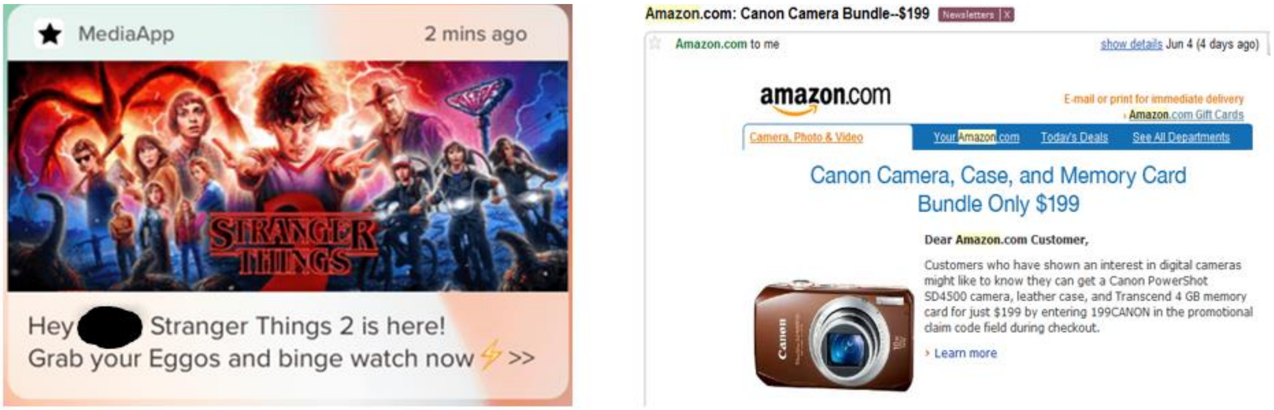
Online Appendix

Figure A1 Examples of Personalized Product Recommendations



Hey [redacted] Stranger Things 2 is here!
Grab your Eggos and binge watch now ⚡ >>

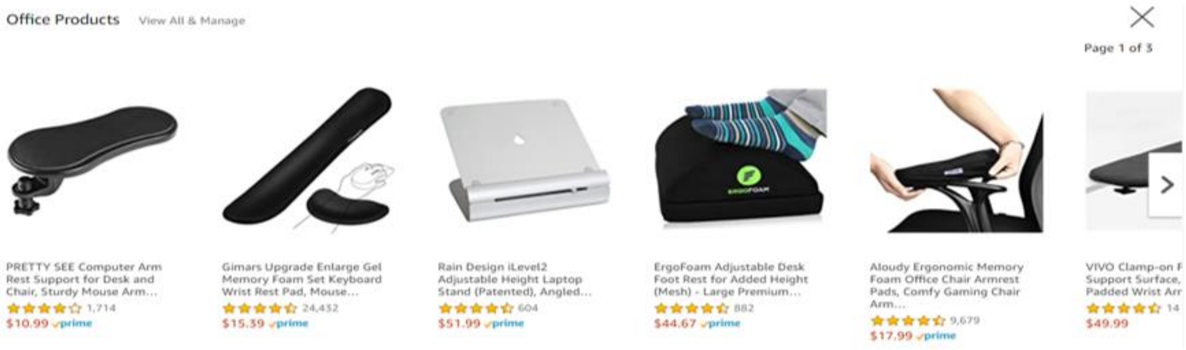
Canon Camera, Case, and Memory Card Bundle Only $199

Dear Amazon.com Customer,

Customers who have shown an interest in digital cameras might like to know they can get a Canon PowerShot SD4500 camera, leather case, and Transcend 4 GB memory card for just $199 by entering 199CANON in the promotional claim code field during checkout.

Recommendation push notifications

Recommendation emails



PRETTY SEE Computer Arm Rest Support for Desk and Chair, Sturdy Mouse Arm... 1,714 $10.99 prime

Gimars Upgrade Enlarge Gel Memory Foam Set Keyboard Wrist Rest Pad, Mouse... 24,432 $15.39 prime

Rain Design iLevel2 Adjustable Height Laptop Stand (Patented), Angled... 604 $51.99 prime

ErgoFoam Adjustable Desk Foot Rest for Added Height (Mesh) - Large Premium... 882 $44.67 prime

Aloudy Ergonomic Memory Foam Office Chair Armrest Pads, Comfy Gaming Chair Arm... 9,679 $17.99 prime

VIVO Clamp-on F Support Surface, Padded Wrist Arr 14 $49.99

On-site recommendations

PROOFS FOR ALL PROPOSITIONS AND LEMMAS

Proof of Proposition 1.

Under the price-neutral recommendation system, seller A's profit is

$$\pi_A^{PN} = r p_A \frac{1}{\beta} \cdot \left[\frac{k}{2}\left(\alpha - p_A + \frac{\sigma t}{2}\right) + (1-k)\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2} p_A + \frac{z}{1-z^2} p_B\right)\right].$$

π_A^{PN} is a quadratic function of p_A, so seller A's best response can be obtained from the first-order condition (FOC):

$$p_A^*(p_B) = \frac{2\alpha(1-z)\big(2-k(1-z)\big) + 4(1-k)p_B z + k(1-z^2)\sigma t}{4(2-k-kz^2)}.$$

Similarly, seller B's best response is:

$$p_B^*(p_A) = \frac{2\alpha(1-z)\big(2-k(1-z)\big) + 4(1-k)p_A z + k(1-z^2)\sigma t}{4(2-k-kz^2)}.$$

Solving these two equations together yields $p^{PN*} = \frac{(1-z)(4\alpha-2k\alpha(1-z)+k(1+z)\sigma t)}{4(2-z-k(1-z+z^2))}$. It is easy to verify that $\frac{\partial p^{PN*}}{\partial \sigma} = \frac{kt(1-z^2)}{4[2-z-k(1-z+z^2)]} > 0$, $\frac{\partial p^{PN*}}{\partial k} = \frac{kt(1-z^2)(2\alpha z+\sigma t(2-z))}{4[2-z-k(1-z+z^2)]^2} > 0$, and $\frac{\partial p^{PN*}}{\partial z} = \frac{-(1-k)[2\alpha(2-(1-z^2)k)-k\sigma t(1-4z+z^2)]}{4[2-z-k(1-z+z^2)]^2} < 0$. The last inequality holds because $\alpha > 2t$. ■

Proof of Corollary 1.

The marketplace's profit under the price-neutral recommendation system is $\Pi^{PN*} = r(p_A^{PN*}D_A^{PN*} + p_B^{PN*}D_B^{PN*}) = \frac{2r(1-z)(2-k-kz^2)[2\alpha(2-k(1-z))+k\sigma t(1+z)]^2}{32(1+z)(2-z+k(1-z+z^2))^2}$, which increases in σ: $\frac{\partial \Pi^{PN*}}{\partial \sigma} > 0$. Similarly, a seller's equilibrium profit is $\pi^{PN*} = \frac{(1-r)(1-z)(2-k-kz^2)[2\alpha(2-k(1-z))+k\sigma t(1+z)]^2}{32(1+z)(2-z+k(1-z+z^2))^2}$, which also increases in σ: $\frac{\partial \pi^{PN*}}{\partial \sigma} > 0$. ■

Proof of Proposition 2.

Define the threshold $G(p_A, p_B) = \frac{(p_A-p_B)(p_A+p_B-\alpha)}{(p_A+p_B)}$. Under the profit-based recommendation system, seller A's profit is

$$\pi_A^{PB*} = (1-r)p_A D_A^{PB}$$
$$= \frac{1-r}{\beta} p_A \cdot \left[\frac{k}{2}\left(1 - \frac{G(p_A,p_B)}{t\sigma}\right)\left[\alpha + \frac{t\sigma}{2} - p_A + \frac{G(p_A,p_B)}{2}\right] + (1-k)\left[\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p_B\right]\right].$$

In a symmetric equilibrium with $p_A = p_B = p$, the recommendation threshold $\hat{t}_0 = \frac{G(p,p)}{\sigma} = 0$. For now, we assume that given the other seller charging the equilibrium price p^*, the focal seller will not deviate its price such that $|\hat{t}_0| > t$. Later, we use Lemma A1 to verify that this condition is satisfied in equilibrium under the assumption of $k > \frac{z^2}{2-z^2}$. For seller A, its FOC when both firms' prices are at the equilibrium level is

FOC: $\frac{\partial \pi_A}{\partial p_A}|_{p_A=p_B=p} = \frac{1-r}{4\beta t\sigma(1-z^2)}\Big[2kp^2(1-z^2) - p\big[4t\sigma(2-z) + k[3\alpha(1-z^2) -$ $4t\sigma(1-z+z^2)]\big] + \alpha^2(1-z)\left[4\frac{t\sigma}{\alpha} + k\left[\left(1+(\frac{t\sigma}{\alpha})^2\right)(1+z) - 2\frac{t\sigma}{\alpha}(1-z)\right]\right]\Big] = 0.$

