A Online Appendix

A.1 Proof of Propositions and Corollary in Section 4

Proposition 1

From Table 1, the only scenario that does not result in differentiated ad competition for sponsored ad is scenario (iv) which occurs if $t < v$ and $\beta \leq \max\left\{0, 1 - \frac{t}{v-t}\right\}$. **(i)** Seller profit is positive and increasing in β in the first three scenarios in Table 1. **(ii)** From a comparison of the equilibrium prices under Theorems 1 and 2, seller prices under dynamic bidding are higher in scenarios (ii) and (iii), which occur if $t < v$ and $\beta > \max\left\{0, 1 - \frac{t}{v-t}\right\}$. Further, seller prices are increasing in β in scenario (iii) which occurs if $t < v$ and $1 - \frac{t}{v} > \beta > \max\left\{0, 1 - \frac{t}{v-t}\right\}$.

Proposition 2

Since the mass of consumers is normalized to one, sales volume and conversion rate are equal. The sales volume under fixed bidding is equal to $\frac{v}{2t}$ if $v < 2t$, and 1 if $2t \leq v$. Under dynamic bidding, the sales volume is equal to $\frac{v}{2t}(1+\beta)$ in scenario (i), $\frac{1+\beta}{2}$ in scenario (ii), $\frac{v+t\beta-v\beta}{2t}$ in scenarios (iii), and 1 in scenario (iv). Comparing the sales volumes across the two regimes gives us the result.

Proposition 3

Let $\Pi_P^{A,D}$, $\Pi_P^{S,D}$ and Π_P^{D}, respectively, denote the platform ad revenue under dynamic bidding, platform sales revenue under dynamic bidding and platform total profit under dynamic bidding, in equilibrium. Similarly, let $\Pi_P^{A,F}$, $\Pi_P^{S,F}$ and Π_P^{F}, respectively, denote the platform ad revenue under fixed bidding, platform sales revenue under fixed bidding and platform total profit under fixed bidding, in equilibrium.

From Table 1, if $v < t$ then $\Pi_P^{A,F} - \Pi_P^{A,D} = \frac{(1-\lambda)\beta v^2}{4t} > 0$. If $t \leq v < \min\left\{2t, t + t\frac{\beta}{1-\beta}\right\}$, then $\Pi_P^{A,F} - \Pi_P^{A,D} = \frac{(1-\lambda)}{4t}\left(\beta v^2 + (1-\beta)(v-t)^2\right) > 0$. If $t + t\frac{\beta}{1-\beta} \leq v < 2t$ then $\Pi_P^{A,F} - \Pi_P^{A,D} = \frac{(1-\lambda)\beta\left(t^2(3-\beta)\beta + \left(2tv - v^2\right)(1-\beta)^2\right)}{4t(1-\beta)^2} > 0$. If $2t \leq v < t + t\frac{\beta}{1-\beta}$ then $\Pi_P^{A,F} - \Pi_P^{A,D} = \frac{1-\lambda}{4}\left((2v-t)(1+\beta) - 2t\right) > \frac{1-\lambda}{4}(2v-3t) > 0$. Lastly, if $\max\left\{2t, t + t\frac{\beta}{1-\beta}\right\} \leq v < 2t + t\frac{\beta}{1-\beta}$, then $\Pi_P^{A,F} - \Pi_P^{A,D} = \frac{(1-\lambda)(t-(v-t)(1-\beta))(t\beta(1+\beta)+(1-\beta)(v+v\beta-2t))}{4t(1-\beta)^2}$, which is greater than zero since $\beta > 1 - \frac{t}{v-t}$. Next, from Table 1, if $v < t$ then $\Pi_P^{S,D} - \Pi_P^{S,F} = \frac{\lambda\beta v^2}{4t} > 0$. If $t \leq v < \min\left\{2t, t + t\frac{\beta}{1-\beta}\right\}$, then $\Pi_P^{S,D} - \Pi_P^{S,F} = \frac{\lambda}{4t}\left(\beta\left(2tv - t^2\right) - (v-t)^2\right)$, hence we need to show that $\beta > \frac{(v-t)^2}{2tv-t^2}$. Since $v < t + t\frac{\beta}{1-\beta}$, it suffices to show that $\frac{v-t}{v} > \frac{(v-t)^2}{2tv-t^2}$ which holds true if and only of $v^2 + t^2 < 3tv$. The result then follows since $t \leq v < 2t$. If $t + t\frac{\beta}{1-\beta} \leq v < 2t$, then $\Pi_P^{S,D} - \Pi_P^{S,F} = \frac{1}{4}\beta\lambda\left(\frac{t\beta}{1-\beta} + \frac{2vt-v^2}{t}\right) > 0$.

If $2t \leq v < t + t\frac{\beta}{1-\beta}$, then $\Pi_P^{S,D} - \Pi_P^{S,F} = \frac{\lambda}{4}\left(2t + (1-\beta)(t-2v)\right)$, hence we need to show that $\beta > \frac{2v-3t}{2v-t}$. Since $v < t + t\frac{\beta}{1-\beta}$, it suffices to show that $\frac{v-t}{v} > \frac{2v-3t}{2v-t}$ which always holds true. Lastly, if $\max\left\{2t, t + t\frac{\beta}{1-\beta}\right\} \leq v < 2t + t\frac{\beta}{1-\beta}$, then $\Pi_P^{S,D} - \Pi_P^{S,F} = \frac{\lambda(v - v\beta + t\beta - 2t)^2}{4t(1-\beta)} > 0$. Now, in the first three scenarios of Table 1, we can verify that $\frac{\partial \Pi_P^D}{\partial \beta} < 0$ if λ is not too high, and $\frac{\partial \Pi_P^D}{\partial \beta} > 0$ otherwise. Lastly, note that if $\lambda = 1$ then the platform total profit is equal to sales revenue, and if $\lambda = 0$ then the total profit is equal to ad revenue. Moreover, the platform profit is linear in λ. Therefore, there must be a $\bar{\lambda} < 1$ such that platform is better off under dynamic bidding if $\lambda > \bar{\lambda}$.

Corollary 1

In equilibrium scenario (i), $\bar{\lambda} = \frac{1}{2}$ and is not affected by t or β. In equilibrium scenario (ii), $\bar{\lambda} = \frac{2(1+\beta)v - (3+\beta)t}{2\beta(2v-t)}$ if $v > 2t$ and $\bar{\lambda} = \frac{v^2 - (1-\beta)(2v-t)t}{2\beta(2v-t)t}$ otherwise. In either case, $\hat{\lambda}$ is decreasing in t and β. In equilibrium scenario (iii), $\bar{\lambda} = \frac{\left(1-\beta^2\right)(v-t) - t(1-3\beta)}{2\beta((1-\beta)v + \beta t)}$ if $v > 2t$ and $\bar{\lambda} = \frac{(1-\beta)^2(v-t)^2 - (1+\beta)t^2}{2((1-\beta)v - (2-\beta)t)((1-\beta)v + \beta t)}$ otherwise. In either case, $\bar{\lambda}$ is decreasing in t. It is increasing in β in the latter case, and increasing in the former case if $\frac{\sqrt{3}-1}{\sqrt{3}}v < t < \frac{1}{2}v$ and $\beta > \frac{\left(2 + \sqrt{\frac{2v-4t}{v-t}}\right)}{3t-v}$.

Proposition 4

Let U_F and U_D, respectively, denote the consumer surplus under fixed and dynamic bidding. Note that under fixed bidding, each consumer is matched with a random seller. Let $u(x)$ be the utility that a consumer at location x receives from a random seller, and let $d(x)$ be the distance from that seller. We have $U_F = \mathbb{E}\left[u(x) \,|\, d(x) \leq \bar{d}(p_F^*)\right] \Pr\left(d(x) \leq \bar{d}(p_F^*)\right)$. From Theorem 1, $p_F^* = \frac{v}{2}$ if $v < 2t$, and $p_F^* = v - t$, otherwise. Therefore, U_F is equal to $\frac{v^2}{8t}$ if $v < 2t$ and $\frac{t}{2}$ otherwise. Consider now dynamic bidding. Note that in scenarios (iv) of Table 1, both biding mechanisms result in the same outcomes. Therefore, we will calculate U_D for the the first three scenarios. Let $u_e(x)$ denote the utility that a consumer at location x receives from the seller with whom she is matched in equilibrium, and let $d_e(x)$ be the distance with that seller. We have $U_D = \mathbb{E}\left[u_e(x) \,|\, d_e(x) \leq \bar{d}(p_D^*)\right] \Pr\left(d_e(x) \leq \bar{d}(p_D^*)\right)$. To calculate the first part, we have $\mathbb{E}\left[u_e(x) \,|\, d_e(x) \leq \bar{d}(p_D^*)\right] = \mathbb{E}\left[v - tx - p_D^* | x \leq \frac{v - p_D^*}{t}\right]$, which is equal to $\frac{v}{4}$ in scenario (i), $\frac{t}{4}$ in scenario (ii), and $\frac{v}{4} - \frac{t\beta}{4(1-\beta)}$ in scenario (iii). To calculate $\Pr\left(d_e(x) \leq \bar{d}(p_D^*)\right)$, note that the platform obtains a signal $\widehat{x}$ regarding the location of the consumer. Let $d_e(\widehat{x})$ be the distance between the consumer and the seller with whom she is matched in equilibrium, in the signal apace. We have

$$\begin{aligned}\Pr\left(d_e(x) \leq \bar{d}(p_D^*)\right) = &\Pr\left(d_e(x) \leq \bar{d}(p_D^*) \,|\, d_e(\widehat{x}) \leq \bar{d}(p_D^*)\right) \Pr\left(d_e(\widehat{x}) \leq \bar{d}(p_D^*)\right) \\ &+ \Pr\left(d_e(x) \leq \bar{d}(p_D^*) \,|\, d_e(\widehat{x}) > \bar{d}(p_D^*)\right) \Pr\left(d_e(\widehat{x}) > \bar{d}(p_D^*)\right)\end{aligned}$$

which is equal to $\left(\beta+(1-\beta)\left(\frac{v-p_D^*}{t}\right)\right)2\left(\frac{v-p_D^*}{t}\right)+(1-\beta)\left(\frac{v-p_D^*}{t}\right)\left(1-2\left(\frac{v-p_D^*}{t}\right)\right)$ in scenarios (i) and (ii), and $\left(\beta+(1-\beta)\left(\frac{v-p_D^*}{t}\right)\right)$ in scenario (iii). Substituting for p_D^*, the term $\Pr\left(d_e(x)\leq\bar{d}(p_D^*)\right)$ is equal to $\frac{v(1+\beta)}{2t}$ in scenario (i), $\frac{1+\beta}{2}$ in scenario (ii) and $\frac{v+t\beta-v\beta}{2t}$ in scenario (iii). Therefore, U_D is equal to $\frac{v^2(1+\beta)}{8t}$ in scenario (i), $\frac{t(1+\beta)}{8}$ in scenario (ii) and $\frac{v+t\beta-v\beta}{2t}\left(\frac{v}{4}-\frac{t\beta}{4(1-\beta)}\right)$ in scenario (iii). Comparing U_D against U_F gives us the result.

A.2 Sequential Pricing then Bidding Decisions Under Fixed Bidding

We show that the equilibrium outcomes and results under fixed bidding remain the same if we assume that seller bids are set after they set prices.

Stage 1a. Sellers simultaneously set their prices.

Stage 1b. After observing prices, sellers simultaneously submit their bids.

Stage 2. Consumers visit the platform and the platform allots the sponsored ad position through a second price auction based on the sellers' bids in Stage 2.

Stage 3. Consumers become aware of the seller in the sponsored ad and click on the ad to visit the seller's product page to learn their preference for the seller's product and make their buying decisions.

We proceed by backward induction. In Stage 3, as before, a consumer at location x who visits the product page of the seller featured in the sponsored ad will buy from the seller if her distance $d_i(x)$ from the seller satisfies the condition $d_i(x)<\overline{d}(p_i)$ where $\overline{d}(p_i)$ is given by equation 1. In Stage 2, as before, the conversion rate $\rho_i(p_i)$ for consumers clicking on seller i's sponsored ad is given by $\rho_i(p_i)=\overline{d}(p_i)$. As before, let $\pi_i(p_i)=(1-\lambda)\rho_i(p_i)p_i$ denote seller i's expected sales profit from a consumer who visits its product page upon seeing the sponsored ad. We denote $\pi_i(p_i)$ as seller i's valuation of the sponsored ad, as before.

We now solve for sellers' optimal bidding decisions in Stage 2. As before, for seller i, let b_j denote the rival seller's bid. Seller i's net expected profit (including the cost of the sponsored ad) in Stage 2 is given by

$$\Pi_i^F=\begin{cases}\pi_i(p_i)-b_j, & \text{if } b_i>b_j;\\ \frac{1}{2}\left(\pi_i(p_i)-b_j\right), & \text{if } b_i=b_j;\\ 0, & \text{if } b_i<b_j.\end{cases}$$

Note that equation above looks the same as equation 2. However, here sellers observe rivals' prices before making bidding decisions.

Lemma 2. *In stage 2, the weakly dominant strategy for a seller is to bid its valuation for the sponsored ad.*

The proof for lemma 2 is straightforward and follows from the standard second price auction results. We now solve for sellers' optimal pricing decisions in Stage 1. Plugging the equilibrium bids obtained in stage 3 into the last equation (i.e., $b_i = \pi_i(p_i), \forall i$), seller i's net expected profit in stage 1 is given by

$$\Pi_i^F = \begin{cases} \pi_i(p_i) - \pi_j(p_j) & \text{if } \pi_i(p_i) > \pi_j(p_j); \\ 0, & o.w. \end{cases}$$

Setting a price that maximizes $\pi_i(p_i)$ is (weakly) optimal as it not only maximizes the profit conditional on winning the auction but also maximizes the scope to outbid the rival. We now show that p_F^* given by theorem 1 is the unique symmetric equilibrium price. As shown in the proof of theorem 1, p_F^* maximizes $\pi_i(p_i), i \in \{1,2\}$. Hence both sellers setting their weakly dominant price p_F^* is an equilibrium. To see the uniqueness, assume there is another symmetric equilibrium $\widehat{p} \neq p_F^*$. By equation above, sellers profits are zero in a symmetric equilibrium. However, seller i can deviate to p_F^*, outbid the rival (since $\pi_i(p_F^*) > \pi_j(\widehat{p})$) and make positive profit.

A.3 Pay-Per-Conversion Auction

We analyze the following game:

Stage 1. Sellers simultaneously set their prices and bids.

Stage 2. Consumers visit the platform and the platform allots the sponsored ad through a second price auction based on the sellers' bids in Stage 1.

Stage 3. Consumers become aware of the seller in the sponsored ad and click on the ad to visit the seller's product page to learn their preference for the seller's product and make their buying decisions. **The platforms collects auction revenue only if a consumer makes a purchase.**

We proceed by backward induction. In Stage 3, as before, a consumer at location x who visits the product page of the seller featured in the sponsored ad will buy from the seller if her distance $d_i(x)$ from the seller satisfies the condition $d_i(x) < \overline{d}(p_i)$, where $\overline{d}(p_i)$ is given by equation 1. In Stage 2, as before, the conversion rate $\rho_i(p_i)$ for consumers clicking on seller i's sponsored ad is given by $\rho_i(p_i) = \overline{d}(p_i)$. We now solve for sellers' optimal pricing and bidding decisions in Stage 1. As before, for seller i, let b_j denote the rival's bid. Seller i's net expected profit (including the cost of the sponsored ad) in Stage 1 is given by

$$\Pi_i^F = \begin{cases} (1-\lambda)\,\rho_i\,(p_i)\,(p_i - b_j)\,, & \text{if } b_i > b_j; \\ \frac{1}{2}\,(1-\lambda)\,\rho_i\,(p_i)\,(p_i - b_j)\,, & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases}$$

Note that the equation above is different from equation 2, as the seller in the sponsored ad position pays the platform only for consumers who make a purchase.

Lemma 3. *If $p_i < v$, then it is weakly dominant for seller i to bid its price p_i.*

A seller's valuation is simply its price p_i. The proof of lemma 3 follows from the standard second price auction results. Notice that if $p_i = v$, the seller profit is zero upon winning the auction (as no consumer buys), and therefore it is indifferent between any two bids.

