Online Appendix to
"On the Value of Coalition Loyalty Programs"

The online appendices provide supplemental materials for the paper, including the full version of lemmas and propositions, as well as additional details. The appendices are organized into five sections:

- Section A provides proofs of lemmas and propositions in the base model.
- Section B provides a detailed analysis of the model extension without customer discounting.
- Section C provides the proof of Proposition 4 in the case with asymmetric firms.
- Section D provides a detailed analysis in which no new reward is earned upon redemption.
- Section E provides a detailed analysis of the discrete-time approximation.

Appendix A: Proofs of Lemmas and Propositions in the Base Model

We first provide a detailed analysis of the customers' decision-making process, including explicit solutions to the optimality equations (1)–(2) and technical derivations. For equations (1)–(2), we consider the discrete-time Markov chain embedded in the continuous-time semi-Markov process and apply the uniformization technique by setting the maximum transition rate to be $\nu = \delta + n\lambda + \mu$ (Puterman 1994). Uniformization is a well-known technique used to formulate continuous-time dynamic programs that simplify the analysis of the resulting dynamic programming model. The transition rates in different states under a continuous-time Markov chain usually differ. For example, in our continuous-time semi-Markov chain, there are two possible transitions in state 1: the customer visits the coalition (with rate $n\lambda$) and decides whether or not to make a purchase, and the reward expires (with rate μ). Hence, the total transition rate in state 1 is $\nu = n\lambda + \mu + \delta$ (where δ is the customer's discount rate). However, there is only one possible transition in state 0: the customer visits the coalition (with rate $n\lambda$) and decides whether or not to make a purchase. Hence, the total transition rate in state 0 is $n\lambda + \delta$, which is less than that in state 1. The key idea of uniformization is to add fictitious transitions such that the transition rates in all states are the same. Here, we add a fictitious transition with rate μ in state 0 to make the total transition rates in the two states the same. Note that the fictitious transition returns to the same state.

Next, we establish the condition under which the customer always makes a purchase in state 0 when she visits a firm in the coalition. Otherwise, the customer will not stay in the market over the long run and therefore does not contribute to the per-firm profit.

Lemma A1. *If*

$$v - p + \frac{n\lambda}{\nu} r \geq 0, \tag{1}$$

then it is optimal for a (v, λ)*-customer to make a purchase whenever she visits a firm in the coalition, and the solution to the optimality equations* (1)–(2) *is given by*

$$u(0) = \frac{n\lambda}{\delta}\left(v - p + \frac{n\lambda}{\nu} r\right), \quad u(1) = \frac{n\lambda}{\delta}\left(v - p + \frac{\delta + n\lambda}{\nu} r\right).$$

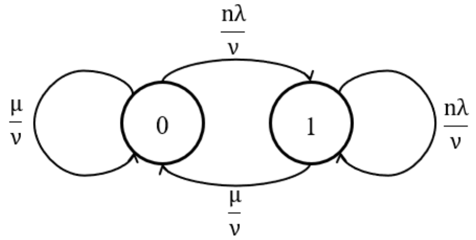

Figure A.1 State Transition Diagram

The state of a customer who makes a purchase whenever she visits a firm in the coalition follows a Markov chain, as illustrated in Figure A.1. In state 1, there is a probability of $\frac{n\lambda}{\delta + n\lambda + \mu}$ that the customer arrives and makes a purchase before reward expiration, in which case she receives a new reward and remains in state 1. Conversely, there is a probability of $\frac{\mu}{\delta + n\lambda + \mu}$ that the reward expires before the customer's arrival, in which case she transits to state 0. The interpretation for state 0 is similar. Let q_0 and q_1 denote the stationary probabilities of states 0 and 1, respectively. By solving the balance equations, we find that $q_0 = \frac{\mu}{n\lambda + \mu}$ and $q_1 = \frac{n\lambda}{n\lambda + \mu}$; that is, on average, there is a $\frac{\mu}{n\lambda + \mu}$ fraction of time that the customer stays in state 0 when purchasing, and a $\frac{n\lambda}{n\lambda + \mu}$ fraction of time that she stays in state 1 when purchasing. Recall that when a customer makes a purchase, she pays a price of $p - r$ in state 1 and a price of p in state 0. Accordingly, when condition (1) is satisfied, the customer contributes a profit to the coalition at rate

$$n\lambda\Big(q_0 p + q_1(p - r)\Big) = n\lambda\Big(p - \frac{n\lambda}{n\lambda + \mu} r\Big). \tag{2}$$

Since there is a $\frac{n\lambda}{n\lambda + \mu}$ fraction of time that the customer stays in state 1 when purchasing, the profit contribution from each purchase is equal to the price p minus the average value of the redeemed reward, which is the reward amount r times the fraction of purchases for which the customer uses a reward.

Proof of Lemma A1

We first show if $v - p + u(1) \geq u(0)$, then $v - p + r + u(1) \geq u(1)$. Suppose for a contradiction that $v - p + r + u(1) < u(1)$. Then,

$$u(1) - u(0) = \frac{n\lambda}{\nu}\Big\{u(1) - [v - p + u(1)]\Big\} = \frac{n\lambda}{\nu}(-v + p) > 0,$$

from which we obtain $v < p$. It follows immediately that

$$v - p + u(1) - u(0) = v - p + \frac{n\lambda}{\nu}(-v + p) = \frac{\delta + \mu}{\nu}(v - p) < 0,$$

contradicting our supposition. Hence, it must be the case that $v - p + r + u(1) \geq u(1)$. Suppose $v - p + u(1) \geq u(0)$. Then, equations (1) and (2) reduce to

$$u(1) = \frac{n\lambda}{\nu}[v - p + r + u(1)] + \frac{\mu}{\nu}u(0), \text{ and } u(0) = \frac{n\lambda}{\nu}[v - p + u(1)] + \frac{\mu}{\nu}u(0).$$

Solving the above set of equations yields that

$$u(1) = \frac{n\lambda}{\delta}(v - p + \frac{\delta + n\lambda}{\nu}r), \text{ and } u(0) = \frac{n\lambda}{\delta}(v - p + \frac{n\lambda}{\nu}r).$$

Plugging $u(1)$ and $u(0)$ back to the supposition $v - p + u(1) \geq u(0)$ yields that

$$v - p + \frac{n\lambda}{\nu}r \geq 0,$$

under which the customer makes a purchase whenever she visits a firm in the coalition. This completes the proof. ■

LEMMA 1. *A CLP with size n improves the per-firm profit relative to no reward program only if*

$$(\alpha - \gamma)\lambda_I\Big(\lambda_F\mu - \lambda_I(\delta + \mu)\Big)(n\lambda_F + \mu) > \beta\lambda_F^2\delta(n\lambda_I + \mu). \tag{3}$$

When condition (3) holds, the optimal price, reward amount, and per-firm profit in the CLP are, respectively,

$$p^* = v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu}r^*,$$

$$r^* = \min\left\{\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}, \frac{(\delta + \mu + n\lambda_F)}{\delta + \mu}v_L\right\},$$

$$\pi^*(n) = \underbrace{\beta\lambda_F\Big(v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu}r^* - \frac{n\lambda_F}{n\lambda_F + \mu}r^*\Big)}_{\textit{profit from HF and LF customers}} + \underbrace{(\alpha - \gamma)\lambda_I\Big(v_L + \frac{n\lambda_F}{\delta + n\lambda_F + \mu}r^* - \frac{n\lambda_I}{n\lambda_I + \mu}r^*\Big)}_{\textit{profit from HI customers}}.$$

Proof of Lemma 1

To derive the per-firm profit in the CLP collected from the four customer segments, we analyze the relationship between price p, $v_H + \frac{n\lambda_F}{\nu_F}r$, $v_H + \frac{n\lambda_I}{\nu_I}r$, $v_L + \frac{n\lambda_F}{\nu_F}r$, and $v_L + \frac{n\lambda_I}{\nu_I}r$ to determine the customer's purchase behavior in each segment. Recall that $\nu_F = \delta + n\lambda_F + \mu$ and $\nu_I = \delta + n\lambda_I + \mu$. One can check that

$$v_L + \frac{n\lambda_I}{\nu_I}r < \min\left\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\right\} \leq \max\left\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\right\} < v_H + \frac{n\lambda_F}{\nu_F}r.$$

Then, depending on the magnitude of price p, we have the following six cases:

Case 1: $p \le v_L + \frac{n\lambda_I}{\nu_I} r < \min\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} \le \max\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < v_H + \frac{n\lambda_F}{\nu_F} r$.

All customers make a purchase. The optimization problem becomes

$$\max_{p \ge r \ge 0} \ \beta\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F + \mu} r\Big) + (1-\beta)\lambda_I\Big(p - \frac{n\lambda_I}{n\lambda_I + \mu} r\Big)$$

$$s.t. \ p \le v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r.$$

Clearly,

$$(p^*, r^*) = (v_L, 0),$$

$$\pi^*(n) = \beta\lambda_F v_L + (1-\beta)\lambda_I v_L = \pi_1,$$

Hence, it is never optimal to adopt the CLP in this case.

Case 2: $v_L + \frac{n\lambda_I}{\nu_I} r < p \le \min\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} \le \max\left\{v_L + \frac{n\lambda_F}{\nu_F} r, v_H + \frac{n\lambda_I}{\nu_I} r\right\} < v_H + \frac{n\lambda_F}{\nu_F} r$.

All customers except LI segment make a purchase. The optimization problem becomes

$$\max_{p \ge r \ge 0} \ \beta\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F + \mu} r\Big) + (\alpha-\gamma)\lambda_I\Big(p - \frac{n\lambda_I}{n\lambda_I + \mu} r\Big)$$

$$s.t. \ v_L + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r < p \le \min\Big\{v_L + \frac{n\lambda_F}{n\lambda_F + \mu + \delta} r, v_H + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r\Big\}.$$

Clearly, $p^* = \min\left\{v_L + \frac{n\lambda_F}{n\lambda_F + \mu + \delta} r, v_H + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r\right\}$. We then have the following two subcases.

Subcase 1: Suppose

$$v_L + \frac{n\lambda_F}{n\lambda_F + \mu + \delta} r \le v_H + \frac{n\lambda_I}{n\lambda_I + \mu + \delta} r.$$

It follows that $p^* = v_L + \frac{n\lambda_F}{n\lambda_F + \mu + \delta} r$. Reorganizing the supposition yields $r \le \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}$. Moreover, $p \ge r$ gives $r \le \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L$. So, the optimization problem can be rewritten as

$$\max_{r \ge 0} \ \beta\lambda_F\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_F}{n\lambda_F + \mu}\big) r\Big] + (\alpha-\gamma)\lambda_I\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_I}{n\lambda_I + \mu}\big) r\Big]$$

$$s.t. \ r \le \min\Big\{\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}, \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L\Big\}.$$

Note that if and only if

$$\beta\lambda_F\big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_F}{n\lambda_F + \mu}\big) + (\alpha-\gamma)\lambda_I\big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_I}{n\lambda_I + \mu}\big) > 0, \tag{4}$$

the coefficient for r above is non-negative. Therefore, at optimality, we have

$$r^* = \min\Big\{\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta + \mu)}, \frac{\delta + n\lambda_F + \mu}{\delta + \mu} v_L\Big\}.$$

Reorganizing inequality (4) gives condition (3) in Lemma 1.

Subcase 2: Suppose

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r \geq v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r.$$

It follows that $p^* = v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r$. Reorganizing the supposition yields $r \geq \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}$. Moreover, $p \geq r$ gives $r \leq \frac{\delta+n\lambda_I+\mu}{\delta+\mu} v_H$. So, the optimization problem can be rewritten as

$$\begin{aligned} \max_{r\geq 0} \;& \beta\lambda_F \Big[v_H + \big(\frac{n\lambda_I}{n\lambda_I+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu}\big) r\Big] + (\alpha-\gamma)\lambda_I \Big[v_H + \big(\frac{n\lambda_I}{n\lambda_I+\mu+\delta} - \frac{n\lambda_I}{n\lambda_I+\mu}\big) r\Big] \\ s.t. \;& \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)} \leq r \leq \frac{\delta+n\lambda_I+\mu}{\delta+\mu} v_H. \end{aligned}$$

Note that $\frac{n\lambda_I}{n\lambda_I+\mu+\delta} < \frac{n\lambda_I}{n\lambda_I+\mu} < \frac{n\lambda_F}{n\lambda_F+\mu}$, so the profit decreases in r, and thus

$$r^* = \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)},$$

under which

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r^* = v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r^*.$$

Hence, the profit in Subcase 2 is dominated by that in Subcase 1.

Case 3: $v_L + \frac{n\lambda_I}{\nu_I} r < v_L + \frac{n\lambda_F}{\nu_F} r < p \leq v_H + \frac{n\lambda_I}{\nu_I} r < v_H + \frac{n\lambda_F}{\nu_F} r$.

Only high-valuation customers make a purchase. The optimization problem becomes

$$\begin{aligned} \max_{p\geq r\geq 0} \;& \gamma\lambda_F \Big(p - \frac{n\lambda_F}{n\lambda_F+\mu} r\Big) + (\alpha-\gamma)\lambda_I \Big(p - \frac{n\lambda_I}{n\lambda_I+\mu} r\Big) \\ s.t. \;& v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r < p \leq v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r. \end{aligned}$$

Clearly,

$$\begin{aligned} (p^*, r^*) &= (v_H, 0), \\ \pi^*(n) &= \gamma\lambda_F v_H + (\alpha-\gamma)\lambda_I v_H = \pi_2, \end{aligned}$$

Hence, it is never optimal to adopt the CLP in this case.