The SOC will also be satisfied:

SOC: $\frac{\partial^2 \pi_A^{PB}(p_A,p)}{\partial p_A^2}|_{p_A=p} = \frac{1-r}{8\alpha p t\sigma(1-z^2)\beta}\big[\alpha^2 k(1-z^2) + 12kp^2(1-z^2) + 2\alpha p\big(-8t\sigma +$ $k(-5+4t\sigma+5z^2+4t\sigma z^2)\big)\big] \leq 0.$

First, we will prove $p^{PB*} = \frac{\alpha}{4k(1-z^2)}\left(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\tilde{\sigma}(1-z+z^2)\big) -\right.$ $\left(\left(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\tilde{\sigma}(1-z+z^2)\big)\right)^2 - 8k(1-z)(1-z^2)\left(4\tilde{\sigma} + k\big((1+\right.\right.$ $\left.\left.\left.\tilde{\sigma}^2)(1+z) - 2\tilde{\sigma}(1-z)\big)\right)\right)^{\frac{1}{2}}\right)$ is the unique solution to FOC and SOC. The proof has three parts. First, we show that FOC has two real roots for p. Second, we show the smaller root satisfies SOC but the larger does not, so the smaller root will be the unique local maximum for π_A^{PB}. It is then obvious that this local maximum will be the global maximum for maximums π_A^{PB}, since $\pi_A^{PB}(0,p) = 0$ and $\pi_A^{PB}(p_A,p)$ will be negative if p_A is sufficiently large.

Part 1: *FOC has two real roots for p.*

With the notation $\tilde{\sigma} = t\sigma/\alpha$, we now show that the discriminant of the quadratic equation of the price p, $\Delta = \left(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\,\tilde{\sigma}(1-z+z^2)\big)\right)^2 - 8k(1-z)(1-z^2)\left(4\tilde{\sigma} + k\big((1+\ \tilde{\sigma}^2)(1+z) - 2\tilde{\sigma}(1-z)\big)\right)$, is positive, so that the equation has two real roots.

$\Delta = k^2(1-z^2)^2 + 8k(1-z^2)\big(2+z-k(1+z+z^2)\big)\tilde{\sigma} + 8\big(2(2-z)^2 - 4k(2-3z+$ $3z^2-z^3) + k^2(1-4z+8z^2-4z^3+z^4)\big)\tilde{\sigma}^2.$

Note that $k^2(1-z^2)^2 \geq 0$ and $8k(1-z^2)\big(2+z-k(1+z+z^2)\big) \geq 0$, so $\Delta \geq 0$ whenever $8\big(2(2-z)^2 - 4k(2-3z+3z^2-z^3) + k^2(1-4z+8z^2-4z^3+z^4)\big) \geq 0$.

By contrast, if $8\big(2(2-z)^2 - 4k(2-3z+3z^2-z^3) + k^2(1-4z+8z^2-4z^3+z^4)\big) < 0$, then Δ is concave in $\tilde{\sigma}^2$. Note that $\tilde{\sigma} = \frac{t\sigma}{\alpha} \leq \frac{1}{2}$, so $\Delta \geq \min\{\Delta|_{\tilde{\sigma}=0}, \Delta|_{\tilde{\sigma}=1/2}\}$. When $0 < z < \frac{1}{2}$ and $0 < k < 1$, both terms in the "min" function are positive: $\Delta|_{\tilde{\sigma}=0} = k^2(1-z^2)^2 \geq 0$, and $\Delta|_{\tilde{\sigma}=1/2} = 4(2-z)^2 - 4k(2-7z+8z^2-z^3) + k^2(-1-12z+14z^2-4z^3+7z^4) \geq \min\left\{\Delta|_{\tilde{\sigma}=\frac{1}{2},k=0}, \Delta|_{\tilde{\sigma}=\frac{1}{2},k=\frac{2-7z+8z^2-z^3}{-1-12z+14z^2-4z^3+7z^4}}, \Delta|_{\tilde{\sigma}=\frac{1}{2},k=1}\right\} = \min\left\{4(2-z)^2, \frac{8(1-z^2)^2(4+8z-3z^2)}{1+12z-14z^2+4z^3-7z^4}, 7(1-z^2)^2\right\} > 0$. Hence, we have proved $\Delta \geq 0$. ■

Part 2. *the smaller root of FOC satisfies SOC, but the larger root does not.*

Let p_1 be the smaller root of FOC. Define

$F_1(p) = 2kp^2(1-z^2) - p\big[4t\sigma(2-z) + k[3\alpha(1-z^2) - 4t\sigma(1-z+z^2)]\big] + \alpha^2(1-z)\left[4\frac{t\sigma}{\alpha} + k\left[\left(1+(\frac{t\sigma}{\alpha})^2\right)(1+z) - 2\frac{t\sigma}{\alpha}(1-z)\right]\right]$ and

$F_2(p) = 12kp^2(1-z^2) + 2\alpha p\big(-8t\sigma + k(-5+4t\sigma+5z^2+4t\sigma z^2)\big) + \alpha^2 k(1-z^2)$.

Furthermore, let

$H(p) = 6F_1(p) - F_2(p) = \alpha(1-z)\left(24t\sigma + k\big(-12t\sigma(1-z) + 5(1+z) + 6(t\sigma)^2(1+z)\big)\right) - 8\left(t\sigma(4-3z) + k\big(1-z^2-t\sigma(2-3z+2z^2)\big)\right)p$, which is linear in p.

Note that $\frac{\partial \pi_A}{\partial p_A}|_{p_A=p_B=p} = \frac{1-r}{4\beta t\sigma(1-z^2)}F_1(p)$ and $\frac{\partial^2 \pi_A^{PB}(p_A,p_B)}{\partial p_A^2}|_{p_A=p_B=p} = \frac{1-r}{8\alpha p t\sigma(1-z^2)\beta}F_2(p)$. Hence $\frac{\partial \pi_A}{\partial p_A}|_{p_A=p_B=p}$ has the same sign as $F_1(p)$, and $\frac{\partial^2 \pi_A^{PB}(p_A,p_B)}{\partial p_A^2}|_{p_A=p_B=p}$ has the same sign as $F_2(p)$. So, to show p satisfies FOC and SOC, it is equivalent to show $F_1(p) = 0$ and $F_2(p) \leq 0$.

In addition, $k \in [0,1]$ implies that $\alpha(1-z)\left(24t\sigma + k\big(-12t\sigma(1-z) + 5(1+z) + 6(t\sigma)^2(1+z)\big)\right) > 0$ and $-8\left(t\sigma(4-3z) + k\big(1-z^2-t\sigma(2-3z+2z^2)\big)\right) < 0$, so $H(0) > 0$ and $H(p)$ linearly strictly decreases with p. Let $p_0 =$

$\frac{(1-z)\left(24t\sigma+k\left(-12t\sigma(1-z)+5(1+z)+6(t\sigma)^2(1+z)\right)\right)}{8\left(t\sigma(4-3z)+k\left(1-z^2-t\sigma(2-3z+2z^2)\right)\right)} > 0$ be the unique root of $H(p) = 0$. Then $H(p) > 0$ if $0 < p < p_0$, and $H(p) < 0$ if $p > p_0$. Furthermore, one can verify that in the parameter region of $z \in (0, \frac{1}{2})$, $k \in (0,1)$, and $t\sigma \in (0, \alpha)$, $F_2(p_0) = 6F_1(p_0) < 0$ is always true.

We can now prove part 2, i.e., the smaller root for FOC (i.e., p_1) satisfies SOC, but the larger root for FOC (p_2) does not. Remember that both $F_1(p)$ is a convex quadratic function and $F_1'(0) < 0$. So $p_2 > p_1 > 0$, $F_1'(p_1) < 0$, and $F_1'(p_2) > 0$. In addition, because $F_1(p_0) < 0$, it must be that $p_1 < p_0 < p_2$. Remember that $H(p) = 6F_1(p) - F_2(p)$ linearly decreases with p, so $H(p_1) > H(p_0) = 0 > H(p_2)$. Finally, since $F_1(p_1) = F_1(p_2) = 0$, it must be that $F_2(p_1) < 0 < F_2(p_2)$, i.e., p_1 satisfies SOC but p_2 does not. ■

Solving for p_1, we obtain

$$p^{PB*} = \frac{\alpha}{4k(1-z^2)}\Bigg(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\tilde{\sigma}(1-z+z^2)\big) \\ - \Big(\big(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\tilde{\sigma}(1-z+z^2)\big)\big)^2 \\ - 8k(1-z)(1-z^2)\left(4\tilde{\sigma} + k\big((1+\tilde{\sigma}^2)(1+z) - 2\tilde{\sigma}(1-z)\big)\right)\Big)^{\frac{1}{2}}\Bigg).$$

Next, we will prove the second part of Proposition 3: p^{PB*} first decreases and then (potentially) increases in σ (or equivalently, in $\tilde{\sigma}$). We show this by first showing that $\lim_{\sigma\to 0}\frac{\partial p^{PB*}}{\partial\tilde{\sigma}} < 0$, and then proving that p^{PB*} is convex in $\tilde{\sigma}$.

