partial(v-\rho)} = 0$, and $\frac{\partial}{\partial\lambda} > 0$;

(b) if $\frac{1}{3} < r \le \frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)} = 0$; and $\frac{\partial}{\partial\lambda} \ge 0$.

(c) if $r > \frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)} = \frac{\partial}{\partial\lambda} = 0$.

(3B): We have,

$$\begin{aligned}\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} &= \xi\left[\left(-\frac{\lambda}{\epsilon}\right)\min\{r,1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r,1-r\}\right], \\ &+(1-\xi)\min\left\{r, 1-\left[(1-r)\left(-\frac{(v-\rho)}{2\epsilon t} + \frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]-r\right\}.\end{aligned}$$

Let $1-A \equiv 1$, and $1-B \equiv \left[1-\left(-\frac{v-\rho}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right)\right]$. Observe, $1-A > 1-B$.

Analyzing different regions of r, we have that (a) if $r \le \frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)} \ge 0$ and $\frac{\partial}{\partial\lambda} \ge 0$, (b) if $r > \frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)} > 0$ and $\frac{\partial}{\partial\lambda} > 0$.

(3C): We have

$$\begin{aligned}\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} &= \xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right), (1-r)\left(-\frac{v-\rho}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right. \\ &\left.+\left(\frac{v-\rho}{2\epsilon t}-\frac{\lambda+\epsilon}{2\epsilon}\right)\min\{r,1-r\} + \left(\frac{-C+2\epsilon+2\lambda}{2\epsilon}\right)\min\{2r,1-r\}\right] \\ &+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right),\ (1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right. \\ &\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\right],\ (1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Examining the different ranges of r, we have (a) if $r \le \frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)} \ge 0$ and $\frac{\partial}{\partial\lambda} \ge 0$; (b) if $r > \frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)} < 0$ and $\frac{\partial}{\partial\lambda} < 0$.

(3D): We have, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}$

$$\begin{aligned}&= \xi\left[\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right), (1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\} + \left(\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}\right)\min\{2r,1-r\}\right] \\ &+(1-\xi)\left[\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right),\ (1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\right. \\ &\left.\min\left\{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right],\ (1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Examining the different ranges of r, we have (a) if $r > \frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)} < 0$ and $\frac{\partial}{\partial\lambda} > 0$; (b) if $r \le \frac{1}{2}$, $\frac{\partial}{\partial\lambda} > 0$ and $\frac{\partial}{\partial(v-\rho)} = \xi\frac{v-\rho}{2\epsilon t}(1-3r) + (1-\xi)\frac{v-\rho}{2\epsilon t}(1-2r) = \frac{v-\rho}{2\epsilon t}(1+(2-5\xi)r)$. Therefore, if $r \le \frac{1}{2}$, $\frac{\partial}{\partial(v-\rho)} \ge 0$ if $\xi \le \frac{2}{5}$ and $\frac{\partial}{\partial(v-\rho)} < 0$ if $\xi > \frac{2}{5}$.

(4A): We have $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\} = \xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\} + \left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right] + (1-\xi)\min\{r,1-r\}$. So we have, $\frac{\partial}{\partial(v-\rho)}=0$ and $\frac{\partial}{\partial\lambda}>0$.

(4B): We have, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=$

$$\xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right]+(1-\xi)\min\left\{r,1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)-r\right\}.$$

Let $1-A\equiv=1$, and $1-B\equiv\left[1-\left(-\frac{v-\rho}{2\epsilon t}+\frac{1}{2}+\frac{C-\lambda}{2\epsilon}\right)\right]$. Observe, $1-A>1-B$. Thus, if $r\leq\frac{1-B}{1-A+1-B}$, $\frac{\partial}{\partial(v-\rho)}=0$ and $\frac{\partial}{\partial\lambda}>0$; if $r>\frac{1-B}{1-A+1-B}$, $\frac{\partial}{\partial(v-\rho)}>0$ and $\frac{\partial}{\partial\lambda}>0$.

(4C): We have,

$$\begin{aligned}\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}\,=\,\xi&\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{\lambda+\epsilon}{2\epsilon}\right),(1-r)\left(-\frac{v-\rho}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.\\&\left.+\left(\frac{v-\rho}{2\epsilon t}-\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\min\{r,1-r\}+\left(\frac{-C+2\epsilon}{2\epsilon}\right)\min\{2r,1-r\}\right]\\&+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right.\\&\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Analyzing all the cases, we have (i) if $r\leq\frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)}>0$ and $\frac{\partial}{\partial\lambda}>0$; (ii) if $r>\frac{1}{2}$ then $\frac{\partial}{\partial(v-\rho)}<0$ and $\frac{\partial}{\partial\lambda}>0$.

(4D): We have, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=$

$$\begin{aligned}&\xi\left[\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\left(\frac{v-\rho}{2\epsilon t}-\frac{(C-\epsilon-\lambda)}{2\epsilon}\right)\min\{2r,1-r\}\right],\\&+(1-\xi)\left[\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\right.\\&\left.\min\left\{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Examining the different ranges of priors r, we have (a) if $r>\frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)}<0$ and $\frac{\partial}{\partial\lambda}>0$; (b) if $r\leq\frac{1}{2}$, $\frac{\partial}{\partial\lambda}\geq 0$, and (b.1) $\frac{\partial}{\partial(v-\rho)}\geq 0$ if $\xi\leq\frac{2}{5}$ and (b.2) $\frac{\partial}{\partial(v-\rho)}<0$ if $\xi>\frac{2}{5}$.

(5A): We have, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=$

$$\xi\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r,1-r\}+\left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r,1-r\}\right]+(1-\xi)\min\{r,1-r\}.$$

Hence, we have $\frac{\partial}{\partial(v-\rho)}=0$ and $\frac{\partial}{\partial\lambda}>0$.

(5B): We have

$$\begin{aligned}\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}\,=\,\xi&\left[\left(\frac{\lambda}{\epsilon}\right)\min\{r,1-r\}+\left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r,1-r\}+\left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r,1-r\}\right],\\&+(1-\xi)\min\left\{r,1-(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)-r\right\}.\end{aligned}$$

For all ranges of r, we have $\frac{\partial}{\partial(v-\rho)}\geq 0$ and $\frac{\partial}{\partial\lambda}\geq 0$.