Case 4: $v_L + \frac{n\lambda_I}{\nu_I} r < v_H + \frac{n\lambda_I}{\nu_I} r < p \leq v_L + \frac{n\lambda_F}{\nu_F} r < v_H + \frac{n\lambda_F}{\nu_F} r$.

Only frequent customers make a purchase. The optimization problem becomes

$$\begin{aligned} \max_{p\geq r\geq 0} \;& \beta\lambda_F \Big(p - \frac{n\lambda_F}{n\lambda_F+\mu} r\Big) \\ s.t. \;& v_H + \frac{n\lambda_I}{n\lambda_I+\mu+\delta} r < p \leq v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r. \end{aligned}$$

Clearly,

$$p^* = v_L + \frac{n\lambda_F}{n\lambda_F+\mu+\delta} r^*,$$

$$\pi^*(n) = \beta\lambda_F\Big[v_L + (\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu})r^*\Big] < \beta\lambda_F v_L < \pi_1.$$

Hence, it is never optimal to adopt the CLP in this case.

Case 5: $v_L + \frac{n\lambda_I}{\nu_I}r < \min\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} \le \max\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} < p \le v_H + \frac{n\lambda_F}{\nu_F}r.$

Only high-valuation frequent customers make a purchase. The optimization problem becomes

$$\max_{p\ge r\ge 0}\ \gamma\lambda_F\Big(p - \frac{n\lambda_F}{n\lambda_F+\mu}r\Big)$$

$$s.t.\ p \le v_H + \frac{n\lambda_F}{n\lambda_F+\mu+\delta}r.$$

Clearly,

$$p^* = v_H + \frac{n\lambda_F}{n\lambda_F+\mu+\delta}r^*,$$

$$\pi^*(n) = \gamma\lambda_F\Big[v_H + (\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu})r^*\Big] < \gamma\lambda_F v_H < \pi_2.$$

Hence, it is never optimal to adopt the CLP in this case.

Case 6: $v_L + \frac{n\lambda_I}{\nu_I}r < \min\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} \le \max\Big\{v_L + \frac{n\lambda_F}{\nu_F}r, v_H + \frac{n\lambda_I}{\nu_I}r\Big\} < v_H + \frac{n\lambda_F}{\nu_F}r < p.$

No customers make a purchase. Clearly, it is not optimal to adopt the CLP in this case.

Taking the above six cases into consideration, we can obtain that only Subcase 1 in Case 2 is possibly optimal. This completes the proof. ■

PROPOSITION 1. *By comparing a CLP and no reward programs, we have the following results:*

(a) When customers' product valuation and shopping intensity are positively correlated, i.e., $\gamma > \alpha\beta$, a CLP, regardless of its size, always yields a lower profit for the firm than no reward programs; that is, $\pi^*(n) < \max\{\pi_1, \pi_2\}$ for any coalition size n.

(b) The aggregate customer surplus increases when firms switch from a partial market coverage strategy with price v_H to forming a CLP, while the aggregate customer surplus decreases when firms switch from a full market coverage strategy with price v_L to forming a CLP.

Proof of Proposition 1

Part (a): The profit $\pi^*(n)$ in (4) consists of the profit collected from frequent customers and HI customers. Note that $\frac{n\lambda_F}{\delta+n\lambda_F+\mu} < \frac{n\lambda_F}{n\lambda_F+\mu}$, so the effective price paid by HF and LF customers is smaller than v_L (i.e., $v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu}r^* - \frac{n\lambda_F}{n\lambda_F+\mu}r^* < v_L$). In contrast, the effective price paid by HI customers can be greater than v_L but smaller than v_H (i.e., $v_L + \frac{n\lambda_F}{\delta+n\lambda_F+\mu}r^* - \frac{n\lambda_I}{n\lambda_I+\mu}r^* \le v_H + \frac{n\lambda_I}{\delta+n\lambda_I+\mu}r^* - \frac{n\lambda_I}{n\lambda_I+\mu}r^* < v_H$). This leads to the upper bound on the per-firm profit that $\pi^*(n) < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H$.

Now, we show $\pi^*(n) < \max\{\pi_1, \pi_2\}$ when $\gamma > \alpha\beta$. We can show that

$$\pi^*(n) - \pi_1 < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H - [\beta\lambda_F + (1-\beta)\lambda_I]v_L = [(\alpha-\gamma)v_H - (1-\beta)v_L]\lambda_I$$

$$<[\alpha(1-\beta)v_H-(1-\beta)v_L]\lambda_I=(1-\beta)(\alpha v_H-v_L)\lambda_I.$$

We can also show that

$$\pi^*(n)-\pi_2<\beta\lambda_F v_L+(\alpha-\gamma)\lambda_I v_H-[\gamma\lambda_F+(\alpha-\gamma)\lambda_I]v_H=[\beta v_L-\gamma v_H]\lambda_F<\beta(v_L-\alpha v_H)\lambda_F.$$

Observe that if $\alpha v_H \geq v_L$, then $\pi^*(n)<\pi_2$; otherwise, $\pi^*(n)<\pi_1$. Hence, $\pi^*(n)<\max\{\pi_1,\pi_2\}$.

Part (b): Table 1 presents customer welfare in each segment under CLPs in both the VD market and the ID market, and compares them with that under the full market coverage strategy with price v_L.

	CLP (VD market)	vs.	Full Market Coverage	vs.	CLP (ID Market)
HF	$v_H-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L$	$>$	v_H-v_L	$<$	$(v_H-v_L)\Big[1+\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]$
LF	$v_L-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L$	$>$	0	$<$	$(v_H-v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}$
HI	$v_H-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L$	$<$	v_H-v_L	$>$	$(v_H-v_L)\Big[1-\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]$
LI	0	$=$	0	$=$	0

Table 1 Customer Welfare Comparison in Each Customer Segment Between CLP and Full Market Coverage with No Reward

When the firm adopts the partial market coverage strategy with price v_H, the aggregate customer surplus is 0. Note that customer surplus of each purchased segment in CLPs is nonnegative, because otherwise, customers will not make a purchase. Indeed, Table 1 shows that the aggregate customer surplus in CLPs is strictly positive. Therefore, CLPs lead to a higher aggregate customer surplus than the partial market coverage strategy with price v_H. Below, we focus on the comparison of the aggregate customer surplus associated with CLPs and that associated with the full market coverage strategy with price v_L. We have the following two cases.

Case 1: Suppose $\frac{v_H}{v_L}\geq\frac{\lambda_F}{\lambda_I}$.

By Table 1, the aggregate customer surplus associated with CLPs in a VD market minus that associated with the full market coverage strategy with price v_L equals

$$\begin{aligned}&\gamma\lambda_F\Big\{v_H-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L-(v_H-v_L)\Big\}+(\beta-\gamma)\lambda_F\Big\{v_L-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L\Big\}\\&\qquad+(\alpha-\gamma)\lambda_I\Big\{v_H-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L-(v_H-v_L)\Big\}\\=&\beta\lambda_F\Big\{v_L-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_F+\mu)(\delta+\mu)}v_L\Big\}+(\alpha-\gamma)\lambda_I\Big\{v_L-\frac{\mu(\delta+n\lambda_F+\mu)}{(n\lambda_I+\mu)(\delta+\mu)}v_L\Big\}\\=&\frac{nv_L}{\delta+\mu}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{n\lambda_F+\mu}-(\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big\}\end{aligned}$$

$$<0,$$

where the last inequality holds because of condition (3).

Case 2: Suppose $\frac{v_H}{v_L} < \frac{\lambda_F}{\lambda_I}$.

By Table 1, the aggregate customer surplus associated with CLPs in an ID market minus that associated with the full market coverage strategy with price v_L equals

$$\begin{aligned}
&\gamma\lambda_F\Big\{(v_H - v_L)\Big[1+\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]-(v_H-v_L)\Big\}\\
&\quad+(\beta-\gamma)\lambda_F\Big\{(v_H-v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
&\quad+(\alpha-\gamma)\lambda_I\Big\{(v_H-v_L)\Big[1-\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]-(v_H-v_L)\Big\}\\
=&\beta\lambda_F\Big\{(v_H-v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}-(\alpha-\gamma)\lambda_I\Big\{(v_H-v_L)\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
=&(v_H-v_L)\frac{\delta+n\lambda_I+\mu}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{n\lambda_F+\mu}-(\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{n\lambda_I+\mu}\Big\}\\
<&0,
\end{aligned}$$

where the last inequality holds because of condition (3). This completes the proof. ■

Before presenting the proof of Proposition 2, we define the following critical threshold on size:

$$\hat{n} := \max\left\{\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}, \frac{\Big(1-\frac{\lambda_I}{\lambda_F}\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}\Big)\mu}{\lambda_I\Big(\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}-1\Big)}\right\}.$$

PROPOSITION 2.

(a) ***[VD Market]*** *When $\frac{v_H}{v_L} \geq \frac{\lambda_F}{\lambda_I}$, the optimal price, reward amount, and per-firm profit in the CLP are*

$$\begin{aligned}
p^* = r^* &= \frac{\delta+\mu+n\lambda_F}{\delta+\mu}v_L;\\
\pi^*(n) &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F\frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I\frac{n[\lambda_F\mu-\lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu}.
\end{aligned}$$

The profit $\pi^*(n)$ increases in the coalition size n; that is, larger CLPs lead to a higher per-firm profit. Therefore, an optimally sized CLP always leads to a higher per-firm profit than a PLP.

(b) ***[ID Market]*** *When $\frac{v_H}{v_L} < \frac{\lambda_F}{\lambda_I}$, if $n < \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$, then the optimal price p^*, reward r^*, and per-firm profit $\pi^*(n)$ are the same as those stated above; otherwise, we have*

$$p^* = \frac{\lambda_F(\delta+\mu+n\lambda_I)v_H - \lambda_I(\delta+\mu+n\lambda_F)v_L}{(\lambda_F-\lambda_I)(\delta+\mu)},$$

$$r^* = \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)},$$

$$\begin{aligned}\pi^*(n) = & \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta+n\lambda_I+\mu}{n\lambda_F+\mu}\frac{\lambda_F\delta(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\\ & + (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu-\lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta+n\lambda_I+\mu}{n\lambda_I+\mu}.\end{aligned}$$

The $\pi^*(n)$ *monotonically increases in the coalition size* n *if* $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \leq 1$*; consequently, larger CLPs lead to a higher per-firm profit. Otherwise, the profit* $\pi^*(n)$ *first increases in* n *for* $n \leq \hat{n}$ *and then decreases in* n*; the optimal size* $n^* = \lfloor \hat{n} \rfloor$ *or* $\lceil \hat{n} \rceil$*, where* $\lfloor \cdot \rfloor$ *and* $\lceil \cdot \rceil$ *are the floor and ceiling functions, respectively. In this case, a PLP yields a higher per-firm profit than an optimally sized CLP when either of the following conditions holds:*

(i) $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} > 1$ *and* $\hat{n} \leq 1$*;*

(ii) $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} > 1$*,* $1 < \hat{n} < 2$*, and* $\pi^*(1) > \pi^*(2)$*.*

Proof of Proposition 2

Part (a): One can show that when either $\frac{v_H}{v_L} \geq \frac{\lambda_F}{\lambda_I}$ or $\frac{v_H}{v_L} < \frac{\lambda_F}{\lambda_I}$ but $n < \frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$ holds,

$$\frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L < \frac{(v_H-v_L)(\delta+n\lambda_F+\mu)(\delta+n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}.$$

Hence, according to Lemma 1,

$$\begin{aligned} r^* &= \frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L,\\ p^* &= v_L + \frac{n\lambda_F}{n\lambda_F+\delta+\mu}\frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L = \frac{\delta+\mu+n\lambda_F}{\delta+\mu}v_L = r^*,\\ \pi^*(n) &= \beta\lambda_F\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_F}{n\lambda_F+\mu}\big)\frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L\Big]\\ &\quad + (\alpha-\gamma)\lambda_I\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F+\mu+\delta} - \frac{n\lambda_I}{n\lambda_I+\mu}\big)\frac{\delta+n\lambda_F+\mu}{\delta+\mu}v_L\Big]\\ &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F\frac{n\lambda_F\delta}{n\lambda_F+\mu}\frac{v_L}{\delta+\mu} + (\alpha-\gamma)\lambda_I\frac{n[\lambda_F\mu-\lambda_I(\delta+\mu)]}{n\lambda_I+\mu}\frac{v_L}{\delta+\mu}.\end{aligned}$$

Taking the derivative of $\pi^*(n)$ with respect to n yields

$$\begin{aligned}\frac{d\pi^*(n)}{dn} &= -\frac{v_L}{\delta+\mu}\Big\{\beta\lambda_F\frac{\lambda_F\delta\mu}{(n\lambda_F+\mu)^2} - (\alpha-\gamma)\lambda_I\frac{\mu[\lambda_F\mu-\lambda_I(\delta+\mu)]}{(n\lambda_I+\mu)^2}\Big\}\\ &= -\frac{v_L\mu}{\delta+\mu}\frac{\beta\lambda_F^2\delta(n\lambda_I+\mu)^2 - (\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)^2}{(n\lambda_F+\mu)^2(n\lambda_I+\mu)^2}\\ &= \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu)^2 - \beta\lambda_F^2\delta(n\lambda_I+\mu)^2}{(n\lambda_F+\mu)^2(n\lambda_I+\mu)^2}\\ &= \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu) - \beta\lambda_F^2\delta(n\lambda_I+\mu)\frac{n\lambda_I+\mu}{n\lambda_F+\mu}}{(n\lambda_F+\mu)(n\lambda_I+\mu)^2}\\ &> \frac{v_L\mu}{\delta+\mu}\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](n\lambda_F+\mu) - \beta\lambda_F^2\delta(n\lambda_I+\mu)}{(n\lambda_F+\mu)(n\lambda_I+\mu)^2}\end{aligned}$$

$$> 0,$$

where the last inequality holds because of condition (3). Hence, the profit $\pi^*(n)$ increases in the coalition size n in this case. To facilitate the discussion, let π_p denote the optimal profit of a PLP. Note that $\pi_p = \pi^*(1)$. It follows immediately that $\pi^*(n^*) \geq \pi_p$.