Lemma 4. *In any equilibrium, both sellers set their prices equal to v.*

Proof. Let $\left(\widetilde{p}_i, \widetilde{b}_i\right)$ denote an equilibrium. Suppose toward a contradiction at least one seller does not set a price v. There are two cases to consider. The first case is when both sellers set their prices less than v, i.e., $\widetilde{p}_i < v, \forall i$. By lemma 3, each seller bids its price as a weakly dominant strategy, therefore $\widetilde{b}_i = \widetilde{p}_i, \forall i$. Then, the seller with the lower price has a profitable deviation by outbidding its rival and setting a price between $\max_i \widetilde{p}_i$ and v. The second case is when only seller i sets $\widetilde{p}_i = v$ and seller j sets $\widetilde{p}_j < v$. Note that seller i's profit is zero. By lemma 3, we must have $\widetilde{b}_j = \widetilde{p}_j$. However, now seller i has a profitable deviation by outbidding its rival by setting a price between $\widetilde{p}_j$ and v. ☐

Lemma 5. *In any equilibrium, at least one of the sellers bids v.*

Proof. Let $\left(\widetilde{p}_i, \widetilde{b}_i\right)$ denote an equilibrium. Suppose toward a contradiction that neither seller bids v. Let $\widehat{b} < v$ denote the highest bid. However, now one of the sellers has a profitable deviation by bidding higher than $\widehat{b}$ and setting a price between $\widehat{b}$ and v. ☐

The result of Proposition 5 follows from Lemmas 4 and 5 above.

A.3.1 Assigning Quality Score in Pay-Per-Conversion Mechanism

Suppose the platform uses a quality by using a seller i's conversion rate $\rho_i\,(p_i)$ as its quality score. Let $b_i^a = \rho_i\,(p_i)\,b_i$ denote the effective bid of seller i. The seller with the highest effective bid wins the auction. If seller i wins the auction then it must pay (on a pay-per-conversion basis) $\frac{b_j^a}{\rho_i(p_i)}$ where b_j^a denotes the rival's effective bis. The seller i's net profit is therefore given by

$$\Pi_i^F = \begin{cases} (1-\lambda)\,\rho_i(p_i)\left(p_i - \frac{b_j^a}{\rho_i(p_i)}\right), & \text{if } b_i > \frac{b_j^a}{\rho_i(p_i)}; \\ \frac{1}{2}(1-\lambda)\,\rho_i(p_i)\left(p_i - \frac{b_j^a}{\rho_i(p_i)}\right), & \text{if } b_i = \frac{b_j^a}{\rho_i(p_i)}; \\ 0, & \text{if } b_i < \frac{b_j^a}{\rho_i(p_i)}. \end{cases} \tag{4}$$

Simplifying the terms and replacing $\pi_i(p_i) = (1-\lambda)\,\rho_i(p_i)\,p_i$, we obtain

$$\Pi_i^F = \begin{cases} \pi_i(p_i) - (1-\lambda)\,b_j^a, & \text{if } b_i > \frac{b_j^a}{\rho_i(p_i)}; \\ \frac{1}{2}\left(\pi_i(p_i) - (1-\lambda)\,b_j^a\right), & \text{if } b_i = \frac{b_j^a}{\rho_i(p_i)}; \\ 0, & \text{if } b_i < \frac{b_j^a}{\rho_i(p_i)}. \end{cases}$$

The last equation resembles equation 2, which is seller i's profit using a pay-per-click mechanism under fixed bidding. The following theorem shows that the equilibrium outcomes are in fact the same with that of theorem 1. Let $\left(p_Q^*, b_Q^*\right)$ denote the equilibrium strategy in prices and bids when quality scores are used.

Theorem 3. *Under pay-per-conversion mechanism with quality scores, the unique weakly dominant strategy in pricing and bidding decisions for each seller is* $p_Q^* = b_Q^* = \arg\max_p \pi(p)$. *Therefore, in equilibrium*

(i) If $t \leq \frac{v}{2}$, *all consumers buy a product. Platform sales revenue is* $\lambda(v-t)$ *and ad revenue is* $(1-\lambda)(v-t)$. *Seller profit is zero.*

(ii) If $t > \frac{v}{2}$, *a fraction* $1 - \frac{v}{2t}$ *of consumers do not buy a product. Platform sales revenue is* $\lambda\frac{v^2}{4t}$ *and ad revenue is* $(1-\lambda)\frac{v^2}{4t}$. *Seller profit is zero.*

Proof. The proof is similar to that of theorem 1. We show that $\left(p_Q^*, b_Q^*\right)$ weakly dominates any other strategy $\left(\overline{p}_i, \overline{b}_i\right) \neq \left(p_Q^*, b_Q^*\right)$. Let π_i^* and $\overline{\pi}_i$, respectively, denote seller i's profit under $\left(p_Q^*, b_Q^*\right)$ and $\left(\overline{p}_i, \overline{b}_i\right)$. There are two cases to consider.

Case (1) $\overline{p}_i \neq p_Q^*$: First note $\pi_i^* \geq 0$. This is because under $\left(p_Q^*, b_Q^*\right)$, the seller bids its price, therefore, the profit upon winning cannot be negative by equation 4. Therefore, if $\overline{\pi}_i \leq 0$, then $\pi_i^* \geq \overline{\pi}_i$. Assume now that under $\left(\overline{p}_i, \overline{b}_i\right)$ seller i wins the auction and the profit is positive, i.e., $\overline{\pi}_i = \overline{p}_i \rho_i(\overline{p}_i) - b_j^a > 0$. We have

$$\begin{aligned} b_F^* \rho\left(p_F^*\right)-b_j^a = & p_F^* \rho\left(p_F^*\right)-b_j^a \\ & =\arg \max_p p \rho(p)-b_j^a \\ & >\bar{p}_i \rho_i\left(\bar{p}_i\right)-b_j^a=\bar{\pi}_i>0 \end{aligned}$$

Therefore, seller i must win the auction under $\left(p_Q^*, b_Q^*\right)$ as well, since $b_F^* \rho\left(p_F^*\right)-b_j^a>0$. Moreover, $\pi_i^*>\bar{\pi}_i$.

Case (2) $\bar{p}_i = p_Q^*$, but $\bar{b}_i \neq b_Q^*$: The proof for this case follows from the standard second price auction results. The price is the same under both strategies, and only the bid is different. Therefore, $\left(p_Q^*, b_Q^*\right)$ weakly dominates $\left(\bar{p}_i, \bar{b}_i\right)$, as under $\left(p_Q^*, b_Q^*\right)$ seller i bids its valuation. □

The result of Proposition 6 follows from the theorem above.

A.4 Multiple Sellers

A.4.1 Benchmark: Exogenous Prices

We first investigate the implications of dynamic bidding for an exogenously given price level, p. As before let $\pi(p)=(1-\lambda) \bar{d}(p)$ denote the seller valuation under fixed bidding where $\bar{d}(p)$ is given by equation 1. Also, as before, let $\pi^H(p)=(1-\lambda) \rho^H(p) p$ denote seller valuation under dynamic bidding for consumer signals for whom the predicted conversion rate is high, (i.e., those for whom $\bar{d}(\widehat{x}) \leq \bar{d}(p)$). Similarly, as before let $\pi^L(p)=(1-\lambda) \rho^L(p) p$ denote seller valuation for consumer signals for whom the predicted conversion rate is low, (i.e., those for whom $\bar{d}(\widehat{x})>\bar{d}(p)$). The expressions for $\rho^H(p)$ and $\rho^L(p)$ are given by equation 3. The next lemma characterizes the weakly dominant bidding strategies.

Lemma 6. *Under fixed bidding, it is weakly dominant for a seller to bid it valuation, $\pi(p)$. Under dynamic bidding, a seller optimized bids for a consumer with signal $\hat{x}$ is $\pi^H(p)$ if $\bar{d}(\widehat{x}) \leq \bar{d}(p)$ and $\pi^L(p)$ if $\bar{d}(\widehat{x})>\bar{d}(p)$.*

The proof follows from lemmas 1 and the standard second price auction results. To avoid trivial cases, we assume $p>v-t$ so that the entire market is not covered by a seller. We compare ad revenue across fixed and dynamic bidding first. The next proposition describes the results.

Proposition 10. *For an exogenously given price $p>v-t$, ad revenue is weakly increasing in number of sellers under dynamic bidding. There exists a threshold n^* such that ad revenue is higher under dynamic bidding than under fixed bidding if and only if $n>n^*$.*

Proof. Following lemma 6, under fixed bidding each seller wins the sponsored ad equally likely for all consumers, therefore, ad revenue under fixed bidding is given by

$$\Pi_P^{A,F} = \pi(p) = (1-\lambda)\left(\frac{v-p}{t}\right)p$$

Under dynamic bidding, following lemma 6, each seller bids high ($\pi^H(p)$) for consumer signals within $\frac{v-p}{t}$ of its location, and bids low ($\pi^L(p)$) for others. Therefore, ad revenue under dynamic bidding is given by

$$\Pi_P^{A,D} = \begin{cases} \pi^L(p) & n < \frac{t}{v-p} \\ n\left(\frac{2}{n} - \frac{v-p}{t}\right)\pi^L(p) + \frac{n}{2}\left(2\left(\frac{v-p}{t}\right) - \frac{2}{n}\right)\pi^H(p) & \frac{t}{v-p} \leq n < \frac{2t}{v-p} \\ \pi^H(p) & \frac{2t}{v-p} < n \end{cases}$$

When $n < \frac{t}{v-p}$, there is no overlap in market coverage between two adjacent sellers. Therefore, the second highest bid is low for all consumers. When $\frac{t}{v-p} \leq n < \frac{2t}{v-p}$, there is an overlap in market coverage between adjacent sellers, however, each seller covers less than its distance with its adjacent sellers. As a result, for consumer signals within $\left(\frac{2}{n} - \frac{v-p}{t}\right)$ of each seller the second highest bid is low, and, for consumer signals in the middle of two adjacent sellers the second highest bid is high. The size of the overlapped region between any two adjacent sellers is equal to $\frac{1}{2}\left(2\left(\frac{v-p}{t}\right) - \frac{2}{n}\right)$. Lastly, when $\frac{2t}{v-p} < n$, each seller covers more than its distance with adjacent sellers. Therefore, the second highest bid is high for all consumers. Now, note that $\pi^L(p) < \pi(p) < \pi^H(p)$ for $\beta > 0$. Moreover, rearranging and collecting the terms for $\frac{t}{v-p} \leq n < \frac{2t}{v-p}$ in the last equation above, we obtain $\Pi_P^{A,D} = 2\pi^L(p) - \pi^H(p) + n\left(\frac{v-p}{t}\right)\left(\pi^H(p) - \pi^L(p)\right)$ which is strictly increasing in n. Therefore, there exists $n^* \in \left(\frac{t}{v-p}, \frac{2t}{v-p}\right)$ such that ad revenue is higher under dynamic bidding than under fixed bidding if and only if $n > n^*$, which is consistent with findings in targeted bidding in auctions. ☐

We next compare the conversion rate across the two bidding regimes. The next proposition describes the results.

Proposition 11. *For an exogenously given price $p > v - t$, conversion rate is higher under dynamic bidding than under fixed bidding.*

Proof. Let $\rho_F(p)$ and $\rho_D(p)$, respectively, denote the average conversion rate under fixed bidding and dynamic bidding for the given price p. Under fixed bidding, each seller wins the sponsored ad equally likely for all consumers, therefore,

$$\rho_F\left(p\right)=\overline{d}\left(p\right)=\left(\frac{v-p}{t}\right)$$

Under dynamic bidding, we have,

$$\rho_D\left(p\right)=\begin{cases}n\left(\frac{v-p}{t}\right)\rho^H\left(p\right)+\frac{n}{2}\left(\frac{2}{n}-2\left(\frac{v-p}{t}\right)\right)\rho^L\left(p\right) & n<\frac{t}{v-p}\\ \rho^H\left(p\right) & \frac{t}{v-p}\leq n\end{cases}$$

As discussed above, when $n<\frac{t}{v-p}$, there is no overlap in market coverage between two adjacent sellers. Therefore, consumers who are within (in the signal space) $\left(\frac{v-p}{t}\right)$ of each seller have high conversion rate for the seller on the sponsored ad. The total size of these consumers is equal to $n\left(\frac{v-p}{t}\right)$. However, consumers between two adjacent sellers who are not covered by either seller (in the signal space) have low conversion rate for the seller on the sponsored ad. The total size of these consumers is equal to $\frac{n}{2}\left(\frac{2}{n}-2\left(\frac{v-p}{t}\right)\right)$. It is straightforward to verify that $\rho_D\left(p\right)$ is strictly increasing in n and $\rho_F\left(p\right)<\rho_D\left(p\right)$ for $n=2$. When $\frac{t}{v-p}\leq n$, there is an overlap in market coverage between two adjacent sellers. Therefore, all consumers have high conversion rate for the seller on the sponsored ad, and $\rho_F\left(p\right)<\rho_D\left(p\right)$. □

Corollary 2. *For an exogenously given price $p>v-t$, sales revenue is higher under dynamic bidding than under fixed bidding.*

Since the price is the same across both bidding regimes, dynamic bidding results in higher sales revenue as it results in higher conversion rate.

A.4.2 Endogenous Prices

Fixed Bidding

Our earlier analysis also applies to the case with $n>2$ sellers. Thus, we have that in equilibrium sellers must charge a price $p_F^*=\arg\max_p \pi\left(p\right)$ and set a bid $b_F^*=\pi\left(p_F^*\right)$, resulting in the same outcomes as in Theorem 1.

Dynamic Bidding

Note that if $\beta=0$, then $\pi^H\left(p_i\right)=\pi^L\left(p_i\right)=\pi_i\left(p_i\right)\forall i$, implying that dynamic bidding converges to fixed bidding. Hence, for a smoother exposition, we assume $\beta>0$ without loss of generality. Also, note that for seller i, any strategy with $p_i=0$ or $p_i\geq v$ is weakly dominated and moreover it is straightforward to verify that neither $p_i=0$ nor $p_i\geq v$ can be part of a symmetric equilibrium.

Therefore, we assume $p_i \in (0, v)\,\forall i$ without loss of generality. Since we are looking for symmetric equilibria, we will write down the profit function for seller i, given that all rivals set the same price denoted by p_{-i}.

Lemma 7. *For any* $p_{-i} \in (0, v)$, *there exists a nonempty interval* $\left(L_{p_{-i}}, U_{p_{-i}}\right) \subseteq (0, v)$ *such that* $\pi^H(p_i) > \pi^L(p_{-i})$ *if and only if* $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$.

Proof. It is straightforward to see that $\pi^H(p_i)$ is continuous and strictly quasi-concave in p_i on $(0, v)$. Since $\pi^H(p_{-i}) > \pi^L(p_{-i})$ for $\beta > 0$, there must a nonempty interval $\left(L_{p_{-i}}, U_{p_{-i}}\right)$ such that $\pi^H(p_i) > \pi^L(p_{-i})$ if and only if $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$. $\square$

Lemma 7 states that for any given price by the rivals, a seller can set a price that guarantees winning the sponsored ad for consumers for whom it has the high conversion rate, but the rivals have the low conversion rate. If $p_i \leq L_{p_{-i}}$ or $p_i \geq U_{p_{-i}}$, seller i's profit would be zero. Following Lemma 7, we restrict $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$ without loss of generality. Essentially, any $p_i \notin \left(L_{p_{-i}}, U_{p_{-i}}\right)$ is weakly dominated for seller i. Moreover, since $p_{-i} \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$, by restricting $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$ we will not lose any symmetric equilibrium. To keep the analysis concise, we will write down the expression for seller i's profit only for cases that are relevant for proving the results. Throughout the analysis, we will denote the set of seller i's rivals by S_{-i} (i.e., $S_{-i} = \{1, 2, ..., i-1, i+1, ..., n\}$).

Scenario A: $\boldsymbol{v - \frac{2t}{n} < p_{-i} < v}$

In this scenario, the market coverage of each seller in S_{-i} is less than its distance with adjacent sellers. The following two cases will be relevant in the proof.

Case A-1: $\boldsymbol{p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right) : \bar{d}(p_i) + \bar{d}(p_{-i}) < \frac{2}{n}}$

Under this case, seller i does not have any overlap in market coverage with adjacent sellers. Following Lemma 7, for any $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$ seller i wins the sponsored ad for consumers with $d_i(\widehat{x}) \leq \bar{d}(p_i)$. Depending on which seller wins the sponsored for consumers with $d_i(\widehat{x}) > \bar{d}(p_i)$, there are different sub-cases. The following two sub-cases will be relevant in the proof.