(5C): We have, $\mathbb{E}\min\{\tilde{r},\, 1-\tilde{r}\}=$

$$\begin{aligned}&\xi\left[\min\left\{r\left(-\frac{v-\rho}{2\epsilon t}+\frac{\lambda+\epsilon}{2\epsilon}\right),(1-r)\left(-\frac{v-\rho}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.\\&\left.+\left(\frac{v-\rho}{2\epsilon t}-\frac{(-\lambda+\epsilon)}{2\epsilon}\right)\min\{r,1-r\}+\left(\frac{C-2\lambda}{2\epsilon}\right)\min\{2r,1-r\}+\left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r,1-r\}\right]\\&+(1-\xi)\left[\min\left\{r\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right\}+\right.\\&\left.\min\left\{r\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(\lambda+\epsilon)}{2\epsilon}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{(C-\lambda+\epsilon)}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

We have, (a) if $r \le \frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)} > 0$ and $\frac{\partial}{\partial\lambda} > 0$; (b) if $r > \frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)} < 0$ and $\frac{\partial}{\partial\lambda} > 0$.

(5D): We have,

$$\begin{aligned}\mathbb{E}\min\{\tilde r,\,1-\tilde r\} = \xi &\left[\min\left\{r\left(1-\frac{v-\rho}{\epsilon t}\right),(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}\right.\\ &\left.+\left(\frac{v-\rho}{2\epsilon t}-\frac{(\lambda-C+\epsilon)}{2\epsilon}\right)\min\{2r,1-r\}+\left(\frac{\lambda-C+\epsilon}{\epsilon}\right)\min\{r,1-r\}\right]\\ &+(1-\xi)\left[\min\left\{r\left(1-\frac{(v-\rho)}{\epsilon t}\right),\,(1-r)\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right\}+\right.\\ &\left.\min\left\{r\left[1-\left(1-\frac{(v-\rho)}{\epsilon t}\right)\right],\,(1-r)\left[1-\left(-\frac{(v-\rho)}{2\epsilon t}+\frac{C-\lambda+\epsilon}{2\epsilon}\right)\right]\right\}\right].\end{aligned}$$

Taking the derivative, we have if $r > \frac{1}{2}$, then $\frac{\partial}{\partial(v-\rho)} < 0$ and $\frac{\partial}{\partial\lambda} > 0$; (ii) if $r \le \frac{1}{2}$, then $\frac{\partial}{\partial\lambda} > 0$ and (ii-a) $\frac{\partial}{\partial(v-\rho)} \ge 0$ if $\xi \le \frac{2}{5}$ and $\frac{\partial}{\partial(v-\rho)} < 0$ if $\xi > \frac{2}{5}$.

Combining the expressions for the slopes, we find that if $r > 0.5$, it lies in the boundary between region (1B) ∪ (2B) ∪ (3B) and region (1C) ∪ (2C) ∪ (3C); if $r < 0.5$, it lies in the boundary between region (1C) ∪ (2C) and region (1D) ∪ (2D). If $r = 0.5$, the regions (1C) ∪ (2C) ∪ (3C) correspond to the optimal test product. □

Proof of Corollary 1. The properties follow directly from the expressions from Proposition 2. □

Proof of Corollary A.1 (Corollary 2 is a special case). The expressions are derived in a straightforward manner by substituting the necessary values to the expressions of $\mathbb{E}_{s_i}\min\{\tilde r, 1-\tilde r\}$ in Appendix B. □

Proof of Proposition 4. (i) Observe that the set of optimal test product's identified in Proposition 2 whose design Λ lie outside the interval $[0, C]$ must pivot longer distance (lower quality V^* and hence demand) than the optimal test product located in $\Lambda = 0$ or $\Lambda = C$. Hence it is optimal to select $\Lambda = 0$ or $\Lambda = C$.

Moreover, if pivoting is costlier for higher quality test product, it is optimal to minimize v while keeping $v-\rho$ constant, which is achieved when $\rho = 0$.

(ii) If $f > t$, it is suboptimal to do pivoting. Thus, the test product design becomes the same as the second stage problem, resulting in $\lambda^* = \Lambda^*$ in Proposition 1. □

Proof of Lemma 4. Taking the first order condition of $\pi^*(\tilde r)$ with respect to p, we have

$$\begin{aligned}\frac{\partial\pi^*(\tilde r)}{\partial p}=0 \iff& \frac{\partial}{\partial p}p\left[\frac{V-p}{\epsilon t}-\left(\frac{V-p}{\epsilon t}-\left(1-\frac{C}{2\epsilon}\right)\right)\min\{\tilde r,1-\tilde r\}\right]=0\\ \iff& p\left(-\frac{1}{\epsilon t}+\frac{\min\{\tilde r,1-\tilde r\}}{\epsilon t}\right)+\frac{V-p}{\epsilon t}-\left(\frac{V-p}{\epsilon t}-\left(1-\frac{C}{2\epsilon}\right)\right)\min\{\tilde r,1-\tilde r\}=0\\ \iff& \frac{V}{\epsilon t}(1-\min\{\tilde r,1-\tilde r\})+\left(1-\frac{C}{2\epsilon}\right)\min\{\tilde r,1-\tilde r\}=\frac{2p}{\epsilon t}(1-\min\{\tilde r,1-\tilde r\})\\ \iff& p^*=\frac{V}{2}+\frac{\min\{\tilde r,1-\tilde r\}}{\max\{\tilde r,1-\tilde r\}}\frac{\epsilon t}{2}\left(1-\frac{C}{2\epsilon}\right).\quad\square\end{aligned}$$

Proof of Proposition 5. Substituting the expression for p^* in Lemma 4,

$$\begin{aligned}\frac{V-p^*}{\epsilon t}&=\frac{V}{2\epsilon t}-\frac{\min\{\tilde r,1-\tilde r\}}{\max\{\tilde r,1-\tilde r\}}\frac{1}{2}\left(1-\frac{C}{2\epsilon}\right),\\ \frac{V-p^*}{\epsilon t}\max\{\tilde r,1-\tilde r\}&=\frac{V}{2\epsilon t}\max\{\tilde r,1-\tilde r\}-\min\{\tilde r,1-\tilde r\}\frac{1}{2}\left(1-\frac{C}{2\epsilon}\right).\end{aligned}$$