Part (b): One can show that when $\frac{v_H}{v_L} < \frac{\lambda_F}{\lambda_I}$ and $n \geq \frac{(v_H - v_L)(\delta+\mu)}{v_L\lambda_F - v_H\lambda_I}$ hold,

$$\frac{\delta + n\lambda_F + \mu}{\delta+\mu} v_L \geq \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta+\mu)}.$$

Hence, according to Lemma 1,

$$\begin{aligned}
r^* &= \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta+\mu)}, \\
p^* &= v_L + \frac{n\lambda_F}{n\lambda_F + \delta + \mu} \frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta+\mu)} = \frac{\lambda_F(\delta + \mu + n\lambda_I)v_H - \lambda_I(\delta + \mu + n\lambda_F)v_L}{(\lambda_F - \lambda_I)(\delta+\mu)}, \\
\pi^*(n) &= \beta\lambda_F\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_F}{n\lambda_F + \mu}\big)\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta+\mu)}\Big] \\
&\quad + (\alpha - \gamma)\lambda_I\Big[v_L + \big(\frac{n\lambda_F}{n\lambda_F + \mu + \delta} - \frac{n\lambda_I}{n\lambda_I + \mu}\big)\frac{(v_H - v_L)(\delta + n\lambda_F + \mu)(\delta + n\lambda_I + \mu)}{n(\lambda_F - \lambda_I)(\delta+\mu)}\Big] \\
&= \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta + n\lambda_I + \mu}{n\lambda_F + \mu} \frac{\lambda_F\delta(v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)} \\
&\quad + (\alpha - \gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H - v_L)}{(\delta+\mu)(\lambda_F - \lambda_I)} \frac{\delta + n\lambda_I + \mu}{n\lambda_I + \mu}.
\end{aligned}$$

Taking the derivative of $\pi^*(n)$ with respect to n yields

$$\begin{aligned}
\frac{d\pi^*(n)}{dn} &= -\frac{v_H - v_L}{(\lambda_F - \lambda_I)(\delta+\mu)}\Big\{\beta\lambda_F^2\delta\frac{\lambda_I\mu - \lambda_F(\delta+\mu)}{(n\lambda_F + \mu)^2} - (\alpha-\gamma)\lambda_I[\lambda_F\mu - \lambda_I(\delta+\mu)]\frac{-\lambda_I\delta}{(n\lambda_I + \mu)^2}\Big\} \\
&= \frac{(v_H - v_L)\delta}{(\lambda_F - \lambda_I)(\delta+\mu)}\Big\{\beta\lambda_F^2\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{(n\lambda_F + \mu)^2} - (\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu - \lambda_I(\delta+\mu)}{(n\lambda_I + \mu)^2}\Big\}.
\end{aligned}$$

Let

$$\bar{n} = \frac{\Big(1 - \frac{\lambda_I}{\lambda_F}\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]}}\Big)\mu}{\lambda_I\Big(\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]}} - 1\Big)}.$$

(i) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu) - \lambda_I\mu]} \leq 1$. Then,

$$(\alpha-\gamma)[\lambda_F\mu - \lambda_I(\delta+\mu)] \leq \beta[\lambda_F(\delta+\mu) - \lambda_I\mu].$$

Note that

$$\frac{\lambda_I^2}{(n\lambda_I + \mu)^2} - \frac{\lambda_F^2}{(n\lambda_F + \mu)^2} = \frac{2n\lambda_F\lambda_I\mu(\lambda_I - \lambda_F) + (\lambda_I^2 - \lambda_F^2)\mu^2}{(n\lambda_I + \mu)^2(n\lambda_F + \mu)^2} < 0,$$

so,

$$(\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(n\lambda_I+\mu)^2}<\beta\lambda_F^2\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{(n\lambda_F+\mu)^2}.$$

Hence, $\frac{d\pi^*}{dn}>0$ for any n. Recall from Part (a) that $\pi^*(n)$ increases in n when $\frac{v_H}{v_L}<\frac{\lambda_F}{\lambda_I}$ and $n<\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$. Therefore, in an *intensity-driven (ID) market*, $\pi^*(n)$ always increases in n in this case.

(ii) Suppose $1<\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<\left(\frac{\lambda_F}{\lambda_I}\right)^2$, then the definition of $\bar{n}$ implies that $\bar{n}$ is positive. One can check that $\frac{d\pi^*}{dn}=0$ if $n=\bar{n}$; $\frac{d\pi^*}{dn}>0$ if $n<\bar{n}$; and $\frac{d\pi^*}{dn}<0$ if $n>\bar{n}$. Therefore, the profit $\pi^*(n)$ first increases in the coalition size n when $n\le\bar{n}$ and then decreases in n in an ID market when $n\ge\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$. Recall from Part (a) that $\pi^*(n)$ increases in n when $n<\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$. To summarize, in an ID market, $\pi^*(n)$ first increases when $n\le\max\{\bar{n},\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}\}=\hat{n}$ and then decreases in this case.

(iii) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\ge\left(\frac{\lambda_F}{\lambda_I}\right)^2$. Then,

$$(\alpha-\gamma)\lambda_I^2[\lambda_F\mu-\lambda_I(\delta+\mu)]>\beta\lambda_F^2[\lambda_F(\delta+\mu)-\lambda_I\mu].$$

Note that $\frac{1}{(n\lambda_I+\mu)^2}>\frac{1}{(n\lambda_F+\mu)^2}$, so

$$(\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(n\lambda_I+\mu)^2}>\beta\lambda_F^2\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{(n\lambda_F+\mu)^2}.$$

Hence, $\frac{d\pi^*}{dn}<0$ for any n. That is, the profit $\pi^*(n)$ decreases in the coalition size n when $n\ge\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$. Recall from Part (a) that $\pi^*(n)$ increases in n when $n<\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$. To summarize, in an ID market, $\pi^*(n)$ first increases when $n\le\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$ and then decreases in this case. Note that in this case, $\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}>\bar{n}$, so it can be rewritten as $\pi^*(n)$ first increases when $n\le\max\{\bar{n},\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}\}=\hat{n}$ and then decreases.

In short, if $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\le 1$, the profit $\pi^*(n)$ monotonically increases in the coalition size n. Then, the optimal size n^* is as large as possible; otherwise, $\pi^*(n)$ first increases in n for $n\le\hat{n}$ and then decreases in n. Therefore, the optimal size $n^*=\lfloor\hat{n}\rfloor$ or $\lceil\hat{n}\rceil$, where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor and ceiling functions, respectively. It follows immediately that $\pi_p>\pi^*(n^*)$ if either $\hat{n}\le 1$ or $1<\hat{n}<2$ and $\pi^*(1)>\pi^*(2)$. This completes the proof. ■

After analyzing the optimal sizing decision, we also compare CLPs (with either endogenous or exogenous coalition size) to no reward programs. The results are summarized in the following lemma.

LEMMA A2. *By comparing a CLP and no reward programs, we have the following results:*[1]

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[1] For the detailed expression of the threshold $\bar{\gamma}$ and the range of γ in Parts (b-i) and Part (b-ii), please refer to the proof of Lemma A2.

(a) There is a threshold $\bar{\gamma}$ *such that under the optimal coalition size* n^*, $\pi^*(n^*) \geq \max\{\pi_1, \pi_2\}$ *when* $\gamma \leq \bar{\gamma}$.

(b) For a given coalition size n,

(i) there exists a range of γ *within* $[0, \bar{\gamma}]$ *such that* $\pi^*(n) < \max\{\pi_1, \pi_2\}$ *when* $n < n'$ *and vice versa for some threshold* n'*;*

(ii) moreover, in an ID market, there exists a range of γ *within* $[0, \bar{\gamma}]$ *such that* $\pi^*(n) \geq \max\{\pi_1, \pi_2\}$ *only when* $n_1 \leq n \leq n_2$ *for some thresholds* n_1 *and* n_2*, with* $n_1 < n_2$.

Lemma A2(a) shows that if γ is sufficiently small (which implies a strong negative correlation), the CLP with the optimal size n^* is more profitable than that with no reward programs, i.e., $\pi^*(n^*) \geq \max\{\pi_1, \pi_2\}$. To understand Lemma A2(b-i), in a VD market, recall that the per-firm profit increases in n for CLPs. Under certain conditions, there exists a threshold size n' such that the per-firm profit in a CLP "crosses" the profit of no reward programs. In an ID market, recall that the per-firm profit may first increase and then decrease in n, so there are two possibilities. First, the per-firm profit may stay above that of no reward program once n is above a threshold as shown by Figures 4(a)–(b). Second, Lemma A2(b-ii) shows that the per-firm profit may only stay above that of no reward program when n falls within a range as illustrated by Figure 4(c). These results highlight the importance of choosing an appropriate size for the coalition.

Proof of Lemma A2

Before presenting the proof, we first introduce the following notations:

$$\bar{\gamma}_v = \min\left\{\alpha - \beta\frac{\delta}{\mu} - (1-\beta)\frac{\lambda_I}{\lambda_F}\frac{\delta+\mu}{\mu}, \frac{(\alpha+\beta)\mu v_L\lambda_F - \alpha(\delta+\mu)v_H\lambda_I}{\mu v_L\lambda_F + (\delta+\mu)v_H(\lambda_F-\lambda_I)}\right\},$$

$$\bar{\gamma}_I = \min\left\{\alpha - \frac{\beta\lambda_F + (1-\beta)\lambda_I}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I + \mu v_L(\lambda_F-\lambda_I)}{\mu v_H(\lambda_F-\lambda_I)} + \beta\frac{\lambda_F}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I + \mu v_L(\lambda_F-\lambda_I)}{(v_H-v_L)\delta\lambda_F + \mu v_H(\lambda_F-\lambda_I)},\right.$$
$$\left.\frac{\frac{v_L}{v_H}[\beta\lambda_F + (1-\beta)\lambda_I] - \alpha\lambda_I}{\lambda_F-\lambda_I}, \alpha - \beta\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{\lambda_F\mu - \lambda_I(\delta+\mu)}\right\},$$

$$\bar{\gamma}_{I,1} = \min\left\{\alpha - \frac{(1-\beta)v_L + \beta\lambda_F\frac{v_H-v_L}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}}{v_L + \frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)}, \frac{\frac{v_L}{v_H}[\beta\lambda_F + (1-\beta)\lambda_I] - \alpha\lambda_I}{\lambda_F-\lambda_I}, \alpha - \beta\frac{\lambda_F(\delta+\mu) - \lambda_I\mu}{\lambda_F\mu - \lambda_I(\delta+\mu)}\right\},$$

$$\underline{\gamma}_v = \max\left\{\alpha - \frac{\beta\lambda_F\delta\frac{\lambda_F(\lambda_I+\mu)}{\lambda_I(\lambda_F+\mu)} + (1-\beta)(\mu+\delta)(\lambda_I+\mu)}{\mu(\lambda_F+\mu+\delta)}, \frac{\alpha\lambda_I(v_L\frac{\lambda_F+\delta+\mu}{\lambda_I+\mu} - v_H\frac{\mu+\delta}{\mu}) + \beta\lambda_F v_L\frac{\lambda_F+\delta+\mu}{\lambda_F+\mu}}{v_H\frac{\mu+\delta}{\mu}(\lambda_F-\lambda_I) + v_L\lambda_I\frac{\lambda_F+\mu+\delta}{\lambda_I+\mu}}\right\},$$

$$\underline{\gamma}_{I,2} = \max\left\{\alpha - \frac{\beta\lambda_F\delta\frac{\lambda_F(\lambda_I+\mu)}{\lambda_I(\lambda_F+\mu)} + (1-\beta)(\mu+\delta)(\lambda_I+\mu)}{\mu(\lambda_F+\mu+\delta)}, \alpha - \frac{(1-\beta)v_L + \beta\lambda_F\frac{v_H-v_L}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}}{v_L + \frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)}\right\}.$$

Part (a): First, we expect to show that in a VD market, if $\gamma \leq \bar{\gamma}_v$, then $\pi^*(n^*) \geq \max\{\pi_1, \pi_2\}$.