Case A-1-1: $\pi^L(p_i) \leq \pi^L(p_{-i})$, which results in the following profit for seller i,

$$\Pi_i^D(p_i, p_{-i}) = f(p_i, p_{-i}) := \bar{d}(p_i)\left(\pi^H(p_i) - \pi^L(p_{-i})\right) \tag{5}$$

Case A-1-2: $\pi^L(p_{-i}) < \pi^L(p_i) \leq \pi^H(p_{-i})$, which results in the following profit for seller i,

$$\begin{aligned}\Pi_i^D(p_i, p_{-i}) =& g(p_i, p_{-i}) := f(p_i, p_{-i}) \\ &+ \left(\frac{2}{n} - \bar{d}(p_i) - \bar{d}(p_{-i})\right)\left(\pi^L(p_i) - \pi^L(p_{-i})\right) \\ &+ \left(\frac{n-2}{2}\right)\max\left\{0, \frac{2}{n} - 2\bar{d}(p_{-i})\right\}\left(\pi^L(p_i) - \pi^L(p_{-i})\right)\end{aligned} \tag{6}$$

Case A-2: $\boldsymbol{p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right) : \bar{d}(p_i) + \bar{d}(p_{-i}) \geq \frac{2}{n}}$ & $\boldsymbol{d(p_i) < \frac{2}{n}}$

Unlike case A-1, in this case seller i has an overlap in market coverage with adjacent sellers, however, its market coverage is less than its distance with adjacent sellers. By Lemma 7, for any $p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)$ seller i wins the sponsored ad for consumers with $d_i(\widehat{x}) < \frac{2}{n} - \bar{d}(p_{-i})$. Depending on which seller wins the sponsored ad for consumers with $d_i(\widehat{x}) \geq \frac{2}{n} - \bar{d}(p_{-i})$, there are different sub-cases. The following two sub-cases will be relevant in the proof.

Case A-2-1: $\pi^H(p_i) \leq \pi^H(p_{-i})$, which results in the following profit for seller i,

$$\begin{aligned}\Pi_i^D(p_i, p_{-i}) = h(p_i, p_{-i}) :=& \left(\frac{2}{n} - \bar{d}(p_{-i})\right)\left(\pi^H(p_i) - \pi^L(p_{-i})\right) \\ &+ \left(\frac{n-2}{2}\right)\max\left\{0, \frac{2}{n} - 2\bar{d}(p_{-i})\right\}\max\left\{0, \left(\pi^L(p_i) - \pi^L(p_{-i})\right)\right\}\end{aligned} \tag{7}$$

Case A-2-2: $\pi^L(p_i) \leq \pi^H(p_{-i}) < \pi^H(p_i)$, which results in the following profit for seller i,

$$\Pi_i^D(p_i, p_{-i}) = k(p_i, p_{-i}) := h(p_i, p_{-i}) + \left(\bar{d}(p_i) + \bar{d}(p_{-i}) - \frac{2}{n}\right)\left(\pi^H(p_i) - \pi^H(p_{-i})\right) \tag{8}$$

Scenario B: $\boldsymbol{p_{-i} \leq v - \frac{2t}{n}}$

In this scenario, each seller in S_{-i} covers a portion of the market which is more than its distance with adjacent sellers. The following two cases will be relevant in the proof.

Case B-1: $\pi^H(p_i) \leq \pi^H(p_{-i})$, which results in the $\Pi_i^D(p_i, p_{-i}) = 0$

Case B-2: $\pi^L(p_i) \leq \pi^H(p_{-i}) < \pi^H(p_i)$, which results in the following profit for seller i,

$$\Pi_i^D(p_i, p_{-i}) = l(p_i, p_{-i}) := \bar{d}(p_i)\left(\pi^H(p_i) - \pi^H(p_{-i})\right) \tag{9}$$

Lemma 8. *The followings hold true.*

(i) For any $p_{-i} \in (0, v)$, $f(p_i, p_{-i})$ (defined by equation 5) is strictly quasi-concave in p_i on $\left(L_{p_{-i}}, U_{p_{-i}}\right)$. Moreover,

(a) If $v \leq 2t$ *and* $p_{-i} = \frac{v}{2}$*, then* $\frac{v}{2} = \arg\max_{p_i \in \left(L_{p_{-i}}, U_{p_{-i}}\right)} f\left(p_i, p_{-i}\right)$.

(b) If $\bar{d}\left(p_{-i}\right) < 1$ *and* $p_{-i} < \frac{v}{2}$*, then* $\frac{\partial f(p_i, p_{-i})}{\partial p_i}\big|_{p_i = p_{-i}} > 0$ *and* $\frac{\partial g(p_i, p_{-i})}{\partial p_i}\big|_{p_i = p_{-i}} > 0$.

(c) if $\bar{d}\left(p_{-i}\right) < 1$ *and* $p_{-i} > \frac{v}{2}$*, then* $\frac{\partial f(p_i, p_{-i})}{\partial p_i}\big|_{p_i = p_{-i}} < 0$ *and* $\frac{\partial g(p_i, p_{-i})}{\partial p_i}\big|_{p_i = p_{-i}} < 0$.

(ii) $\pi^H\left(p_i\right)$ *is strictly quasi-concave on* $(0, v)$*. Let* $I = [a, b] \subseteq (0, v)$ *and* $m = \max\left\{v - t, \frac{v}{2} + \frac{t\beta}{2(1-\beta)}\right\}$*. Then* $\pi^H\left(p_i\right)$ *is maximized at* $p_i = b$ *if* $b < m$*, is maximized at* $p_i = a$ *if* $m < a$*, and is maximized at* $p_i = m$ *if* $a \leq m \leq b$*.*

The proof of Lemma 8 is straightforward and is skipped for conciseness. We are now ready to prove the equilibrium outcomes of Table 1.

Scenario (i): $\boldsymbol{v < \frac{2t}{n}}$

We first show that $p_D^* = \frac{v}{2}$ is an equilibrium and then show that this is the only one. Fix $p_{-i} = p_D^* = \frac{v}{2}$. Since $v < \frac{2t}{n}$, the profit of seller i comes from scenario A. Let $R_1 = \left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i > 2v - \frac{2t}{n} - p_D^*\right\}$ and $R_2 = \left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i \leq 2v - \frac{2t}{n} - p_D^*\right\}$. If $p_i \in R_1$, seller i's profit comes from case A-1-1 (i.e., given by function f), since $\pi^L\left(p_i\right) \leq \pi^L\left(p_D^*\right) \forall p_i \in \left(L_{p_D^*}, U_{p_D^*}\right)$. If $p_i \in R_2$, seller i's profit comes from case A-2. Note that for any $p_i \in R_2$, we have $p_i \leq 2v - \frac{2t}{n} - p_D^* < p_D^*$. Therefore, by Lemmas 8-(ii), $\pi^H\left(p_i\right) \leq \pi^H\left(p_D^*\right) \forall p_i \in R_2$. As a result, for $p_i \in R_2$ seller i's profit comes from case A-2-1 (i.e., given by function h). Now, by Lemmas 8-(ii), $\Pi_i^D\left(p_i, p_D^*\right)$ is strictly increasing in p_i on R_2. Also, since $p_D^* > 2v - \frac{2t}{n} - p_D^*$ and $p_D^* \in \left(L_{p_D^*}, U_{p_D^*}\right)$, we have $p_D^* \in R_1$. Therefore, by Lemmas 8-(i), $\Pi_i^D\left(p_i, p_D^*\right)$ reaches its maximum on R_1 at $p_i = p_D^*$. To prove the uniqueness, suppose by contradiction that there exists a symmetric equilibrium $p_D^{**} \neq \frac{v}{2}$. If $p_D^{**} > \frac{v}{2}$ then $p_D^{**} > v - \frac{t}{n}$, since $v < \frac{2t}{n}$. Therefore, following the conditions in case A-1-2, there exists some $\epsilon > 0$ such that $\Pi_i^F\left(p_i, p_D^{**}\right) = g\left(p_i, p_D^{**}\right) \forall p_i \in \left(p_D^{**} - \epsilon, p_D^{**}\right]$, however following Lemmas 8-(i), $\frac{\partial g\left(p_i, p_D^{**}\right)}{\partial p_i}\Big|_{p_i = \left(p_D^{**}\right)^-} < 0$. Now assume $p_D^{**} < \frac{v}{2}$, then two cases may happen: (a) If $p_D^{**} < v - \frac{t}{n}$, then following the conditions in case A-2-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i\left(p_i, p_D^{**}\right) = k\left(p_i, p_D^{**}\right) \forall p_i \in \left[p_D^{**}, p_D^{**} + \epsilon\right)$. However by Lemmas 8-(ii), $\frac{\partial k\left(p_i, p_D^{**}\right)}{\partial p_i}\Big|_{p_i = \left(p_D^{**}\right)^+} > 0$. (b) If $v - \frac{t}{n} \leq p_D^{**} < \frac{v}{2}$, then following the conditions in case A-1-2, there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = g\left(p_i, p_D^{**}\right) \forall p_i \in \left[p_D^{**}, p_D^{**} + \epsilon\right)$. However, by Lemmas 8-(i), $\frac{\partial g\left(p_i, p_D^{**}\right)}{\partial p_i}\Big|_{p_i = \left(p_D^{**}\right)^+} > 0$.

Scenario (ii): $\boldsymbol{\frac{2t}{n} \leq v < \frac{2t}{n} + t\frac{\beta}{1-\beta}}$

We first show that $p_D^* = v - \frac{t}{n}$ is an equilibrium and then prove the uniqueness. Fix $p_{-i} = p_D^* = v - \frac{t}{n}$. Note that $\frac{v - p_D^*}{t} < \frac{2}{n}$, hence the profit of seller i comes from scenario A. Let $R_1 =$

$\left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i > p_D^*\right\}$ and $R_2 = \left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i \leq p_D^*\right\}$. If $p_i \in R_1$, then the profit of seller i comes from case A-1-1 (i.e., given by function f). This is because $\frac{v}{2} \leq v - \frac{t}{n} = p_D^*$, hence $\pi^L\left(p_i\right) \leq \pi^L\left(p_D^*\right) \forall p_i \in R_1$. If $p_i \in R_2$, seller i's profit comes from case A-2-1 (i.e., function h). This is because $p_D^* < \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$, hence $\pi^H\left(p_i\right) \leq \pi^H\left(p_D^*\right) \forall p_i \in R_2$ by Lemmas 8-(ii). Now, by Lemmas 8, $\Pi_i^D\left(p_i, p_D^*\right)$ is strictly decreasing in p_i on R_1 and strictly increasing in p_i on R_2. Therefore, $\Pi_i^D\left(p_i, p_D^*\right)$ reaches its maximum at $p_i = p_D^*$. To prove the uniqueness, suppose by contradiction that there exists a symmetric equilibrium $p_D^{**} \neq v - \frac{t}{n}$. If $p_D^{**} > v - \frac{t}{n}$, then following the conditions in case A-1-2, there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = g\left(p_i, p_D^{**}\right) \forall p_i \in (p_D^{**} - \epsilon, p_D^{**}]$. However, by Lemmas 8-(i), $\left.\frac{\partial g\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$. If $v - \frac{2t}{n} < p_D^{**} < v - \frac{t}{n}$ then following the conditions in case A-2-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = k\left(p_i, p_D^{**}\right) \forall p_i \in [p_D^{**}, p_D^{**} + \epsilon)$, however, by Lemmas 8-(ii), $\left.\frac{\partial k\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^+} > 0$. If $p_D^{**} \leq v - \frac{2t}{n}$ then following the conditions in case B-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = l\left(p_i, p_D^{**}\right) \forall p_i \in [p_D^{**}, p_D^{**} + \epsilon)$, however, by Lemmas 8-(ii), $\left.\frac{\partial l\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^+} > 0$.

Scenario (iii): $\boldsymbol{\frac{2t}{n} + t\frac{\beta}{1-\beta} \leq v < \frac{4t}{n} + t\frac{\beta}{1-\beta}}$

Fix $p_{-i} = p_D^* = \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$. The given condition implies that $v - \frac{2t}{n} < p_D^* \leq v - \frac{t}{n}$. Therefore, the profit of seller i comes from scenario A. Let $R_1 = \left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i > 2v - \frac{2t}{n} - p_D^*\right\}$ and $R_2 = \left\{p_i \in \left(L_{p_D^*}, U_{p_D^*}\right) : p_i \leq 2v - \frac{2t}{n} - p_D^*\right\}$. If $p_i \in R_1$, the profit of seller i comes from case A-1-1 (i.e., give by function f), since $\frac{v}{2} < p_D^* \leq 2v - \frac{2t}{n} - p_D^*$, hence $\pi^L\left(p_i\right) < \pi^L\left(p_D^*\right) \forall p_i \in R_1$. If $p_i \in R_2$, seller i's profit comes from case A-2-1 (i.e., give by function h), since $\pi^H\left(p_i\right) \leq \pi^H\left(p_D^*\right) \forall p_i \in \left(L_{p_D^*}, U_{p_D^*}\right)$ by Lemmas 8-(ii). Now, by Lemmas 8-(i), $\Pi_i^D\left(p_i, p_D^*\right)$ is strictly decreasing in p_i on R_1. Moreover, since $p_D^* \leq 2v - \frac{2t}{n} - p_D^*$ and $p_D^* \in \left(L_{p_D^*}, U_{p_D^*}\right)$, $p_D^* \in R_2$. Therefore, $\Pi_i^D\left(p_i, p_D^*\right)$ reaches its maximum at $p_i = \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$ by Lemmas 8-(ii). To prove the uniqueness, suppose by contradiction that there exists a symmetric equilibrium p_D^{**} such that $p_D^{**} \neq \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$. If $p_D^{**} \leq v - \frac{2t}{n}$ then following the conditions in case B-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = l\left(p_i, p_D^{**}\right) \forall p_i \in [p_D^{**}, p_D^{**} + \epsilon)$, however, by Lemmas 8-(ii), $\left.\frac{\partial l\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^+} > 0$. If $v - \frac{2t}{n} < p_D^{**} < \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$, then following the conditions in case A-2-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = k\left(p_i, p_D^{**}\right) \forall p_i \in [p_D^{**}, p_D^{**} + \epsilon)$, however, by Lemmas 8-(ii) $\left.\frac{\partial k\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^+} > 0$. If $\frac{v}{2} + \frac{t\beta}{2(1-\beta)} < p_D^{**} \leq v - \frac{t}{n}$, then following the conditions of case A-2-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = k\left(p_i, p_D^{**}\right) \forall p_i \in (p_D^{**} - \epsilon, p_D^{**}]$, however, by Lemmas 8-(ii) $\left.\frac{\partial k\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$. If $p_D^{**} > v - \frac{t}{n}$, then following the conditions of case A-1-2, there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = g\left(p_i, p_D^{**}\right) \forall p_i \in \left(p_D^{**} - \epsilon, p_D^{**}\right]$, however, by Lemmas 8-(i), $\left.\frac{\partial g\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$.

