Appendix B: Technical Appendix for "Influencing and Influencers"

B.1. The independent influencer never withholds a signal.

Claim B.1. *The information value from the independent influencer's review, conditional on s, is always non-negative: $VOI(0,0)|g \geq 0$ and $VOI(0,0)|b \geq 0$. Since $VOI(0,0) = 0$ if there is no review posted, this implies that the independent influencer who receives s always posts a review.*

Proof. The independent influencer posts $m = \mathcal{G}$ if she receives $s = g$ and $m = \mathcal{B}$ if she receives $s = b$. We use the results in Lemma 1 and Lemma 2 with $a = 0$ in this proof.

1. $\rho_0 \in [0, \underline{\rho}(0))$. If the influencer posts a review, $VOI(0,0)|g = VOI(0,0)|b = 0$ since the consumer does not buy in this region.

2. $\rho_0 \in [\underline{\rho}(0), k)$. Suppose that the influencer receives $s = g$. The consumer's utility with no review is 0, since he does not buy based on his prior. If the influencer posts $m = \mathcal{G}$, the consumer buys, and his expected utility is $P(G|\mathcal{G})v - p > 0$. Therefore, $VOI(0,0)|g = P(G|\mathcal{G})v - p > 0$. Suppose the influencer receives $s = b$. If she posts $m = \mathcal{B}$, the consumer does not buy the product, and his expected utility is 0. Therefore, $VOI(0,0)|b = 0$.

3. $\rho_0 \in [k, \bar{\rho})$. Suppose that the influencer receives $s = g$. Conditional on this information, the consumer's expected utility (from the influencer's perspective) with no review is $\lambda\Big[P(G|g)v - p\Big] > 0$ since he buys the product based on his prior alone. If the influencer does post $m = \mathcal{G}$, the consumer's utility is $P(G|\mathcal{G})v - p > 0$. Therefore, $VOI(0,0)|g = \Big[P(G|\mathcal{G})v - p\Big] - \lambda\Big[P(G|g)v - p\Big] > 0$. However, since we assume that the influencer posts $\mathcal{G}$ if she receives $s = g$, $P(G|g) = P(G|\mathcal{G})$. Hence, $VOI(0,0)|g = (1 - \lambda)\Big[P(G|\mathcal{G})v - p\Big] > 0$.
Suppose that the influencer receives $s = b$. Conditional on this information, the consumer's expected utility (from the influencer's perspective) with no review is $\lambda[P(G|\mathcal{B})v - p] < 0$. If the influencer does post $m = \mathcal{B}$, the consumer would not buy the product and have a utility of 0. Therefore, $VOI(0,0)|b = 0 - \lambda[P(G|\mathcal{B})v - p] = \lambda[p - P(G|\mathcal{B})v] > 0$.

4. $\rho_0 \in [\bar{\rho}, 1]$. The consumer always buys regardless of m. Suppose $s = g$. Conditional on this information, the consumer's expected utility (from the influencer's perspective) with no review is $\lambda\Big[P(G|g)v - p\Big] \equiv \lambda\Big[P(G|\mathcal{G})v - p\Big] > 0$. If the influencer posts $m = \mathcal{G}$, the consumer's expected utility is $P(G|\mathcal{G})v - p$. Therefore, $VOI(0,0)|g = (1 - \lambda)\Big[P(G|\mathcal{G})v - p\Big] > 0$. Similarly, suppose $s = b$. The consumer's expected utility (from the influencer's perspective) with no review is $\lambda\Big[P(G|b)v - p\Big] > 0$. If the influencer posts $m = \mathcal{B}$, the consumer would still buy the product, and his expected utility is $P(G|\mathcal{B})v - p$. We assume that the independent influencer posts $m = \mathcal{B}$ if she receives $s = b$, $P(G|b) = P(G|\mathcal{B})$. Therefore, $VOI(0,0)|b = (1 - \lambda)\Big[P(G|\mathcal{B})v - p\Big] > 0$.

□

B.2. Allowing awareness-only posts

Here we expand the strategy space of the influencer by allowing "awareness-only" posts. These posts are not a binary signal, but are just messages that inform the consumer of the existence of the product. We assume that the influencer bears a cost of 0 of posting an awareness-only post. (We can also assume a non-zero but small cost which complicates the analysis slightly). We refer to the awareness-only post as m_a.