Substituting the expression for p^* and $\frac{V-p^*}{\epsilon t}\max\{\tilde{r},1-\tilde{r}\}$, we have

$$\begin{aligned}\pi^*(\tilde{r}) &= p\left[\frac{V-p}{\epsilon t}-\left(\frac{V-p}{\epsilon t}-\left(1-\frac{C}{2\epsilon}\right)\right)\min\{\tilde{r},1-\tilde{r}\}\right]=p\left[\frac{V-p}{\epsilon t}\max\{\tilde{r},1-\tilde{r}\}+\left(1-\frac{C}{2\epsilon}\right)\min\{\tilde{r},1-\tilde{r}\}\right]\\ &= \left[\frac{V}{2}+\frac{\min\{\tilde{r},1-\tilde{r}\}}{\max\{\tilde{r},1-\tilde{r}\}}\frac{\epsilon t}{2}\left(1-\frac{C}{2\epsilon}\right)\right]\cdot\left[\frac{V}{2\epsilon t}\max\{\tilde{r},1-\tilde{r}\}+\min\{\tilde{r},1-\tilde{r}\}\frac{1}{2}\left(1-\frac{C}{2\epsilon}\right)\right]\\ &= \frac{1}{4}\left[V+\frac{\min\{\tilde{r},1-\tilde{r}\}}{\max\{\tilde{r},1-\tilde{r}\}}\epsilon t\left(1-\frac{C}{2\epsilon}\right)\right]\cdot\left[\frac{V}{\epsilon t}\max\{\tilde{r},1-\tilde{r}\}+\min\{\tilde{r},1-\tilde{r}\}\left(1-\frac{C}{2\epsilon}\right)\right]\\ &= \frac{\epsilon t}{4}\left[\frac{V}{\epsilon t}+\frac{\min\{\tilde{r},1-\tilde{r}\}}{\max\{\tilde{r},1-\tilde{r}\}}\left(1-\frac{C}{2\epsilon}\right)\right]\cdot\left[\frac{V}{\epsilon t}-\left[\frac{V}{\epsilon t}-\left(1-\frac{C}{2\epsilon}\right)\right]\min\{\tilde{r},1-\tilde{r}\}\right]\end{aligned}$$

Thus, taking the expectation with respect to $\tilde{r}$, we have the expression for $\mathbb{E}_{s_i}\pi^*(\tilde{r},\lambda,v)$. □

LEMMA C.1. *Let $v\equiv v-\rho$ for notational simplicity, and let us denote the probability that the test product results in sale when $\lambda=W$ and $\lambda\neq W$ respectively as*

$$q_1(v)=\min\left(1,\frac{v}{\epsilon t}\right),\quad q_2(v)=\min\left(1,\frac{v}{2\epsilon t}-\frac{C-\epsilon}{2\epsilon}\right). \tag{C-1}$$

Then $v^*\in\{\epsilon t,\,(C-\epsilon)t\}$ maximizes $\delta q_1(v)-(1-\delta)q_2(v)$ for any $\delta\in[0,1]$. Moreover, if $C<2\epsilon$, then $q_2(\epsilon t)<1-q_1((C-\epsilon)t)$. In other words, $\frac{q_2(\epsilon t)}{1-q_1((C-\epsilon)t)+q_2(\epsilon t)}<0.5$.

Proof of Lemma C.1. We have,

$$\delta q_1(v)-(1-\delta)q_2(v)=\min\left(\delta,\frac{\delta}{\epsilon t}v\right)-\min\left(1-\delta,\frac{1-\delta}{2\epsilon t}v-\frac{(1-\delta)(C-\epsilon)}{2\epsilon}\right).$$

Thus, the piecewise linear function can achieve the maximum difference in the point where either $q_1(v)$ or $q_2(v)$ hit the inflection points $v=\epsilon t$ and $v=(C-\epsilon)t$ respectively. Moreover, since $q_2(\epsilon t)=1-\frac{C}{2\epsilon}$ and $q_1((C-\epsilon)t)=\frac{C-\epsilon}{\epsilon}$,

$$q_2(\epsilon t)<1-q_1((C-\epsilon)t)\Leftrightarrow 1-\frac{C}{2\epsilon}<1-\frac{C-\epsilon}{\epsilon}\Leftrightarrow\frac{C-\epsilon}{\epsilon}<\frac{C}{2\epsilon}\Leftrightarrow\epsilon>C/2.\qquad\square$$

Proof of Proposition 5 and Corollary 3. To simplify notations, let

$$D_1\triangleq\int_{-\frac{V-p}{t}+\Lambda}^{\frac{V-p}{t}+\Lambda}h(x|W=\Lambda)dx,\qquad D_2\triangleq\int_{-\frac{V-p}{t}+\Lambda}^{\frac{V-p}{t}+\Lambda}h(x|W\neq\Lambda)dx,$$

so that D_1 and D_2 are demand from launching the ideal product and non-ideal product, respectively. Then, the optimal final stage profit given posterior $\tilde{r}$ is

$$\pi^*(\tilde{r})=\max\{\tilde{r}D_1+(1-\tilde{r})D_2,\,(1-\tilde{r})D_1+\tilde{r}D_2\}.$$

We now examine the launch of test products. To simplify notation, let $v\equiv v-\rho$. Let n denote the period to go. We will prove by induction (a) the expression for $(\lambda_n^*,v_n^*)\ \forall n$, (b) $\pi_n^*(\lambda^*,v^*)=A_n+B_n|0.5-q_2(\epsilon t)^n|$, $A_n,B_n>0\ \forall n$, and (c) the expression for $\beta_n(0.5)$.

(Base Case). Suppose $n=1$. The expected profit is

$$\begin{aligned}\pi(\lambda=0,v|r) &= [q_1(v)r+q_2(v)(1-r)]\cdot\max\left\{\frac{q_1(v)rD_1+q_2(v)(1-r)D_2}{q_1(v)r+q_2(v)(1-r)},\ \frac{q_2(v)(1-r)D_1+q_1(v)rD_2}{q_1(v)r+q_2(v)(1-r)}\right\}\\ &\quad+[(1-q_1(v))r+(1-q_2(v))(1-r)]\cdot\max\left\{\frac{(1-q_2(v))(1-r)D_1+(1-q_1(v))rD_2}{(1-q_1(v))r+(1-q_2(v))(1-r)},\right.\\ &\qquad\left.\frac{(1-q_1(v))rD_1+(1-q_2(v))(1-r)D_2}{(1-q_1(v))r+(1-q_2(v))(1-r)}\right\}\\ &= \max\{q_1(v)rD_1+q_2(v)(1-r)D_2,\ q_2(v)(1-r)D_1+q_1(v)rD_2\} \qquad\qquad \text{(C-2)}\\ &\quad+\max\{(1-q_2(v))(1-r)D_1+(1-q_1(v))rD_2,\ (1-q_1(v))rD_1+(1-q_2(v))(1-r)D_2\}.\end{aligned}$$

If the prior belief $r \geq 0.5$, staying course is optimal in the event of a sale of test product since $q_1(v) \geq q_2(v)$ for all v. Thus,

$$\begin{aligned}\pi(\lambda=0,v|r\geq 0.5) &= q_1(v)rD_1 + q_2(v)(1-r)D_2 \\ &\quad + \max\{(1-q_2(v))(1-r)D_1 + (1-q_1(v))rD_2,\ (1-q_1(v))rD_1 + (1-q_2(v))(1-r)D_2\} \\ &= \max\{[q_1(v)r + (1-q_2(v))(1-r)]D_1 + [q_2(v)(1-r) + (1-q_1(v))r]D_2, rD_1 + (1-r)D_2\} \\ &= rD_1 + (1-r)D_2 + \max\{0, [(1-2r) + \{rq_1(v) - (1-r)q_2(v)\}](D_1 - D_2)\}.\end{aligned}$$