The first condition $\lim_{\sigma\to 0}\frac{\partial p^{PB*}}{\partial\tilde{\sigma}} < 0$ is true since

$$\frac{\partial p^{PB*}}{\partial\tilde{\sigma}} = \frac{\alpha}{4k(1-z^2)}\Bigg[8 - 4k - 4z + 4kz - 4kz^2 - \\ \frac{4\left(4\tilde{\sigma}(-2+z)^2 + k\left(2+z-2z^2-z^3+8\tilde{\sigma}\left(-2+3z-3z^2+z^3\right)\right) + k^2\left(-1-z+z^3+z^4+2\tilde{\sigma}\left(1-4z+8z^2-4z^3+z^4\right)\right)\right)}{\Delta^{1/2}}\Bigg].$$

Hence, $\lim_{\sigma \to 0} \frac{\partial p^{PB*}}{\partial \tilde{\sigma}} = -\frac{2(1-k)z}{k(1-z^2)} < 0$.

Next, we will show $\frac{\partial^2 p^{PB*}}{\partial \tilde{\sigma}^2} > 0$.

$$\frac{\partial^2 p^{PB*}}{\partial \tilde{\sigma}^2} = \frac{2\alpha k(1-z^2)\left(16z - 8kz(3+z^2) + k^2(1+8z-2z^2+8z^3+z^4)\right)}{\Delta^{3/2}}$$

$$= \frac{2\alpha k(1-z^2)\left(16z - 8kz(3+z^2) + k^2(1+8z-2z^2+8z^3+z^4)\right)}{\Delta^{3/2}}.$$

We only need to show $16z - 8kz(3+z^2) + k^2(1+8z-2z^2+8z^3+z^4) > 0$. Because $0 < z < \frac{1}{2}$, $1+8z-2z^2+8z^3+z^4 > 0$. So, $16z - 8kz(3+z^2) + k^2(1+8z-2z^2+8z^3+z^4) \geq 16z - 8z(3+z^2) \cdot \frac{4z(3+z^2)}{1+8z-2z^2+8z^3+z^4} + (1+8z-2z^2+8z^3+z^4)\left[\frac{4z(3+z^2)}{1+8z-2z^2+8z^3+z^4}\right]^2 = \frac{16(1-z)^3 z(1+z)^2}{1+8z-2z^2+8z^3+z^4} \geq 0$.

Finally, Figure 2 in the main paper provides an example that p^{PB*} can indeed strictly increase with σ in some parameter regions. ■

The following lemma ensures the existence of a symmetric equilibrium by considering a seller's potential incentive to deviate from the equilibrium price.

Lemma A1. If $k > \frac{z^2}{2-z^2}$, then a seller will not set its price such that $|\hat{t}_0| = \frac{G(p_A, p_B)}{\sigma} > t$, given the other seller's price being p^{PB*}. This will guarantee that a symmetric equilibrium exists.

Proof. By symmetricity, it is without loss of generality to consider whether seller A will deviate from the equilibrium price $p_A^* = p^{PB*}$, given $p_B = p^{PB*}$.

Conditional on $p_B = p^{PB*}$, seller A's profit is

$$\pi_A(p_A, p^{PB*}) = \begin{cases} H_1(p_A, p^{PB*}), \text{if } G(p_A, p^{PB*}) > t\sigma \\ H_2(p_A, p^{PB*}), \text{if } |G(p_A, p^{PB*})| \leq t\sigma \\ H_3(p_A, p^{PB*}), \text{if } G(p_A, p^{PB*}) < -t\sigma \end{cases},$$

where $H_1(p_A, p^{PB*}) \triangleq (1-r)p_A\left[\frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{PB*}\right)\right]$,

$$H_2(p_A, p^{PB*}) \triangleq (1-r)p_A\left\{\frac{k}{2\beta}\left(1 - \frac{G(p_A, p^{PB*})}{t\sigma}\right)\left(\alpha - p_A + t\sigma\frac{1+\frac{G(p_A, p^{PB*})}{t\sigma}}{2}\right) + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{PB*}\right)\right\},$$

and $H_3(p_A, p^{PB*}) \triangleq (1-r)p_A\left\{\frac{k}{\beta}(\alpha - p_A) + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{PB*}\right)\right\}$.

We can solve the symmetric equilibrium in the following two steps. First, we find p^{PB*} such that $p^{PB*} \in argmax_{p_A>0}H_2(p_A, p^{PB*})$. Second, we verify that $\pi_A(p^{PB*}, p^{PB*}) > \pi_A(p', p^{PB*})$ for any p' satisfying $|G(p', p^{PB*})| > t\sigma$. If p^{PB*} satisfies the conditions in the second step, then there exists a symmetric equilibrium.

The first step has been accomplished in the proof of Proposition 3, which shows that there exists a unique p^{PB*} satisfying the first requirement, where

$$\begin{aligned} p^{PB*} = \frac{\alpha}{4k(1-z^2)}\Bigg(& 4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\tilde{\sigma}(1-z+z^2)\big) \\ & -\Big(\big(4\tilde{\sigma}(2-z) + k\big(3(1-z^2) - 4\,\tilde{\sigma}(1-z+z^2)\big)\big)^2 \\ & - 8k(1-z)(1-z^2)\left(4\tilde{\sigma} + k\big((1+\tilde{\sigma}^2)(1+z) - 2\tilde{\sigma}(1-z)\big)\right)\Big)^{\frac{1}{2}}\Bigg). \end{aligned}$$

The rest of this proof will show that when $k > \frac{z^2}{2-z^2}$ and $z < \frac{1}{2}$, there will not exist p' such that $|G(p', p^{PB*})| > t\sigma$ and $\pi_A(p^{PB*}, p^{PB*}) < \pi_A(p', p^{PB*})$.

We prove this result by contradiction. Suppose the above statement is false, then one of the following two (mutually exclusive) conditions must be true.

(A1) $G(p', p^{PB*}) < -t\sigma$ and $\pi_A(p', p^{PB*}) > \pi_A(p^{PB*}, p^{PB*}) = \frac{1-r}{\beta} p^{PB*} \left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{PB*}) + \frac{kt\sigma}{2}\right]$.

(A2) $G(p', p^{PB*}) > t\sigma$ and $\pi_A(p', p^{PB*}) > \pi_A(p^{PB*}, p^{PB*}) = \frac{1-r}{\beta} p^{PB*} \left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{PB*}) + \frac{kt\sigma}{2}\right]$.

Case 1: (A1) is true.

When this is true, the marketplace will recommend product A to all uninformed consumers. Note that $G(p', p^{PB*}) < -t\sigma$ is equivalent to $\frac{\alpha-\sigma t-\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2} < p' < \frac{\alpha-\sigma t+\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2}$.

By the definition of p^{PB*}, $\pi_A\left(\frac{\alpha-\sigma t-\sqrt{\alpha^2-2\alpha(2p^*+\sigma t)+(2p^*-\sigma t)^2}}{2}, p^{PB*}\right) \le \pi_A(p^{PB*}, p^{PB*})$ and $\pi_A\left(\frac{\alpha-\sigma t+\sqrt{\alpha^2-2\alpha(2p^*+\sigma t)+(2p^*-\sigma t)^2}}{2}, p^{PB*}\right) \le \pi_A(p^{PB*}, p^{PB*})$. Since $\pi_A(p_A, p^{PB*})$ is a concave quadratic function of p_A when $\frac{\alpha-\sigma t-\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2} < p_A < \frac{\alpha-\sigma t+\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2}$ and $\pi_A(p', p^{PB*}) > \pi_A(p^{PB*}, p^{PB*})$, it follows that $\pi_A(p_A, p^*)$ first increases then decreases in p_A when

$p_A \in [\frac{\alpha-\sigma t-\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2}, \frac{\alpha-\sigma t+\sqrt{\alpha^2-2\alpha(2p^{PB*}+\sigma t)+(2p^{PB*}-\sigma t)^2}}{2}]$. The maximum of $\pi_A(p_A, p^{PB*})$ is reached at $p_{A,3} = argmax_{p_A} H_3(p_A, p^{PB*})$, where $p_{A,3} = \frac{(1-k)p^{PB*}z+\alpha(1-z)(1+kz)}{2(1-kz^2)}$. But under the assumption of $t < \min\{\frac{(1-z)\alpha}{2(1+z)}, \frac{2\alpha(1-z)(2-k(1+z^2))}{(1+z)[4(2-z)-k(3-2z+3z^2)]}\}$, one can verify that $\pi_A(p', p^{PB*}) \le \pi_A(p_{A,3}, p^{PB*}) = H_3(p_{A,3}, p^{PB*}) < H_2(p_{A,3}, p^{PB*}) \le H_2(p^{PB*}, p^{PB*}) = \pi_A(p^{PB*}, p^{PB*})$, which contradicts the assumption that $\pi_A(p', p^{PB*}) > \pi_A(p^{PB*}, p^{PB*})$. As a result, (A1) cannot be true.