Scenario (iv): $\boldsymbol{\frac{4t}{n} + t\frac{\beta}{1-\beta} \leq v < 2t + t\frac{\beta}{1-\beta}}$

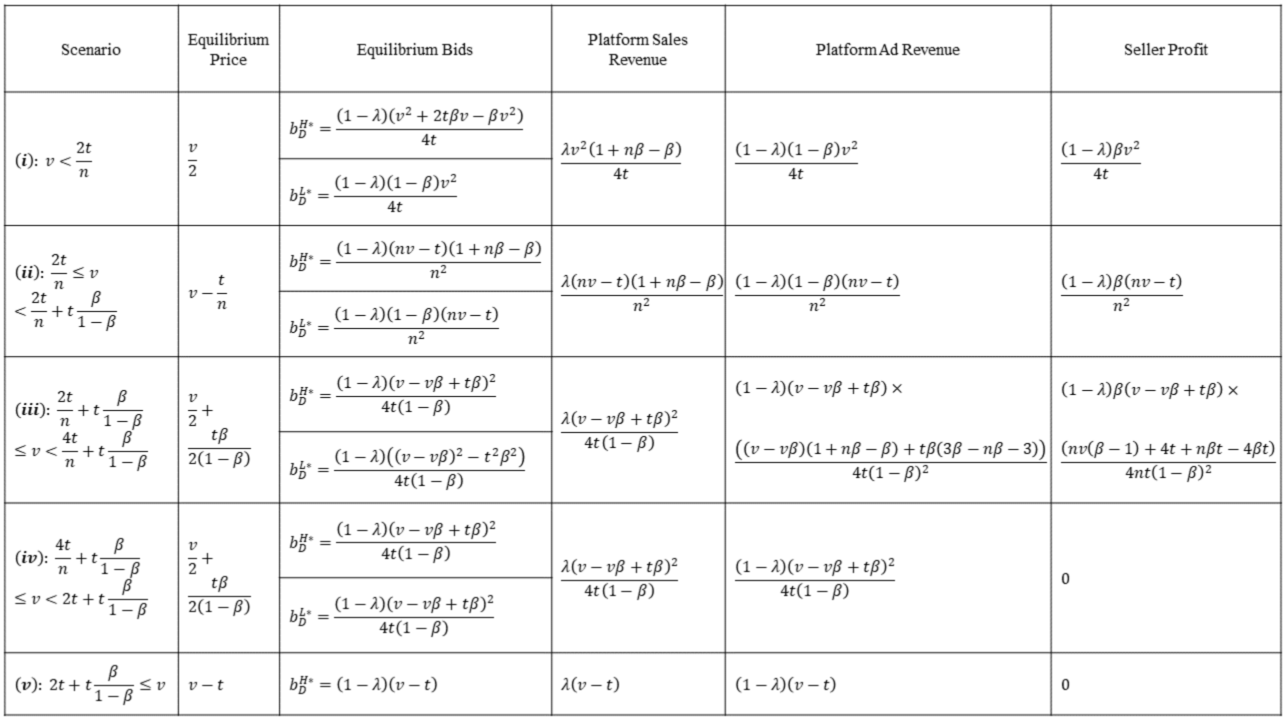
Fix $p_{-i} = p_D^* = \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$. The given condition implies that $v - t < p_D^* \leq v - \frac{2t}{n}$. Therefore, the profit of seller i comes from case B-1, i.e., $\Pi_i^D\left(p_i, p_D^*\right) = 0, \forall p_i \in \left(L_{p_i}, U_{p_i}\right)$, since $\pi^H\left(p_i\right) \leq \pi^H\left(p_D^*\right) \forall p_i \in \left(L_{p_i}, U_{p_i}\right)$ by Lemmas 8-(ii). To prove the uniqueness, suppose by contradiction that there exists a symmetric equilibrium p_D^{**} such that $p_D^{**} \neq \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$. If $p_D^{**} < \frac{v}{2} + \frac{t\beta}{2(1-\beta)}$, then following the conditions in case B-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = l\left(p_i, p_D^{**}\right) \forall p_i \in \left[p_D^{**}, p_D^{**} + \epsilon\right)$, however, by Lemmas 8-(ii) $\left.\frac{\partial l\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^+} > 0$. If $\frac{v}{2} + \frac{t\beta}{2(1-\beta)} < p_D^{**} \leq v - \frac{2t}{n}$, then following the conditions in case B-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = l\left(p_i, p_D^{**}\right) \forall p_i \in \left(p_D^{**} - \epsilon, p_D^{**}\right]$, however, by Lemmas 8-(ii) $\left.\frac{\partial l\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$. If $v - \frac{2t}{n} < p_D^{**} \leq v - \frac{t}{n}$, then following the conditions of case A-2-2 and Lemmas 8-(ii), there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = k\left(p_i, p_D^{**}\right) \forall p_i \in \left(p_D^{**} - \epsilon, p_D^{**}\right]$, however, by Lemmas 8-(ii) $\left.\frac{\partial k\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$. If $p_D^{**} > v - \frac{t}{n}$, then following the conditions of case A-1-2, there exists some $\epsilon > 0$ such that $\Pi_i^D\left(p_i, p_D^{**}\right) = g\left(p_i, p_D^{**}\right) \forall p_i \in \left(p_D^{**} - \epsilon, p_D^{**}\right]$, however, by Lemmas 8-(i), $\left.\frac{\partial g\left(p_i, p_D^{**}\right)}{\partial p_i}\right|_{p_i = \left(p_D^{**}\right)^-} < 0$.

Scenario (v): $\boldsymbol{2t + t\frac{\beta}{1-\beta} \leq v}$

The proof of this scenario is similar to that of scenario (iv) and is skipped for conciseness.

We next derive the equilibrium outcomes. By Lemma 1, the high and low bids, respectively, are given by $\pi^H\left(p_D^*\right) = (1-\lambda)\,\rho^H\left(p_D^*\right)p_D^*$ and $\pi^L\left(p_D^*\right) = (1-\lambda)\,\rho^L\left(p_D^*\right)p_D^*$, where $\rho^H\left(p_D^*\right)$ and $\rho^L\left(p_D^*\right)$ are given by equation 3 in which p_i is replaced with p_D^*. Under scenarios (i) and (ii), platform sales and ad revenue, respectively, are $\frac{\lambda n}{2}\left(2\bar{d}\left(p_D^*\right)\rho^H\left(p_D^*\right)p_D^* + \left(\frac{2}{n} - 2\bar{d}\left(p_D^*\right)\right)\rho^L\left(p_D^*\right)p_D^*\right)$ and $(1-\lambda)\,\rho^L\left(p_D^*\right)p_D^*$. Under scenario (iii), platform sales and ad revenue, respectively, are $\lambda\rho^H\left(p_D^*\right)p_D^*$ and $\frac{n(1-\lambda)}{2}\left(2\left(\frac{2}{n} - \bar{d}\left(p_D^*\right)\right)\rho^L\left(p_D^*\right)p_D^* + \left(\frac{2}{n} - 2\left(\frac{2}{n} - \bar{d}\left(p_D^*\right)\right)\right)\rho^H\left(p_D^*\right)p_D^*\right)$. Under scenario (iv) and (v), platform sales and ad revenue, respectively, are $\lambda\rho^H\left(p_D^*\right)p_D^*$ and $(1-\lambda)\,\rho^H\left(p_D^*\right)p_D^*$. Lastly, in all scenarios, seller profit is equal to $\frac{1}{n}\left(\frac{\text{platform sales revenue}}{\lambda} - \text{platform sales revenue} - \text{platform ad revenue}\right)$. Table 3 summarizes the equilibrium results and outcomes under dynamic bidding for the general case $n \geq 2$.

Table 3: Equilibrium Strategies and Outcomes under Dynamic Bidding for $n \geq 2$ Sellers

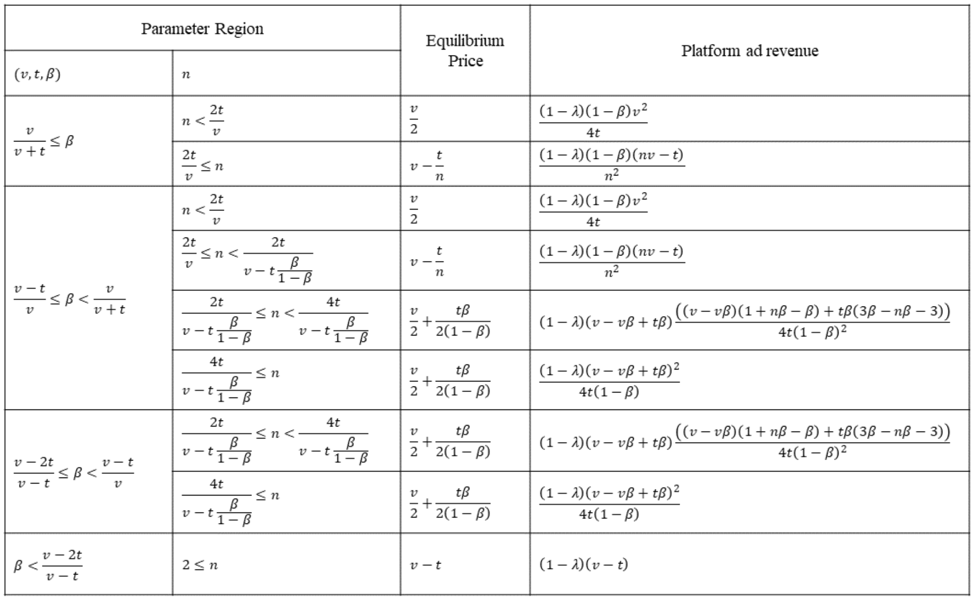


Scenario	Equilibrium Price	Equilibrium Bids	Platform Sales Revenue	Platform Ad Revenue	Seller Profit
(*i*): $v < \frac{2t}{n}$	$\frac{v}{2}$	$b_D^{H*} = \frac{(1-\lambda)(v^2+2t\beta v-\beta v^2)}{4t}$ $b_D^{L*} = \frac{(1-\lambda)(1-\beta)v^2}{4t}$	$\frac{\lambda v^2(1+n\beta-\beta)}{4t}$	$\frac{(1-\lambda)(1-\beta)v^2}{4t}$	$\frac{(1-\lambda)\beta v^2}{4t}$
(*ii*): $\frac{2t}{n} \leq v < \frac{2t}{n} + t\frac{\beta}{1-\beta}$	$v - \frac{t}{n}$	$b_D^{H*} = \frac{(1-\lambda)(nv-t)(1+n\beta-\beta)}{n^2}$ $b_D^{L*} = \frac{(1-\lambda)(1-\beta)(nv-t)}{n^2}$	$\frac{\lambda(nv-t)(1+n\beta-\beta)}{n^2}$	$\frac{(1-\lambda)(1-\beta)(nv-t)}{n^2}$	$\frac{(1-\lambda)\beta(nv-t)}{n^2}$
(*iii*): $\frac{2t}{n} + t\frac{\beta}{1-\beta} \leq v < \frac{4t}{n} + t\frac{\beta}{1-\beta}$	$\frac{v}{2} + \frac{t\beta}{2(1-\beta)}$	$b_D^{H*} = \frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$ $b_D^{L*} = \frac{(1-\lambda)\left((v-v\beta)^2-t^2\beta^2\right)}{4t(1-\beta)}$	$\frac{\lambda(v-v\beta+t\beta)^2}{4t(1-\beta)}$	$(1-\lambda)(v-v\beta+t\beta)\times$ $\frac{\left((v-v\beta)(1+n\beta-\beta)+t\beta(3\beta-n\beta-3)\right)}{4t(1-\beta)^2}$	$(1-\lambda)\beta(v-v\beta+t\beta)\times$ $\frac{(nv(\beta-1)+4t+n\beta t-4\beta t)}{4nt(1-\beta)^2}$
(*iv*): $\frac{4t}{n} + t\frac{\beta}{1-\beta} \leq v < 2t + t\frac{\beta}{1-\beta}$	$\frac{v}{2} + \frac{t\beta}{2(1-\beta)}$	$b_D^{H*} = \frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$ $b_D^{L*} = \frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$	$\frac{\lambda(v-v\beta+t\beta)^2}{4t(1-\beta)}$	$\frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$	0
(*v*): $2t + t\frac{\beta}{1-\beta} \leq v$	$v - t$	$b_D^{H*} = (1-\lambda)(v-t)$	$\lambda(v-t)$	$(1-\lambda)(v-t)$	0

A.4.3 Proof of Proposition 7

We first show that dynamic bidding improves platform's sales revenue. Following proposition 3, it suffices to show that sales revenue under dynamic bidding is (weakly) increasing in n. This is straightforward to see in scenarios (i), (iii) and (iv) in Table 3. Under scenario (ii), taking derivative of $\frac{\lambda(nv-t)(1+n\beta-\beta)}{n^2}$with respect to n, we obtain $\frac{\lambda}{n^3}\left(\beta\left(nv+nt-2t\right)-\left(nv-2t\right)\right)$ which is positive given the conditions of scenario (ii). We next show that under scenario (ii) of Table 3, ad revenue is decreasing in n, and under scenario (iii) ad revenue is increasing in n. Under scenario (ii), taking derivative with respect to n, we get $(1-\lambda)(1-\beta)\left(\frac{2t-nv}{n^3}\right)$ which is negative given the conditions of scenario (ii). Under scenario (iii), taking derivative with respect to n, we get $\frac{(1-\lambda)\beta}{4t(1-\beta)^2}\left(v^2(1-\beta)^2-t^2\beta^2\right)$ which is positive since $\frac{t\beta}{1-\beta} < v$ given the conditions of scenario (iii). Lastly, in the first four scenarios of Table 3, we can verify that the ratio of sales to ad revenue is higher under dynamic bidding than under fixed bidding.

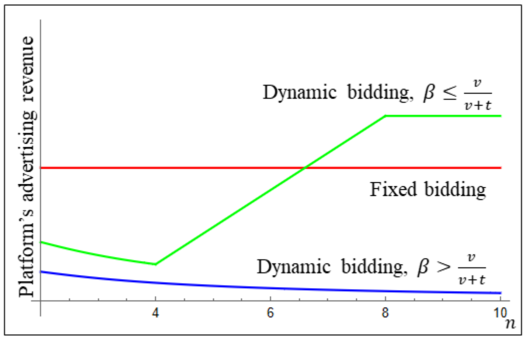
Table 4: Equilibrium Price and Advertising Revenue under Dynamic Bidding



Parameter Region		Equilibrium Price	Platform ad revenue
(v,t,β)	n		
$\frac{v}{v+t} \leq \beta$	$n < \frac{2t}{v}$	$\frac{v}{2}$	$\frac{(1-\lambda)(1-\beta)v^2}{4t}$
	$\frac{2t}{v} \leq n$	$v - \frac{t}{n}$	$\frac{(1-\lambda)(1-\beta)(nv-t)}{n^2}$
$\frac{v-t}{v} \leq \beta < \frac{v}{v+t}$	$n < \frac{2t}{v}$	$\frac{v}{2}$	$\frac{(1-\lambda)(1-\beta)v^2}{4t}$
	$\frac{2t}{v} \leq n < \frac{2t}{v-t\frac{\beta}{1-\beta}}$	$v - \frac{t}{n}$	$\frac{(1-\lambda)(1-\beta)(nv-t)}{n^2}$
	$\frac{2t}{v-t\frac{\beta}{1-\beta}} \leq n < \frac{4t}{v-t\frac{\beta}{1-\beta}}$	$\frac{v}{2}+\frac{t\beta}{2(1-\beta)}$	$(1-\lambda)(v-v\beta+t\beta)\frac{\left((v-v\beta)(1+n\beta-\beta)+t\beta(3\beta-n\beta-3)\right)}{4t(1-\beta)^2}$
	$\frac{4t}{v-t\frac{\beta}{1-\beta}} \leq n$	$\frac{v}{2}+\frac{t\beta}{2(1-\beta)}$	$\frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$
$\frac{v-2t}{v-t} \leq \beta < \frac{v-t}{v}$	$\frac{2t}{v-t\frac{\beta}{1-\beta}} \leq n < \frac{4t}{v-t\frac{\beta}{1-\beta}}$	$\frac{v}{2}+\frac{t\beta}{2(1-\beta)}$	$(1-\lambda)(v-v\beta+t\beta)\frac{\left((v-v\beta)(1+n\beta-\beta)+t\beta(3\beta-n\beta-3)\right)}{4t(1-\beta)^2}$
	$\frac{4t}{v-t\frac{\beta}{1-\beta}} \leq n$	$\frac{v}{2}+\frac{t\beta}{2(1-\beta)}$	$\frac{(1-\lambda)(v-v\beta+t\beta)^2}{4t(1-\beta)}$
$\beta < \frac{v-2t}{v-t}$	$2 \leq n$	$v-t$	$(1-\lambda)(v-t)$

We next rewrite Table 3 in the form of Table 4 so that we can observe how ad revenue is changing with n more clearly. From Table 4, we can verify that ad revenue with respect to n is decreasing if $\beta \geq \frac{v}{v+t}$, initially decreasing and then increasing if $\frac{v}{v+t} > \beta \geq \frac{v-t}{v}$, and increasing if $\frac{v-t}{v} > \beta \geq \frac{v-2t}{v-t}$. Now, following proposition 3, if $\beta \geq \frac{v}{v+t}$ ad revenue is lower under dynamic bidding than under fixed bidding. For $\frac{v-2t}{v-t} \leq \beta < \frac{v}{v+t}$, we will show that there exists a threshold on n above which ad revenue is higher under dynamic bidding than under fixed bidding. Using Tables 1 and 4, we have $\Pi_P^{S,D} = \frac{\lambda}{1-\lambda}\Pi_P^{A,D}$ if $\frac{v-2t}{v-t} \leq \beta < \frac{v}{v+t}$ and $\frac{4t}{v-t\frac{\beta}{1-\beta}} \leq n$. The result then follows by the fact that sales revenue is higher under dynamic bidding than under fixed bidding for $\frac{v-2t}{v-t} \leq \beta$. Figure 4 is an illustrative example of how ad revenue is compared across both bidding regimes.