Similarly, if $r \leq 0.5$, pivoting is optimal in the event of a no sale of test product since $q_1(v) \geq q_2(v)$ for all v. Thus,

$$\begin{aligned}\pi(\lambda=0,v|r\leq 0.5) &= (1-q_2(v))(1-r)D_1 + (1-q_1(v))rD_2 \\ &\quad + \max\{q_1(v)rD_1 + q_2(v)(1-r)D_2,\ q_2(v)(1-r)D_1 + q_1(v)rD_2\} \\ &= \max\{[(1-q_2(v))(1-r) + q_1(v)r]D_1 + [(1-q_1(v))r + q_2(v)(1-r)]D_2, (1-r)D_1 + rD_2\} \\ &= (1-r)D_1 + rD_2 + \max\{0, \{q_1(v)r - (1-r)q_2(v)\}(D_1 - D_2)\}.\end{aligned}$$

Thus, we have

$$\pi(\lambda=0,v|r) = \begin{cases} rD_1 + (1-r)D_2 + \max\{0, [(1-2r) + \{rq_1(v) - (1-r)q_2(v)\}](D_1 - D_2)\}, & r \geq 0.5, \\ (1-r)D_1 + rD_2 + \max\{0, \{q_1(v)r - (1-r)q_2(v)\}(D_1 - D_2)\}, & r \leq 0.5. \end{cases}$$

Similarly, we have

$$\pi(\lambda=C,v|r) = \begin{cases} rD_1 + (1-r)D_2 + \max\{0, \{(1-r)q_1(v) - rq_2(v)\}(D_1 - D_2)\}, & r \geq 0.5, \\ (1-r)D_1 + rD_2 + \max\{0, [(2r-1) + \{(1-r)q_1(v) - rq_2(v)\}](D_1 - D_2)\}, & r \leq 0.5. \end{cases}$$

By Lemma C.1, we need to only consider $v \in \{\epsilon t,\ (C-\epsilon)t\}$ when evaluating for optimal v^*. If $v^* = \epsilon t$, then $q_1(v^*) = 1$ and $q_2(v^*) < 1$; if $v^* = (C-\epsilon)t$, then $q_1(v^*) > 0$ and $q_2(v^*) = 0$. Identifying the optimal development level of the test product v^* require comparing the weighted difference $\rho q_1(v) - (1-\rho)q_2(v)$, $\rho \in [0,1]$.

$$\begin{aligned}&\rho - (1-\rho)q_2(\epsilon t) \gtrless_{v^*=(C-\epsilon)t}^{v^*=\epsilon t} \rho q_1((C-\epsilon)t) \ \Leftrightarrow\ \rho(1 - q_1((C-\epsilon)t)) \gtrless_{v^*=(C-\epsilon)t}^{v^*=\epsilon t} (1-\rho)q_2(\epsilon t) \\ &\Leftrightarrow\ \rho \gtrless_{v^*=(C-\epsilon)t}^{v^*=\epsilon t} \frac{q_2(\epsilon t)}{1 - q_1((C-\epsilon)t) + q_2(\epsilon t)}.\end{aligned} \tag{C-3}$$

For the case $r \geq 0.5$, we have, $\pi(\lambda=0,v|r) = rD_1 + (1-r)D_2 + \max\{0, [(1-2r) + \{rq_1(v) - (1-r)q_2(v)\}](D_1 - D_2)\}$, which involves maximizing $rq_1(v) - (1-r)q_2(v)$. Since, $r > 0.5 > \frac{q_2(\epsilon t)}{1-q_1((C-\epsilon)t)+q_2(\epsilon t)}$ (Lemma C.1), $v^* = \epsilon t$, i.e., $\pi(\lambda=0, v=\epsilon t|r) > \pi(\lambda=0, v=(C-\epsilon)t|r)$.

We next show that $\pi(\lambda=0,v=\epsilon t|r) \geq \max\{\pi(\lambda=C,v=\epsilon t|r), \pi(\lambda=C, v=(C-\epsilon)t|r)\}$. We have, $\pi(\lambda=0,v=\epsilon t|r) > \pi(\lambda=C,v=\epsilon t|r) \Leftrightarrow 1 - r - (1-r)q_2(\epsilon t) > (1-r) - rq_2(\epsilon t) \ \Leftrightarrow\ r > 0.5$; and $\pi(\lambda=0,v=\epsilon t|r) > \pi(\lambda=C, v=(C-\epsilon)t|r) \ \Leftrightarrow\ 1 - r - (1-r)q_2(\epsilon t) > (1-r)q_1((C-\epsilon)t) \ \Leftrightarrow\ 1 - q_1((C-\epsilon)t) > q_2(\epsilon t)$. Thus, if $r > 0.5$, it is optimal to launch the test product with $\lambda = 0$, and at development level $v = \epsilon t$.

Similarly, one can show that it is optimal to launch the test product with $\lambda = C$ at development level $v = \epsilon t$ when $r < 0.5$. Thus, $(\lambda^*, v^*) = (0, \epsilon t)$ if $r \geq 0.5$ and $(\lambda^*, v^*) = (C, \epsilon t)$ if $r < 0.5$.

Thus, combining the expression for $v^* = \epsilon t$, we have

$$\pi(\lambda^*(v), v=\epsilon t|r) = \begin{cases} (rD_1+(1-r)D_2)p+[(1-2r)+r-(1-r)q_2(\epsilon t)](D_1-D_2)p, & r\geq 0.5, \\ ((1-r)D_1+rD_2)p+[(2r-1)+(1-r)-rq_2(\epsilon t)](D_1-D_2)p, & r\leq 0.5. \end{cases}$$

$$= \left[D_1 p - \frac{q_2(\epsilon t)}{2}(D_1-D_2)p\right] + q_2(\epsilon t)(D_1-D_2)p\left|r-\frac{1}{2}\right| \quad \text{(C-4)}$$

$$\equiv A_0 + B_0|r-0.5|. \quad \text{(C-5)}$$

Moreover, observe that

$$\beta_0(v=\epsilon t|r) = \begin{cases} (r+\beta(v=\epsilon t|r))D_1 p+(1-r-\beta(v=\epsilon t|r))D_2 p, & r\geq 0.5, \\ (1-r+\beta(v=\epsilon t|r))D_1+(r-\beta(v=\epsilon t|r))D_2 p, & r\leq 0.5. \end{cases}$$

$$= \begin{cases} (1-r)(1-q_2(\epsilon t)), & r\geq 0.5, \\ r(1-q_2(\epsilon t)), & r\leq 0.5. \end{cases}$$