Recall that $\pi^*(n)$ increases in n in this market, so it suffices to show $\lim_{n\to\infty}\pi^*(n)\geq\max\{\pi_1,\pi_2\}$. One can check that

$$\lim_{n\to\infty}\pi^*(n)=(\alpha+\beta-\gamma)\lambda_F v_L\frac{\mu}{\delta+\mu}.$$

Comparing $\lim_{n\to\infty}\pi^*(n)$ with π_1 yields that if $\gamma\leq\alpha-\beta\frac{\delta}{\mu}-(1-\beta)\frac{\lambda_I}{\lambda_F}\frac{\delta+\mu}{\mu}$, then $\lim_{n\to\infty}\pi^*(n)\geq\pi_1$. Similarly, comparing $\lim_{n\to\infty}\pi^*(n)$ with π_2 yields that if $\gamma\leq\frac{(\alpha+\beta)\mu v_L\lambda_F-\alpha(\delta+\mu)v_H\lambda_I}{\mu v_L\lambda_F+(\delta+\mu)v_H(\lambda_F-\lambda_I)}$, then $\lim_{n\to\infty}\pi^*(n)\geq\pi_2$. Therefore, if $\gamma\leq\bar{\gamma}_v$, then $\pi^*(n^*)\geq\max\{\pi_1,\pi_2\}$.

Second, we expect to show that in an ID market, if $\gamma\leq\bar{\gamma}_I$, then $\pi^*(n^*)\geq\max\{\pi_1,\pi_2\}$. Note that $\gamma<\alpha-\beta\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{\lambda_F\mu-\lambda_I(\delta+\mu)}$ is equivalent to $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}>1$, under which the profit $\pi^*(n)$ increases in n when $n<\hat{n}$ and decreases in n otherwise. Recall that

$$\hat{n}=\max\left\{\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I},\frac{\left(1-\frac{\lambda_I}{\lambda_F}\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}\right)\mu}{\lambda_I\left(\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}-1\right)}\right\}.$$

Given $\pi^*(n^*)\geq\pi^*(n=\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I})$, in order to show $\pi^*(n^*)\geq\max\{\pi_1,\pi_2\}$, it suffices to show $\pi^*(n=\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I})$ is greater than $\max\{\pi_1,\pi_2\}$. One can check when $n=\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}$,

$$\pi^*(n)=\mu v_L v_H(\lambda_F-\lambda_I)\Big[\frac{\beta\lambda_F}{(v_H-v_L)\delta\lambda_F+\mu v_H(\lambda_F-\lambda_I)}+\frac{(\alpha-\gamma)\lambda_I}{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}\Big].\tag{5}$$

Comparing the profit in (5) with π_1 yields that if

$$\gamma\leq\alpha-\frac{\beta\lambda_F+(1-\beta)\lambda_I}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}{\mu v_H(\lambda_F-\lambda_I)}+\beta\frac{\lambda_F}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}{(v_H-v_L)\delta\lambda_F+\mu v_H(\lambda_F-\lambda_I)},$$

then $\pi^*(n)\geq\pi_1$. Moreover, if

$$\gamma\leq\frac{\frac{v_L}{v_H}[\beta\lambda_F+(1-\beta)\lambda_I]-\alpha\lambda_I}{\lambda_F-\lambda_I},$$

then $\pi_1\geq\pi_2$, and thus $\pi^*(n)\geq\pi_1\geq\pi_2$. Therefore, if $\gamma\leq\bar{\gamma}_I$, then $\pi^*(n^*)\geq\max\{\pi_1,\pi_2\}$.

Part (b-i): First, we expect to show that in a VD market, if $\gamma\in(\underline{\gamma}_v,\bar{\gamma}_v)$, then there exists a threshold n'_v such that $\pi^*(n)<\max\{\pi_1,\pi_2\}$ when $n<n'_v$ and vice versa. Note that $\pi^*(n)$ increases in n and we have shown $\lim_{n\to\infty}\pi^*(n)>\max\{\pi_1,\pi_2\}$ if $\gamma<\bar{\gamma}_v$ in Part (a). It thus suffices to show $\pi^*(1)<\max\{\pi_1,\pi_2\}$ if $\gamma>\underline{\gamma}_v$. One can check

$$\pi^*(1)=\frac{\mu}{\mu+\delta}v_L(\lambda_F+\mu+\delta)\Big\{\beta\frac{\lambda_F}{\lambda_F+\mu}+(\alpha-\gamma)\frac{\lambda_I}{\lambda_I+\mu}\Big\}.$$

Comparing $\pi^*(1)$ with π_1 yields that if

$$\gamma>\alpha-\frac{\beta\lambda_F\delta\frac{\lambda_F(\lambda_I+\mu)}{\lambda_I(\lambda_F+\mu)}+(1-\beta)(\mu+\delta)(\lambda_I+\mu)}{\mu(\lambda_F+\mu+\delta)},$$

then $\pi^*(1) < \pi_1$. Comparing $\pi^*(1)$ with π_2 yields that if

$$\gamma > \frac{\alpha\lambda_I(v_L\frac{\lambda_F+\delta+\mu}{\lambda_I+\mu} - v_H\frac{\mu+\delta}{\mu}) + \beta\lambda_F v_L\frac{\lambda_F+\delta+\mu}{\lambda_F+\mu}}{v_H\frac{\mu+\delta}{\mu}(\lambda_F-\lambda_I) + v_L\lambda_I\frac{\lambda_F+\mu+\delta}{\lambda_I+\mu}},$$

then $\pi^*(1) < \pi_2$. Therefore, if $\gamma \in (\underline{\gamma}_v, \bar{\gamma}_v)$, then $\pi^*(1) < \max\{\pi_1, \pi_2\} < \lim_{n\to\infty}\pi^*(n)$. Given that $\pi^*(n)$ increases in n, there must exist a threshold n'_v such that $\pi^*(n) < \max\{\pi_1, \pi_2\}$ when $n < n'_v$ and vice versa.

Second, we expect to show in an ID market, if $\gamma \in (\underline{\gamma}_v, \bar{\gamma}_{I,1})$, then there exists a threshold n'_I such that $\pi^*(n) < \max\{\pi_1, \pi_2\}$ when $n < n'_I$ and vice versa. Note that $\gamma < \alpha - \beta\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{\lambda_F\mu-\lambda_I(\delta+\mu)}$ is equivalent to $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} > 1$, under which the profit $\pi^*(n)$ increases in the coalition size n when $n < \hat{n}$ and decreases in n otherwise. Similar to the VD market, it suffices to show $\pi^*(1) < \max\{\pi_1, \pi_2\} < \lim_{n\to\infty}\pi^*(n)$. One can check

$$\pi^*(1) = \frac{\mu}{\mu+\delta}v_L(\lambda_F+\mu+\delta)\Big\{\beta\frac{\lambda_F}{\lambda_F+\mu} + (\alpha-\gamma)\frac{\lambda_I}{\lambda_I+\mu}\Big\},$$

$$\lim_{n\to\infty}\pi^*(n) = \beta\lambda_F[v_L - \frac{\lambda_I(v_H-v_L)}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}] + (\alpha-\gamma)\lambda_I[v_L + \frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)].$$

Comparing $\pi^*(1)$ with π_1 yields that if

$$\gamma > \alpha - \frac{\beta\lambda_F\delta\frac{\lambda_F(\lambda_I+\mu)}{\lambda_I(\lambda_F+\mu)} + (1-\beta)(\mu+\delta)(\lambda_I+\mu)}{\mu(\lambda_F+\mu+\delta)},$$

then $\pi^*(1) < \pi_1$. Comparing $\pi^*(1)$ with π_2 yields that if

$$\gamma > \frac{\alpha\lambda_I(v_L\frac{\lambda_F+\delta+\mu}{\lambda_I+\mu} - v_H\frac{\mu+\delta}{\mu}) + \beta\lambda_F v_L\frac{\lambda_F+\delta+\mu}{\lambda_F+\mu}}{v_H\frac{\mu+\delta}{\mu}(\lambda_F-\lambda_I) + v_L\lambda_I\frac{\lambda_F+\mu+\delta}{\lambda_I+\mu}},$$

then $\pi^*(1) < \pi_2$.

On the other hand, if

$$\gamma < \alpha - \frac{(1-\beta)v_L + \beta\lambda_F\frac{v_H-v_L}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}}{v_L + \frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)},$$

then $\lim_{n\to\infty}\pi^*(n) > \pi_1$. Moreover, if

$$\gamma < \frac{\frac{v_L}{v_H}[\beta\lambda_F + (1-\beta)\lambda_I] - \alpha\lambda_I}{\lambda_F-\lambda_I},$$

then $\pi_1 > \pi_2$, and thus $\lim_{n\to\infty}\pi^*(n) > \pi_1 > \pi_2$. Combining the above yields that if $\gamma \in (\underline{\gamma}_v, \bar{\gamma}_{I,1})$, then $\pi^*(1) < \max\{\pi_1, \pi_2\} < \lim_{n\to\infty}\pi^*(n)$, and thus there exists a threshold n'_I such that $\pi^*(n) < \max\{\pi_1, \pi_2\}$ when $n < n'_I$ and vice versa.

Part (b-ii): We expect to show that in an ID market, if $\gamma \in (\underline{\gamma}_{I,2}, \bar{\gamma}_I)$, then there exists a range $[n_1, n_2]$ such that $\pi^*(n) \geq \max\{\pi_1, \pi_2\}$ only when $n \in [n_1, n_2]$.

Note that $\gamma < \alpha - \beta\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{\lambda_F\mu-\lambda_I(\delta+\mu)}$ is equivalent to $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} > 1$, under which the profit $\pi^*(n)$ increases in the coalition size n when $n<\hat{n}$ and decreases in n otherwise. Therefore, it suffices to show

$$\pi^*(1) < \max\{\pi_1,\pi_2\}, \lim_{n\to\infty}\pi^*(n) < \max\{\pi_1,\pi_2\}, \text{ and } \pi^*(n^*) > \max\{\pi_1,\pi_2\}.$$

One can check

$$\begin{aligned}
\pi^*(1) &= \frac{\mu}{\mu+\delta}v_L(\lambda_F+\mu+\delta)\Big\{\beta\frac{\lambda_F}{\lambda_F+\mu}+(\alpha-\gamma)\frac{\lambda_I}{\lambda_I+\mu}\Big\},\\
\lim_{n\to\infty}\pi^*(n) &= \beta\lambda_F[v_L-\frac{\lambda_I(v_H-v_L)}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}]+(\alpha-\gamma)\lambda_I[v_L+\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)],\\
\pi^*(n^*) &\geq \pi^*(n=\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I})\\
&= \mu v_L v_H(\lambda_F-\lambda_I)\Big[\frac{\beta\lambda_F}{(v_H-v_L)\delta\lambda_F+\mu v_H(\lambda_F-\lambda_I)}+\frac{(\alpha-\gamma)\lambda_I}{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}\Big].
\end{aligned}$$

We can show that $\pi_1 > \pi_2$ if

$$\gamma < \frac{\frac{v_L}{v_H}[\beta\lambda_F+(1-\beta)\lambda_I]-\alpha\lambda_I}{\lambda_F-\lambda_I}.$$

Comparing $\pi^*(1)$ with π_1 yields that $\pi^*(1) < \pi_1$ if

$$\gamma > \alpha - \frac{\beta\lambda_F\delta\frac{\lambda_F(\lambda_I+\mu)}{\lambda_I(\lambda_F+\mu)}+(1-\beta)(\mu+\delta)(\lambda_I+\mu)}{\mu(\lambda_F+\mu+\delta)}.$$

Comparing $\lim_{n\to\infty}\pi^*(n)$ with π_1 yields that $\lim_{n\to\infty}\pi^*(n) < \pi_1$ if

$$\gamma > \alpha - \frac{(1-\beta)v_L+\beta\lambda_F\frac{v_H-v_L}{\lambda_F-\lambda_I}\frac{\delta}{\delta+\mu}}{v_L+\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}(v_H-v_L)}.$$

We can further show that $\pi^*(n=\frac{(v_H-v_L)(\delta+\mu)}{v_L\lambda_F-v_H\lambda_I}) > \pi_1$ if

$$\gamma < \alpha - \frac{\beta\lambda_F+(1-\beta)\lambda_I}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}{\mu v_H(\lambda_F-\lambda_I)}+\beta\frac{\lambda_F}{\lambda_I}\frac{(v_H-v_L)\delta\lambda_I+\mu v_L(\lambda_F-\lambda_I)}{(v_H-v_L)\delta\lambda_F+\mu v_H(\lambda_F-\lambda_I)}.$$

Combining the above establishes the result. ■

Appendix B: No Customer Discounting

Note that the solution for $h(\cdot)$ is not unique. By fixing $h(0)=0$, we solve the customer's dynamic programming equations (5)–(6) and derive a solution for the optimal average customer surplus g^* and the bias function $h(1)$. We summarize the results as follows. If

$$v-p+\frac{n\lambda}{n\lambda+\mu}r \geq 0,$$

then it is optimal for a (v,λ)-customer to make a purchase whenever she visits a firm in the coalition, and a solution to the optimality equations (5)–(6) is given by

$$g^* = \frac{n\lambda}{n\lambda+\mu}(v-p+\frac{n\lambda}{n\lambda+\mu}r), \quad h(1)=\frac{n\lambda}{n\lambda+\mu}r, \quad h(0)=0.$$

Proposition 3. *Suppose that customers do not discount the future surplus. For any fixed coalition size n, the optimal per-firm profit in the CLP is*

$$\begin{aligned} p^* &= v_L + \frac{n\lambda_F}{n\lambda_F+\mu} r^*, \\ r^* &= \min\Big\{ \frac{(v_H - v_L)(n\lambda_F+\mu)(n\lambda_I+\mu)}{n\mu(\lambda_F-\lambda_I)}, \frac{(\mu+n\lambda_F)}{\mu} v_L \Big\}, \\ \pi^*(n) &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I \min\Big\{ v_H, \frac{n\lambda_F+\mu}{n\lambda_I+\mu} v_L \Big\}. \end{aligned}$$

Moreover, in a VD market, the per-firm profit $\pi^*(n)$ monotonically increases in the coalition size n, while in an ID market, $\pi^*(n)$ first increases in n and then remains constant for $n \geq \frac{\mu(v_H-v_L)}{v_L\lambda_F - v_H\lambda_I}$.