Case 2: (A2) is true.

If (A2) is true, then $H_1(p', p^{PB*}) > H_2(p^{PB*}, p^{PB*})$. In this case, the marketplace will recommend product B to all uninformed consumers. Note that $H_1(p_A, p^*)$ is maximized at

$p_{A,1} = \frac{zp^{PB*}+(1-z)\alpha}{2}$, and $H_1(p', p^{PB*}) \le H_1\left(p_{A,1}, p^{PB*}\right) = \frac{(1-r)(1-k)}{4\beta(1-z^2)}(zp^{PB*} + (1-z)\alpha)^2$.

We next show that when $k > \frac{z^2}{2-z^2}$, $H_1\left(p_{A,1}, p^{PB*}\right) \le \pi_A(p^*, p^*)$.

$\pi_A(p^{PB*}, p^{PB*}) - H_1\left(p_{A,1}, p^{PB*}\right) = \frac{1-r}{4\beta(1-z^2)}\Big[-\alpha^2(1-k)(1-z)^2 + p^{PB*}\Big(2k\sigma t(1-z^2) + 2\alpha(1-z)\big(2-z-k(1-2z)\big)\Big) - {p^{PB*}}^2\big((2-z)^2 - k(2-4z+3z^2)\big)\Big]$,

which is positive if and only if $\frac{1}{(2-z)^2+k(2-4z+3z^2)}\Bigg(k\sigma t(1-z^2) + \alpha(1-z)\big(2-z-k(1-2z)\big) - \sqrt{k(1-z)^2(1+z)\left(\alpha^2\big(2-k(1-z)\big) + k\sigma^2t^2(1+z) + 2\alpha\sigma t\big(2-z-k(1-2z)\big)\right)}\Bigg) <$ $p^{PB*} < \frac{1}{(2-z)^2+k(2-4z+3z^2)}\Bigg(k\sigma t(1-z^2) + \alpha(1-z)\big(2-z-k(1-2z)\big) + \sqrt{k(1-z)^2(1+z)\left(\alpha^2\big(2-k(1-z)\big) + k\sigma^2t^2(1+z) + 2\alpha\sigma t\big(2-z-k(1-2z)\big)\right)}\Bigg)$.

One can verify that when $k > \frac{z^2}{2-z^2}$ and $z < \frac{1}{2}$, p^{PB*} always satisfies the above inequality. ■

Proof of Proposition 3.

Without loss of generality, we focus on seller A's profit, because it will be proportional to the marketplace's profit. In equilibrium, seller A's profit under the profit-based recommendation system is

$$\pi_A(p^{PB*}, p^{PB*}) = \frac{(1-r)p^{PB*}}{\beta}\left\{\frac{k}{2}\left(\alpha + \frac{t\sigma}{2} - p^{PB*}\right) + \frac{1-k}{1+z}\left(\alpha - p^{PB*}\right)\right\}.$$

One can show that $\frac{\partial \pi_A\left(p^{PB*},\ p^{PB*}\right)}{\partial \sigma} = \frac{1-r}{\beta}\left\{\left[\frac{k}{2}\left[\alpha + \frac{t\sigma}{2} - p^{PB*}\right] + \frac{1-k}{1+z}\left(\alpha - p^{PB*}\right)\right]\frac{\partial p^{PB*}}{\partial \sigma} + p^{PB*}\left[\frac{tk}{4} - \frac{\partial p^{PB*}}{\partial \sigma}\cdot\left(\frac{k}{2} + \frac{1-k}{1+z}\right)\right]\right\} = \frac{1-r}{\beta}\left\{\left(\frac{k}{2} + \frac{1-k}{1+z}\right)\left(\alpha - 2p^{PB*}\right)\frac{\partial p^{PB*}}{\partial \sigma} + \frac{kt}{4}p^{PB*}\left(\frac{\partial p^{PB*}/p^{PB*}}{\partial \sigma/\sigma} + 1\right)\right\}$.

Proposition 3 has shown that p^{PB*} is concave in σ. Moreover $p^{PB*} \to \frac{\alpha}{2}$ when $\sigma \to 0$, so $\frac{\partial p^{PB*}/p^{PB*}}{\partial \sigma/\sigma} < 0$ must imply $p^{PB*} < \frac{\alpha}{2}$.

Hence, when $\frac{\partial p^{PB*}/p^{PB*}}{\partial \sigma/\sigma} < -1$, $\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - 2p^{PB*})\frac{\partial p^{PB*}}{\partial \sigma} < 0$ and $\frac{kt}{4} \cdot p^{PB*}\left(\frac{\partial p^{PB*}/p^{PB*}}{\partial \sigma/\sigma} + 1\right) < 0$, so $\frac{\partial \pi_A(p^{PB*},p^{PB*})}{\partial \sigma} < 0$. ■

Proof of Proposition 4.

Let $\Delta p^* = p^{PB*} - p^{PN*}$ be the difference between the equilibrium prices under the profit-based recommendation system and under the fit-based one. We need to show $\Delta p^* > 0$ when $\sigma < \bar{\sigma}$, and $\Delta p^* < 0$ when $\sigma > \bar{\sigma}$. Because p^{PB*} is convex in σ and p^{PN*} is linear in σ, Δp^* is also convex in σ. It is important to observe that $\Delta p^*|_{\sigma \to 0} \to \frac{\alpha}{2} - \frac{\alpha}{2} \cdot \frac{(1-z)(2-k(1-z))}{2-z-k(1-z+z^2)} > 0$, and $\Delta p^*|_{\sigma=\bar{\sigma}} = \frac{\alpha}{2} - \frac{\alpha}{2} = 0$ when $\sigma = \bar{\sigma} = \frac{2(1-k)z}{tk(1-z^2)}\alpha$. So, for any $\sigma' = \phi\bar{\sigma} < \bar{\sigma}$, where $0 < \phi < 1$, $\Delta p^*|_{\sigma=\sigma'} \leq \phi \Delta p^*|_{\sigma \to 0} + (1-\phi)\Delta p^*|_{\sigma=\bar{\sigma}} < 0$; For any $\sigma' = \phi\bar{\sigma} > \bar{\sigma}$, where $\phi > 1$, $0 = \Delta p^*|_{\sigma=\bar{\sigma}} \leq \frac{1}{\phi}\Delta p^*|_{\sigma \to 0} + \frac{\phi-1}{\phi}\Delta p^*|_{\sigma=\sigma'}$, which implies $\Delta p^*|_{\sigma=\sigma'} \geq -\frac{1}{\phi-1}\Delta p^*|_{\sigma \to 0} > 0$. ■

Proof of Proposition 5.

Under both types of recommender systems, when both sellers set their prices to be $p_j = p$, a seller's equilibrium profit can be written as $\pi^0(p) = \frac{(1-r)p}{\beta}\left\{\frac{k}{2}\left(\alpha + \frac{t\sigma}{2} - p\right) + \frac{1-k}{1+z}(\alpha - p)\right\}$, $j \in \{A, B\}$. Therefore, a seller's equilibrium profit under the profit-based recommendation system is $\pi^{PB*} = \pi^0(p^{PB*})$, and its profit under the price-neutral recommendation system is $\pi^{PN*} = \pi^0(p^{PN*})$.

$\pi^0(p)$ increases with p when $p \leq p^{**} \overset{\text{def}}{=} \frac{\alpha}{2} + \frac{k\sigma t(1+z)}{4(2-k(1-z))}$, and decreases with p when $p > p^{**}$. When $\sigma < \bar{\sigma}$, $p^{PB*} < p^{PN*} \leq p^{**}$, so $\pi^{PN*} < \pi^{PB*}$. By contrast, when $\sigma > \bar{\sigma}$, $p^{PB*} < p^{PN*} \leq p^{**}$, so $\pi^{PB*} < \pi^{PN*}$.

The proof for the marketplace's profit is similar because the marketplace's equilibrium profit is proportional to a seller's equilibrium profit. ■

Proof of Proposition 6.