(Induction Step) Suppose claims (a), (b), and (c) are true for $n-1$ periods to go. Then, after launching a test product and observing sales, the expected profit $\pi(\lambda, v|r)$ for the choice of test product with n periods to go is

$$\pi_n(\lambda=0, v|r) = [q_1(v)r+q_2(v)(1-r)]\pi^*_{n-1}\left(\frac{q_1(v)r}{q_1(v)r+q_2(v)(1-r)}\right)$$
$$+[(1-q_1(v))r+(1-q_2(v))(1-r)]\pi^*_{n-1}\left(\frac{(1-q_1(v))r}{(1-q_1(v))r+(1-q_2(v))(1-r)}\right),$$
$$\pi_n(\lambda=C, v|r) = [q_1(v)(1-r)+q_2(v)r]\pi^*_{n-1}\left(\frac{q_1(v)(1-r)}{q_1(v)(1-r)+q_2(v)r}\right)$$
$$+[(1-q_1(v))(1-r)+(1-q_2(v))r]\pi^*_{n-1}\left(\frac{(1-q_1(v))(1-r)}{(1-q_1(v))(1-r)+(1-q_2(v))r}\right).$$

Examining the first expression, we have

$$\pi(\lambda=0, v|r) = [q_1(v)r+q_2(v)(1-r)]\left\{A+B\left|\frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}-\frac{1}{2}\right|\right\}$$
$$+[(1-q_1(v))r+(1-q_2(v))(1-r)]\left\{A+B\left|\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)}\right|\right\}$$
$$= A+B\left\{[q_1(v)r+q_2(v)(1-r)]\left|\frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}-\frac{1}{2}\right|\right.$$
$$\left.+\,[(1-q_1(v))r+(1-q_2(v))(1-r)]\left|\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)}-\frac{1}{2}\right|\right\}.$$

Observe that $\frac{(1-q_1(v))r}{(1-q_1(v))r+(1-q_2(v))(1-r)} < r < \frac{q_1(v)r}{q_1(v)r+q_2(v)(1-r)}$, so we have three cases:

(i) If $\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)} < \frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)} < \frac{1}{2}$ (equivalently, $r < \frac{q_2(v)}{q_1(v)(2q_2(\epsilon t)^{n-1}-1)+q_2(v)}$), then

$$\pi(\lambda=0, v|r) = A+B\left\{-[q_1(v)r+q_2(v)(1-r)]\left(\frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}-\frac{1}{2}\right)\right.$$
$$\left.-\,[(1-q_1(v))r+(1-q_2(v))(1-r)]\left(\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)}-\frac{1}{2}\right)\right\}$$
$$= A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-\frac{q_1(v)r+q_2(v)(1-r)}{2}+(1-q_1(v))rq_2(\epsilon t)^{n-1}-\frac{(1-q_1(v))r+(1-q_2(v))(1-r)}{2}\right\}$$
$$= A+B\left\{\frac{1}{2}-rq_2(\epsilon t)^{n-1}\right\}.$$

(ii) If $\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)} < \frac{1}{2} < \frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}$ (equivalently, $r \in \left[\frac{q_2(v)}{q_1(v)(2q_2(\epsilon t)^{n-1}-1)+q_2(v)}, \frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}\right]$), then

$$\begin{aligned}\pi(\lambda=0,v|r) &= A+B\left\{[q_1(v)r+q_2(v)(1-r)]\left(\frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}-\frac{1}{2}\right)\right.\\ &\quad\left.-[(1-q_1(v))r+(1-q_2(v))(1-r)]\left(\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)}-\frac{1}{2}\right)\right\}\\ &= A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-\frac{q_1(v)r+q_2(v)(1-r)}{2}-(1-q_1(v))rq_2(\epsilon t)^{n-1}+\frac{(1-q_1(v))r+(1-q_2(v))(1-r)}{2}\right\}\\ &= A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-q_2(v)(1-r)+\frac{1}{2}-r\right\}.\end{aligned}$$

(iii) If $\frac{1}{2} < \frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)} < \frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}$ (equivalently, $r \geq \frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}$), then

$$\begin{aligned}\pi(\lambda=0,v|r) &= A+B\left\{[q_1(v)r+q_2(v)(1-r)]\left(\frac{q_1(v)rq_2(\epsilon t)^{n-1}}{q_1(v)r+q_2(v)(1-r)}-\frac{1}{2}\right)\right.\\ &\quad\left.+[(1-q_1(v))r+(1-q_2(v))(1-r)]\left(\frac{(1-q_1(v))rq_2(\epsilon t)^{n-1}}{(1-q_1(v))r+(1-q_2(v))(1-r)}-\frac{1}{2}\right)\right\}\\ &= A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-\frac{q_1(v)r+q_2(v)(1-r)}{2}+(1-q_1(v))rq_2(\epsilon t)^{n-1}-\frac{(1-q_1(v))r+(1-q_2(v))(1-r)}{2}\right\}\\ &= A+B\left\{rq_2(\epsilon t)^{n-1}-\frac{1}{2}\right\}.\end{aligned}$$

Thus, after algebraic simplifications, we have

$$\pi^*(\lambda=0,v|r) = \begin{cases} A+B\left\{\frac{1}{2}-rq_2(\epsilon t)^{n-1}\right\}, & r<\frac{q_2(v)}{q_1(v)(2q_2(\epsilon t)^{n-1}-1)+q_2(v)},\\ A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-q_2(v)(1-r)+\frac{1}{2}-r\right\}, & r\in\left[\frac{q_2(v)}{q_1(v)+q_2(v)}, \frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}\right],\\ A+B\left\{rq_2(\epsilon t)^{n-1}-\frac{1}{2}\right\}, & r\geq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}\end{cases}$$

Using the same logic, we have

$$\pi^*(\lambda=C,v|r) = \begin{cases} A+B\left\{\frac{1}{2}-(1-r)q_2(\epsilon t)^{n-1}\right\}, & (1-r)<\frac{q_2(v)}{q_1(v)(2q_2(\epsilon t)^{n-1}-1)+q_2(v)},\\ A+B\left\{q_1(v)(1-r)q_2(\epsilon t)^{n-1}-q_2(v)r+\frac{1}{2}-(1-r)\right\}, & (1-r)\in\left[\frac{q_2(v)}{q_1(v)(2q_2(\epsilon t)^{n-1}-1)+q_2(v)}, \frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}\right],\\ A+B\left\{(1-r)q_2(\epsilon t)^{n-1}-\frac{1}{2}\right\}, & (1-r)\geq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)},\end{cases}$$