Proof of Proposition 3

To derive the per-firm profit in the CLP collected from the four customer segments, we analyze the relationship between price p, $v_H + \frac{n\lambda_F}{n\lambda_F+\mu} r$, $v_H + \frac{n\lambda_I}{n\lambda_I+\mu} r$, $v_L + \frac{n\lambda_F}{n\lambda_F+\mu} r$, and $v_L + \frac{n\lambda_I}{n\lambda_I+\mu} r$ to determine the customer's purchase behavior in each segment. The analysis is very similar to the proof of Lemma 1. Therefore, to avoid repetition, we omit the analysis of the non-optimal cases (one can verify that the profits in the non-optimal cases are dominated by those with not offering reward programs). Below, we just present the details of the optimal case.

Suppose

$$v_L + \frac{n\lambda_F}{n\lambda_F+\mu} r \leq v_H + \frac{n\lambda_I}{n\lambda_I+\mu} r.$$

It follows that $p^* = v_L + \frac{n\lambda_F}{n\lambda_F+\mu} r$. Reorganizing the supposition gives $r \leq \frac{(v_H-v_L)(n\lambda_F+\mu)(n\lambda_I+\mu)}{n\mu(\lambda_F-\lambda_I)}$. Moreover, $p \geq r$ yields $r \leq \frac{n\lambda_F+\mu}{\mu} v_L$. Hence,

$$r \leq \min\Big\{ \frac{(v_H - v_L)(n\lambda_F+\mu)(n\lambda_I+\mu)}{n\mu(\lambda_F-\lambda_I)}, \frac{n\lambda_F+\mu}{\mu} v_L \Big\}. \tag{6}$$

The profit in this case becomes

$$\beta\lambda_F v_L + (\alpha-\gamma)\lambda_I \Big[v_L + (\frac{n\lambda_F}{n\lambda_F+\mu} - \frac{n\lambda_I}{n\lambda_I+\mu}) r \Big].$$

Note that

$$\frac{n\lambda_F}{n\lambda_F+\mu} - \frac{n\lambda_I}{n\lambda_I+\mu} > 0,$$

so, by inequality (6), we have

$$r^* = \min\Big\{ \frac{(v_H - v_L)(n\lambda_F+\mu)(n\lambda_I+\mu)}{n\mu(\lambda_F-\lambda_I)}, \frac{n\lambda_F+\mu}{\mu} v_L \Big\}.$$

Plugging r^* into the profit function yields

$$\pi^*(n) = \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I \min\Big\{ v_H, \frac{n\lambda_F+\mu}{n\lambda_I+\mu} v_L \Big\}.$$

In a VD market ($\frac{v_H}{v_L} \geq \frac{\lambda_F}{\lambda_I}$), one can check that

$$\pi^*(n) = \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L.$$

Taking the derivative of $\pi^*(n)$ with respect to n yields

$$\frac{d\pi^*}{dn} = (\alpha - \gamma)\lambda_I v_L \frac{\mu(\lambda_F - \lambda_I)}{(n\lambda_I + \mu)^2} > 0.$$

Therefore, the profit increases in the coalition size n.

In an ID market ($\frac{v_H}{v_L} < \frac{\lambda_F}{\lambda_I}$),

$$\pi^* = \begin{cases} \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I \frac{n\lambda_F + \mu}{n\lambda_I + \mu} v_L, & \text{if } n < \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}, \\ \beta\lambda_F v_L + (\alpha - \gamma)\lambda_I v_H, & \text{otherwise.} \end{cases} \tag{7}$$

Clearly, the profit first increases in the coalition size n and then remains a constant for $n \geq \frac{\mu(v_H - v_L)}{v_L\lambda_F - v_H\lambda_I}$. This completes the proof. ■

Appendix C: Proof of Proposition 4

A generic customer visits firms 1 and 2 with shopping intensities $n_1\lambda$ and $n_2\lambda$, where $n_1 \neq n_2$ are positive integers, respectively. Since firms 1 and 2 are independent, the total shopping intensity for the two firms in the coalition is thus $n_1\lambda + n_2\lambda$. Hence, the optimality equations can be written as

$$u(1) = \frac{n_1\lambda + n_2\lambda}{\delta + n_1\lambda + n_2\lambda + \mu} \max\Big\{v - p + r + u(1), u(1)\Big\} + \frac{\mu}{\delta + n_1\lambda + n_2\lambda + \mu} u(0), \tag{8}$$

$$u(0) = \frac{n_1\lambda + n_2\lambda}{\delta + n_1\lambda + n_2\lambda + \mu} \max\Big\{v - p + u(1), u(0)\Big\} + \frac{\mu}{\delta + n_1\lambda + n_2\lambda + \mu} u(0). \tag{9}$$

Following the same procedure as that for the base model, we can obtain the explicit solution to the above optimality equations. We then have that if

$$v - p + \frac{(n_1 + n_2)\lambda}{\delta + (n_1 + n_2)\lambda + \mu} r \geq 0, \tag{10}$$

then it is optimal for a customer to make a purchase whenever she visits a firm in the coalition, and a solution to the optimality equations (8)–(9) is given by

$$u(1) = \frac{(n_1 + n_2)\lambda}{\delta}\Big\{v - p + \frac{\delta + (n_1 + n_2)\lambda}{\delta + (n_1 + n_2)\lambda + \mu} r\Big\}, u(0) = \frac{(n_1 + n_2)\lambda}{\delta}\Big\{v - p + \frac{(n_1 + n_2)\lambda}{\delta + (n_1 + n_2)\lambda + \mu} r\Big\}.$$

Based on the customer's purchase behavior, we derive the optimal price, reward amount, and coalition's total profit as follows. Let π_1^A and π_2^A denote the respective optimal profits of firm 1 and firm 2. Given any exogenous expiration rate μ, offering a CLP makes the two asymmetric firms better off only if

$$(\alpha - \gamma)\lambda_I \Big(\mu\lambda_F - \lambda_I(\delta + \mu)\Big)((n_1 + n_2)\lambda_F + \mu) > \beta\lambda_F^2\delta((n_1 + n_2)\lambda_I + \mu),$$

under which the optimal price, reward amount, and the coalition's profit are

$$p^* = v_L + \frac{(n_1+n_2)\lambda_F}{\delta+(n_1+n_2)\lambda_F+\mu} r^*, \tag{11}$$

$$r^* = \min\Big\{\frac{(v_H-v_L)(\delta+(n_1+n_2)\lambda_F+\mu)(\delta+(n_1+n_2)\lambda_I+\mu)}{((n_1+n_2)\lambda_F-(n_1+n_2)\lambda_I)(\delta+\mu)}, \frac{(\delta+\mu+(n_1+n_2)\lambda_F)}{\delta+\mu} v_L\Big\}, \tag{12}$$

$$\begin{aligned}\pi_1^A+\pi_2^A = {} & \beta(n_1+n_2)\lambda_F\Big(v_L + \frac{(n_1+n_2)\lambda_F}{\delta+(n_1+n_2)\lambda_F+\mu} r^* - \frac{(n_1+n_2)\lambda_F}{(n_1+n_2)\lambda_F+\mu} r^*\Big)\\ & + (\alpha-\gamma)(n_1+n_2)\lambda_I\Big(v_L + \frac{(n_1+n_2)\lambda_F}{\delta+(n_1+n_2)\lambda_F+\mu} r^* - \frac{(n_1+n_2)\lambda_I}{(n_1+n_2)\lambda_I+\mu} r^*\Big).\end{aligned}$$

Otherwise, a CLP should not be offered.

According to Lemma 1, for a CLP consisting of n_1+n_2 symmetric firms, the price and reward are the same as those stated in (11) and (12), and the per-firm profit is

$$\begin{aligned}\pi^*(n_1+n_2) = {} & \beta\lambda_F\Big(v_L + \frac{(n_1+n_2)\lambda_F}{\delta+(n_1+n_2)\lambda_F+\mu} r^* - \frac{(n_1+n_2)\lambda_F}{(n_1+n_2)\lambda_F+\mu} r^*\Big)\\ & + (\alpha-\gamma)\lambda_I\Big(v_L + \frac{(n_1+n_2)\lambda_F}{\delta+(n_1+n_2)\lambda_F+\mu} r^* - \frac{(n_1+n_2)\lambda_I}{(n_1+n_2)\lambda_I+\mu} r^*\Big).\end{aligned}$$

Clearly, $\pi^*(n_1+n_2) = \frac{\pi_1^A+\pi_2^A}{n_1+n_2}$. This completes the proof. ■

Appendix D: No New Reward upon Redemption

Following a similar process as in the base model, we first derive the customer's optimal purchasing decision and then analyze the coalition's pricing and reward decisions, by assuming that the coalition size n is given.

LEMMA D1. *If*

$$v - p + \frac{n\lambda}{\delta+2n\lambda+\mu} r \geq 0, \tag{13}$$

then it is optimal for a (v,λ)*-customer to make a purchase whenever she visits a firm in the coalition, and the solution to the optimality equations* (8)–(9) *is given by*

$$u(0) = \frac{n\lambda}{\delta}\left(v - p + \frac{n\lambda}{\delta+2n\lambda+\mu} r\right), \quad u(1) = \frac{n\lambda}{\delta}\left(v - p + \frac{\delta+n\lambda}{\delta+2n\lambda+\mu} r\right).$$

Proof of Lemma D1

We first show if $v-p+u(1) \geq u(0)$, then $v-p+r+u(0) \geq u(1)$. Suppose for a contradiction that $v-p+r+u(0) < u(1)$. Then,

$$u(1) - u(0) = \frac{n\lambda}{\nu}\Big\{u(1) - [v-p+u(1)]\Big\} = \frac{n\lambda}{\nu}(-v+p) > 0,$$

from which we obtain $v < p$. It follows immediately that

$$v-p+u(1)-u(0)=v-p+\frac{n\lambda}{\nu}(-v+p)=\frac{\delta+\mu}{\nu}(v-p)<0,$$

contradicting our supposition. Hence, it must be the case that $v-p+r+u(0)\geq u(1)$. Suppose $v-p+u(1)\geq u(0)$. Then, equations (8) and (9) reduce to

$$u(1)=\frac{n\lambda}{\nu}[v-p+r+u(0)]+\frac{\mu}{\nu}u(0),\text{ and } u(0)=\frac{n\lambda}{\nu}[v-p+u(1)]+\frac{\mu}{\nu}u(0).$$

Solving the above set of equations yields that

$$u(1)=\frac{n\lambda}{\delta}\left(v-p+\frac{\delta+n\lambda}{\delta+2n\lambda+\mu}r\right),\text{ and } u(0)=\frac{n\lambda}{\delta}\left(v-p+\frac{n\lambda}{\delta+2n\lambda+\mu}r\right).$$

Plugging $u(1)$ and $u(0)$ back to the supposition $v-p+u(1)\geq u(0)$ yields that

$$v-p+\frac{n\lambda}{\delta+2n\lambda+\mu}r\geq 0,$$

under which the customer makes a purchase whenever she visits a firm in the coalition. This completes the proof. ■

By solving the balance equations $q_0\frac{n\lambda}{\nu}=q_1\frac{n\lambda+\mu}{\nu}$ and $q_0+q_1=1$, we find that $q_0=\frac{n\lambda+\mu}{2n\lambda+\mu}$ and $q_1=\frac{n\lambda}{2n\lambda+\mu}$. Accordingly, when condition (13) is satisfied, the customer contributes a profit to the coalition at rate

$$n\lambda\Big(q_0p+q_1(p-r)\Big)=n\lambda\Big(p-\frac{n\lambda}{2n\lambda+\mu}r\Big). \tag{14}$$

To derive the per-firm profit in the CLP collected from the four customer segments, we analyze the relationship between price p, $v_H+\frac{n\lambda_F}{\delta+2n\lambda_F+\mu}r$, $v_H+\frac{n\lambda_I}{\delta+2n\lambda_I+\mu}r$, $v_L+\frac{n\lambda_F}{\delta+2n\lambda_F+\mu}r$, and $v_L+\frac{n\lambda_I}{\delta+2n\lambda_I+\mu}r$ to determine the customer's purchase behavior in each segment. Lemma D2 summarizes the optimal price, reward amount, and per-firm profit.