Based on Lemma 2, the VOI of the m_a message in regions 3 and 4 is $VOI(m_a) = (1-\lambda)\Big[\rho_0 v - p\Big]$. We can use Lemmas 1-3 to derive the following. (The VOI of m_a is zero in regions 1 and 2. Hence, the independent influencer does not post awareness-only messages in regions 1 and 2).

1. If $\rho_0 \in [k, \bar{\rho})$, the independent influencer posts a review rather than an awareness message if $\alpha\Big(P(\mathcal{G})\Big[P(G|\mathcal{G})v - p\Big] - \Big[\rho_0 v - p\Big]\Big) \geq c$. Otherwise, the influencer posts an awareness message.
2. If $\rho_0 \in [\bar{\rho}, 1]$, the independent influencer posts an awareness-only message.

The modified Proposition 1 is the following:

Proposition B.1. *The following describes the unique PBNE equilibrium of the three-player game.*

1. *Suppose* $\rho_0 \in [0, \underline{\rho}(0))$*, there is no affiliation, no independent review and no purchase.*
2. *Suppose* $\rho_0 \in [\underline{\rho}(0), k)$*,*
   (a) *If* $c \leq \bar{C}_1$*, the firm pays* $S(\bar{a}) > 0$ *to affiliate at* $a^* = \bar{a}$*,* $0 < \bar{a} < 1$*,* $P(G|\mathcal{B}, \bar{a}) < P(G|\mathcal{G}, \bar{a}) = k$*. Here the consumer buys if* $\mathcal{G}$ *and does not buy if* $\mathcal{B}$*. Finally,* $VOI(1, \bar{a}) = 0$*.*
   (b) *Otherwise, there is no affiliation, no independent review, and no purchase.*
3. *Suppose* $\rho_0 \in [k, \bar{\rho})$*,*
   (a) *If* $c \leq \Big(P(\mathcal{G})\Big[P(G|\mathcal{G})v - p\Big] - \Big[\rho_0 v - p\Big]\Big)$*,* $a^* = 1$*,* $S(1) > 0$*,* $P(G|\mathcal{B}, 1) < k \leq P(G|\mathcal{G}, 1) = \rho_0$*. Here the consumer buys if* $\mathcal{G}$ *and does not buy if* $\mathcal{B}$*.* $VOI(1,1) = (1-\lambda)(\rho_0 v - p)$*.*
   (b) *Otherwise, there is no affiliation, and the independent influencer posts an awareness-only message.* $VOI(0,0) = (1-\lambda)(\rho_0 v - p)$*.*
4. *Suppose* $\rho_0 \in [\bar{\rho}, 1]$*, There is no affiliation, and the independent influencer posts an awareness-only message.* $VOI(0,0) = (1-\lambda)(\rho_0 v - p)$

Note that if we allow (costless) awareness-only messages, awareness no longer plays a role in the decision to affiliate. Of course, if we allow a cost to be associated with awareness-only messages, then the firm in some cases may want to affiliate to achieve awareness. Otherwise, the results are fairly similar to what we obtain in Proposition 1 in the main body of the paper. The two big differences is that in region 3 there is an awareness-only message that is posted by the independent influencer if the cost of reviewing the product, c, is too high. Also, there is no longer any affiliation in region 4.

B.3. Proof of Proposition 2

For Proposition 2 and Proposition 3, we make the following assumption on the out-of-equilibiurm beliefs:

Assumption B.1. *If the consumer expects no affiliation in equilibrium but observes affiliation or the consumer expects full affiliation,* $\tilde{a} = 1$*, but observes* $m = \mathcal{B}$*, then the consumer updates his belief using* $\tilde{a} = 1 - \epsilon$*.*

To prove the Proposition, we need to re-do all the Lemmas from Section 3.1 to Section 3.4 in the new set-up where the consumer does not directly observe a.