Figure 4: An Illustrative Example Comparing Ad Revenue Across Both Bidding Regimes



A.5 Strategic Consumers

A.5.1 Fixed Bidding

Under fixed bidding, the profit of seller i is given by the following equation. Note that strategic consumers visit the seller with lower price under fixed bidding.

$$\Pi_i^F(p_i, b_i; p_j, b_j) = \begin{cases} r(\pi_i(p_i) - b_j) & p_i > p_j \quad \& \quad b_i > b_j \\ 0 & p_i > p_j \quad \& \quad b_i < b_j \\ \frac{1}{2} r(\pi_i(p_i) - b_j) & p_i > p_j \quad \& \quad b_i = b_j \\ \pi_i(p_i) - b_j & p_i < p_j \quad \& \quad b_i > b_j \\ (1-r)\pi_i(p_i) & p_i < p_j \quad \& \quad b_i < b_j \\ \frac{1}{2}(\pi_i(p_i) - b_j) + \frac{1}{2}(1-r)\pi_i(p_i) & p_i < p_j \quad \& \quad b_i = b_j \\ \left(r + \frac{1-r}{2}\right)(\pi_i(p_i) - b_j) & p_i = p_j \quad \& \quad b_i > b_j \\ \left(\frac{1-r}{2}\right)\pi_i(p_i) & p_i = p_j \quad \& \quad b_i < b_j \\ \frac{1}{2}\left(r + \frac{1-r}{2}\right)(\pi_i(p_i) - b_j) + \frac{1}{2}\left(\frac{1-r}{2}\right)\pi_i(p_i) & p_i = p_j \quad \& \quad b_i = b_j \end{cases} \tag{10}$$

Lemma 9. *If $r = 0$, the unique symmetric pure equilibrium is $(p_F^*, b_F^*) = (0, 0)$. If $r = 1$, the unique symmetric pure equilibrium is $(p_F^*, b_F^*) = \left(\frac{v}{2}, \frac{(1-\lambda)v^2}{4t}\right)$ for $v \leq 2t$ and $(p_F^*, b_F^*) = (v - t, (1-\lambda)(v-t))$ for $v > 2t$. If $r \in (0, 1)$, there is no equilibrium in pure strategies.*

Proof. If $r = 0$, all consumers are strategic and visit the seller with lower price. Fix $(p_j, b_j) = (p_F^*, b_F^*) = (0, 0)$. Since all consumers visit the seller with lower price, seller i's profit would be zero for all $(p_i, b_i) \in \mathbb{R}_+^2$. To prove the uniqueness, suppose there exists a symmetric equilibrium $(p_F^{**}, b_F^{**}) \neq (0, 0)$. Let Π^{**} denote the seller's profit in this equilibrium, hence by equation 10, $\Pi^{**} = \frac{1}{2}\pi(p_F^{**}) - \frac{1}{4}b_F^{**}$. If $0 < p_F^{**} < v$ and $b_F^{**} > 0$, a seller can deviate to $(p_F^{**}, 0)$ and make $\frac{1}{2}\pi(p_F^{**})$. If $0 < p_F^{**} < v$ and $b_F^{**} = 0$, then $\Pi^{**} = \frac{1}{2}\pi(p_F^{**})$, however, a seller can deviate to $(p_F^{**} - \epsilon, 0)$ and make $\pi(p_F^{**} - \epsilon)$ which is strictly greater than Π^{**} for ϵ small enough. If $p_F^{**} \geq v$ and $b_F^{**} = 0$, then $\Pi^{**} = 0$, however, a seller can deviate to $(p, 0)$ for any $p \in (0, v)$, and make positive profit. If $p_F^{**} \geq v$ and $b_F^{**} > 0$, then $\Pi^{**} = \frac{-1}{4}b_F^{**} < 0$, however a seller can deviate to $(0, 0)$ and make zero profit. Lastly, if $p_F^{**} = 0$ and $b_F^{**} > 0$, then $\Pi^{**} = \frac{-1}{4}b_F^{**} < 0$, however a seller can deviate to $(0, 0)$ and make zero profit. If $r = 1$, all consumers are ad-driven and the proof follows from Theorem 1. For $r \in (0, 1)$, suppose by contradiction there exists an equilibrium in pure strategies, denoted by $\left(\left(\hat{p}_1, \hat{b}_1\right), \left(\hat{p}_2, \hat{b}_2\right)\right)$. Let $\overline{p} = \max\{\hat{p}_1, \hat{p}_2\}$, $\underline{p} = \min\{\hat{p}_1, \hat{p}_2\}$, $\overline{b} = \max\left\{\hat{b}_1, \hat{b}_2\right\}$ and $\underline{b} = \min\left\{\hat{b}_1, \hat{b}_2\right\}$. We prove the results in the following steps. First note that in this equilibrium, it must be that $\overline{p} \leq p_F^* = \arg\max_p \pi(p)$. To see this, suppose $\overline{p} > p_F^*$. Then it is straightforward to see that we must have $\underline{p} = p_F^*$. However, now the seller whose price is $\overline{p}$ can lower its price to p_F^*. Doing so, it obtains more profit from strategic consumers, and it may obtain more profit from ad-driven consumers as well. Next, note that it must be that $\underline{p} = \overline{p}$, since if $\underline{p} < \overline{p}$ the seller whose price is $\underline{p}$ can raise its price and obtain more profit from strategic consumers, and it may also obtain more profit from ad-driven consumers. Hence, let $\widehat{p}_F := \underline{p} = \overline{p}$. It is straightforward to see it must be that $\widehat{p}_F > 0$, therefore $\widehat{p}_F \in (0, p_F^*]$. If $\underline{b} < \overline{b}$, by equation 10, the profit of the seller whose bid is $\overline{b}$ is equal to $\left(r + \frac{1-r}{2}\right)\left(\pi\left(\widehat{p}_F\right) - \underline{b}\right)$. If $\pi\left(\widehat{p}_F\right) \leq \underline{b}$, the profit of the seller whose bid is $\overline{b}$ is less than or equal to zero, however, this seller can lower its bid to zero and make positive profit. If $\underline{b} < \pi\left(\widehat{p}_F\right)$, the seller whose bid is $\overline{b}$ can lower its price slightly to $\widehat{p}_F - \epsilon$ and make $\pi\left(\widehat{p}_F - \epsilon\right) - \underline{b}$ which is greater than $\left(r + \frac{1-r}{2}\right)\left(\pi\left(\widehat{p}_F\right) - \underline{b}\right)$ for ϵ small enough. Therefore, it must be that $\underline{b} = \overline{b}$. Let $\widehat{b}_F := \underline{b} = \overline{b}$. Now, by deviating to the following strategy a seller can increases its profit: choosing $\left(\widehat{p}_F, \widehat{b}_F + \epsilon\right)$ with probability half and $\left(\widehat{p}_F - \epsilon, \widehat{b}_F - \epsilon\right)$ with probability half, where ϵ is small enough. □

Since there is no equilibrium in pure strategies for $r \in (0,1)$, we look for a symmetric mixed equilibrium. In general, the mixed equilibrium is defined by a probability distribution over a two dimensional support (for prices and bids). However, to make the analysis more straightforward, we focus on a family of mixed equilibria with the following form. We assume that sellers mix on prices over some support and for any given price in the support, there is a specific bid given by a function. Put differently, in such an equilibrium, the strategy of each seller is given by (F, S, b) where $F : S \to [0,1]$ is the probability distribution according to which price is mixed, S is the support of this distribution and $b : S \to \mathbb{R}_+$ is the bidding function that maps every price in the support S to a corresponding bid. The following lemma characterizes some properties of such a mixed equilibrium.

Lemma 10. *In any symmetric mixed equilibrium: (a) the equilibrium profit is greater then zero, (b)* $0 \notin S$, *(c)* F *does not admit a mass point in* S, *(d) the supremum of* S *is equal to* $p_F^* = \arg\max_p \pi(p)$, *and (e)* F *does not admit a hole in* S.

Proof. **(a)** First, it is straightforward to show that $\Pr(p_i \geq v) = 0$ for $i = 1, 2$. Then, by definition of mixed equilibrium, $\Pr(p_i > 0) > 0$ for $i = 1, 2$. This means that there exists a positive $\epsilon \in S$ such that $\Pr(p_i \geq \epsilon) > 0$ for $i = 1, 2$, which then implies that at price equal to ϵ, a positive profit from strategic consumers is guaranteed (note that $\epsilon < v$). Therefore, the equilibrium profit must be positive. **(b)** Immediate from part (a) since the profit is zero when price is zero. **(c)** Assume there is a mass point in S. Therefore, there must be a $m \in S$ such that $\Pr(p_i = m) = q > 0$ for $i = 1, 2$. From part (a), $m > 0$. We show that $(m, b(m))$ is strictly dominated by $(m - \epsilon, b(m))$ for ϵ small enough, implying that m cannot be in the support. First note that since the bid is the same at two points, the difference between profit from ad-driven consumers at two points converges to zero as $\epsilon \to 0$. Also, since there is a mass point at m (by contradiction assumption), $(m - \epsilon, b(m))$ generates more clicks from strategic consumers than $(m, b(m))$ by at least $\frac{1-r}{2}q$. Therefore, for ϵ small enough $(m, b(m))$ is strictly dominated by $(m - \epsilon, b(m))$. **(d)** Suppose by contradiction that $\overline{p} := \sup(S) \neq p_F^*$. We show that $(\overline{p}, b(\overline{p}))$ is strictly dominated by $(p_F^*, b(\overline{p}))$. Assume first that $p_F^* < \overline{p}$. Therefore, $\Pr(p_F^* < p_i < \overline{p}) > 0$ for $i = 1, 2$. Note that the size of clicks from ad-driven consumers is the same at both points, while the size of clicks from strategic consumers is higher at $(p_F^*, b(\overline{p}))$ than $(\overline{p}, b(\overline{p}))$, since $\Pr(p_F^* < p_i < \overline{p}) > 0$ (In fact at $(\overline{p}, b(\overline{p}))$ no clicks is generated from strategic consumers, since there is no mass point in S) . Therefore, $(\overline{p}, b(\overline{p}))$ is strictly dominated by $(p_F^*, b(\overline{p}))$ since $\pi(p_F^*) > \pi(\overline{p})$ by definition of p_F^*. Assume now $\overline{p} < p_F^*$, therefore, the profit from strategic consumers is zero at both points (note that there is no mass point). By part (a), the profit from ad-driven consumers is positive at $(\overline{p}, b(\overline{p}))$. This implies that the profit from ad-driven consumers is higher at $(p_F^*, b(\overline{p}))$ than at $(\overline{p}, b(\overline{p}))$, since the bid is the same at both points and $\pi(p_F^*) > \pi(\overline{p})$ by definition of p_F^*. **(e)** Assume by contradiction that there are $l, u \in S$ such that $\Pr(l < p_i < u) = 0$ for $i = 1, 2$. We show that $(l, b(l))$ is strictly dominated by $(u, b(l))$. First note that since the bid is the same at both prices, the size of clicks from ad-driven consumers is equal at two points. Moreover, since there is no mass point in S, the size of clicks from strategic consumers is also equal at two points. The result then follows since $\pi(l) < \pi(u) < \pi(p_F^*)$ by part (d). $\square$

Note that $\pi(p)$ is increasing in p on $[0, p_F^*]$. Therefore given that $\pi(p)$ is a seller's sales profit from a consumer visiting its product page, Lemma 10 suggests that the bidding function $b : S \to \mathbb{R}_+$ should be increasing in p on S. The next lemma formalizes the results. Further, it shows that there is a family of feasible bidding functions.

Lemma 11. *In any symmetric mixed equilibrium, the bidding function* $b : S \to \mathbb{R}_+$ *must be strictly increasing in price. Further, any function* b *such that* $r\pi(p) \leq b(p) \leq \pi(p)$ *can be a symmetric mixed equilibrium.*

Proof. Consider that seller i follows the strategy of the form $(F, S; b)$, where support $S = \left[\underline{p}, p_F^*\right]$. Suppose towards a contradiction $b(p_1) > b(p_2)$ for $p_1, p_2 \in S$ where $p_1 < p_2$. The seller wins fewer sponsored ads if it bids $b(p_2)$ than $b(p_1)$. Then it must either be profitable for the seller to bid $b(p_2)$ instead of $b(p_1)$ for $p_i = p_1$ or bid $b(p_1)$ instead of $b(p_2)$ for $p_i = p_2$. Specifically, suppose that it is not profitable for the seller to deviate from $b(p_1)$ to $b(p_2)$ for $p_i = p_1$. Then, it must be that the additional auctions that the seller wins by bidding higher provide incremental profit. However, if that were the case then it must be also be profitable to bid $b(p_1)$ instead of $b(p_2)$ for $p_i = p_2$: the seller wins all of the auctions it does and at the same profit as when it bids $b(p_2)$ and, in addition, wins additional auctions that are incrementally profitable at a price $p_i = p_2$ since they were incrementally profitable at a price $p_i = p_1$ and $\pi(p_2) > \pi(p_1)$ (which means they are even more profitable at this price). Next, suppose that $b(p_1) = b(p_2)$. Then the seller wins the same auctions at both prices and for the same cost (losing bid). But the seller's profit on each auction it wins is higher for $p_i = p_2$ since $\pi(p_1) < \pi(p_2)$, which is the desired contradiction. Therefore, $b(p)$ must be strictly increasing in p.

Now in any symmetric equilibrium, given that seller i follows the strategy of the form $(F, S; b)$, it has to be optimal for seller j to bid $b(p_j)$ at any price $p_j \in S$. Let b_j denote the bid submitted at price p_j. Then the profit of seller j at p_j is as follows

If $b_j \leq b(p_j)$, (i.e., $b^{-1}(b_j) \leq p_j$), then

$$\Pi_j^F(p_j) = \int_{\underline{p}}^{b^{-1}(b_j)} r\left(\pi_j(p_j) - b(p_i)\right) dF(p_i) + \int_{b^{-1}(b_j)}^{p_j} (0)\, dF(p_i) + \int_{p_j}^{p_F^*} (1-r)\,\pi_j(p_j)\, dF(p_i)$$

and if $b_j \geq b(p_j)$, (i.e., $b^{-1}(b_j) \geq p_j$), then

$$\begin{aligned}
\Pi_j^F(p_j) &= \int_{\underline{p}}^{p_j} r\left(\pi_j(p_j) - b(p_i)\right) dF(p_i) + \int_{p_j}^{b^{-1}(b_j)} \left(\pi_j(p_j) - b(p_i)\right) dF(p_i) \\
&\quad + \int_{b^{-1}(b_j)}^{p_F^*} (1-r)\,\pi_j(p_j)\, dF(p_i) \\
&= \int_{\underline{p}}^{p_j} r\left(\pi_j(p_j) - b(p_i)\right) dF(p_i) + \int_{p_j}^{p_F^*} (1-r)\,\pi_j(p_j)\, dF(p_i) \\
&\quad + \int_{p_j}^{b^{-1}(b_j)} \left(r\pi_j(p_j) - b(p_i)\right) dF(p_i)
\end{aligned}$$

Let $b_j^*(p_j)$ denote the optimal bid at price p_j. In a symmetric equilibrium, we must have $b_j^*(p_j) = b(p_j)$. Therefore, $\Pi_j^F(p_j)$ must be increasing in b_j for $b_j \leq b(p_j)$ and decreasing for $b_j \geq b(p_j)$. From two equations above, we can see that this is the case if and only if $r\pi_j(p_j) \leq b(p_j) \leq \pi_j(p_j)$. □

Lemma 11 shows that the bidding function is strictly increasing in price. Consequently, in equilibrium, the seller with higher price wins the auction. It hence obtains the visits from ad-driven consumers, but no visit from strategic consumers. Lemma 11 further implies that the bidding function is not unique and a continuum of functions satisfy the equilibrium conditions. To continue, we choose $b(p) = \pi(p)$, namely, the most aggressive bidding function. This will be equivalent to selecting the equilibrium in which the platform earns the most profit, and sellers the least profit. We note that the equilibrium F can be obtained for any strictly increasing function b satisfying the conditions in Lemma 11 and our main insights remain qualitatively the same.