Lemma D2. *A CLP with size n improves the per-firm profit relative to no reward program only if*

$$(\alpha-\gamma)\lambda_I\Big(\lambda_F\mu-\lambda_I(\delta+\mu)\Big)(2n\lambda_F+\mu)>\beta\lambda_F^2\delta(2n\lambda_I+\mu). \tag{15}$$

When condition(15) *is satisfied, the optimal price, reward amount, and per-firm profit in the CLP are as follows:*

(a) When $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}<\frac{v_H}{v_L}$, *the optimal price, reward amount, and per-firm profit in the CLP are*

$$p^*=r^*=\frac{\delta+2n\lambda_F+\mu}{\delta+n\lambda_F+\mu}v_L;$$

$$\pi_a^*=\beta\lambda_F v_L+(\alpha-\gamma)\lambda_I v_L-\beta\lambda_F\frac{n\lambda_F\delta}{2n\lambda_F+\mu}\frac{v_L}{\delta+n\lambda_F+\mu}+(\alpha-\gamma)\lambda_I\frac{n[\lambda_F\mu-\lambda_I(\delta+\mu)]}{2n\lambda_I+\mu}\frac{v_L}{\delta+n\lambda_F+\mu}.$$

(b) When $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)} \geq \frac{v_H}{v_L}$*, then the optimal price, reward amount, and per-firm profit are*

$$\begin{aligned}
p^* &= \frac{\lambda_F(\delta+2n\lambda_I+\mu)v_H - \lambda_I(\delta+2n\lambda_F+\mu)v_L}{(\lambda_F-\lambda_I)(\delta+\mu)},\\
r^* &= \frac{(v_H-v_L)(\delta+2n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)},\\
\pi_b^* &= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F \frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu}\frac{\lambda_F\delta(v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\\
&\quad + (\alpha-\gamma)\lambda_I \frac{[\lambda_F\mu - \lambda_I(\delta+\mu)](v_H-v_L)}{(\delta+\mu)(\lambda_F-\lambda_I)}\frac{\delta+2n\lambda_I+\mu}{2n\lambda_I+\mu}.
\end{aligned}$$

The proof of Lemma D2 is very similar to that of Lemma 1. To avoid repetition, we omit the analysis of the non-optimal cases (one can verify that the profits in the non-optimal cases are dominated by those with not offering reward programs). Below, we just present the details of the optimal case. Suppose

$$v_L + \frac{n\lambda_F}{\delta+2n\lambda_F+\mu}r \leq v_H + \frac{n\lambda_I}{\delta+2n\lambda_I+\mu}r.$$

It follows that $p^* = v_L + \frac{n\lambda_F}{\delta+2n\lambda_F+\mu}r$. Reorganizing condition (13) gives $r \leq \frac{(v_H-v_L)(\delta+2n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}$. Moreover, $p \geq r$ yields $r \leq \frac{\delta+2n\lambda_F+\mu}{\delta+n\lambda_F+\mu}v_L$. So,

$$r \leq \min\Big\{\frac{(v_H-v_L)(\delta+2n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}, \frac{\delta+2n\lambda_F+\mu}{\delta+n\lambda_F+\mu}v_L\Big\}. \tag{16}$$

The profit in this case becomes

$$\beta\lambda_F\Big[v_L + (\frac{n\lambda_F}{\delta+2n\lambda_F+\mu} - \frac{n\lambda_F}{2n\lambda_F+\mu})r\Big] + (\alpha-\gamma)\lambda_I\Big[v_L + (\frac{n\lambda_F}{\delta+2n\lambda_F+\mu} - \frac{n\lambda_I}{2n\lambda_I+\mu})r\Big]. \tag{17}$$

Note that if and only if

$$\beta\lambda_F(\frac{n\lambda_F}{\delta+2n\lambda_F+\mu} - \frac{n\lambda_F}{2n\lambda_F+\mu}) + (\alpha-\gamma)\lambda_I(\frac{n\lambda_F}{\delta+2n\lambda_F+\mu} - \frac{n\lambda_I}{2n\lambda_I+\mu}) > 0, \tag{18}$$

$r^* \neq 0$. Therefore, if condition (18) holds, by inequality (16),

$$r^* = \min\Big\{\frac{(v_H-v_L)(\delta+2n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}{n(\lambda_F-\lambda_I)(\delta+\mu)}, \frac{\delta+2n\lambda_F+\mu}{\delta+n\lambda_F+\mu}v_L\Big\}.$$

Reorganizing inequality (18) yields condition (15). Comparing the two terms within r^* and plugging the smaller one into p^* and (17) lead to the two cases in Lemma D2.

We next compare CLPs with no reward programs in terms of per-firm profit and customer surplus. Lemma D3 shows that the comparison result between CLPs and no reward programs in the base model still holds when no reward is earned upon redemption.

Lemma D3. *By comparing a CLP and no reward programs, we have the following:*

(a) When customers' product valuation and shopping intensity are positively correlated, i.e., $\gamma > \alpha\beta$, a CLP, regardless of its size, always yields a lower profit for the firm than no reward programs; that is, $\pi^*(n) < \max\{\pi_1, \pi_2\}$ for any coalition size n.

(b) The aggregate customer surplus increases when firms switch from a partial market coverage strategy with price v_H to forming a CLP, while the aggregate customer surplus decreases when firms switch from a full market coverage strategy with price v_L to forming a CLP.

Proof of Lemma D3

Part (a): The profit $\pi^*(n)$ in (17) consists of the profit collected from frequent customers and HI customers. Note that $\frac{n\lambda_F}{\delta+2n\lambda_F+\mu} < \frac{n\lambda_F}{2n\lambda_F+\mu}$, so the effective price paid by HF and LF customers is smaller than v_L (i.e., $v_L + \frac{n\lambda_F}{\delta+2n\lambda_F+\mu} r^* - \frac{2n\lambda_F}{n\lambda_F+\mu} r^* < v_L$). In contrast, the effective price paid by HI customers can be greater than v_L but smaller than v_H (i.e., $v_L + \frac{n\lambda_F}{\delta+2n\lambda_F+\mu} r^* - \frac{n\lambda_I}{2n\lambda_I+\mu} r^* \le v_H + \frac{n\lambda_I}{\delta+2n\lambda_I+\mu} r^* - \frac{n\lambda_I}{2n\lambda_I+\mu} r^* < v_H$). This leads to an upper bound on the per-firm profit, $\pi^*(n) < \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_H$, which is the same as that in the base model. Similar to the proof of Proposition 1(a), one can show that $\pi^*(n) < \max\{\pi_1, \pi_2\}$ when $\gamma > \alpha\beta$.

Part (b): Table 2 presents customer welfare in each segment under CLPs, and compares them with that under the full market coverage strategy with price v_L.

	CLP ($\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)} < \frac{v_H}{v_L}$)	vs.	Full Market Coverage	vs.	CLP ($\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)} \ge \frac{v_H}{v_L}$)
HF	$v_H - \frac{(n\lambda_F+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)} v_L$	$>$	$v_H - v_L$	$<$	$(v_H - v_L)\left[1 + \frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu} \frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\right]$
LF	$v_L - \frac{(n\lambda_F+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)} v_L$	$>$	0	$<$	$(v_H - v_L)\frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu} \frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}$
HI	$v_H - \frac{(n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_I+\mu)(\delta+n\lambda_F+\mu)} v_L$	$<$	$v_H - v_L$	$>$	$(v_H - v_L)\left[1 - \frac{\delta+2n\lambda_I+\mu}{2n\lambda_I+\mu} \frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\right]$
LI	0	$=$	0	$=$	0

Table 2 Customer Welfare Comparison in Each Customer Segment Between CLP and Full Market Coverage when No New Reward upon Redemption

When the firm adopts the partial market coverage strategy with price v_H, the aggregate customer surplus is 0. Note that customer surplus of each purchased segment in CLPs is nonnegative, because otherwise, customers will not make a purchase. Indeed, Table 2 shows that the aggregate customer surplus in CLPs is strictly positive. Therefore, CLPs lead to a higher aggregate customer surplus than the partial market coverage strategy with price v_H. Below, we focus on the comparison between CLPs and full market coverage strategy with price v_L. We have the following two cases.

Case 1: Suppose $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)} < \frac{v_H}{v_L}$. By Table 2, the aggregate customer surplus associated with CLPs in this case minus that associated with the full market coverage strategy with price v_L equals

$$
\begin{aligned}
&\gamma\lambda_F\Big\{v_H-\frac{(n\lambda_F+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)}v_L-(v_H-v_L)\Big\}+(\beta-\gamma)\lambda_F\Big\{v_L-\frac{(n\lambda_F+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)}v_L\Big\}\\
&\quad+(\alpha-\gamma)\lambda_I\Big\{v_H-\frac{(n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_I+\mu)(\delta+n\lambda_F+\mu)}v_L-(v_H-v_L)\Big\}\\
=&\beta\lambda_F\Big\{v_L-\frac{(n\lambda_F+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)}v_L\Big\}+(\alpha-\gamma)\lambda_I\Big\{v_L-\frac{(n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(2n\lambda_I+\mu)(\delta+n\lambda_F+\mu)}v_L\Big\}\\
=&\frac{nv_L}{\delta+n\lambda_F+\mu}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{2n\lambda_F+\mu}-(\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{2n\lambda_I+\mu}\Big\}\\
<&0,
\end{aligned}
$$

where the last inequality holds because of condition (15).

Case 2: Suppose $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}\geq\frac{v_H}{v_L}$. Similarly, we have

$$
\begin{aligned}
&\gamma\lambda_F\Big\{(v_H-v_L)\Big[1+\frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]-(v_H-v_L)\Big\}\\
&\quad+(\beta-\gamma)\lambda_F\Big\{(v_H-v_L)\frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
&\quad+(\alpha-\gamma)\lambda_I\Big\{(v_H-v_L)\Big[1-\frac{\delta+2n\lambda_I+\mu}{2n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big]-(v_H-v_L)\Big\}\\
=&\beta\lambda_F\Big\{(v_H-v_L)\frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu}\frac{\lambda_F\delta}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}-(\alpha-\gamma)\lambda_I\Big\{(v_H-v_L)\frac{\delta+2n\lambda_I+\mu}{2n\lambda_I+\mu}\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\}\\
=&(v_H-v_L)\frac{\delta+2n\lambda_I+\mu}{(\delta+\mu)(\lambda_F-\lambda_I)}\Big\{\beta\lambda_F\frac{\lambda_F\delta}{2n\lambda_F+\mu}-(\alpha-\gamma)\lambda_I\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{2n\lambda_I+\mu}\Big\}\\
<&0,
\end{aligned}
$$

where the last inequality holds because of condition (15). This completes the proof. ■

PROPOSITION 5. *Suppose that customers do not earn a new reward upon redeeming an existing one. The following results hold:*

(a) When $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}<\frac{v_H}{v_L}$*, the profit* $\pi^*(n)$ *does not monotonically increase in the coalition size* n*. In particular, the profit decreases in* n *when* n *is sufficiently large.*

(b) When $\frac{(\delta+n\lambda_I+\mu)(\delta+2n\lambda_F+\mu)}{(\delta+n\lambda_F+\mu)(\delta+2n\lambda_I+\mu)}\geq\frac{v_H}{v_L}$*, the profit* $\pi^*(n)$ *monotonically increases in the coalition size* n *if* $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\leq 1$*. The profit* $\pi^*(n)$ *monotonically decreases in the coalition size* n *if* $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\geq\left(\frac{\lambda_F}{\lambda_I}\right)^2$*. Otherwise, the profit* $\pi^*(n)$ *first increases and then decreases in* n*. Similar to the base model, there exist conditions under which a PLP yields a higher per-firm profit than an optimally sized CLP.*

Proof of Proposition 5

Part (a): Taking the derivative of π_a^* with respect to n yields

$$\begin{aligned}
\frac{d\pi_a^*}{dn} &= -\beta\lambda_F^2\delta v_L\Big(\frac{n}{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)}\Big)' \\
&\quad + (\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]v_L\Big(\frac{n}{(2n\lambda_I+\mu)(\delta+n\lambda_F+\mu)}\Big)' \\
&= -\beta\lambda_F^2\delta v_L\frac{(2n\lambda_F+\mu)(\delta+n\lambda_F+\mu)-n[2\lambda_F(\delta+n\lambda_F+\mu)+\lambda_F(2n\lambda_F+\mu)]}{(2n\lambda_F+\mu)^2(\delta+n\lambda_F+\mu)^2} \\
&\quad + (\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]v_L \\
&\qquad \times \frac{(2n\lambda_I+\mu)(\delta+n\lambda_F+\mu)-n[2\lambda_I(\delta+n\lambda_F+\mu)+\lambda_F(2n\lambda_I+\mu)]}{(2n\lambda_I+\mu)^2(\delta+n\lambda_F+\mu)^2} \\
&= -\beta\lambda_F^2\delta v_L\frac{\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)}{(2n\lambda_F+\mu)^2(\delta+n\lambda_F+\mu)^2} \\
&\quad + (\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]v_L\frac{\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_I+\mu)}{(2n\lambda_I+\mu)^2(\delta+n\lambda_F+\mu)^2} \\
&= \frac{v_L}{(\delta+n\lambda_F+\mu)^2} \\
&\quad \times\Big(\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)][\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_I+\mu)](2n\lambda_F+\mu)^2}{(2n\lambda_F+\mu)^2(2n\lambda_I+\mu)^2} \\
&\qquad - \frac{\beta\lambda_F^2\delta[\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)](2n\lambda_I+\mu)^2}{(2n\lambda_F+\mu)^2(2n\lambda_I+\mu)^2}\Big) \qquad (19) \\
&> \frac{v_L}{(\delta+n\lambda_F+\mu)^2} \\
&\quad \times\Big(\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)][\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)](2n\lambda_F+\mu)}{(2n\lambda_F+\mu)(2n\lambda_I+\mu)^2} \\
&\qquad - \frac{\beta\lambda_F^2\delta[\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)](2n\lambda_I+\mu)\frac{2n\lambda_I+\mu}{2n\lambda_F+\mu}}{(2n\lambda_F+\mu)(2n\lambda_I+\mu)^2}\Big) \\
&> \frac{v_L}{(\delta+n\lambda_F+\mu)^2}[\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)] \\
&\quad \times\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](2n\lambda_F+\mu)-\beta\lambda_F^2\delta(2n\lambda_I+\mu)}{(2n\lambda_F+\mu)(2n\lambda_I+\mu)^2} \\
&> 0,
\end{aligned}$$

where the first inequality holds because $2n\lambda_I+\mu<2n\lambda_F+\mu$, the second inequality holds because $\frac{2n\lambda_I+\mu}{2n\lambda_F+\mu}<1$ and $\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)>0$, and the last inequality holds because of condition (15) and $\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)>0$ when $n<\frac{\sqrt{2\mu(\delta+\mu)}}{2\lambda_F}$. This establishes that when n is relatively small (i.e., $n<\frac{\sqrt{2\mu(\delta+\mu)}}{2\lambda_F}$), the profit π_a^* is increasing in size n.