Lemma B.1. *Consider the aware consumer's posterior beliefs and actions at* $t = 5$*, given* $\psi' = (m, a)$*. Let* $\underline{\rho}(\tilde{a}) \equiv \dfrac{k(1-\gamma(1-\tilde{a}))}{(1-k)(\gamma + \tilde{a}(1-\gamma)) + k(1-\gamma(1-\tilde{a}))}$ *and* $\bar{\rho} \equiv \dfrac{k\gamma}{(1-k)(1-\gamma) + k\gamma}$*.*

1. *If* $\rho_0 \in [0, \underline{\rho}(\tilde{a}))$*, the consumer never buys the product.*

2. *If* $\rho_0 \in [\underline{\rho}(\tilde{a}), k)$*, the consumer only buys the product if he sees a positive review,* $m = \mathcal{G}$.

3. *If* $\rho_0 \in [k, \bar{\rho})$*, the consumer buys the product if he sees a positive review or no review,* $m = \{\mathcal{G}, \emptyset\}$*, but he does not buy if he sees a negative review,* $m = \mathcal{B}$.

4. *If* $\rho_0 \in [\bar{\rho}, 1]$*, the consumer always buy the product.*

Proof. The proof is identical to the proof of Lemma 1 in the main text, where we substitute $\tilde{a}$ for a since the consumer does not observe a. □

Lemma B.2. *The value of information of the influencer's review from the perspective of the influencer is as follows,*

1. *If* $\rho_0 \in [0, \underline{\rho}(\tilde{a}))$, $VOI(\theta, a) = 0$
2. *If* $\rho_0 \in [\underline{\rho}(\tilde{a}), k)$, $VOI(\theta, a) = P(\mathcal{G}|a)\Big[P(G|\mathcal{G}, a)v - p\Big] \geq 0$ *and* $\dfrac{\partial VOI(1,a)}{\partial a} < 0$.
3. *If* $\rho_0 \in [k, \bar{\rho})$, $VOI(\theta, a) = P(\mathcal{G}|a)\Big[P(G|\mathcal{G}, a)v - p\Big] - \lambda\Big[P(G)v - p\Big] > 0$ *and* $\dfrac{\partial VOI(1,a)}{\partial a} < 0$.
4. *If* $\rho_0 \in [\bar{\rho}, 1]$, $VOI(\theta, a) = (1-\lambda)\Big[P(G)v - p\Big] > 0$

Proof. The consumer's decision rule on whether to buy or not to buy following m is a function of $\tilde{a}$. That is why the regions are a function of $\tilde{a}$. (See Lemma 1 in the Technical Appendix). However, the expected utility of the consumer is calculated here from the perspective of the influencer who observes the true a. That is why a is being used in the calculation. We could also calculate the VOI from the perspective of the consumer, and in this case we would use $\tilde{a}$. However, since the contract is between the firm and the influencer, the relevant VOI is from the perspective of the influencer. The rest of the proof is identical to the proof of Lemma 2 in the main body of the text. □

Lemma B.3. *The independent influencer's expected utility is* $\alpha VOI(0,0)$ *since* $c = 0$*. She always acquires information at* $t = 3$ *and posts a review at* $t = 4$ *(which we simplify to "posts a review") since* $\alpha VOI(0,0) \geq 0$.

We now redo the proof of Proposition 1 with a few changes accounting for partial observability of a. The firm's problem is,

$$\begin{aligned} \Pi^1(a^*) = \max_{a, S(a)} \quad & P(\mathcal{G}|a) \cdot \mathbb{1}\{P(G|\mathcal{G}, \tilde{a}) \geq k\} \cdot p + P(\mathcal{B}|a) \cdot \mathbb{1}\{P(G|\mathcal{B}, \tilde{a}) \geq k\} \cdot p - S(a) \\ \text{s.t.} \quad & S(a) + \alpha VOI(1,a) - \alpha VOI(0,0) \geq 0 \end{aligned} \tag{B.1}$$