Under both types of recommender systems, when both sellers set their prices to be $p_j = p$, the consumer surplus can be written as

$$CS^0(p) = k\left[\int_{-t}^{0} \frac{1}{2t} \cdot \left[\frac{\sigma+1}{2} \cdot \frac{(\alpha-p-t_i)^2}{2\beta} + \frac{1-\sigma}{2} \cdot \frac{(\alpha-p+t_i)^2}{2\beta}\right] dt_i + \int_{0}^{t} \frac{1}{2t} \cdot \left[\frac{\sigma+1}{2} \cdot \frac{(\alpha-p+t_i)^2}{2\beta} + \frac{1-\sigma}{2} \cdot \frac{(\alpha-p-t_i)^2}{2\beta}\right] dt_i\right] + (1-k)\int_{-t}^{t} \frac{1}{2t} \cdot \frac{2\left((2-z)(\alpha-p)^2+(2+z)t_i^2\right)}{(4-z^2)\beta} dt_i$$

$$= \frac{1}{\beta}\left[\frac{k}{2} \cdot \left[\frac{t^2}{3} + (\alpha-p+\sigma t)(\alpha-p)\right] + (1-k)\left(\frac{2(\alpha-p)^2}{2+z} + \frac{2t^2}{3(2-z)}\right)\right]$$

$$= \frac{1}{\beta}\left[\frac{t^2}{3}\left(\frac{k}{2} + \frac{2(1-k)}{2-z}\right) + \frac{k}{2}(\alpha-p+\sigma t)(\alpha-p) + \frac{2(1-k)}{2+z}(\alpha-p)^2\right].$$

Suppose the equilibrium price is p^* and the consumer surplus is CS^*, then

$$\frac{dCS^*}{d\sigma} = \frac{\partial CS^0(p^*)}{\partial\sigma} + \frac{\partial CS^0(p^*)}{\partial p^*} \cdot \frac{\partial p^*}{\partial\sigma},$$

and $\frac{d^2CS^*}{d\sigma^2} = \frac{\partial^2 CS^0(p^*)}{\partial\sigma^2} + \frac{\partial^2 CS^0(p^*)}{\partial {p^*}^2} \cdot \frac{\partial p^*}{\partial\sigma} + \frac{\partial CS^0(p^*)}{\partial p^*} \cdot \frac{\partial^2 p^*}{\partial\sigma^2} = 0 - \frac{4-k(2-z)}{\beta(2+z)} \cdot \frac{\partial p^*}{\partial\sigma} - \frac{2(\alpha-p^*)\left(4-k(2-z)\right)+k\sigma t(2+z)}{2\beta(2+z)} \cdot \frac{\partial^2 p^*}{\partial\sigma^2}$. If $\frac{\partial p^*}{\partial\sigma} < 0$, then $\frac{dCS^*}{d\sigma} > 0$ must be true.

Under the fit-based recommendation system, the equilibrium consumer surplus is $CS^{FB*} = CS^0(p^{FB*})$. Note that our parameter region implies $t < \frac{\alpha}{2}$, so

$$\frac{dCS^{FB*}}{d\sigma} = \frac{kt\Big(3k\sigma t(1-z^2)\big(4-k(2-z+2z^2+z^3)\big)-2\alpha\Big(4(2+z^2)-2k(4-z+7z^2+z^3+z^4)+k^2(2-z+6z^2+4z^4+z^5)\Big)\Big)}{16\beta(2+z)\big(2-z-k(1-z+z^2)\big)^2}$$

$$< \frac{kt\Big(3k\sigma t(1-z^2)\big(4-k(2-z+2z^2+z^3)\big)-2\alpha\Big(4(2+z^2)-2k(4-z+7z^2+z^3+z^4)+k^2(2-z+6z^2+4z^4+z^5)\Big)\Big)}{16\beta(2+z)\big(2-z-k(1-z+z^2)\big)^2}\Big|_{\sigma=1,t=\frac{\alpha}{2}}$$

$$= \frac{\alpha^2 k(16(2+z^2)-4k(11-2z+11z^2+2z^3+2z^4)+k^2(14-7z+24z^2+6z^3+10z^4+z^5))}{64\beta(2+z)(2-z-k(1-z+z^2))^2} > 0.$$

Under the profit-based recommendation system, the equilibrium consumer surplus is $CS^{PB*} = CS^0(p^{PB*})$.

$$\frac{\partial CS^{PB*}}{\partial \sigma} = \frac{\partial CS^0(p^{PB*})}{\partial \sigma} + \frac{\partial CS^0(p^{PB*})}{\partial p^{PB*}} \cdot \frac{\partial p^{PB*}}{\partial \sigma},$$

and $\frac{\partial^2 CS^{PB*}}{\partial \sigma^2} = \frac{\partial^2 CS^0(p^{PB*})}{\partial \sigma^2} + \frac{\partial^2 CS^0(p^{PB*})}{\partial {p^{PB*}}^2} \cdot \frac{\partial p^{PB*}}{\partial \sigma} + \frac{\partial CS^0(p^{PB*})}{\partial p^{PB*}} \cdot \frac{\partial^2 p^{PB*}}{\partial \sigma^2} = 0 - \frac{4-k(2-z)}{\beta(2+z)} \cdot \frac{\partial p^{PB*}}{\partial \sigma} - \frac{2(\alpha-p^{PB*})\big(4-k(2-z)\big)+k\sigma t(2+z)}{2\beta(2+z)} \cdot \frac{\partial^2 p^{PB*}}{\partial \sigma^2}$.

Let σ^* be implicitly defined by $\frac{\partial p^{PB*}}{\partial \sigma}\big|_{\sigma=\sigma^*} = 0$. σ^* is uniquely defined because Proposition 3 has shown that $\frac{\partial^2 p^{PB*}}{\partial \sigma^2} > 0$. As a result, $\frac{\partial p^{PB*}}{\partial \sigma} < 0$ when $\sigma < \sigma^*$, and $\frac{\partial p^{PB*}}{\partial \sigma} > 0$ when $\sigma > \sigma^*$.

When $\sigma < \sigma^*$, $\frac{dCS^{PB*}}{d\sigma} > 0$ must be true since $\frac{\partial p^{PB*}}{\partial \sigma} < 0$.

When $\sigma > \sigma^*$, because $\frac{\partial^2 p^{PB*}}{\partial \sigma^2} > 0$, $\frac{\partial^2 CS^{PB*}}{\partial \sigma^2} < 0$, i.e., $\frac{\partial CS^{PB*}}{\partial \sigma}$ decreases with σ.

Hence, to show that $\frac{\partial CS^{PB*}}{\partial \sigma} \geq 0$ for all σ, we only need to verify $\frac{\partial CS^{PB*}}{\partial \sigma} > 0$ when $\sigma = 1$. Similarly, we only need to verify $\frac{\partial CS^{PB*}}{\partial \sigma} > 0$ when $t = \frac{\alpha}{2}$.

$$\frac{\partial CS^{PB*}}{\partial \sigma}\Big|_{\sigma=1,t=\frac{\alpha}{2}}$$

$$= \frac{\alpha^2}{16(1-z^2)^2\beta}\Bigg(-\frac{1}{k}\Big(-2+k+z-kz+kz^2+\frac{2(-2+z)^2+k^2z(-5+8z-3z^2+2z^3)+k(-6+13z-14z^2+3z^3)}{\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}}\Big)\Big(4-2z+k(-5+2z+z^2)-\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}\Big)+\frac{1}{k^2(2+z)}8(1-k)\Big(-2+k+z-kz+kz^2+\frac{2(-2+z)^2+k^2z(-5+8z-3z^2+2z^3)+k(-6+13z-14z^2+3z^3)}{\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}}\Big)\Big(-4+2z+k(3-2z+z^2)+\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}\Big)-(-1+z^2)\Big(-4+2z+k(3-2z+z^2)+\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}\Big)\Bigg(1-\frac{-2+k+z-kz+kz^2+\frac{2(-2+z)^2+k^2z(-5+8z-3z^2+2z^3)+k(-6+13z-14z^2+3z^3)}{\sqrt{4(-2+z)^2+4k(-2+7z-8z^2+z^3)+k^2(-1-12z+14z^2-4z^3+7z^4)}}}{k(-1+z^2)}\Bigg)\Bigg).$$

One can verify that when $0 < z < \frac{1}{2}$ and $\frac{z^2}{2-z^2} < k < 1$, the above expression is always positive.

■

Proof of Proposition 7

Part (1)

Let us consider an uninformed consumer with signal $\hat{t}_i$. This consumer's expected consumer surplus is $E\left[CS_{iA,U}\middle|\hat{t}_i\right] = \sigma\frac{(\alpha-p_A+\hat{t}_i)^2}{2\beta} + (1-\sigma)\int_{t_i\in[-t,t]\setminus\{\hat{t}_i\}}\frac{1}{2t}\frac{(\alpha-p_A+t_i)^2}{2\beta}dt_i$ if product A is recommended, and is $E\left[CS_{iB,U}\middle|\hat{t}_i\right] = \sigma\frac{(\alpha-p_B-\hat{t}_i)^2}{2\beta} + (1-\sigma)\int_{t_i\in[-t,t]\setminus\{\hat{t}_i\}}\frac{1}{2t}\frac{(\alpha-p_B-t_i)^2}{2\beta}dt_i$ if product B is recommended. Thus, the marketplace will recommend product A if and only if $E\left[CS_{iA,U}\middle|\hat{t}_i\right] > E\left[CS_{iB,U}\middle|\hat{t}_i\right]$, i.e., $\hat{t}_i > \hat{t}_{0,CS}(p_A, p_B) = \frac{p_A-p_B}{2\sigma}$.