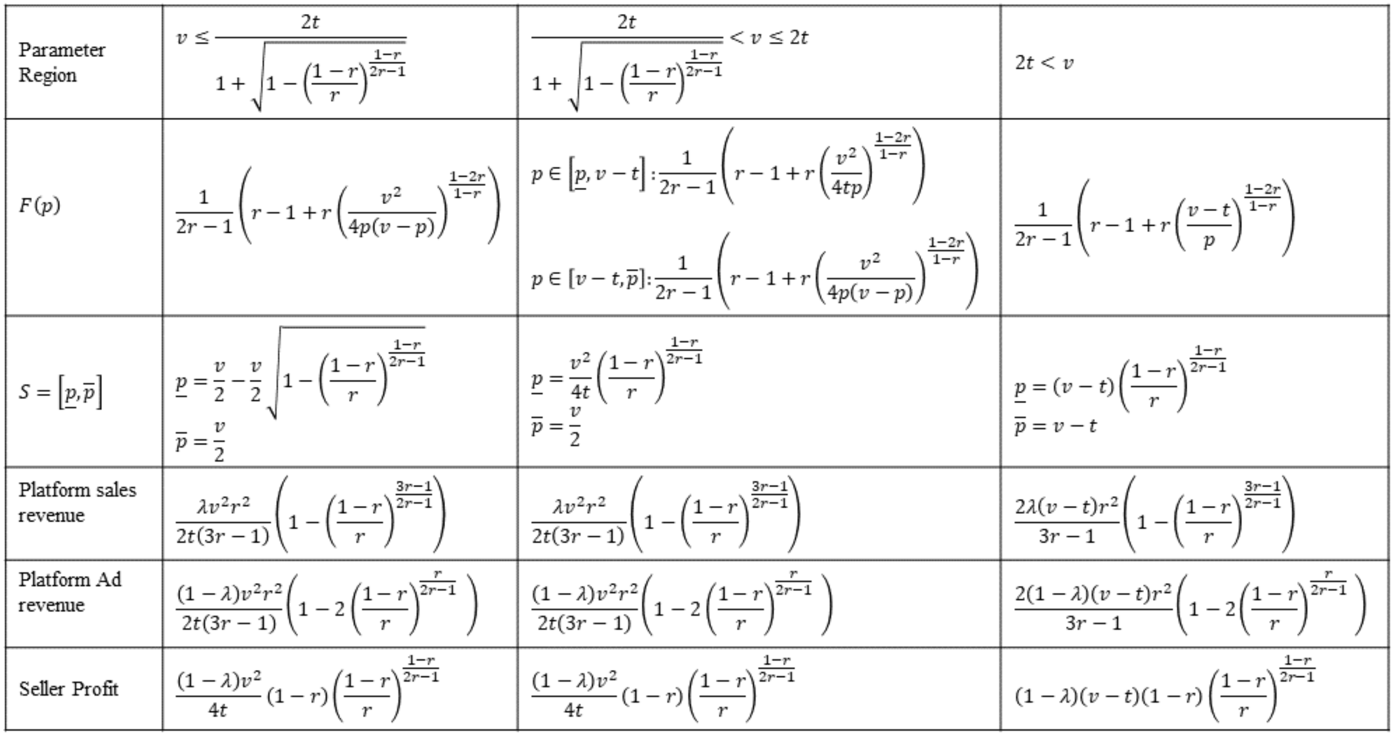
Given that seller i follows the strategy $(F, S; b)$ with $b(p_i) = \pi_i(p_i)\,\forall p_i \in S$, seller j must be indifferent to any price in S. The indifference equation is:

$$\int_{\underline{p}}^{p_j} r\left(\pi_j(p_j) - \pi_i(p_i)\right) f(p_i)\, dp_i + \int_{p_j}^{p_F^*} (1-r)\,\pi_j(p_j) f(p_i)\, dp_i = \Pi_F^*, \forall p_j \in S \tag{11}$$

where $\underline{p}$ is the infimum of S, f is the density function corresponding to distribution F and Π_F^* is the equilibrium seller profit. The first term on the left hand side of equation above corresponds to the situation where the price of seller j is higher then the rival's price in which case seller j wins the auction and obtains the visits from ad-driven consumers, but no visit from strategic consumers. The second term corresponds to the situation where the price of seller j is lower than the rival's price in which case seller j obtains the visits from strategic consumers, but no visit from ad-driven consumers. The next theorem gives the solution to equation above and summarizes the equilibrium results and outcomes.

Theorem 4. *Under fixed bidding, there is a symmetric equilibrium in mixed strategies in which sellers mix on prices on the support* $S = \left[\underline{p}, \overline{p}\right]$ *according to the distribution* $F : S \to [0,1]$ *and for any* $p \in S$*, the equilibrium bid is given by* $b(p) = \pi(p)$*. Table 5 characterizes equilibrium results and outcomes.*

Table 5: Equilibrium Results and Outcomes under Fixed Bidding Considering Strategic Consumers



Parameter Region	$v \leq \frac{2t}{1+\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}}$	$\frac{2t}{1+\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}} < v \leq 2t$	$2t < v$
$F(p)$	$\frac{1}{2r-1}\left(r-1+r\left(\frac{v^2}{4p(v-p)}\right)^{\frac{1-2r}{1-r}}\right)$	$p \in \left[\underline{p}, v-t\right]: \frac{1}{2r-1}\left(r-1+r\left(\frac{v^2}{4tp}\right)^{\frac{1-2r}{1-r}}\right)$ $p \in [v-t, \bar{p}]: \frac{1}{2r-1}\left(r-1+r\left(\frac{v^2}{4p(v-p)}\right)^{\frac{1-2r}{1-r}}\right)$	$\frac{1}{2r-1}\left(r-1+r\left(\frac{v-t}{p}\right)^{\frac{1-2r}{1-r}}\right)$
$S = \left[\underline{p}, \bar{p}\right]$	$\underline{p} = \frac{v}{2} - \frac{v}{2}\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}$ $\bar{p} = \frac{v}{2}$	$\underline{p} = \frac{v^2}{4t}\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$ $\bar{p} = \frac{v}{2}$	$\underline{p} = (v-t)\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$ $\bar{p} = v-t$
Platform sales revenue	$\frac{\lambda v^2 r^2}{2t(3r-1)}\left(1-\left(\frac{1-r}{r}\right)^{\frac{3r-1}{2r-1}}\right)$	$\frac{\lambda v^2 r^2}{2t(3r-1)}\left(1-\left(\frac{1-r}{r}\right)^{\frac{3r-1}{2r-1}}\right)$	$\frac{2\lambda(v-t)r^2}{3r-1}\left(1-\left(\frac{1-r}{r}\right)^{\frac{3r-1}{2r-1}}\right)$
Platform Ad revenue	$\frac{(1-\lambda)v^2 r^2}{2t(3r-1)}\left(1-2\left(\frac{1-r}{r}\right)^{\frac{r}{2r-1}}\right)$	$\frac{(1-\lambda)v^2 r^2}{2t(3r-1)}\left(1-2\left(\frac{1-r}{r}\right)^{\frac{r}{2r-1}}\right)$	$\frac{2(1-\lambda)(v-t)r^2}{3r-1}\left(1-2\left(\frac{1-r}{r}\right)^{\frac{r}{2r-1}}\right)$
Seller Profit	$\frac{(1-\lambda)v^2}{4t}(1-r)\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$	$\frac{(1-\lambda)v^2}{4t}(1-r)\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$	$(1-\lambda)(v-t)(1-r)\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$

Proof. We solve the equation 11 for different parameter regions.

Scenario A: $v \leq 2t$ and $\underline{p} \geq v - t$:

Note that $v \leq 2t$ implies $p_F^* = \frac{v}{2}$. Under this scenario, for any $p_j \in S$, the indifference equation is given by:

$$\int_{\underline{p}}^{p_j} r\left(p_j\left(\frac{v-p_j}{t}\right) - p_i\left(\frac{v-p_i}{t}\right)\right) f(p_i)\, dp_i + \int_{p_j}^{p_F^*} (1-r)\, p_j\left(\frac{v-p_j}{t}\right) f(p_i)\, dp_i = \frac{\Pi_F^*}{1-\lambda}, \forall p_j \in S$$

Taking derivative of equation above with respect to p_j and simplifying and rearranging the terms, we have

$$f(p_j) + \frac{(1-2r)(v-2p_j)}{(1-r)\, p_j (v-p_j)} F(p_j) = \frac{(v-2p_j)}{p_j (v-p_j)}, \quad \forall p_j \in \left[\underline{p}, p_F^*\right]$$

Equation above is a first order differential equation. We will use the integrating factor method to solve equation above. Let $\gamma(p_j) = \frac{(1-2r)(v-2p_j)}{(1-r)p_j(v-p_j)}$ and $\omega(p_j) = \frac{(v-2p_j)}{p_j(v-p_j)}$. Multiplying equation above by $e^{\int \gamma(p_j) dp_j}$, we get

$$f(p_j)\, e^{\int \gamma(p_j) dp_j} + \gamma(p_j)\, F(p_j)\, e^{\int \gamma(p_j) dp_j} = \omega(p_j)\, e^{\int \gamma(p_j) dp_j}$$

Integrating both sides with respect to p_j, we get

$$F\left(p_j\right) e^{\int \gamma(p_j) dp_j} = \frac{r-1}{2r-1} e^{\int \gamma(p_j) dp_j} + K$$

Where K is a constant. Next, multiplying both sides by $e^{-\int \gamma(p_j) dp_j}$ results in

$$F\left(p_j\right) = \frac{r-1}{2r-1} + C \quad \left(p_j\left(v-p_j\right)\right)^{\frac{2r-1}{1-r}} \quad \forall p_j \in \left[\underline{p}, p_F^*\right] \tag{12}$$

Where C is a constant. Lastly, applying the boundary conditions $F\left(p_F^* = \frac{v}{2}\right) = 1$ and $F\left(\underline{p}\right) = 0$, we get $C = \frac{r}{2r-1}\left(\frac{v^2}{4}\right)^{\frac{1-2r}{1-r}}$ and $\underline{p} = \frac{v}{2} - \frac{v}{2}\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}$. Next, applying the condition $\underline{p} \geq v-t$ results in $v \leq \frac{2t}{1+\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}}$.

Scenario B: $\boldsymbol{v \leq 2t}$ and $\boldsymbol{\underline{p} < v-t}$:

Under this scenario, for any $p_j \geq v-t$, the indifference equation is

$$\begin{aligned} &\int_{\underline{p}}^{v-t} r\left(p_j\left(\frac{v-p_j}{t}\right) - p_i\right) f\left(p_i\right) dp_i \\ &+ \int_{v-t}^{p_j} r\left(p_j\left(\frac{v-p_j}{t}\right) - p_i\left(\frac{v-p_i}{t}\right)\right) f\left(p_i\right) dp_i \\ &+ \int_{p_j}^{p_F^*} \left(1-r\right) p_j\left(\frac{v-p_j}{t}\right) f\left(p_i\right) dp_i = \frac{\Pi_F^*}{1-\lambda}, \forall p_j \in \left[v-t, p_F^*\right] \end{aligned}$$

Taking derivative with respect to p_j and simplifying and rearranging the terms, we get

$$f\left(p_j\right) + \frac{\left(1-2r\right)\left(v-2p_j\right)}{\left(1-r\right) p_j\left(v-p_j\right)} F\left(p_j\right) = \frac{\left(v-2p_j\right)}{p_j\left(v-p_j\right)} \quad \forall p_j \in \left[v-t, p_F^*\right]$$

Using the same integrating factor used above, we get

$$F\left(p_j\right) = \frac{r-1}{2r-1} + C_1 \quad \left(p_j\left(v-p_j\right)\right)^{\frac{2r-1}{1-r}} \quad \forall p_j \in \left[v-t, p_F^*\right]$$

Where C_1 is a constant. Next, for any $p_j < v-t$ the indifference equation is:

$$\int_{\underline{p}}^{p_j} r\left(p_j - p_i\right) f\left(p_i\right) dp_i + \int_{p_j}^{p_F^*} \left(1-r\right) p_j f\left(p_i\right) dp_i = \frac{\Pi_F^*}{1-\lambda}, \forall p_j \in \left[\underline{p}, v-t\right)$$

Taking derivative with respect to p_j and simplifying and rearranging the terms, we get

$$f\left(p_j\right) + \frac{\left(1-2r\right)}{\left(1-r\right) p_j} F\left(p_j\right) = \frac{1}{p_j} \quad \forall p_j \in \left[\underline{p}, v-t\right)$$

Using the same integrating factor, we get

$$F\left(p_j\right) = \frac{r-1}{2r-1} + C_2 \quad \left(p_j\right)^{\frac{2r-1}{1-r}} \quad \forall p_j \in p_j \in \left[\underline{p}, v-t\right)$$

Where C_2 is a constant. Note that since there is no mass point in S, from equations A.5.1 and A.5.1, we have $C_1 \left(t\left(v-t\right)\right)^{\frac{2r-1}{1-r}} = C_2(v-t)^{\frac{2r-1}{1-r}}$. This combined with boundary conditions $F\left(p_F^* = \frac{v}{2}\right) = 1$ and $F\left(\underline{p}\right) = 0$ gives us $C_1 = \frac{r}{2r-1}\left(\frac{v^2}{4}\right)^{\frac{1-2r}{1-r}}$, $C_2 = \frac{r}{2r-1}\left(\frac{v^2}{4t}\right)^{\frac{1-2r}{1-r}}$ and $\underline{p} = \frac{v^2}{4t}\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$. Lastly applying the condition $\underline{p} < v-t$ results in $\frac{2t}{1+\sqrt{1-\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}}} < v$.

Scenario C: $2t \leq v$

Note that under $v > 2t$ implies that $p_F^* = v - t$. Under this scenario, for any $p_j \in S$, the indifference equation is:

$$\int_{\underline{p}}^{p_j} r\left(p_j - p_i\right) f\left(p_i\right) dp_i + \int_{p_j}^{p_F^*} \left(1-r\right) p_j f\left(p_i\right) dp_i = \frac{\Pi_F^*}{1-\lambda}, \forall p_j \in S \tag{13}$$

Taking derivative with respect to p_j and simplifying and rearranging the terms, we get

$$f\left(p_j\right) + \frac{\left(1-2r\right)}{\left(1-r\right)p_j} F\left(p_j\right) = \frac{1}{p_j} \quad \forall p_j \in \left[\underline{p}, v-t\right]$$

Using the same integrating factor method, we get

$$F\left(p_j\right) = \frac{r-1}{2r-1} + C \quad \left(p_j\right)^{\frac{2r-1}{1-r}} \quad \forall p_j \in \left[\underline{p}, v-t\right] \tag{14}$$

Where C is a constant. Lastly, applying the boundary conditions $F\left(v-t\right) = 1$ and $F\left(\underline{p}\right) = 0$, we get $C = \frac{r}{2r-1}\left(v-t\right)^{\frac{1-2r}{1-r}}$ and $\underline{p} = \left(v-t\right)\left(\frac{1-r}{r}\right)^{\frac{1-r}{2r-1}}$.