Note that, as we argued in Lemma 2, VOI is a function of a since the influencer observes a. The consumer inference in the objective function is a function of $\tilde{a}$. The posterior beliefs are defined as follows,

$$P(G \mid \mathcal{G}, \tilde{a}) = \frac{[\gamma + \tilde{a}(1-\gamma)]\rho_0}{[\gamma + \tilde{a}(1-\gamma)]\rho_0 + [(1-\gamma) + \tilde{a}\gamma](1-\rho_0)} \tag{B.2}$$

$$P(G \mid \mathcal{B}, \tilde{a}) = \frac{(1-\gamma)\rho_0}{(1-\gamma)\rho_0 + \gamma(1-\rho_0)} \tag{B.3}$$

The firm's affiliation decision depends on the relative magnitudes of Π^0 and $\Pi^1(a^*)$,

$$\theta^* = \begin{cases} 1, & \text{if } \Pi^1(a^*) \geq \Pi^0 \\ 0, & \textit{otherwise} \end{cases}$$

We use Lemma 1 with $a = 0$ to determine the consumer behavior in each region. Similarly, we use Lemma 2 to obtain the value of information and Lemma 3 to determine the independent influencer's decision to post.

Case 1 $\rho_0 \in [0, \underline{\rho}(0))$. Here the consumer's prior is so low such that no message will convince him to buy. The firm is not able to make any profit and is not willing to pay for affiliation. Based on Lemma 3, the independent influencer does not post a review.

Case 2 $\rho_0 \in [\underline{\rho}(0), k)$. Let's first consider a potential equilibrium with affiliation, $\theta^* = 1$. What will be the equilibrium level of affiliation, a^*? In this region, the consumer buys only if he observes $\mathcal{G}$. As the firm increases a, it increases $P(\mathcal{G}|a)$, but $P(G|\mathcal{G}, \tilde{a})$ does not change since a is not observed by the consumer. The consumer buys the product if $P(G|\mathcal{G}, \tilde{a}) \geq k$. Hence, the firm's maximization problem becomes,

$$\begin{aligned} \max_{a} \quad & \Pi^1 = \mathbb{1}\{\ P(G|\mathcal{G}, \tilde{a}) \geq k\} P(\mathcal{G}|a)p - S(a) \\ \text{s.t.} \quad & S(a) + \alpha VOI(1,a) - \alpha VOI(0,0) \geq 0 \qquad (1) \\ & 0 \leq a \leq 1 \qquad (2) \end{aligned}$$

The consumer buys the product as long as

$$\tilde{a} \leq \frac{\gamma(1-k)\rho_0 - k(1-\rho_0)(1-\gamma)}{\gamma k(1-\rho_0) - (1-\gamma)(1-k)\rho_0} \equiv \bar{a}$$

We have shown before that $0 \leq \bar{a} < 1$.

We can re-write the firm's problem as follows:

$$\begin{aligned} \max_{a} \quad & \Pi^1 = P(\mathcal{G}|a)p - S(a) \\ \text{s.t.} \quad & S(a) + \alpha VOI(1,a) - \alpha VOI(0,0) \geq 0 \qquad (1) \\ & \tilde{a} \leq \bar{a} \qquad (2) \\ & a \geq 0 \qquad (3) \end{aligned}$$

Note that this is identical to the problem that we had under full full commitment, except for constraint (2). But again, an increase in a does not change $\tilde{a}$. As before, constraint (1) is binding, and the expression for $S(a)$ can be substituted back into the objective function. We can show that

$$\Pi^1 = (1-\alpha)P(\mathcal{G}|a)p + \alpha P(G)P(\mathcal{G}|G, a)v - \alpha VOI(0,0)$$

In the expression above, only the first two terms are functions of a. Both terms are increasing functions of a. Hence, $a^* = 1$ — the firm fully affiliates with the influencer. That is, if we had a candidate equilibrium with affiliation ($\theta^* = 1$), the consumer would infer that $\tilde{a} = 1$. But this results in a contradiction since for this equilibrium to hold, it must be the case that $\tilde{a} \leq \bar{a} < 1$. Hence, there cannot exist an equilibrium with affiliation in this region.

Let's next consider an equilibrium with no affiliation, $\theta^* = 0$. Here the firm makes $\Pi^0 = P(\mathcal{G})p$ since the independent influencer posts a review. If the firm deviates to $\theta = 1$, our assumption on the out-of-equilibrium beliefs implies that $\tilde{a} = 1 - \epsilon$. Hence, there would be no purchase since $\tilde{a} > \bar{a}$. This implies that the the firm has no incentive to deviate.