Hence, seller A's demand is $D_A = \underbrace{k\cdot\int_{\hat{t}_{0,CS}}^{t} E\left[q_{iA,U}\middle|\hat{t}_i\right]\mathrm{d}F_{\hat{t}}(\hat{t}_i)}_{\text{Demand from uninformed consumers}} + \underbrace{(1-k)\cdot\int_{-t}^{t} q_{iA,I}\,\mathrm{d}F_t(t_i)}_{\text{Demand from informed consumers}}$,

and its profit is $(1-r)p_A D_A$.

Conditional on $p_B = p^{CB*}$, seller A's profit is

$$\pi_A(p_A, p^{CB*}) = \begin{cases} H_1(p_A, p^{CB*}), \text{if } p_A - p^{CS*} > 2t\sigma \\ H_2(p_A, p^{CB*}), \text{if } |p_A - p^{CS*}| \le 2t\sigma\ , \\ H_3(p_A, p^{CB*}), \text{if } p_A - p^{CS*} < -2t\sigma \end{cases}$$

where $H_1(p_A, p^{CB*}) \triangleq (1-r)p_A\left[\frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{CB*}\right)\right]$,

$H_2(p_A, p^{CB*}) \triangleq (1-r)p_A\left\{\frac{k}{2\beta}\left(1 - \frac{p_A-p^{CB*}}{2t\sigma}\right)\left(\alpha - p_A + t\sigma\frac{1+\frac{p_A-p^{CB*}}{2t\sigma}}{2}\right) + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{CB*}\right)\right\}$,

and $H_3(p_A, p^{CB*}) \triangleq (1-r)p_A\left\{\frac{k}{\beta}(\alpha - p_A) + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p_A + \frac{z}{1-z^2}p^{CB*}\right)\right\}$.

In the symmetric equilibrium, $p_A = p_B = p^{CB*}$ and FOC is satisfied. The FOC can be expressed as

$$\frac{2\alpha(2-k(1-z))(1-z)-4p^{CB*}(2-k-z+kz-kz^2)+kt(1-z^2)\sigma}{4(1-z^2)\beta} = \frac{k}{\beta}p^{CB*}(\alpha - p^{CB*})\frac{\partial\hat{t}_{0,CS}\left(p^{CB*}, p^{CB*}\right)}{\partial p_A} = \frac{k}{\beta}p^{CB*}(\alpha - p^{CS*})\cdot\frac{1}{2\sigma}$$

From this, one can solve the equilibrium price:

$$p^{CB*} = \frac{\alpha}{2k(1-z^2)}\left(k(1-z^2) + 4\tilde{\sigma}\big(2-z-k(1-z+z^2)\big) - \left(\Big(k(1-z^2) + 4\tilde{\sigma}\big(2-z- k(1-z+z^2)\big)\Big)^2 - 4\tilde{\sigma}k(1-z)(1-z^2)\big(2\big(2-k(1-z)\big)+k(1+z)\tilde{\sigma}\big)\right)^{\frac{1}{2}}\right).$$

One can also show that the SOC is also satisfied under the condition of $k > \frac{z^2}{2-z^2}$ and $z < \frac{1}{2}$.

The above analysis assumes FOC generates global optimal solutions. We need to verify that there does not exist p' such that $|\frac{p'-p^{CB*}}{2\sigma}| > t$ and $\pi_A(p', p^{CB*}) > \pi_A(p^{CB*}, p^{CB*})$.

Suppose the above statement is false, then one of the following two (mutually exclusive) conditions must be true.

(A1): $p' - p^{CB*} > 2t\sigma$ and $(1-r)p'\left[\frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p' + \frac{z}{1-z^2}p^{CB*}\right)\right] > \frac{1-r}{\beta}p^{CB*}\left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{CB*}) + \frac{kt\sigma}{4}\right]$.

(A2): $p' - p^{CB*} < -2t\sigma$ and $(1-r)p'\left\{\frac{k}{\beta}(\alpha - p') + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p' + \frac{z}{1-z^2}p^{CB*}\right)\right\} > \frac{1-r}{\beta}p^{CB*}\left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{CB*}) + \frac{kt\sigma}{4}\right]$.

<u>Case 1: (A1) is true</u>

Note that $(1-r)p'\left[\frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p' + \frac{z}{1-z^2}p^{CB*}\right)\right] - \frac{1-r}{\beta}p^{CS*}\left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{CB*}) + \frac{kt\sigma}{2}\right]$ increases with p' when $p' < \frac{1}{2}((1-z)\alpha + zp^{CB*})$, and decreases with p' when $p' > \frac{1}{2}((1-z)\alpha + zp^{CB*})$. In addition, one can show that $H_1(p', p^{CB*}) < 0$ when $p' = p^{CB*} + 2\sigma t$. So, (A1) is true if and only if (a) $H_1(p', p^{CB*}) > 0$ at $p' = \frac{1}{2}((1-z)\alpha + zp^{CB*})$ and (b) $\frac{1}{2}\big((1-z)\alpha + zp^{CB*}\big) > p^{CB*} + 2t\sigma$.

Condition (a) and (b) are equivalent to

$$p^{CB*} < \min\left\{\frac{(1-z)\alpha}{2[(2-z)^2 - k(2-4z+3z^2)]}\cdot\left[2\big(2-z-k(1-2z)\big) + (1+z)k\tilde{\sigma} - \sqrt{(1+z)\left(k\left(4\big(2-k(1-z)\big)\right) + 4\tilde{\sigma}\big(2-z-k(1-2z)\big) + k(1+z)\tilde{\sigma}^2\right)}\right], \alpha\frac{(1-z)-4\tilde{\sigma}}{2-z}\right\}.$$

It can be verified that the right hand side strictly decreases with σ, but p^{CB*} strictly increases with σ. So, there exists $\underline{\sigma}$ such that (A1) is true if and only if $\sigma < \underline{\sigma}$. In other words, when $\sigma \geq \underline{\sigma}$, (A1) is false.

Case 2: (A2) is true

Note that and $(1-r)p'\left\{\frac{k}{\beta}(\alpha - p') + \frac{1-k}{\beta}\left(\frac{\alpha}{1+z} - \frac{1}{1-z^2}p' + \frac{z}{1-z^2}p^{CS*}\right)\right\} - \frac{1-r}{\beta}p^{CB*}\left[\left(\frac{k}{2} + \frac{1-k}{1+z}\right)(\alpha - p^{CB*}) + \frac{kt\sigma}{4}\right]$ increases with p' when $p' < \frac{1}{2(1-kz^2)}((1-z)(1+kz)\alpha + z(1-k)p^{CB*})$ and decreases with p' when $p' > \frac{1}{2(1-kz^2)}((1-z)(1+kz)\alpha + z(1-k)p^{CB*})$. In addition, one can show that $H_2(p', p^{CB*}) < 0$ when $p' = p^{CB*} - 2\sigma t$. So, (A2) is true if and only if (a) $H_2(p', p^{CB*}) > 0$ when $p' = \frac{1}{2(1-kz^2)}((1-z)(1+kz)\alpha + z(1-k)p^{CB*})$ and (b) $\frac{1}{2(1-kz^2)}\left((1-z)(1+kz)\alpha + z(1-k)p^{CB*}\right) < p^{CB*} - 2t\sigma.$

One can show (a) is always true but (b) is always false. So (A2) cannot be true.