On the other hand, by (19), we have

$$\frac{d\pi_a^*}{dn}=\frac{v_L}{(\delta+n\lambda_F+\mu)^2}\times\Big(\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)][\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_I+\mu)](2n\lambda_F+\mu)^2}{(2n\lambda_F+\mu)^2(2n\lambda_I+\mu)^2}$$

$$
\begin{aligned}
&\qquad\qquad - \frac{\beta\lambda_F^2\delta[\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)](2n\lambda_I+\mu)^2}{(2n\lambda_F+\mu)^2(2n\lambda_I+\mu)^2}\Big) \\
&= \frac{v_L}{(\delta+n\lambda_F+\mu)^2}\times\Big(\frac{(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)][\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_I+\mu)]}{(2n\lambda_I+\mu)^2} \\
&\qquad\qquad - \frac{\beta\lambda_F^2\delta[\mu(\delta+n\lambda_F+\mu)-n\lambda_F(2n\lambda_F+\mu)]}{(2n\lambda_F+\mu)^2}\Big) \\
&\to \frac{v_L}{(\delta+n\lambda_F+\mu)^2}\times\Big((\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)](-\frac{\lambda_F}{2\lambda_I})-\beta\lambda_F^2\delta(-\frac{1}{2})\Big) \qquad\qquad [\text{as } n\to\infty] \\
&= \frac{v_L}{2(\delta+n\lambda_F+\mu)^2}\times\Big(-(\alpha-\gamma)\lambda_F[\lambda_F\mu-\lambda_I(\delta+\mu)]+\beta\lambda_F^2\delta\Big) \\
&< 0,
\end{aligned}
$$

where the last inequality holds because as $n\to\infty$, condition (15) becomes

$$
\begin{aligned}
&(\alpha-\gamma)\lambda_I\Big(\lambda_F\mu-\lambda_I(\delta+\mu)\Big)(2n\lambda_F+\mu) > \beta\lambda_F^2\delta(2n\lambda_I+\mu) \\
&(\alpha-\gamma)\lambda_I\Big(\lambda_F\mu-\lambda_I(\delta+\mu)\Big)\frac{2n\lambda_F+\mu}{2n\lambda_I+\mu} > \beta\lambda_F^2\delta \\
&(\alpha-\gamma)\lambda_I\Big(\lambda_F\mu-\lambda_I(\delta+\mu)\Big)\frac{\lambda_F}{\lambda_I} > \beta\lambda_F^2\delta \\
&(\alpha-\gamma)\lambda_F\Big(\lambda_F\mu-\lambda_I(\delta+\mu)\Big) > \beta\lambda_F^2\delta
\end{aligned}
$$

This implies that when n is sufficiently large, the profit π_a^* decreases in n.

Part (b): Taking the derivative of π_b^* with respect to n yields

$$
\begin{aligned}
\frac{d\pi_b^*}{dn} &= -\frac{v_H-v_L}{(\lambda_F-\lambda_I)(\delta+\mu)}\Big\{\beta\lambda_F^2\delta[\frac{\delta+2n\lambda_I+\mu}{2n\lambda_F+\mu}]'-(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)][\frac{\delta+2n\lambda_I+\mu}{2n\lambda_I+\mu}]'\Big\} \\
&= -\frac{v_H-v_L}{(\lambda_F-\lambda_I)(\delta+\mu)}\Big\{2\beta\lambda_F^2\delta\frac{\lambda_I\mu-\lambda_F(\delta+\mu)}{(2n\lambda_F+\mu)^2}-2(\alpha-\gamma)\lambda_I[\lambda_F\mu-\lambda_I(\delta+\mu)]\frac{-\lambda_I\delta}{(2n\lambda_I+\mu)^2}\Big\} \\
&= \frac{2(v_H-v_L)\delta}{(\lambda_F-\lambda_I)(\delta+\mu)}\Big\{\beta\lambda_F^2\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{(2n\lambda_F+\mu)^2}-(\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(2n\lambda_I+\mu)^2}\Big\}.
\end{aligned}
$$

Let

$$
\tilde{n} = \frac{\Big(1-\frac{\lambda_I}{\lambda_F}\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}\Big)\mu}{2\lambda_I\Big(\sqrt{\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}}-1\Big)} = \frac{1}{2}\hat{n}.
$$

(i) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]} \le 1$. Then,

$$
(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)] \le \beta[\lambda_F(\delta+\mu)-\lambda_I\mu].
$$

Note that

$$
\frac{\lambda_I^2}{(2n\lambda_I+\mu)^2}-\frac{\lambda_F^2}{(2n\lambda_F+\mu)^2} = \frac{[4n\lambda_F\lambda_I+\mu(\lambda_I+\lambda_F][\mu(\lambda_I-\lambda_F]]}{(2n\lambda_I+\mu)^2(2n\lambda_F+\mu)^2} < 0,
$$

so,

$$(\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(2n\lambda_I+\mu)^2}<\beta\lambda_F^2\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{(2n\lambda_F+\mu)^2}.$$

Hence, $\frac{d\pi_b^*}{dn}>0$ for any n.

(ii) Suppose $1<\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<\left(\frac{\lambda_F}{\lambda_I}\right)^2$, then the definition of $\tilde{n}$ implies that $\tilde{n}$ is positive. One can check that $\frac{d\pi_b^*}{dn}=0$ if $n=\tilde{n}$; $\frac{d\pi_b^*}{dn}>0$ if $n<\tilde{n}$; and $\frac{d\pi_b^*}{dn}<0$ if $n>\tilde{n}$. Therefore, the profit π_b^* first increases in the coalition size n when $n\le\tilde{n}$ and then decreases in n.

(iii) Suppose $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\ge\left(\frac{\lambda_F}{\lambda_I}\right)^2$. Then,

$$(\alpha-\gamma)\lambda_I^2[\lambda_F\mu-\lambda_I(\delta+\mu)]>\beta\lambda_F^2[\lambda_F(\delta+\mu)-\lambda_I\mu].$$

Note that $\frac{1}{(2n\lambda_I+\mu)^2}>\frac{1}{(2n\lambda_F+\mu)^2}$, so

$$(\alpha-\gamma)\lambda_I^2\frac{\lambda_F\mu-\lambda_I(\delta+\mu)}{(2n\lambda_I+\mu)^2}>\beta\lambda_F^2\frac{\lambda_F(\delta+\mu)-\lambda_I\mu}{(2n\lambda_F+\mu)^2}.$$

Hence, $\frac{d\pi^*}{dn}<0$ for any n. That is, the profit π_b^* decreases in the coalition size n.

To summarize, if $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\le 1$, the profit π_b^* monotonically increases in the coalition size n; if $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\ge\left(\frac{\lambda_F}{\lambda_I}\right)^2$, the profit π_b^* monotonically decreases in the coalition size n. Otherwise, π_b^* first increases in n for $n\le\tilde{n}$ and then decreases in n, implying that the optimal size $n^*=\lfloor\tilde{n}\rfloor$ or $\lceil\tilde{n}\rceil$. Thus, a PLP yields a higher per-firm profit than an optimally sized CLP when either of the following conditions holds:

(i) $1<\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<(\frac{\lambda_F}{\lambda_I})^2$ and $\tilde{n}\le 1$;

(ii) $1<\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}<(\frac{\lambda_F}{\lambda_I})^2$, $1<\tilde{n}<2$, and $\pi^*(1)>\pi^*(2)$;

(iii) $\frac{(\alpha-\gamma)[\lambda_F\mu-\lambda_I(\delta+\mu)]}{\beta[\lambda_F(\delta+\mu)-\lambda_I\mu]}\ge(\frac{\lambda_F}{\lambda_I})^2$.

This completes the proof. ■

Appendix E: A Discrete-Time Approximation

In the base model, the reward expiration term is assumed to be exponentially distributed for analytical tractability. In practice, reward expiration terms are usually deterministic. This section considers such a case and constructs a discrete-time model in which the expiration term is a constant. Consider a market with n "small" and independent firms, where the customer composition in the market remains the same as that in the base model. In each period, a customer visits firm j ($j\in\{1,2,\ldots,n\}$) with probability λ. We assume that λ is sufficiently small, i.e., $\lambda\ll 1$. Even though there is a positive probability that a customer visits more than one firm in a period, the probability is quite small and will be ignored in our analysis. Such an arrival process can be the binomial approximation to the Poisson process, which is quite common in the literature (Lautenbacher and Stidham Jr 1999). Specifically, let X denote the number of firms that the customer visits in a period. Then, X follows a Binomial distribution with parameters (n,λ). We assume that λ is sufficiently small $(\lambda \ll 1)$ such that the probability that $X \geq 2$ is negligible. According to Poisson limit theorem in probability theory, the binomial distribution can be approximated by a Poisson distribution with parameter $n\lambda$. Then, the probability that a customer doesn't visit any firm in the coalition is $P(X=0)=e^{-n\lambda}\frac{(n\lambda)^0}{0!}=e^{-n\lambda}$. Performing a Taylor expansion at $X=0$ gives

$$P(X=0)=1+(-n\lambda)+\frac{(-n\lambda)^2}{2!}+\cdots+\frac{(-n\lambda)^n}{n!}+o((-n\lambda)^n)\approx 1-n\lambda.$$

That is, the probability that the customer visits at least one firm in the coalition can be approximated by $n\lambda$, and the probability of not visiting any firm can be approximated by $1-n\lambda$. We note here that such an approximation is common in the revenue management literature (see, e.g., Adelman 2007).

Suppose that these n firms form a CLP, where each purchase earns the customer a reward r that will expire in K periods and can be used at any of the n firms before its expiration. Customers' future surplus is discounted by a constant per-period factor of $\delta \in (0,1)$. Consider a generic customer with product valuation v and shopping probability $n\lambda$. The customer's purchase decision problem can be formulated as an infinite-horizon discounted-reward dynamic program, and her objective is to maximize the total discounted surplus. Let $i \in \{0,1,\ldots,K\}$ denote the state, where $i \geq 1$ denotes that a customer's reward will expire in i periods, and $i=0$ denotes that the customer does not have a valid reward on hand when she visits a firm in the coalition. Let $u(\cdot)$ denote the value function.

Then, the optimality equations can be written as

$$u(i)=\begin{cases} n\lambda\max\{v-p+r+\delta u(K),\delta u(i-1)\}+(1-n\lambda)\delta u(i-1), & \text{if } i=1,2,...K,\\ n\lambda\max\{v-p+\delta u(K),\delta u(0)\}+(1-n\lambda)\delta u(0), & \text{if } i=0. \end{cases} \tag{20}$$

Following a similar process as in the base model, we first derive the customer's optimal purchasing decision and then analyze the coalition's pricing and reward decisions, by assuming that the coalition size n is given. Next, we solve a generic customer's decision problem. We first introduce several preliminary results.

Lemma E1. *The value function $u(x)$ is increasing in x.*

Proof of Lemma E1

We show this result by induction. Note that

$$u(0)=n\lambda\max\{v-p+\delta u(K),\delta u(0)\}+(1-n\lambda)\delta u(0),$$

$$u(1) = n\lambda \max\{v - p + r + \delta u(K), \delta u(0)\} + (1 - n\lambda)\delta u(0).$$

Clearly, $u(1) \geq u(0)$. Suppose $u(i-1) \geq u(i-2)$. We now show $u(i) \geq u(i-1)$. One can show that

$$u(i) = n\lambda \max\{v - p + r + \delta u(K), \delta u(i-1)\} + (1 - n\lambda)\delta u(i-1),$$
$$u(i-1) = n\lambda \max\{v - p + r + \delta u(K), \delta u(i-2)\} + (1 - n\lambda)\delta u(i-2).$$

Then,

$$\begin{aligned} u(i) - u(i-1) = n\lambda\Big\{ &\max\{v - p + r + \delta u(K), \delta u(i-1)\} - \max\{v - p + r + \delta u(K), \delta u(i-2)\}\Big\} \\ &+ (1 - n\lambda)\delta[u(i-1) - u(i-2)] \\ \geq 0, \end{aligned}$$

where the last inequality holds because $u(i-1) \geq u(i-2)$. This completes the proof. ■

LEMMA E2. *Suppose* $v - p + \delta[u(K) - u(0)] \geq 0$, *then for each* $i \in \{1, 2, \ldots, K\}$,

$$v - p + r + \delta u(K) \geq \delta u(i-1).$$

Proof of Lemma E2

Suppose for a contradiction that $v - p + r + \delta u(K) < \delta u(i-1)$ for some i. Then,

$$\begin{aligned} u(i) &= n\lambda \max\{v - p + r + \delta u(K), \delta u(i-1)\} + (1 - n\lambda)\delta u(i-1) \\ &= n\lambda\delta u(i-1) + (1 - n\lambda)\delta u(i-1) \\ &= \delta u(i-1) < u(i-1), \end{aligned}$$

which contradicts Lemma E1. This completes the proof. ■

Now, we are ready to introduce a generic customer's optimal purchasing behavior for a given CLP.