The platform sales revenue, ad revenue and seller profit, respectively, are given by the following equations, ☐

$$\begin{aligned}\Pi_P^{S,F} =& \frac{\lambda}{1-\lambda}\int_{\underline{p}}^{\overline{p}}\left[\int_{\underline{p}}^{p_i}\left(r\pi_i\left(p_i\right) + \left(1-r\right)\pi_j\left(p_j\right)\right) f\left(p_j\right) dp_j \right. \\ &\left. + \int_{p_i}^{\overline{p}}\left(\left(1-r\right)\pi_i\left(p_i\right) + r\pi_j\left(p_j\right)\right) f\left(p_j\right) dp_j\right] f\left(p_i\right) dp_i \\ \Pi_P^{A,F} =& \int_{\underline{p}}^{\overline{p}}\left[\int_{\underline{p}}^{p_i} r\pi_j\left(p_j\right) f\left(p_j\right) dp_j + \int_{p_i}^{\overline{p}} r\pi_i\left(p_i\right) f\left(p_j\right) dp_j\right] f\left(p_i\right) dp_i\end{aligned}$$

$$\Pi_S^F = \left(1-r\right)\pi\left(\underline{p}\right)$$

A.5.2 Dynamic Bidding

We look for a perfect Bayesian equilibrium (PBE) in pure strategies which is symmetric in prices. A PBE consists of sellers strategies and strategic consumers' beliefs at every information set. An information set consists of two prices: the price of the seller on the sponsored position and the price of the other seller. Upon observing both prices, strategic consumers form a belief regarding their expected utility from visiting each seller. Given their beliefs, they proceed rationally and visit the seller with higher expected utility. Also, taking consumers' beliefs into account, each seller best responds to its rival and consumers' beliefs must be consistent with sellers' actions. We will show that in any pure equilibrium where there is differentiated competition for sponsored ads, the sponsored ad is informative and strategic consumers will therefore always visit the seller in the sponsored ad. Moreover, the equilibrium outcome is the same as if all consumers were ad-driven. We lastly verify the conditions that such a pure strategy equilibrium indeed exists and is sustained as a PBE.

As before, let p_D^* denote the symmetric equilibrium price under dynamic bidding.

Informativeness of position: We define the position of sellers to be informative, if there exists $0 < t_1 \leq t_2 < 1$ such that strategic consumers believe that for consumers located in $[2 - t_1, t_1]$ seller 1 bids higher than seller 2, and for consumers located in $[t_2, 2 - t_2]$ seller 2 bids higher than seller 1.

If strategic consumers believe that one seller wins the sponsored ad for all consumers, or each seller wins the sponsored ad for every consumer with probability half, then the position of sellers is not informative. Consequently, they visit the seller with lower price (or each seller with probability half if prices are equal).

Lemma 12. *Suppose there exists a pure symmetric equilibrium in which $p_D^* > \max\{0, v - t\}$. If $r > 0$, then in equilibrium strategic consumers believe the position of sellers is informative.*

Proof. Suppose by contradiction that this is not true. Since prices are the same for both sellers, strategic consumers visit each seller with probability half. Therefore, by being on the sponsored and non-sponsored positions, each seller receives $r + \frac{1-r}{2}$ and $\frac{1-r}{2}$ clicks, respectively, implying that the size of incremental clicks received by being on the sponsored position is r for each seller. Therefore, the optimal high and low bids for each seller, respectively, are $b_D^H = (1 - \lambda) \frac{r}{r + \frac{(1-r)}{2}} p_D^*$ and $b_D^L = 0$. Since $p_D^*, r > 0$, we have $b_D^H > b_D^L = 0$. Therefore, the position must be informative since $p_D^* > v - t$. $\square$

Corollary 3. *Suppose there exists a pure symmetric equilibrium in which* $p_D^* > \max\{0, v-t\}$*. If* $r > 0$*, then in equilibrium strategic consumers visit the seller on the sponsored position.*

The proof of Corollary 3 is straightforward following Lemma 12. We next need to put structure on consumers' beliefs off the equilibrium path. One can simplify the analysis by assuming that consumers have passive beliefs, meaning that upon observing off equilibrium prices, they hold the same belief as upon observing the equilibrium prices. However, such a belief may not be consistent with sellers' optimal actions if those prices were to be chosen. Therefore, following a more stringent approach, we require that consumers' beliefs upon observing off-equilibrium prices to be consistent with sellers' optimal bidding decisions, if those prices were to be chosen by sellers. Under this assumption, we can extend Lemma 12 as follows.

Lemma 13. *Assume* $r > 0$ *and let* p_S *and* p_{NS}*, respectively, be the observed prices on the sponsored and non-sponsored positions such that* $p_S > p_{NS} > 0$ *and* $p_S > v - t$*. Then, strategic consumers believe that the position of sellers is informative.*

Proof. For convenience, let seller 1 be on the sponsored position and seller 2 be on the non-sponsored position. Suppose by contradiction that strategic consumers believe the position is not informative. Therefore, they visit seller 2, since it has a lower price. Note that upon observing p_{NS} on the sponsored position and p_S on the non-sponsored position, strategic consumers visit the seller on the sponsored position, whether or not they believe the position is informative. Therefore, the size of incremental clicks that seller 1 obtains by being on the sponsored position is $r - 0 = r$, and the size of incremental clicks that seller 2 obtains by being on the sponsored position is $1 - (1 - r) = r$. As a result, for seller 1 the optimal high and low bids, respectively, are $b_1^H = (1 - \lambda)\frac{r}{r}p_S$ and $b_1^L = 0$. Also, for seller 2, the optimal high and low bids, respectively, are $b_2^H = (1 - \lambda)\frac{r}{1}p_{NS}$ and $b_2^L = 0$. Since $P_S > 0$, $P_{NS} > 0$ and $r > 0$, we have $b_1^H > b_2^L = 0$ and $b_2^H > b_1^L = 0$. Therefore, the position of sellers must be informative, since $p_S > v - t$. □

Corollary 4. *Assume* $r > 0$ *and let* p_S *and* p_{NS}*, respectively, be the observed prices on the sponsored and non-sponsored positions, such that* $p_S > p_{NS} > 0$ *and* $p_S > v - t$*. There exists an* $\epsilon > 0$ *such that if* $p_S - p_{NS} < \epsilon$*, then strategic consumers visit the seller on the sponsored position.*

Proof. The proof is straightforward following Lemma 13. □

Corollary 4 has an important implication. If under dynamic bidding a pure equilibrium exists in which $p_D^* > v - t$, it must be the same equilibrium as under $r = 1$. This is because by Corollary 4, for a local deviation from the candidate equilibrium price, all consumers visit the sponsored seller.

Theorem 5. *There exists $\bar{r} < 1$ such that for $r \geq \bar{r}$ a unique symmetric equilibrium in pure strategies exists under dynamic bidding, where $\bar{r} = 0$ for $t \geq \frac{2}{3}v$ and $\bar{r} = \frac{2v-3t}{2(v-t)}$ otherwise. In equilibrium, all consumers visit the seller on the sponsored position. The equilibrium outcomes are given by theorem 2 for $\beta = 1$.*

Proof. Following Corollary 4 and Theorem 2, the candidate symmetric equilibrium price is $p_D^* = \frac{v}{2}$ if $v < t$ and $p_D^* = v - \frac{t}{2}$ if $t \leq v$.[20] Let p_d denote the price of a seller who deviates from the candidate equilibrium price. For the deviation to be profitable, strategic consumers must expect at least as much utility from visiting the seller on the non-sponsored position as from visiting the seller on the sponsored position. Note that if $r = 1$, no deviation is profitable by Theorem 2. Therefore, to show the existence of $\bar{r}$, it suffices to show that the profit under deviation is decreasing in r. Let $\underline{p_d}$ be the price on the non-sponsored position that makes strategic consumers indifferent between visiting either seller. Note that $\underline{p_d} < p_D^*$. For any $p_d < \underline{p_d}$, the deviation profit would be

$$\pi_d = \begin{cases} \frac{1}{2}p_d + \left(\bar{d}\left(p_d\right) - \frac{1}{2}\right)\left(1-r\right)p_d & \text{if} \quad p_D^* = v - \frac{t}{2} \\ \bar{d}\left(p_d\right)p_d & \text{if} \quad p_D^* = \frac{v}{2} \quad \& \quad \bar{d}\left(p_d\right) + \bar{d}\left(p_D^*\right) \leq 1 \\ \left(1 - \bar{d}\left(p_D^*\right)\right)p_d + \left(\bar{d}\left(p_d\right) + \bar{d}\left(p_D^*\right) - 1\right)\left(1-r\right)p_d & \text{if} \quad p_D^* = \frac{v}{2} \quad \& \quad \bar{d}\left(p_d\right) + \bar{d}\left(p_D^*\right) > 1 \end{cases}$$

Note that if $v < t$ (in which case the equilibrium price is $\frac{v}{2}$), the deviation profit π_d can be written as $\bar{d}\left(p_d\right)p_d - rp_d\left(\max\left\{0, \bar{d}\left(p_d\right) + \bar{d}\left(p_D^*\right) - 1\right\}\right)$. Moreover, $\bar{d}\left(p_d\right)p_d$ is maximized if $p_d = \frac{v}{2}$. Therefore, deviation is never profitable in this case. If $t \leq v$ (in which case the equilibrium price is $v - \frac{t}{2}$), the deviation can be profitable. Let us first calculate $\underline{p_d}$. The expected utility from visiting the seller on the sponsored ad is equal to $v - p_D^* - t\frac{1}{4} = \frac{t}{4}$. Also, for any $p < p_D^* = v - \frac{t}{2}$ the expected utility from visiting the deviating seller is equal to $\left(\frac{\frac{v-p}{t} - \frac{1}{2}}{\frac{1}{2}}\right)\left(v - p - \frac{\frac{1}{2} + \frac{v-p}{t}}{2}\right)$ which is less than $\frac{t}{4}$ if $p < v - t$. Therefore $\underline{p_d} = v - t$ and the deviation profit is bounded above by $\frac{1}{2}\left(v-t\right) + \frac{1}{2}\left(v-r\right)\left(1-r\right)$ which is less than the equilibrium profit, $\left(\frac{v}{2} - \frac{t}{4}\right)$, if $v \leq \frac{3t}{2}$ or, $v > \frac{3t}{2}$ and $r > \frac{3t-2v}{2t-2v}$. $\square$

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[20] Note that $p_D^* \leq v - t$ cannot be an equilibrium. This is because in such equilibrium sellers' profits would be zero. However, a seller can deviate by setting a price greater than $v - t$ and obtain positive profit.

Proof of Proposition 8

Follows from Theorems 4 and 5.

Proof of Proposition 9

Note that the platform does not extract the entire surplus under fixed bidding with the presence of strategic consumers, moreover the sales revenue per consumer is hurt because of intensified price competition. Also, sellers make positive profit under fixed bidding with the presence of strategic consumers. The result then follows by the fact that the equilibrium outcomes under dynamic bidding remain the same with the presence of strategic consumers.

A.6 Discrete Segments of Consumers

In this section, we model product fit with consumer tastes using a discrete segments of consumers along the lines of the model in Zhou and Zou (2025). There are two sellers, 1 and 2, and two groups of consumers. One group with size α has a match probability q_h with sellers 1 and q_l with seller 2, where $q_h > q_l$. The other group with size $1-\alpha$ has a match probability q_l with sellers 1 and q_h with seller 2. Therefore, consumers in group 1 have a higher match probability with seller 1 than with seller 2; conversely, consumers in group 2 have a higher match probability with seller 2 than with seller 1. Consumers obtain a valuation v from purchasing a product upon a match; and valuation zero, otherwise. Therefore, the utility that a consumer obtains from purchasing the product of seller $i \in \{1,2\}$ is equal to $u_i = v - p_i$ upon a match; and $u_i = -p_i$, otherwise. We assume $\alpha > \frac{1}{2}$, implying that more consumers are matched with seller 1 than with seller 2. We call seller 1 the *superior seller* and seller 2 the *inferior seller*. The platform has granular information about consumers' match likelihood with a seller, which is used in optimizing the seller's bid under dynamic bidding. In particular, the platform has granular information regarding which group a consumer belongs to, which we model as follows. Let $g \in \{1,2\}$ denote a group of consumers. For each consumer, the platform receives a signal $s \in \{1,2\}$ about which group the consumer belongs to. Similar to the main model, we allow for the signal to be imperfect. With probability $\beta \in [0,1]$, the signal received by the platform is accurate, i.e., $s = g$; and, with probability $1-\beta$, the signal reveals no information beyond the prior. Therefore, $\Pr(s=1|g=1) = \beta + (1-\beta)\alpha$ and $\Pr(s=2|g=2) = \beta + (1-\beta)(1-\alpha)$. We next analyze the equilibrium outcomes and results under each bidding regime.

A.6.1 Fixed Bidding

The sequence of moves is the same as before. In Stage 3, a consumer who visits the product page of the seller featured in the sponsored ad will buy from the seller if she obtains positive utility from the seller's product. A consumer obtains positive utility from seller i if it is matched with seller i and $p_i < v$. In Stage 2, seller i wins the sponsored ad auction for all consumers if $b_i > b_j$, and for $\frac{1}{2}$ proportion of consumers if $b_i = b_j$. For seller 1, the conversion rate $\rho_1(p_1)$ for consumers clicking on seller 1's sponsored ad is $\rho_1(p_1) = \alpha q_h + (1-\alpha) q_l$ if $p_1 \leq v$, and $\rho_1(p_1) = 0$ otherwise. Similarly, the conversion rate $\rho_2(p_2)$ for consumers clicking on seller 2's sponsored ad is $\rho_2(p_2) = \alpha q_l + (1-\alpha) q_h$ if $p_2 \leq v$, and $\rho_2(p_2) = 0$ otherwise. As before, let $\pi_i(p_i) = (1-\lambda)\rho_i(p_i) p_i$ denote seller i's expected sales profit from a consumer who visits its product page upon seeing the sponsored ad, which we will refer to as seller i's *valuation* of the sponsored ad, as before. We now solve for sellers' optimal pricing and bidding strategies in Stage 1. Seller i's net expected profit (including the cost of the sponsored ad) is given by

$$\Pi_i^F(p_i) = \begin{cases} \pi_i(p_i) - b_j, & \text{if } b_i > b_j; \\ \frac{1}{2}(\pi_i(p_i) - b_j), & \text{if } b_i = b_j; \\ 0, & \text{if } b_i < b_j. \end{cases} \tag{15}$$

The following theorem establishes that there is a unique equilibrium. Let p_F^{1*} and p_F^{2*}, respectively, denote the equilibrium price of seller 1 and 2 under fixed bidding. Also, let b_F^{1*} and b_F^{2*}, respectively, denote the equilibrium bid of seller 1 and 2 under fixed bidding.

Theorem 6. *Under fixed bidding, there is a unique equilibrium in pricing and bidding strategies as follows.* $p_F^{1*} = p_F^{2*} = v$, $b_F^{1*} = (1-\lambda)(\alpha q_h + (1-\alpha) q_l) v$ *and* $b_F^{2*} = (1-\lambda)(\alpha q_l + (1-\alpha) q_h) v$. *Seller 1 wins the sponsored ad auction. All consumers make a purchase. Platform sales revenue is* $\lambda(\alpha q_h + (1-\alpha) q_l) v$ *and ad revenue is* $(1-\lambda)(\alpha q_l + (1-\alpha) q_h) v$. *Seller 1 profit is* $(1-\lambda)(2\alpha - 1)(q_h - q_l) v$ *and seller 2 profit is zero.*

Proof. First note given its price, it is weakly dominant for seller i to bid its valuation, $\pi_i(p_i)$, following the standard results in second price auctions. Moreover, the conversion rate $\rho_i(p_i)$ does not depend on the price and $\pi_i(p_i)$ is increasing in p_i. Therefore, it is weakly dominant for a seller to set its price equal to v, since by doing so it simultaneously maximizes the scope of outbidding the rival and the profit upon winning the auction. Therefore, seller 1 bid in equilibrium is $b_F^{1*} =$

$\pi_1(v) = (1-\lambda)(\alpha q_h + (1-\alpha) q_l) v$ which is greater than seller 2 in equilibrium $b_F^{2*} = \pi_2(v) = (1-\lambda)(\alpha q_l + (1-\alpha) q_h) v$, since $\alpha > \frac{1}{2}$ and $q_h > q_l$. The profits are calculated accordingly. $\square$

A.6.2 Dynamic Bidding

The sequence of moves is as before. In stage 4, as before, a consumer who visits the product page of the seller featured in the sponsored ad will buy from the seller if she obtains positive utility from the seller's product. A consumer obtains positive utility from seller i if it is matched with seller i and $p_i < v$. In stage 3, consider a consumer with signal $s \in \{1,2\}$ who visits seller i's product page by clicking on the seller's sponsored ad. Using the Bayes' rule, we have $\Pr(g=1|s=1) = \beta + (1-\beta)\alpha$ and $\Pr(g=2|s=2) = \beta + (1-\beta)(1-\alpha)$. Let $\rho_i(p_i, s)$ denote the platform's predicted conversion rate for this consumer upon clicking on seller i's sponsored ad. we have

$$\rho_1(p_1, s) = \begin{cases} q_h(\beta + (1-\beta)\alpha) + q_l(1-\beta)(1-\alpha), & \text{if } s = 1 \text{ and } p_i \le v; \\ q_l(\beta + (1-\beta)(1-\alpha)) + q_h(1-\beta)\alpha, & \text{if } s = 2 \text{ and } p_i \le v; \\ 0, & \text{if } p_i > v. \end{cases} \tag{16}$$

and

$$\rho_2(p_2, s) = \begin{cases} q_l(\beta + (1-\beta)\alpha) + q_h(1-\beta)(1-\alpha), & \text{if } s = 1 \text{ and } p_i \le v; \\ q_h(\beta + (1-\beta)(1-\alpha)) + q_l(1-\beta)\alpha, & \text{if } s = 2 \text{ and } p_i \le v; \\ 0, & \text{if } p_i > v. \end{cases} \tag{17}$$

Let $b_D^{1*}(s)$ and $b_D^{2*}(s)$, respectively, denote the optimized bids of seller 1 and 2 for a consumer with signal s. The following lemma characterizes the equilibrium bids in Stage 2.