In summary, there exists a symmetric equilibrium when $\sigma \geq \underline{\sigma}$. In the equilibrium, $p^{CB*} = \frac{\alpha}{2k(1-z^2)}\left(k(1-z^2) + 4\tilde{\sigma}\left(2 - z - k(1-z+z^2)\right) - \left(\left(k(1-z^2) + 4\tilde{\sigma}\left(2 - z - k(1-z+z^2)\right)\right)^2 - 4\tilde{\sigma}k(1-z)(1-z^2)\left(2\left(2-k(1-z)\right) + k(1+z)\tilde{\sigma}\right)\right)^{\frac{1}{2}}\right).$

Part (2)

$\frac{dCS^*}{d\sigma} = \frac{\partial CS}{\partial\sigma} + \frac{\partial CS}{\partial p^{CB*}} \cdot \frac{\partial p^{CB*}}{\partial\sigma} = \frac{1}{\beta}\left[\frac{kt}{2}(\alpha - p^{CB*}) - \frac{\partial p^{CB*}}{\partial\sigma} \cdot \left(\frac{k}{2}\sigma t + \left(k + \frac{4(1-k)}{2+z}\right)(\alpha - p^{CB*})\right)\right]$. A higher profiling accuracy can hurt the consumers and the marketplace if

$$\frac{\partial p^{CB*}}{\partial\sigma} > \frac{kt(\alpha - p^{CB*})}{\sigma kt + 2\left(k + \frac{4(1-k)}{2+z}\right)(\alpha - p^{CB*})}, \text{ i.e., } \frac{\partial p^{CB*}/p^{CB*}}{\partial\sigma/\sigma} > \left(\frac{p^{CB*}}{\alpha - p^{CB*}} + \left(1 + \frac{4(1-k)}{k(2+z)}\right)\left(\frac{p^{CB*}}{t\sigma}\right)\right)^{-1}.$$

Proof of Proposition 8.

Let us consider an uninformed consumer with signal $\hat{t}_i$. The value of the recommendation system's objective function for product A is $g_{iA}(\hat{t}_i) = r^w \beta^{w-1} p_A^w (\alpha + \sigma\hat{t}_i - p_A)^{1-w}$, and for product B is $g_{iB}(\hat{t}_i) = r^w \beta^{w-1} p_B^w (\alpha - \sigma \hat{t}_i - p_B)^{1-w}$. Hence, the marketplace will recommend product A if and only if $\hat{t}_i > \hat{t}_0(w) = \frac{\alpha\left(p_B^{\frac{w}{1-w}} - p_A^{\frac{w}{1-w}}\right) - \left(p_B^{\frac{1}{1-w}} - p_A^{\frac{1}{1-w}}\right)}{\sigma\left(p_A^{\frac{w}{1-w}} + p_B^{\frac{w}{1-w}}\right)}$. Hence, seller A's demand is

$$D_A(w) = \underbrace{k \cdot \int_{\hat{\boldsymbol{t}}_{\mathbf{0}}(\boldsymbol{w})}^{t} E\left[q_{iA,U} \middle| \hat{t}_i\right] \mathrm{d}F_{\hat{t}}(\hat{t}_i)}_{\text{Demand from uninformed consumers}} + \underbrace{(1-k) \cdot \int_{-t}^{t} q_{iA,I}\, \mathrm{d}F_t(t_i)}_{\text{Demand from informed consumers}},$$

and its profit is $(1-r)p_A D_A(w)$.

In the symmetric equilibrium, the FOC must be satisfied:

$$p^2 \cdot \frac{k}{t\sigma(1-w)} - p \cdot \left[\frac{4(1-k)}{1+z} - \frac{4kz^2}{1-z^2} + \frac{\alpha k(1+w)}{t\sigma(1-w)} + 4t(1-w)\right] + \frac{2\alpha[2-k(1-z)]}{1+z} + kt\sigma + \frac{\alpha^2 kw}{t\sigma(1-w)} = 0.$$

The equilibrium price is

$$\begin{aligned} p^*(w) = \frac{\alpha}{2k(1-z^2)} \Bigg(& k(1+w)(1-z^2) + 4\tilde{\sigma}(1-w)\big(2-z-k(1-z+z^2)\big) - \Big((1- \\ & w)\Big(k^2(1-w)(1-z^2)^2 + 8k\tilde{\sigma}(1-z^2)\Big((1-k)z + w\big(2-z-k(1-z+z^2)\big)\Big) + \\ & 4\tilde{\sigma}^2\big(4(1-w)(2-z)^2 - 8k(1-w)(2-3z+3z^2-z^3) + k^2(3-8z+14z^2-8z^3+3z^4- \\ & 4w(1-z+z^2)^2)\big)\Big)\Big)^{\frac{1}{2}}\Bigg). \end{aligned}$$

It can be numerically verified that p^* strictly increases with w, and $p^*(1) = \alpha$. Note that $p^*(\frac{1}{2}) = p^{PB*}$ and $p^{PB*} < p^{**} < \alpha$, so there exists a unique $w^{**} > 1/2$ such that $p^*(w^{**}) = p^{**}$.

ANALYSES FOR ALTERNATIVE SPECIFICATIONS

The Case of Unit-demand

Price-neutral recommendation system

The marketplace will recommend product A to an uninformed consumer if and only if her signal $\hat{t} \geq 0$; otherwise the marketplace will recommend product B.

Product A's demand is

$$D_A^{PN}(p_A, p_B) = k \cdot \int_0^{\frac{1}{2}} q_U(A|\hat{t})\, d\hat{t} + (1-k)\left[\frac{1}{2} q_I(A|l = a) + \frac{1}{2} q_I(A|l = b)\right],$$

and the product B's demand is

$$D_B^{PN}(p_A, p_B) = k \cdot \int_{-\frac{1}{2}}^{0} q_U(B|\hat{t})\, d\hat{t} + (1-k)\left[\frac{1}{2} q_I(B|l = a) + \frac{1}{2} q_I(B|l = b)\right].$$

Seller A's profit is $(1-r)p_A D_A^{FB}(p_A, p_B)$ and seller B's profit is $(1-r)p_B D_B^{FB}(p_A, p_B)$.

Maximizing their respective profits yields the equilibrium price, p^{PN*}. We numerically solve the symmetric equilibrium.

To verify whether Proposition 1 and 2 in the main model, i.e., the equilibrium price and the equilibrium profits for the sellers and the marketplace will strictly increase with σ, still qualitatively hold in the discrete-choice model, we have numerically verified that $\frac{\partial p^{PN*}}{\partial \sigma} > 0$ and $\frac{\partial \pi^{PN*}}{\partial \sigma} > 0$ at every grid points of $(\sigma, k, V_L, V_H) = (0.005z_1, 0.1z_2, 0.5z_3, 0.5z_4)$, where z_1, z_2, z_3, z_4 are integers, $0 \le \sigma < 1$, $0 < k < 1$, $-3 \le V_L \le 3$, and $V_L < V_H \le 3$. (It is easy to see that h is only a scale parameter for prices and profits that does not qualitatively influence the results.) In this numerical examination, we have used $\frac{f(\sigma+10^{-5})-f(\sigma)}{10^{-5}}$ to approximate the value of $\frac{\partial f}{\partial \sigma}$, where $f \in \{p^{PN*}, \pi^{PN*}\}$ and the expression π represents either the seller's profit or the marketplace's. Hence, our numerical experiments show that our qualitative results in Proposition 1 and 2 are robust to when consumers have unit demand for the products.

Profit-based recommendation system

Product A's demand is

$$D_A^{PB}(p_A, p_B) = k \cdot \int_{\hat{t}_0}^{\frac{1}{2}} q_U(A|\hat{t})\, d\hat{t} + (1-k)\left[\frac{1}{2} q_I(A|l = a) + \frac{1}{2} q_I(A|l = b)\right],$$

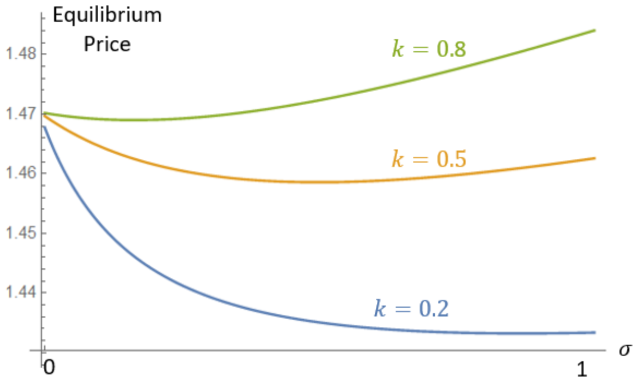
Product B's demand is

$$D_B^{PB}(p_A, p_B) = k \cdot \int_{-\frac{1}{2}}^{\hat{t}_0} q_U(B|\hat{t})\, d\hat{t} + (1-k)\left[\frac{1}{2} q_I(B|l = a) + \frac{1}{2} q_I(B|l = b)\right].$$

Seller A's profit is $(1-r)p_A D_A^{PB}(p_A, p_B)$ and seller B's profit is $(1-r)p_B D_B^{PB}(p_A, p_B)$.

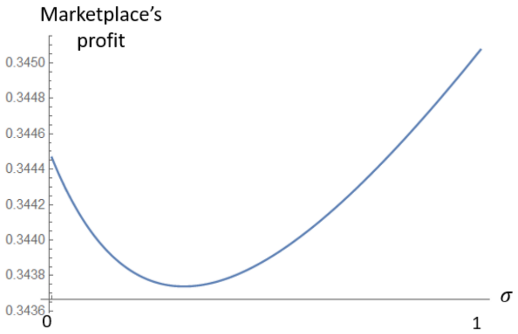
We numerically solve the symmetric equilibrium. Let p^{PB*} be the equilibrium price under the profit-based recommendation system. Figure A1XX shows how the equilibrium price changes with σ. (In this plot, $V_H = 1$, $V_L = 0$.) Similar to Proposition 3 in the main model, the equilibrium price first decreases and then can increase with σ.