Case 3 $\rho_0 \in [k, \bar{\rho})$. Here the consumer buys if he sees no review or a positive review, but he does not buy if he sees a negative review. Let's consider a candidate equilibrium with affiliation, $\theta^* = 1$. The firm's affiliation problem is,

$$\begin{aligned} \max_a \quad & \Pi^1 = Pr(\mathcal{G}|a)p - S(a) & \\ \text{s.t.} \quad & S(a) + \alpha VOI(1,a) - \alpha VOI(0,0) \geq 0 & (1) \\ & \tilde{a} \leq 1 & (2) \\ & a \geq 0 & (3) \end{aligned}$$

Again, we can show that $a^* = 1$ and hence $\tilde{a} = 1$. Note that at $\tilde{a} = 1$, the signal has no information above and beyond awareness. That is, the signal does not change the consumer's prior belief, ρ_0. This also implies that $VOI(1,1) = (1-\lambda)\Big[P(G)v - p\Big]$. The firm has no incentive to deviate to $\theta = 0$ since that yields strictly less profit for the firm.

Next, let's consider another candidate equilibrium where $\theta^* = 0$. We can show that this cannot be an equilibrium since the firm would make strictly more profit by deviating to $\theta^* = 1$, where by assumption $\tilde{a} = 1 - \epsilon$.

Case 4 $\rho_0 \in [\bar{\rho}, 1]$. In this region the consumer always buys regardless the valence of review. Hence, affiliation is only important for the purpose of awareness. The rest of the proof is almost identical to the proof of Proposition 1, case 4. The only difference is in the out-of-equilibrium beliefs, but the results remain the same.

B.4. Proof of Proposition 3

The proof replicates most of the steps of the Proof of Proposition 2. Here we highlight the differences:

Case 2 Here the consumer buys following a positive review. Let's consider a candidate equilibrium with affiliation, $\theta^* = 1$. We follow the steps in the Proof of Proposition 2 to conclude that it must be the case that $a^* = \tilde{a} = 1$. But this implies that the consumer would not choose to buy following a positive review, which is a contradiction.

Next, consider a candidate equilibrium with no affiliation, $\theta^* = 0$. Since no disclosure is possible, the influencer could deviate to affiliation without any detection by the consumer. Again, using the same steps as in case 2 in the Proof of Proposition 2, we can show that the optimal deviation would be $\theta = 1$ and $a = 1$. This leads to a contradiction: the consumer's belief in equilibrium is not the same as the influencer's action in equilibrium. Hence, no pure strategy equilibrium exists.

Case 3 We can show that the unique equilibrium is one where $\theta^* = 1$ and $a^* = 1$. By assumption, the influencer is not able to disclose anything about the affiliation, but the consumer infers $\tilde{\theta} = 1$ and $\tilde{a} = 1$.

B.5. Voluntary Disclosure

We make disclosure optional: the influencer can choose to reveal more or less information about her affiliation. There are three revelation levels, $\sigma = \{\emptyset, \theta, f(a)\}$. Here $\emptyset$ – no disclosure about affiliation, θ – the influencer discloses whether she is affiliated, $\theta = 1$, or is independent, $\theta = 0$, and $f(a)$ – full disclosure. While we don't require any level of disclosure, we do require that the disclosure is truthful. That is, the influencer who is affiliated with the firm cannot falsely reveal that $\theta = 0$. We also assume that the firm specifies the level of disclosure, σ, in its contract with the influencer.

The consumer makes an inference on the affiliation level given the disclosed level of affiliation, σ, and the review m. We denote the consumer's inference on the affiliation level as $\tilde{a}(\sigma,m)$. The belief on the product fit is $P(G|m,\tilde{a}(\sigma,m))$. The firm also conditions the payment to the influencer on the disclosure level as well as the affiliation level: $S(a,\sigma)$, and $VOI(\theta,a,\sigma)$.