Lemma 14. *Under dynamic bidding, seller i's optimized bids for a consumer with signal s is* $b_D^{i*} = (1-\lambda)\rho_i(p_i, s) p_i$.

Proof. Consider a consumer for whom the signal s is received by the platform. Let $b_i(s)$ denote seller i's bid for this consumer. Then seller i's expected profit for this consumer (including the cost of the sponsored ad) is

$$\pi_i(p_i, s) = \begin{cases} (1-\lambda)\rho_i(p_i, s) p_i - b_j(s), & \text{if } b_i(s) > b_j(s); \\ \frac{1}{2}((1-\lambda)\rho_i(p_i, s) p_i - b_j(s)), & \text{if } b_i(s) = b_j(s); \\ 0, & \text{o.w.} \end{cases}$$

The result then follows from standard second price auction results. □

We next analyze sellers' pricing decisions in Stage 1. The following theorem summarizes the results under dynamic bidding. Let p_D^{1*} and p_D^{2*}, respectively, denote the equilibrium prices of seller 1 and 2 under dynamic bidding.

Theorem 7. *Under dynamic bidding, there is a unique equilibrium in pricing and bid strategies as follows.* $p_D^{1*} = p_D^{2*} = v$ *and* $b_D^{i*}(s) = (1-\lambda)\rho_i(v,s)v$. *In equilibrium:*

(i) If $\beta \leq \frac{2\alpha-1}{2\alpha}$, *seller 1 wins the auction for all consumers. Seller 1 profit is* $(1-\lambda)(2\alpha-1)(q_h-q_l)v$ *and seller 2 profit is zero. Platform sales revenue is* $\lambda(\alpha q_h+(1-\alpha)q_l)v$ *and ad revenue is* $(1-\lambda)(\alpha q_l+(1-\alpha)q_h)v$.

(ii) If $\beta > \frac{2\alpha-1}{2\alpha}$, *seller 1 wins the auction for consumers with signal* $s=1$ *and seller 2 wins the auction for consumers with signal* $s=2$. *Seller 1 profit is* $(1-\lambda)\alpha(2\beta-1+2\alpha(1-\beta))v(q_h-q_l)$ *and seller 2 profit is* $(1-\lambda)(1-\alpha)(1-2\alpha(1-\beta))v(q_h-q_l)$. *Platform sales revenue is* $\lambda v((1+2\alpha(-1+\alpha+\beta-\beta\alpha))q_h+2\alpha(-1+\alpha)(-1+\beta)q_l)$ *and ad revenue is*$(1-\lambda)v(2(-1+\alpha)\alpha(-1+\beta)(q_h-q_l)+q_l)$.

Proof. From equations 16 and 17, we observe that the conversion rates do not depend on prices. Therefore, it is weakly dominant for a seller to set its price equal to v, since by doing so it simultaneously maximizes the scope of outbidding the rival and the profit upon winning the auction. Therefore, by lemma 14, we have $b_D^{i*}(s) = (1-\lambda)\rho_i(v,s)v$. Comparing $b_D^{1*}(s) = (1-\lambda)\rho_i(v,s)v$ and $b_D^{2*}(s) = (1-\lambda)\rho_i(v,s)v$ by using equations 16 and 17 gives us the rest of the results. □

Notice that when there is no information advantage under dynamic bidding ($\beta = 0$), seller 1, the superior seller, is expected to be a better match for a consumer. Therefore, when the accuracy of prediction under dynamic bidding is sufficiently low ($\beta \leq \frac{2\alpha-1}{2\alpha}$), even for consumers with signal $s=2$, seller 1 is predicted to be a better match. As seller 1 wins the auction for all consumers in this case, the equilibrium outcomes are the same across both bidding regimes. When the prediction accuracy under dynamic bidding is high ($\beta > \frac{2\alpha-1}{2\alpha}$), there is a high probability that a consumer with signal $s=2$ belongs to group 2 who have higher match likelihood with seller 2 than with seller 1. Therefore, in this case seller 2 wins the auction for consumers with signal $s=2$.

A.6.3 Implications of Dynamic Bidding

First note that for $\beta \leq \frac{2\alpha-1}{2\alpha}$ both bidding mechanisms result in the same outcomes, since the equilibrium price is the same and seller 1 wins the auction for all consumers under both regimes. When $\beta > \frac{2\alpha-1}{2\alpha}$, the bids are more targeted. Seller 1 wins the auction for consumers with signal $s = 1$, and seller 2 wins the auction for consumers with signal $s = 2$. Due to better match between sellers and consumers, both sellers are better off under dynamic bidding. The platform sales revenue is also improves under dynamic bidding because of better match. However, we find that ad revenue is lower under dynamic bidding than under fixed bidding. This is because the bids are more targeted under dynamic bidding. Each seller bids high for consumers for whom it has high predicted conversion rate, and low for consumers for whom it has low predicted conversion rate. In a second price auction, ad revenue is generated from consumers with low conversion rate, which is lower than the average bid for all consumers.[21] As a result, similar to the findings in the main model, we find that dynamic bidding shifts the revenue mix away from ad revenue, and towards sales revenue. Lastly, we find that platform's total revenue is higher under dynamic bidding than under fixed bidding if the revenue sharing rate λ is high enough. The following proposition summarizes the results.

Proposition 12. *When $\beta > \frac{2\alpha-1}{2\alpha}$, sellers are better off under dynamic bidding. The platform sales revenue is higher under dynamic bidding than under fixed bidding, whereas, its ad revenue is lower under dynamic bidding than under fixed bidding. The platform's total revenue is higher under dynamic bidding than under fixed bidding if $\lambda > \frac{1}{2}$.*

Proof. All the results can be verified by comparing the expressions in theorems 6 and 7. $\square$

A.7 Reserve Price in Dynamic Bidding

We focus on the case $\beta = 1$.[22] We consider that, under dynamic bidding, the platform sets its reserve price R in Stage 1, simultaneously with sellers setting their price.[23] The rest of the game proceeds as before. We will show that in the unique symmetric equilibrium sellers set too high a

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[21] Note that for an intermediate range prediction accuracy ($\frac{2\alpha-1}{2\alpha} < \beta < \frac{2\alpha-1}{\alpha}$), ad revenue is higher under fixed bidding than under dynamic bidding for consumers with signal $s = 2$. However, ad revenue is lower under dynamic bidding than under fixed for consumers with signal $s = 1$, and it turns out that the total ad revenue is also lower under dynamic bidding.

[22] This is mainly for ease of exposition and conveying the intuition. We have verified that the analysis and results readily extend to the case where β is sufficiently high, i.e., we do not obtain knife-edge results.

[23] We also analyzed the case where the platform sets its reserve price after sellers set their prices. We obtain the same results for $\beta = 1$.

price. Let $b_i(p_i, x)$ denote seller i's optimized bid for a consumer with location x on the Salop circle. The following lemma characterizes sellers' optimized bids.

Lemma 15. *Seller i's optimized bid for a consumer with location x is*

$$b_i(p_i, x) = \begin{cases} p_i, & \text{if } d_i(x) \le \bar{d}(p_i); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider a consumer with location x. Let $\pi_i(p_i; x)$ denote seller i's expected profit from sales conditional on winning the sponsored ad for this consumer. Thus, $\pi_i(p_i; x)$ represents seller i's value of the sponsored ad for this consumer. Let $b_j(x)$ be the rival bid for this consumer. Then seller i's expected profit for this consumer (including the cost of the sponsored ad) is

$$\Pi_i^F(p_i; x) = \begin{cases} \pi_i(p_i; x) - \max\{R, b_j(x)\}, & \text{if } b_i > \max\{R, b_j(x)\}; \\ \pi_i(p_i; x) - R, & \text{if } b_i = R \text{ and } b_j(x) < R; \\ \frac{1}{2}(\pi_i(p_i; x) - b_j(x)) & \text{if } b_i = b_j(x) \text{ and } R \le b_j(x); \\ 0, & \text{if } b_i < \max\{R, b_j(x)\}. \end{cases}$$

We observe that it is weakly dominant for seller i to bid $\pi_i(p_i; x)$ for this consumer. The proof follows from the standard second price auction results. Since $\beta = 1$, we have $\pi_i(p_i; x) = p_i$ if $d_i(x) \le \bar{d}(p_i)$ and $\pi_i(p_i; x) = 0$ otherwise. □

We next solve for a pure strategy equilibrium which is symmetric in prices. Let $p_R^* > 0$ denote the equilibrium price and R^* denote the equilibrium reserve price. The following lemma is useful in proving our main result.

Lemma 16. *If $p_R^* < v$, then $R^* = p_R^*$.*

Proof. By lemma 15, each seller bids p_R^* for consumers within $\overline{d}(p_R^*)$ of its location, and bids zero for others. Since $p_R^* < v$, $\overline{d}(p_R^*) > 0$. Therefore, there is a positive measure of consumers for whom the bid is p_R^*. As a result, the unique optimal reserve price for the platform is to set $R^* = p_R^*$ and extract all the surplus. □

We now obtain the equilibrium.

Proposition 13. *When reserve price is used in sponsored ad auctions under dynamic bidding, there is a unique equilibrium where $p_R^* = R^* = v$.*

Proof. We first show that $p_R^* = R^* = v$ is an equilibrium . The seller and platform profits in equilibrium are zero. Further, neither the sellers nor the platform have a profitable deviation. Taking $p_1 = p_2 = v$ as given, the platform profit is zero regardless of the reserve price it sets. And, taking $p_i = R = v$ as given, then in order to win the sponsored ad for any consumer, seller j's bid must be at least v, which is not profitable for any price $p_j < v$.

Next, suppose towards a contradiction there exists another equilibrium p_R^{**}, R^{**} such that $p_R^{**} < v$ or $R^{**} < v$. If $p_R^{**} < v$, then by lemma 16, we must have $R^{**} = p_R^{**}$ which results in zero profit for sellers. However, seller i can deviate to $p_i = p^{**} + \epsilon < v$ and make positive profit for consumers with $d_i(x) \leq \bar{d}(p^{**} + \epsilon)$. If $p_R^{**} = v$ but $R^{**} < v$, then sellers' profits in equilibrium are zero. However, seller i can deviate to price $p_i = v - \epsilon > R^{**}$ and make positive profit for consumers with $d_i(x) \leq \bar{d}(v - \epsilon)$. $\square$

A.8 Quality Score under Dynamic Bidding

We investigate a quality score scheme under dynamic bidding with $\frac{1}{p_i}$ as the quality score for seller i. That is, if seller i bids b_i, then its effective bid is $\widehat{b_i} = \frac{1}{p_i} b_i$. The seller with the highest effective bid wins the auction. If seller i wins the auction then it pays $\frac{\hat{b}_j}{\frac{1}{p_i}}$, where $\hat{b}_j = \frac{1}{p_j} b_j$ is the rival's effective bid. We focus on the case $\beta = 1$.[24] We will show that the equilibrium without a quality score is also an equilibrium with the quality score. Additionally, there may be an equilibrium where the sellers set the price too low such that all consumers will buy and, consequently, the outcome is the same as or worse than under fixed bidding for the platform. Thus, the quality score scheme is not effective.

Let $b_i(p_i, x)$ denote seller i's optimized bid for a consumer at location x. As before, we have $b_i(p_i, x) = \pi^H(p_i) = p_i$ if $d_i(x) \leq \overline{d}(p_i)$, and $b_i(p_i, x) = \pi^L(p_i) = 0$ otherwise. Therefore, seller i's effective bid is

$$\widehat{b_i}(p_i, x) = \begin{cases} 1 & d_i(x) \leq \overline{d}(x) \\ \\ 0 & d_i(x) > \overline{d}(x) \end{cases} \tag{18}$$

We solve for an equilibrium which is symmetric in prices. Let p_Q^* denote the symmetric equilibrium price. The following lemmas proves some properties of the equilibrium.

Lemma 17. *If the effective bids for a consumer at location x are $\widehat{b_i}(p_i, x) = \widehat{b_j}(p_j, x) = 1$, then*

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[24] This is mainly for ease of exposition and conveying the intuition. We have verified that the analysis and results readily extend to the case where β is sufficiently high, i.e., we do not obtain knife-edge results.

sellers make zero profit from this consumer.

Proof. Let $\Pi_i(p_i, x)$ be seller i's profit (including the cost of sponsored ad) from such consumer. Each seller wins the sponsored ad equally likely for this consumer. Therefore, $\Pi_i(p_i, x) = p_i - \frac{\widehat{b_j}(p_j,x)}{\frac{1}{p_i}} = 0$. ☐

Lemma 18. *There is no equilibrium in which* $\frac{1}{2} < \overline{d}\left(p_Q^*\right) < 1$.

Proof. Suppose by contradiction that such an equilibrium exists. Since $\frac{1}{2} < \overline{d}\left(p_Q^*\right) < 1$, there is a positive measure of consumers for whom the adjusted bid for both sellers is 1 according to the equation 18. Now, take $p_j = p_Q^*$ as given. By lemma 17, seller i's profit for $p_i \in [p^*, p^* + \epsilon]$ would be

$$\Pi_i(p_i) = \left(1 - \overline{d}\left(p_Q^*\right)\right) p_i + \left(\overline{d}(p_i) + \overline{d}\left(p_Q^*\right) - 1\right)(0) = \left(1 - \overline{d}\left(p_Q^*\right)\right) p_i.$$

Therefore, seller i has the incentive to raise its price and make strictly higher profit. ☐

Lemma 19. *Any* $p_Q^* > 0$ *such that* $p_Q^* \leq v - t$ *is an equilibrium.*

Proof. Seller profits are zero on the equilibrium path. Take $p_j = p_Q^*$ as given. Therefore, $\widehat{b_j}(p_j, x) = 1, \forall x$ and following lemma 17, seller i's profit is zero regardless of its price. Hence, the seller has no incentive to deviate. ☐

The next proposition characterizes the equilibrium results.

Proposition 14. *If* $v < t$*, the unique pure equilibrium is* $p_Q^* = \frac{v}{2}$*. If* $t < v$*, the set of equilibria is* $\left\{p_Q^* : p_Q^* = v - \frac{t}{2} \quad \text{or} \quad p_Q^* \leq v - t\right\}$.

Proof. Following 19, $p_Q^* : p_Q^* \leq v - t$ is an equilibrium if $v > t$. Now consider the case where $\overline{d}\left(p_Q^*\right) < 1$. By lemma 17, we must have $\overline{d}\left(p_Q^*\right) < \frac{1}{2}$ and, therefore, $p_Q^* \geq v - \frac{t}{2}$. Take any candidate equilibrium $p_Q^* \geq v - \frac{t}{2}$ and fix $p_j = p_Q^*$. Then following lemma 18, seller i's profit is given by

$$\Pi_i^D = \begin{cases} \overline{d}(p_i)\, p_i & p_Q^* \leq p_i \leq v; \\ \overline{d}(p_i)\, p_i & 2v - p_Q^* - t \leq p_i < p_Q^*; \\ \left(1 - \overline{d}\left(p_Q^*\right)\right) p_i & p_i < 2v - p_Q^* - t. \end{cases}$$

Solving for seller i's optimal price, we obtain $p_Q^* = \frac{v}{2}$ if $v \leq t$ and $p_Q^* = v - \frac{t}{2}$ if $v > t$ as in the case without a quality score. ☐