Figure A1 Effect of σ on the equilibrium price



Similarly, Figure A2XX shows that it is possible that an increase in σ can decrease the equilibrium profits for the marketplace. (In this plot, $V_H = 1$, $V_L = 0.5$, and $k = 0.2$.) Similarly, a seller's equilibrium profit can also decrease with σ. This is qualitatively the same with Proposition 4 in the main model.

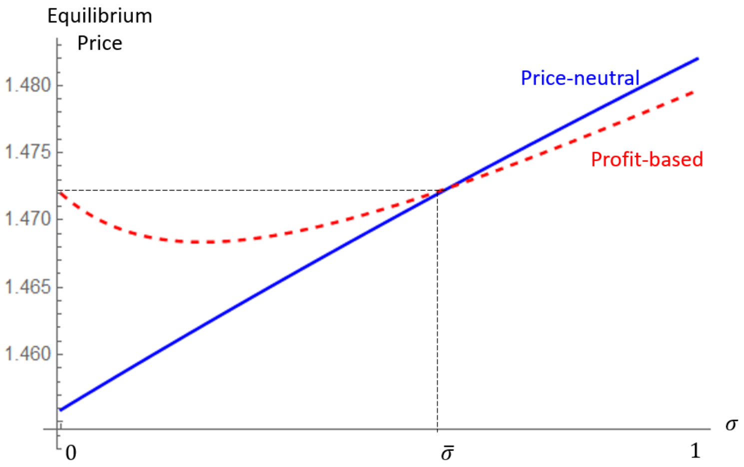
Figure A2 Effect of σ on the equilibrium profit



Comparison between systems

Figure A3 compares the equilibrium prices under the price-neutral recommendation system (p^{PN*}) and under the profit-based system (p^{PB*}). (In this plot, $V_H = 1$, $V_L = -0.5$, $k = 0.5$). This figure shows that $p^{PN*} > p^{PB*}$ if and only if $\sigma > \bar{\sigma}$. Moreover, the equilibrium price under the profit-based recommendation system, p^{PB*}, is the same when $\sigma = \bar{\sigma}$ and when $\sigma \to 0$. This shows that under the profit-based recommendation system, the recommendation-competition effect will drive the seller to set a medium price, which is the equilibrium price when $\sigma \to 0$. The result is qualitatively the same as Proposition 5 in the main model. Similarly, we can verify Proposition 6 also qualitatively hold.

Figure A3 Equilibrium prices under price-neutral and profit-based recommendation systems



Consumer Surplus

Finally, we investigate how consumer surplus changes with σ. It is well known that a consumer's expected surplus is $E[CS_i] = \ln(1 + \sum_{j \in C_i} e^{V_{ij} - hp_j})$, where $j \in \{A, B\}$ and C_i is consumer i's consideration set for the products.

Note that under both the profit-based and the price-neutral recommendation systems, in equilibrium the marketplace will recommend product A (B) to an uninformed consumer if and only if her signal $\hat{t}_i > 0$ ($\hat{t}_i < 0$). Suppose that the equilibrium price is p^* for both firms. For an uninformed consumer with signal $\hat{t}$, the marketplace will recommended the product of expected valuation V_H with probability $Pr_H(\hat{t}) = \frac{1+2\sigma|\hat{t}|}{2}$, and the product of expected valuation V_L with probability $Pr_L(\hat{t}) = \frac{1-2\sigma|\hat{t}|}{2}$. Hence, her expected surplus is $E[CS_{U,i}(\hat{t})] = \frac{1+2\sigma|\hat{t}|}{2}\ln(1 + e^{V_H - hp^*}) + \frac{1-2\sigma|\hat{t}|}{2}\ln(1 + e^{V_L - hp^*})$.

For an informed consumer, since she knows both products, her expected surplus is $E[CS_{I,i}] = \ln(1 + e^{V_H - hp^*} + e^{V_L - hp^*})$.

As a result, the expected total consumer surplus is

$$CS(p^*, \sigma) = k \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} E[CS_{U,i}(\hat{t})] d\hat{t} + (1-k) \cdot E[CS_{I,i}]$$
$$= k\left[\frac{2+\sigma}{4}\ln\left(1 + e^{V_H - hp^*}\right) + \frac{2-\sigma}{4}\ln\left(1 + e^{V_L - hp^*}\right)\right] + (1-k)\ln(1 + e^{V_H - hp^*} + e^{V_L - hp^*}).$$

To verify whether the consumer surplus increases with σ under both types of recommendation systems, we have numerically verified that $\frac{dCS(p^{PN*}, \sigma)}{d\sigma} > 0$ and $\frac{dCS(p^{PB*}, \sigma)}{d\sigma} > 0$ at every grid points of $(\sigma, k, V_L, V_H) = (0.005z_1, 0.1z_2, 0.5z_3, 0.5z_4)$, where z_1, z_2, z_3, z_4 are integers, $0 < \sigma < 1$, $0 < k < 1$, $-3 \leq V_L \leq 3$, and $V_L < V_H \leq 3$,. In this numerical examination, we have used $\frac{f(\sigma+10^{-5})-f(\sigma)}{10^{-5}}$ to approximate the value of $\frac{\partial f}{\partial \sigma}$. Hence, our numerical experiments show that our qualitative results in Proposition 7 are robust to when consumers have unit demand for the products.

In summary, all the results in the main paper hold qualitatively in this alternative discrete choice model. Hence, our insights equally apply to both when consumers can buy multiple units of the products and when consumers have unit demands for the products.

The case of dual-orientation marketplace

Dual-goal recommendation system:

We consider the dual-goal recommendation system that recommends the product with the higher expected dual-goal payoff $w_i = s \cdot CS_i + (1-s) \cdot \Pi_{\mathrm{i}}$ to consumer i.

Conditional on an uninformed consumer's signal $\hat{t}_i$, the marketplace's dual-goal payoff from this consumer will be

$$w_i(A|\hat{t}_i) = \frac{1}{2\beta}[2(1-s)(\alpha - p_A + \sigma\hat{t}_i)rp_A + s[(1-\sigma)(\alpha - p_A)^2 + (1-\sigma)^2 \cdot \frac{t^2}{3} + \sigma(\alpha - p_A + \hat{t}_i)^2]]$$

if product A is recommended. The dual-goal payoff from this consumer will be

$$w_i(B|\hat{t}_i) = \frac{1}{2\beta}[2(1-s)(\alpha - p_B - \sigma\hat{t}_i)rp_B + s[(1-\sigma)(\alpha - p_B)^2 + (1-\sigma)^2 \cdot \frac{t^2}{3} + \sigma(\alpha - p_B - \hat{t}_i)^2]]$$

If product B is recommended.

Let $H(p_A, p_B; s) = \frac{p_A - p_B}{2} \cdot \frac{2(p_A + p_B - \alpha) + \frac{s}{(1-s)r}(2\alpha - p_A - p_B)}{(p_A + p_B) + \frac{s}{(1-s)r} \cdot (2\alpha - p_A - p_B)}$. The marketplace will recommend product A if and only if $\hat{t}_i = \hat{t}_{0,s} \geq \frac{1}{\sigma} \cdot H(p_A, p_B; s)$.

It can be shown that H is convex in p_A and concave in p_B. In a symmetric equilibrium with $p_A^* = p_B^* = p^*$, if $s < \frac{r}{1+r}$, $\hat{t}_{0,s}$ will increase with p_A (at $p_A = p^*$) if and only if $p^* > \frac{1-\frac{s}{(1-s)r}}{2-\frac{s}{(1-s)r}}$; if $s > \frac{r}{1+r}$; $\hat{t}_{0,s}$ strictly decreases with p_A (at $p_A = p^*$) for all $p^* > 0$.

Seller A's profit under the dual-goal recommendation system will be

$$\pi_A^{DG} = \frac{1-r}{\beta} p_A \cdot \left[\frac{k}{2} \left(1 - \frac{H(p_A, p_B; s)}{t\sigma} \right) \left[\alpha + \frac{t\sigma}{2} - p_A + \frac{H(p_A, p_B; s)}{2} \right] + (1 - k) \left[\frac{\alpha}{1+z} - \frac{1}{1-z^2} p_A + \frac{z}{1-z^2} p_B \right] \right]$$

The game can be numerically solved. We numerically checked that there will be a unique solution to FOC that satisfies SOC and yields positive profit. In addition, a sufficient condition for the solution constitutes a symmetric equilibrium is that $\sigma > \underline{\sigma}$.