Observe that $\pi^*(\lambda=0,v|r)$ and $\pi^*(\lambda=C,v|r)$ are symmetric around $r=0.5$, and regardless of v, it is always optimal to launch $\lambda=0$ if $r\geq 0.5$ and $\lambda=C$ if $r\leq 0.5$. This results in

$$\pi^*(\lambda=0,v|r\geq 0.5) = \begin{cases} A+B\left\{q_1(v)rq_2(\epsilon t)^{n-1}-q_2(v)(1-r)+\frac{1}{2}-r\right\}, & r\leq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)},\\ A+B\left\{rq_2(\epsilon t)^{n-1}-\frac{1}{2}\right\}, & r\geq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}\end{cases}$$

$$\pi^*(\lambda=C,v|r< 0.5) = \begin{cases} A+B\left\{q_1(v)(1-r)q_2(\epsilon t)^{n-1}-q_2(v)r+\frac{1}{2}-(1-r)\right\}, & (1-r)\leq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)},\\ A+B\left\{(1-r)q_2(\epsilon t)^{n-1}-\frac{1}{2}\right\}, & (1-r)\geq\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)},\end{cases}$$

Since both expression involve optimizing $\rho q_1(v)-(1-\rho)q_2(v)$ with respect to v for $\rho>0.5$, by (C-3), $v^*=\epsilon t$. Furthermore, if $v^*=\epsilon t$, the $q_1(\epsilon t)=1$, so the inequalities become $\frac{1-q_2(v)}{(1-q_1(v))(2q_2(\epsilon t)^{n-1}-1)+1-q_2(v)}=1$. Hence, we have

$$\begin{aligned}\pi^*(\lambda^*(v),v^*=\epsilon t|r) &= \begin{cases}\left[D_1p-\frac{q_2(\epsilon t)}{2}(D_1-D_2)p\right]+q_2(\epsilon t)(D_1-D_2)p\left\{\frac{1}{2}-q_2(\epsilon t)^n(1-r)\right\}, & r\geq 0.5,\\ \left[D_1p-\frac{q_2(\epsilon t)}{2}(D_1-D_2)p\right]+q_2(\epsilon t)(D_1-D_2)p\left\{\frac{1}{2}-q_2(\epsilon t)^n r\right\}, & r\leq 0.5.\end{cases}\\ &= A_n+B_n|q_2(\epsilon t)^n r-0.5|.\end{aligned}$$

Furthermore, the expression becomes

$$\begin{aligned}\pi^*(\lambda^*(v),v^*=\epsilon t|r) &= \begin{cases}\left[D_1p-\frac{q_2(\epsilon t)}{2}(D_1-D_2)p\right]+q_2(\epsilon t)(D_1-D_2)p\left\{\frac{1}{2}-q_2(\epsilon t)(1-r)\right\}, & r\geq 0.5,\\ \left[D_1p-\frac{q_2(\epsilon t)}{2}(D_1-D_2)p\right]+q_2(\epsilon t)(D_1-D_2)p\left\{\frac{1}{2}-q_2(\epsilon t)r\right\}, & r\leq 0.5.\end{cases}\\ &= \begin{cases} rD_1p+(1-r)D_2p+(1-r)(1-[q_2(\epsilon t)]^n)(D_1-D_2)p, & r\geq 0.5,\\ (1-r)D_1p+rD_2p+r(1-[q_2(\epsilon t)]^n)(D_1-D_2)p, & r\leq 0.5.\end{cases}\end{aligned}$$

where

$$\beta_n^* = \begin{cases}(1-r)(1-[q_2(\epsilon t)]^n), & r\geq 0.5,\\ r(1-[q_2(\epsilon t)]^n), & r\leq 0.5.\end{cases} \qquad \square$$

Proof of Proposition 6. Suppose the entrepreneur chooses $\Lambda=0$. Then, if $W=0$, he will earn D_1. If $W=C$, he will earn D_2. If $W=-C$, he will also earn D_2 due to the fact that there are two overlaps. Thus, the expected profit is

$$\pi(\Lambda=0) = r_0D_1+r_CD_2+(1-r_0-r_C)D_2=r_0D_1+(1-r_0)D_2.$$

Suppose the entrepreneur chooses $\Lambda=C$. Then, if $W=C$, he will earn D_1. If $W=0$, he will earn D_2. However, if $W=-C$, he will earn 0 since there is no overlap between C and $-C$. Thus, the expected profit is

$$\pi(\Lambda=C) = r_CD_1+r_0D_2.$$

Similarly, suppose the entrepreneur chooses $\Lambda=-C$. Then, if $W=-C$, he will earn D_1. If $W=0$, he will earn D_2. However, if $W=C$, he will earn 0 since there is no overlap between C and $-C$. Thus, the expected profit is

$$\pi(\Lambda=-C) = (1-r_0-r_C)D_1+r_0D_2.$$

The optimal $\pi^*=\max\{\pi(\Lambda=0),\ \pi(\Lambda=C),\ \pi(\Lambda=-C)\}$. Moreover, $\Lambda^*=0$ if $\pi(\Lambda=0)>\pi(\Lambda=C)$ and $\pi(\Lambda=0)>\pi(\Lambda=-C)$; $\Lambda^*=C$ if $\pi(\Lambda=C)>\pi(\Lambda=0)$ and $\pi(\Lambda=C)>\pi(\Lambda=-C)$; and $\Lambda^*=-C$ if $\pi(\Lambda=-C)>\pi(\Lambda=0)$ and $\pi(\Lambda=-C)>\pi(\Lambda=C)$. Working out the inequalities we have our result. $\square$

Proof of Proposition 7. We let $v\equiv v-\rho$ to simplify notation. The Bayesian posterior beliefs are as follows:

$$\begin{aligned}
\bar{r}_0 &\triangleq P(W=0|\text{Sale of }(\lambda,v))=\frac{P(\text{Sale of }(\lambda,v)|W=0)\cdot r_0}{P(\text{Sale of }(\lambda,v))},\\
\bar{r}_C &\triangleq P(W=C|\text{Sale of }(\lambda,v))=\frac{P(\text{Sale of }(\lambda,v)|W=C)\cdot r_C}{P(\text{Sale of }(\lambda,v))},\\
\underline{r}_0 &\triangleq P(W=0|\text{No sale of }(\lambda,v))=\frac{(1-P(\text{No sale of }(\lambda,v)|W=0))\cdot r_0}{1-P(\text{No sale of }(\lambda,v))},\\
\underline{r}_C &\triangleq P(W=C|\text{No sale of }(\lambda,v))=\frac{(1-P(\text{No Sale of }(\lambda,v)|W=C))\cdot r_C}{1-P(\text{No sale of }(\lambda,v))},
\end{aligned}$$

where

$$\begin{aligned}
P(\text{Sale of }(\lambda,v)|W=0)&=\int_{-v/t+\lambda}^{v/t+\lambda}h(x|W=0)dx,\\
P(\text{Sale of }(\lambda,v)|W=C)&=\int_{-v/t+\lambda}^{v/t+\lambda}h(x|W=C)dx,\\
P(\text{Sale of }(\lambda,v))&=\sum_{w\in\{-C,0,C\}}r_wP(\text{Sale of }(\lambda,v)|W=w).
\end{aligned}$$