Consider region 2 where the consumer buys only after observing $\{\mathcal{G},\sigma^*\}$. The affiliated firm's problem becomes,

$$\begin{aligned}\Pi^1(a^*) = \max_{a,S(a,\sigma),\sigma} \quad & P(\mathcal{G}|a)p - S(a,\sigma) \\ \text{s.t.} \quad & S(a,\sigma) + \alpha VOI(1,a,\sigma) - VOI(0,0,f(a)) \geq 0 \quad (1) \\ & 0 \leq \tilde{a}(\sigma,m) \leq \bar{a} \quad (2)\end{aligned} \tag{B.4}$$

We also need to specify out-of-equilibrium consumer beliefs following deviation in disclosure. We assume that, since dislosure claims are assumed to be credible, the consumer updates his belief using the disclosed information structure.

We can show that the independent influencer can not do better than fully disclosing $f(a)$, or, equivalently, disclose $\{\theta = 0, a = 0\}$, which is reflected in the VOI for the independent influencer.

Next, consider the candidate equilibrium $\{\sigma^* = f(a), a = \bar{a}, S = S(\bar{a})\}$. The amount of affiliation and the payment is the same as in the full commitment case, and we showed in Proposition 1 that this yields the highest profit to the firm under full commitment. We can show that no player wants to deviate, and that, in particular, deviations by the firm to more coarse disclosure will yield 1) lower profit in the case of $\sigma = \theta$ and 2) zero profit in the case of $\sigma = \emptyset$. We can also show that the firm would deviate from other candidate equilibria (i.e. partial or no disclosure) to $\{\sigma^* = f(a), a = \bar{a}, S = S(\bar{a})\}$. Hence, full disclosure is the unique equilibrium here.

B.6. Proof of Proposition 4

The analysis for Regions 1 and 4 is the same as in Proposition 1.

- Region 2: Here the independent influencer always posts a review.

Conditional on affiliation with the influencer, the firm's optimal affiliation problem is

$$\begin{aligned}\max_{a,S_c} \quad & P(\mathcal{G}|a)p - P(\mathcal{G}|a)S_c \\ \text{s.t.} \quad & P(\mathcal{G}|a)S_c + \alpha VOI(1,a) \geq \alpha VOI(0,0) \quad (IR) \\ & 0 \leq a \leq \bar{a} \quad (A) \\ & S_c + \alpha\left[P(G|g)v - p\right] \geq 0 \quad (IC_g) \\ & S_c + \alpha\left[P(G|b)v - p\right] = 0 \quad (IC_b) \\ & S_c \geq 0 \quad (LL)\end{aligned} \tag{B.5}$$

We solve the constrained maximization. First, let's turn to the optimal compensation contract. Since $P(G|b)v - p < 0$, (IC_b) implies that $S_c > 0$. In addition, since $P(G|g)v - p \geq 0$ and $S_c > 0$, (IC_g) must be slack when (IC_b) holds. From (IC_b), we have $S_c = \alpha[p - P(G|b)v] = \alpha[p - P(G|\mathcal{B},a)v]$ since $P(G|b) = P(G|\mathcal{B}) = P(G|\mathcal{B},a)$.

Let's consider the following candidate solution: $S_c = \alpha[p - P(G|\mathcal{B},a)v]$. This is the *minimal* compensation that satisfies the IC constraints and the limited liability constraint. If this compensation also satisfies the

(IR) constraint, then this is the optimal compensation from the point of view of the firm. We next substitute the candidate solution into (IR) to ensure that it is satisfied:

$$P(\mathcal{G}|a)S_c + \alpha VOI(1,a) - \alpha VOI(0,0) \geq 0 \tag{B.6}$$

$$= \alpha P(\mathcal{G}|a)[p - P(G|\mathcal{B})v] + \alpha P(\mathcal{G}|a)[P(G|\mathcal{G},a)v - p] - \alpha P(\mathcal{G})[P(G|\mathcal{G})v - p] \geq 0 \tag{B.7}$$

$$= P(\mathcal{G}|a)P(G|\mathcal{G},a)v - P(\mathcal{G}|a)P(G|\mathcal{B},a)v - P(\mathcal{G})P(G|\mathcal{G})v + P(\mathcal{G})p \geq 0 \tag{B.8}$$

$$= P(\mathcal{G}|a)P(G|\mathcal{G},a)v - (1 - P(\mathcal{B}|a))P(G|\mathcal{B},a)v - P(\mathcal{G})P(G|\mathcal{G})v + P(\mathcal{G})p \geq 0 \tag{B.9}$$

$$= P(\mathcal{G}|a)P(G|\mathcal{G},a)v + P(\mathcal{B}|a)P(G|\mathcal{B},a)v - P(G|\mathcal{B},a)v - P(\mathcal{G})P(G|\mathcal{G})v + P(\mathcal{G})p \geq 0 \tag{B.10}$$