LEMMA E3. *Let* $G(i, \lambda) = \delta n\lambda \frac{1-\delta^i(1-n\lambda)^i}{1-\delta(1-n\lambda)}$, *where* $i = 1, 2, \ldots, K$. *If* $v - p + G(K, \lambda)r \geq 0$, *then it is optimal for a* (v, λ)*-customer to make a purchase whenever she visits a firm in the coalition, and a solution to the optimality equations* (20) *is given by*

$$u(i) = \frac{n\lambda}{1-\delta}\Big\{v - p + G(K, \lambda)r\Big\} + \frac{G(i, \lambda)}{\delta} r,$$
$$u(0) = \frac{n\lambda}{1-\delta}\Big\{v - p + G(K, \lambda)r\Big\}.$$

Proof of Lemma E3

Suppose $v-p+\delta[u(K)-u(0)]\geq 0$, then according to Lemma E2, $v-p+r+\delta u(K)\geq \delta u(i-1)$ for each $i\in\{1,2,\dots,K\}$. Hence, equation (20) reduces to

$$\begin{aligned}
&u(i)=n\lambda[v-p+r+\delta u(K)]+(1-n\lambda)\delta u(i-1), \quad \text{if } i=1,2,\dots,K,\\
&u(0)=n\lambda[v-p+\delta u(K)]+(1-n\lambda)\delta u(0).
\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}
u(i)&=n\lambda[v-p+r+\delta u(K)]+(1-n\lambda)\delta u(i-1)\\
&=n\lambda[v-p+r+\delta u(K)]+(1-n\lambda)\delta\Big\{n\lambda[v-p+r+\delta u(K)]+(1-n\lambda)\delta u(i-2)\Big\}\\
&=[1+(1-n\lambda)\delta]n\lambda[v-p+r+\delta u(K)]+[(1-n\lambda)\delta]^2u(i-2)\\
&=\dots\\
&=\big\{1+(1-n\lambda)\delta+\cdots+[(1-n\lambda)\delta]^{i-1}\big\}n\lambda[v-p+r+\delta u(K)]+[(1-n\lambda)\delta]^iu(0)\\
&=\frac{1-[(1-n\lambda)\delta]^i}{1-(1-n\lambda)\delta}n\lambda[v-p+r+\delta u(K)]+[(1-n\lambda)\delta]^iu(0).
\end{aligned} \tag{22}$$

Therefore,

$$u(K)=\frac{1-[(1-n\lambda)\delta]^K}{1-(1-n\lambda)\delta}n\lambda[v-p+r+\delta u(K)]+[(1-n\lambda)\delta]^Ku(0). \tag{23}$$

Solving the set of equations (23) and (21) yields

$$u(K)=\frac{n\lambda}{1-\delta}\Big\{v-p+G(K,\lambda)r\Big\}+\frac{G(K,\lambda)}{\delta}r,\ \text{and}\ u(0)=\frac{n\lambda}{1-\delta}\Big\{v-p+G(K,\lambda)r\Big\}.$$

Plugging $u(K)$ and $u(0)$ into (22) yields

$$u(i)=\frac{n\lambda}{1-\delta}\Big\{v-p+G(K,\lambda)r\Big\}+\frac{G(i,\lambda)}{\delta}r.$$

Plugging $u(K)$ and $u(0)$ back to our supposition $v-p+\delta[u(K)-u(0)]\geq 0$ leads to the condition

$$v-p+G(K,\lambda)r\geq 0.$$

This completes the proof. ■

Let q_0 and q_i denote the stationary probabilities of states 0 and i. We can obtain that $q_0=(1-n\lambda)^K$ and $q_i=n\lambda(1-n\lambda)^{K-i}$. Therefore, if $p\leq v+G(K,\lambda)r$, a generic customer's profit contribution is $n\lambda\Big(q_0p+(1-q_0)(p-r)\Big)=n\lambda\Big(p-[1-(1-n\lambda)^K]r\Big)$.

The following lemma summarizes the equilibrium outcomes.

Lemma E4. *For any given exogenous expiration term K, a CLP with size n improves the per-firm profit, compared to no reward programs, only if*

$$(\alpha-\gamma)\lambda_I\Big[(1-n\lambda_I)^K-1+G(K,\lambda_F)\Big]>\beta\lambda_F\Big[1-(1-n\lambda_F)^K-G(K,\lambda_F)\Big]. \tag{24}$$

When condition(24) is satisfied, the optimal price, reward amount, and per-firm profit in the CLP are as follows:

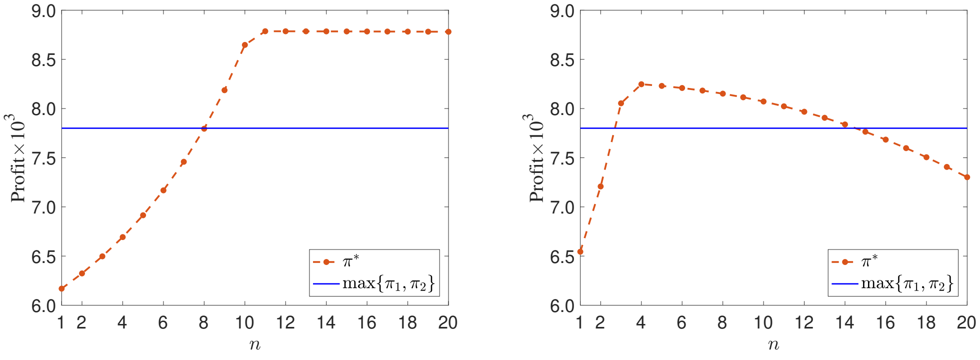
(a) If $v_H G(K,\lambda_F)-v_L G(K,\lambda_I)\le v_H-v_L$, then

$$p^*=r^*=\frac{v_L}{1-G(K,\lambda_F)},\quad \pi^*=\beta\lambda_F\frac{(1-n\lambda_F)^K}{1-G(K,\lambda_F)}v_L+(\alpha-\gamma)\lambda_I\frac{(1-n\lambda_I)^K}{1-G(K,\lambda_F)}v_L.$$

(b) Otherwise, we have

$$\begin{aligned}
p^*&=\frac{G(K,\lambda_F)v_H-G(K,\lambda_I)v_L}{G(K,\lambda_F)-G(K,\lambda_I)},\\
r^*&=\frac{v_H-v_L}{G(K,\lambda_F)-G(K,\lambda_I)},\\
\pi^*&=\beta\lambda_F v_L+(\alpha-\gamma)\lambda_I v_L-\beta\lambda_F\frac{[1-(1-n\lambda_F)^K]-G(K,\lambda_F)}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H-v_L)\\
&\quad+(\alpha-\gamma)\lambda_I\frac{G(K,\lambda_F)-[1-(1-n\lambda_I)^K]}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H-v_L).
\end{aligned} \tag{25}$$

Lemma E4 shows that the general structure of the optimal policy in the discrete-time approximation is very similar to the one in the base model (see Lemma 1 and Proposition 2). This is very encouraging, as it indicates that our base model not only allows a tractable analysis but also preserves the essence of the problem in a continuous-time framework.



(a)$K=2$ (b)$K=9$

Figure E.1 The Optimal Per-firm Profit of the Coalition vs. Size n under the Discrete-time Model; The Parameters $(\alpha,\beta,\gamma,v_H,v_L,\lambda_F,\lambda_I,\delta)=(0.6,0.2,0.06,1,0.45,0.04,0.01,0.98)$.

Unfortunately, we cannot derive any analytical results regarding the relationship between the per-firm profit π^* stated in (25) (whose expression becomes very complicated) and the coalition size n. We thus have to rely on numerical experiments to examine their relationship. When the expiration term is short, Figure 1(a) shows that the per-firm profit keeps increasing as the coalition size n increases, due to the strong positive effect of reducing the breakage rate and alleviating customer discounting. By contrast, when the expiration term is long, Figure 1(b) shows that the per-firm profit starts to decrease when the coalition expands beyond a certain threshold size. These observations are consistent with those observed in Figure 4 under the base model. Again, Figure E.1 shows that a CLP may be dominated by no reward programs if an inappropriate size is chosen. Our numerical results demonstrate that the main findings in the base model still hold under the discrete-time approximation.

Proof of Lemma E4

To derive the per-firm profit in the CLP collected from the four customer segments, we analyze the relationship between price p, $v_H + G(K,\lambda_F)r$, $v_H + G(K,\lambda_I)r$, $v_L + G(K,\lambda_F)r$, and $v_L + G(K,\lambda_I)r$ to determine the customer's purchase behavior in each segment. The analysis is very similar to the proof of Lemma 1. Therefore, to avoid repetition, we omit the analysis of the non-optimal cases (one can verify that the profits in the non-optimal cases are dominated by those with not offering reward programs). Below, we just present the details of the optimal case. Suppose

$$v_L + G(K,\lambda_F)r \le v_H + G(K,\lambda_I)r.$$

It follows that $p^* = v_L + G(K,\lambda_F)r$. Reorganizing the supposition gives $r \le \frac{v_H - v_L}{G(K,\lambda_F) - G(K,\lambda_I)}$. Moreover, $p \ge r$ yields $r \le \frac{v_L}{1-G(K,\lambda_F)}$. So,

$$r \le \min\left\{\frac{v_H - v_L}{G(K,\lambda_F) - G(K,\lambda_I)}, \frac{v_L}{1-G(K,\lambda_F)}\right\}. \tag{26}$$

The profit in this case becomes

$$\beta\lambda_F\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_F)^K]\Big)r\Big] + (\alpha-\gamma)\lambda_I\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_I)^K]\Big)r\Big].$$

Note that if and only if

$$\beta\lambda_F\Big(G(K,\lambda_F) - [1-(1-n\lambda_F)^K]\Big) + (\alpha-\gamma)\Big(G(K,\lambda_F) - [1-(1-n\lambda_I)^K]\Big) > 0, \tag{27}$$

$r^* \ne 0$. Therefore, if condition (27) holds, by inequality (26),

$$r^* = \min\left\{\frac{v_H - v_L}{G(K,\lambda_F) - G(K,\lambda_I)}, \frac{v_L}{1-G(K,\lambda_F)}\right\}.$$

Reorganizing inequality (27) yields condition (24) in Lemma E4.

Part (a): One can show that if $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) \le v_H - v_L$ holds,

$$\frac{v_L}{1-G(K,\lambda_F)} \le \frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)}.$$

Hence,

$$\begin{aligned}
r^* &= \frac{v_L}{1-G(K,\lambda_F)},\\
p^* &= v_L + G(K,\lambda_F)\frac{v_L}{1-G(K,\lambda_F)} = \frac{v_L}{1-G(K,\lambda_F)} = r^*,\\
\pi^* &= \beta\lambda_F\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_F)^K]\Big)\frac{v_L}{1-G(K,\lambda_F)}\Big]\\
&\quad + (\alpha-\gamma)\lambda_I\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_I)^K]\Big)\frac{v_L}{1-G(K,\lambda_F)}\Big]\\
&= \beta\lambda_F\frac{(1-n\lambda_F)^K}{1-G(K,\lambda_F)}v_L + (\alpha-\gamma)\lambda_I\frac{(1-n\lambda_I)^K}{1-G(K,\lambda_F)}v_L.
\end{aligned}$$

Part (b): One can show that if $v_H G(K,\lambda_F) - v_L G(K,\lambda_I) > v_H - v_L$ holds,

$$\frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)} < \frac{v_L}{1-G(K,\lambda_F)}.$$

Hence,

$$\begin{aligned}
r^* &= \frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)},\\
p^* &= v_L + G(K,\lambda_F)\frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)} = \frac{G(K,\lambda_F)v_H - G(K,\lambda_I)v_L}{G(K,\lambda_F)-G(K,\lambda_I)},\\
\pi^* &= \beta\lambda_F\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_F)^K]\Big)\frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)}\Big]\\
&\quad + (\alpha-\gamma)\lambda_I\Big[v_L + \Big(G(K,\lambda_F) - [1-(1-n\lambda_I)^K]\Big)\frac{v_H - v_L}{G(K,\lambda_F)-G(K,\lambda_I)}\Big]\\
&= \beta\lambda_F v_L + (\alpha-\gamma)\lambda_I v_L - \beta\lambda_F\frac{[1-(1-n\lambda_F)^K] - G(K,\lambda_F)}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H - v_L)\\
&\quad + (\alpha-\gamma)\lambda_I\frac{G(K,\lambda_F) - [1-(1-n\lambda_I)^K]}{G(K,\lambda_F)-G(K,\lambda_I)}(v_H - v_L).
\end{aligned}$$

This completes the proof. ■

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