To simplify notations, we let

$$q_1(v)\triangleq\int_{-\frac{v}{t}+\lambda}^{\frac{v}{t}+\lambda}h(x|W=\lambda)dx,\qquad q_2(v)\triangleq\int_{-\frac{v}{t}+\lambda}^{\frac{v}{t}+\lambda}h(x|\,|W-\lambda|=C)dx,$$

so that $q_1(v)$ and $q_2(v)$ are probability of sales that the test product will resul in sales when launching the ideal design and adjacent-to-ideal design, respectively. (If $\lambda=C$ and $W=-C$, then the probability is zero.) The $q_1(v)$ and $q_2(v)$ is identical to those in (C-1). We will suppress $q_1\equiv q_1(v)$ and $q_2\equiv q_2(v)$.

Suppose $\lambda = 0$.

$$\begin{aligned}
\mathbb{E}\pi(v) &= P(sale)\pi(\bar{r_0}, \bar{r_C}) + (1 - P(sale))\pi(\underline{r_0}, \underline{r_C}) \\
&= P(sale)p\max\left\{\frac{q_1 r_0}{P(sale)}D_1 + (1 - \frac{q_1 r_0}{P(sale)}D_2,\ \frac{q_2 r_C}{P(sale)}D_1 + \frac{q_1 r_0}{P(sale)})D_2,\ (1 - \frac{q_1 r_0 + q_2 r_C}{P(sale)}D_1 + \frac{q_1 r_0}{P(sale)})D_2\right\} \\
&\quad + P(nosale)p\max\left\{\frac{(1-q_1)r_0}{P(nosale)}D_1 + (1 - \frac{(1-q_1)r_0}{P(nosale)}D_2,\ \frac{(1-q_2)r_C}{P(nosale)}D_1 + \frac{(1-q_1)r_0}{P(nosale)})D_2,\ (1 - \frac{(1-q_1)r_0 + (1-q_2)r_C}{P(nosale)}D_1 + \frac{(1-q_1)r_0}{P(nosale)})D_2\right\} \\
&= p\max\{q_1 r_0 D_1 + (1 - r_0)q_2 D_2,\ q_2 r_C D_1 + r_0 q_1 D_2,\ (1 - r_0 - r_C)q_2 D_1 + r_0 q_1 D_2\} \\
&\quad + p\max\{(1-q_1)r_0 D_1 + (1-r_0)(1-q_2)D_2,\ (1-q_2)r_C D_1 + r_0(1-q_1)D_2,\ (1-r_0-r_C)(1-q_2)D_1 + r_0(1-q_1)D_2\}
\end{aligned}$$

Suppose $\lambda = C$.

$$\begin{aligned}
\mathbb{E}\pi(v) &= P(sale)\pi(\bar{r_0}, \bar{r_C}) + (1 - P(sale))\pi(\underline{r_0}, \underline{r_C}) \\
&= P(sale)p\max\left\{\frac{q_2 r_0}{P(sale)}D_1 + (1 - \frac{q_2 r_0}{P(sale)}D_2,\ \frac{q_1 r_C}{P(sale)}D_1 + \frac{q_2 r_0}{P(sale)})D_2,\ (1 - \frac{q_2 r_0 + q_1 r_C}{P(sale)}D_1 + \frac{q_2 r_0}{P(sale)})D_2\right\} \\
&\quad + P(nosale)p\max\left\{\frac{(1-q_2)r_0}{P(nosale)}D_1 + (1 - \frac{(1-q_1)r_0}{P(nosale)}D_2,\ \frac{(1-q_1)r_C}{P(nosale)}D_1 + \frac{(1-q_2)r_0}{P(nosale)})D_2,\ (1 - \frac{(1-q_2)r_0 + (1-q_1)r_C}{P(nosale)}D_1 + \frac{(1-q_2)r_0}{P(nosale)})D_2\right\} \\
&= p\max\{q_2 r_0 D_1 + r_C q_1 D_2,\ q_1 r_C D_1 + r_0 q_2 D_2,\ 0 + r_0 q_2 D_2\} \\
&\quad + p\max\{(1-q_2)r_0 D_1 + [(1-r_0) - r_C q_1]D_2,\ (1-q_1)r_C D_1 + r_0(1-q_2)D_2,\ (1-r_0-r_C)D_1 + r_0(1-q_2)D_2\}.
\end{aligned}$$

Suppose $\lambda = -C$.

$$\begin{aligned}
\mathbb{E}\pi(v) &= P(sale)\pi(\bar{r_0}, \bar{r_C}) + (1 - P(sale))\pi(\underline{r_0}, \underline{r_C}) \\
&= P(sale)p\max\left\{\frac{q_2 r_0}{P(sale)}D_1 + (1 - \frac{q_2 r_0}{P(sale)}D_2,\ 0 + \frac{q_2 r_0}{P(sale)})D_2,\ (1 - \frac{q_2 r_0}{P(sale)})D_1 + \frac{q_2 r_0}{P(sale)})D_2\right\} \\
&\quad + P(nosale)p\max\left\{\frac{(1-q_2)r_0}{P(nosale)}D_1 + (1 - \frac{(1-q_1)r_0}{P(nosale)}D_2,\ \frac{(r_C}{P(nosale)}D_1 + \frac{(1-q_2)r_0}{P(nosale)})D_2,\ (1 - \frac{(1-q_2)r_0 + r_C}{P(nosale)}D_1 + \frac{(1-q_2)r_0}{P(nosale)})D_2\right\} \\
&= p\max\{q_2 r_0 D_1 + (1 - r_0 - r_C)q_1 D_2,\ 0 + r_0 q_2 D_2,\ q_1(1 - r_0 - r_C)D_1 + r_0 q_2 D_2\} \\
&\quad + p\max\{(1-q_2)r_0 D_1 + [1 - r_0 - (1 - r_0 - r_C)q_1]D_2,\ r_C D_1 + r_0(1-q_2)D_2,\ (1-r_0-r_C)(1-q_1)D_1 + r_0(1-q_2)D_2\}.
\end{aligned}$$

By Lemma C.1, it suffices to check $v = \epsilon t$ and $v = (C - \epsilon)t$. If $v = \epsilon t$, then $q_1 = 1$ and $q_2 < 1$. If $v = (C - \epsilon)t$, then $q_1 < 1$ and $q_2 = 0$. Comparing the values, we find that $v^* = \epsilon t$, thus for simplicity, we will focus on analyzing this case without writing down the expression for $v = (C - \epsilon)t$.