By the Martingale property, beliefs do not change in expectation, which implies that

$$P(\mathcal{G}|a)P(G|\mathcal{G},a) + P(\mathcal{B}|a)P(G|\mathcal{B},a) = P(G) \tag{B.11}$$

Using Bayes' Theorem, $P(G)P(\mathcal{B}|G) = P(\mathcal{B})P(G|\mathcal{B})$.
Making these substitutions,

$$= P(G)v - P(G|\mathcal{B},a)v - P(G)P(\mathcal{G}|G)v + P(\mathcal{G})p \geq 0 \tag{B.12}$$

$$= P(G)[1 - P(\mathcal{G}|G)]v - P(G|\mathcal{B})v + P(\mathcal{G})p \geq 0 \tag{B.13}$$

$$= P(G)P(\mathcal{B}|G)v - P(G|\mathcal{B})v + P(\mathcal{G})p \geq 0 \tag{B.14}$$

$$= P(\mathcal{B})P(G|\mathcal{B})v - P(G|\mathcal{B})v + P(\mathcal{G})p \geq 0 \tag{B.15}$$

$$= P(G|\mathcal{B})[P(\mathcal{B}) - 1]v + P(\mathcal{G})p \geq 0 \tag{B.16}$$

$$= -P(G|\mathcal{B})P(\mathcal{G})v + P(\mathcal{G})p \geq 0 \tag{B.17}$$

$$= P(\mathcal{G})[p - P(G|\mathcal{B})v] > 0 \tag{B.18}$$

Hence (IR) is slack. Next we substitute $S_c = \alpha[p - P(G|\mathcal{B},a)v] \equiv \alpha[p - P(G|b)v]$ into the firm's objective function in order to obtain the optimal affiliation level:

$$\begin{aligned} &\max_a \quad P(\mathcal{G}|a)(p - S_c) \\ &\max_a \quad P(\mathcal{G}|a)[p(1-\alpha) + \alpha P(G|b)v] \end{aligned} \tag{B.19}$$

Only the term $P(\mathcal{G}|a)$ is a function of a and also increasing in a. Since $[p(1-\alpha) + \alpha P(G|b)v] > 0$, the objective function is also increasing in a. Therefore the (A) is binding, and $a^* = \bar{a}$.
Lastly, the firm prefers affiliation to no affiliation if

$$\Pi^0 \leq \Pi^1 \tag{B.20}$$

$$P(\mathcal{G})p \leq P(\mathcal{G}|\bar{a})[p - \alpha(p - P(G|b)v)] \tag{B.21}$$

$$\alpha \leq \frac{[P(\mathcal{G}|\bar{a}) - P(\mathcal{G})]p}{P(\mathcal{G}|\bar{a})[p - P(G|b)v]} \tag{B.22}$$

Both the numerator and the denominator of this fraction are positive. If $\gamma = 1$, this reduces to

$$\alpha \leq \frac{P(\mathcal{G}|\bar{a}) - P(\mathcal{G})}{P(\mathcal{G}|\bar{a})} \leq 1 \tag{B.23}$$

If this condition does not hold, then the firm prefers not to affilate.

- Region 3: The firm's maximization problem is the same as in Region 2 except for constraint (A): $0 \leq a \leq 1$.

Therefore, $a^* = 1$. The condition for affiliation becomes

$$\alpha \leq \frac{[1 - P(\mathcal{G})]p}{[p - P(G|b)v]} \tag{B.24}$$

If $\gamma = 1$, this reduces to

$$\alpha \leq P(\mathcal{B}) \tag{B.25}$$

If this condition is not satisfied, the firm prefers not to affiliate.