(i) Suppose $r_0 = 0$. Then,

$$\begin{aligned}
\pi(\lambda = 0) &= p\max\{q_2 D_2,\ q_2 r_C D_1,\ (1 - r_C)q_2 D_1\} + p\max\{(1 - q_2)D_2,\ (1 - q_2)r_C D_1,\ (1 - r_C)(1 - q_2)D_1\}, \\
\pi(\lambda = C) &= p\max\{rCq_1 D_2,\ q_1 r_C D_1,\ 0\} + p\max\{(1 - r_C q_1)D_2,\ (1 - q_1)r_C D_1,\ (1 - r_C)D_1\}, \\
\pi(\lambda = -C) &= p\max\{(1 - rC)q_1 D_2,\ 0,\ q_1(1 - r_C)D_1\} + p\max\{(1 - (1 - r_C)q_1)D_2,\ r_C D_1,\ (1 - r_C)(1 - q_1)D_1\}.
\end{aligned}$$

When $v = \epsilon t$,

$$\begin{aligned}
\pi(\lambda = 0) &= p\max\{r_C D_1, (1 - r_C)D_1\}, \\
\pi(\lambda = C) &= pD_1, \\
\pi(\lambda = -C) &= pD_1.
\end{aligned}$$

Hence, $\lambda^* = \pm C$, $v^* = \epsilon t$.

(ii) Suppose $r_C = 0$ (or $1 - r_0 - r_C = 0$. For simplicity, we will omit this case). Then,

$$\begin{aligned}
\pi(\lambda = 0) &= p\max\{q_1 r_0 D_1 + (1 - r_0)q_2 D_2,\ r_0 q_1 D_2,\ (1 - r_0)q_2 D_1 + r_0 q_1 D_2\} \\
&\quad + p\max\{(1 - q_1)r_0 D_1 + (1 - q_2)(1 - r_0)D_2,\ (1 - q_1)r_0 D_2,\ (1 - q_2)(1 - r_0)D_1 + r_0(1 - q_1)D_2\}, \\
\pi(\lambda = C) &= p\max\{q_2 r_0 D_1,\ q_2 r_0 D_2,\ q_2 r_0 D_2\} \\
&\quad + p\max\{(1 - q_2)r_0 D_1 + (1 - r_0)D_2,\ (1 - q_2)r_0 D_2,\ (1 - r_0)D_1 + (1 - q_2)r_0 D_2\}, \\
\pi(\lambda = -C) &= p\max\{q_2 r_0 D_1 + (1 - r_0)q_1 D_2,\ r_0 q_2 D_2,\ q_1(1 - r_0)D_1 + r_0 q_2 D_2\} \\
&\quad + p\max\{(1 - q_2)r_0 D_1 + (1 - r_0)(1 - q_1)D_2,\ (1 - q_2)r_0 D_2,\ (1 - r_0)(1 - q_1)D_1 + (1 - q_2)r_0 D_2\}.
\end{aligned}$$

When $v = \epsilon t$,

$$\begin{aligned}
\pi(\lambda=0) &= p[r_0 D_1 + (1-r_0)[q_2 D_1 + (1-q_2)D1],\\
\pi(\lambda=C) &= p[q_2 r_0 D_1 + (1-q_2)(1-r_0)D_1],\\
\pi(\lambda=-C) &= p[(1-r_0)D_1 + r_0[q_2 D_1 + (1-q2)D1].
\end{aligned}$$

Hence, $\lambda^* = C$ is ruled out. and $\lambda^* = 0$ if $r_0 > 0.5$ and $\lambda^* = -C$ otherwise.

(iii) Finally, suppose $r_0 = 1/3$. Let $r_C = 1/3$.

$$\begin{aligned}
\pi(\lambda=0) &= p\max\{\frac{1}{3}q_1 D_1 + \frac{2}{3}q_2 D_2,\ \frac{1}{3}q_2 + \frac{1}{3}q_1 D_2,\ \frac{1}{3}q_2 D_1 + \frac{1}{3}q_1 D_2\}\\
&\quad + p\max\{\frac{1}{3}(1-q_1)D_1 + \frac{2}{3}(1-q_2)D_2,\ \frac{1}{3}(1-q_2)D_1 + \frac{1}{3}(1-q_1)D_2,\ \frac{1}{3}(1-q_2)D_1 + \frac{1}{3}(1-q_1)D_2\},\\
\pi(\lambda=C) &= p\max\{\frac{1}{3}q_2 D_1 + \frac{1}{3}q_1 D_2,\ \frac{1}{3}q_1 D_1 + \frac{1}{3}q_2 D_2,\ \frac{1}{3}q_2 D_2\}\\
&\quad + p\max\{\frac{1}{3}(1-q_2)D_1 + (\frac{2}{3} - \frac{1}{3}q_1)D_2,\ \frac{1}{3}(1-q_1)D_1 + \frac{1}{3}(1-q_2)D_2,\ \frac{1}{3}D_1 + \frac{1}{3}(1-q_2)D_2\},\\
\pi(\lambda=-C) &= p\max\{\frac{1}{3}q_2 D_1 + \frac{1}{3}q_1 D_2,\ \frac{1}{3}q_2 D_2,\ \frac{1}{3}q_1 D_1 + \frac{1}{3}q_2 D_2\}\\
&\quad + p\max\{\frac{1}{3}(1-q_2)D_1 + (\frac{2}{3} - \frac{1}{3}q_1)D_2,\ \frac{1}{3}D_1 + \frac{1}{3}(1-q_2)D_2,\ \frac{1}{3}(1-q_1)D_1 + \frac{1}{3}(1-q_2)D_2\}.
\end{aligned}$$

Observe that $\pi(\lambda = C) = \pi(\lambda = -C)$. Also, when $v^* = \epsilon t$, then $q_1 = 1$ and we have

$$\begin{aligned}
\pi(\lambda=0) &= p(\frac{2}{3}D_1 + \frac{2}{3}q_2 D_2 - \frac{1}{3}q_2 D_1),\\
\pi(\lambda=C) = \pi(\lambda=-C) &= p(\frac{2}{3}D_1 + \frac{1}{3}D_2).
\end{aligned}$$

Since $q_2 < 0.5$, $\lambda = 0$ is ruled out as optimal. Thus, if $r_C > 1/3$, then $\lambda^* = C$, otherwise $\lambda^* = -C$. $\square$