The Impact of Heterogeneity and Ill-Conditioning on Diffusion Model Parameter Estimates

Albert C. Bemmaor*

Janghyuk Lee*

October 2001

* Albert C. Bemmaor is a Professor (e-mail: bemmaor@essec.fr, tel: 33 1 34 43 30 43, fax: 33 1 34 43 30 01), Ecole Supérieure des Sciences Economiques et Commerciales (ESSEC), 95021 Cergy-Pontoise Cedex, France. Janghyuk Lee is an Assistant Professor, Groupe HEC, 78351 Jouy-en-Josas Cedex, France (e-mail: lee@hec.fr, tel: 33 1 39 67 94 40, fax: 33 1 39 67 70 87). Janghyuk Lee thanks the doctoral program at ESSEC for its financial assistance. The authors are grateful to the anonymous reviewers for their comments on an earlier draft of this study and the participants of the Marketing Science Conference in Syracuse, NY and the joint ESSEC/HEC/INSEAD seminar.

Appendix 1

Mean and Variance of the Individual-level Shifted Gompertz Model

The mean and variance of the individual-level shifted Gompertz model are, respectively:

$$E(t \mid \eta, b) = (-1/b)[E(\ln T) - \ln \eta]$$

$$V(t \mid \eta, b) = (1/b^2)[E(\ln^2 T) - E^2(\ln T)]$$

with $T = \eta e^{-bt}$, $E(\ln T) = [1 + (1/\eta)] \int_0^{\eta} e^{-T} \ln T \, dT - (1/\eta) \int_0^{\eta} T e^{-T} \ln T \, dT$ and

$$E(\ln^2 T) = [1 + (1/\eta)] \int_0^{\eta} e^{-T} \ln^2 T \, dT - (1/\eta) \int_0^{\eta} T e^{-T} \ln^2 T \, dT.$$

Note: The formula for $E(\ln T)$ corrects an error in Bemmaor (1994, Table 2).

Appendix 2
The Gamma/Shifted Gompertz Model: Key Formulas

Components	Characteristics	Analytical expression
Individual-level Model	Cumulative distribution function	$F(t \mid \eta, b) = (1 - e^{-bt})\exp(-\eta e^{-bt}),\ t > 0,\ b,\ \eta > 0$
	Probability density function	$f(t \mid \eta, b)= be^{-bt} \exp(-\eta e^{-bt})[1 + \eta(1 - e^{-bt})]$
	Abscissa of the mode of the probability density function	$t^* = 0,\ 0 < \eta \leq 0.5$ $t^* = (-1/b)\ln x^*,\ 0 < x^* < 1,$ with $x^* = [3 + \eta - (\eta^2 + 2\eta + 5)^{1/2}]/(2\eta),\ \eta > 0.5$
Model of Heterogeneity	Probability density function	$h(\eta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \eta^{\alpha - 1} e^{-(1/\beta)\eta},\ \eta > 0,\ \alpha, \beta > 0$
	Mean	$E(\eta) = \alpha\beta$
	Variance	$V(\eta) = \alpha\beta^2$
Mixture	Cumulative distribution function	$F(t) = (1 - e^{-bt})/(1 + \beta e^{-bt})^{\alpha}$
	Probability density function	$f(t) = be^{-bt}(1 + \beta e^{-bt})^{-(\alpha + 1)}\ [1 + \alpha\beta + \beta e^{-bt}(1 - \alpha)]$
	Abscissa of the mode of the probability density function[a]	$t^* = (-1/b)\ln x^*,\ 0 < x^* < 1$ with $x^* = (-B + \sqrt{\Delta})/(2A)$, $A = -\beta(\alpha - 1)^2,\ B = \beta\alpha^2 + 3\alpha - 2$ $\Delta = \alpha(\beta^2\alpha^3 + 2\beta\alpha^2 + 4\beta\alpha + 5\alpha - 4\beta - 4),\ \Delta \geq 0$

[a] With the Bass model, $t^* = (1/b)\ln\beta$

<u>Appendix 3</u>

Nonlinear Least Square Parameter Estimates: Bass Model[a]

Product category	Number of observations t^+	$\hat{p}$	$\hat{q}$	Market size $\hat{c}$	Predicted time to peak sales $\hat{t}*$	Actual time to peak sales $t*$
Corn (1948)	11	.00115	.747	1.0	8.65	
	12	.00105	.766	.989	8.59	
	13	.000857	.796	.966	8.58	
	14	.000810	.804	.961	8.57	9
Ultrasound	11	.00238	.504	1.0	10.58	
	12	.00313	.477	1.0	10.46	
	13	.00202	.560	.860	10.01	
	14	.00132	.621	.801	9.90	10.5
Foreign language	9	.00323	.602	.411	8.64	
	10	.00351	.569	.446	8.88	
	11	.00328	.588	.430	8.78	
	12	.00189	.697	.376	8.46	8
Mammography	11	7.29×10^{-4}	.774	.576	8.99	
	12	3.83×10^{-4}	.870	.529	8.88	
	13	4.35×10^{-4}	.853	.536	8.89	
	14	4.10×10^{-4}	.861	.533	8.88	10
Corn (1943)	12	4.00×10^{-5}	.940	.929	10.71	
	13	7.62×10^{-5}	.862	.999	10.83	
	14	5.16×10^{-5}	.907	.965	10.78	
	15	4.11×10^{-5}	.932	.948	10.76	11
Tetracycline	10	.0739	.323	.777	3.72	
	11	.0753	.294	.802	3.69	
	12	.0757	.286	.809	3.67	
	13	.0767	.273	.820	3.63	
	14	.0786	.246	.842	3.52	
	15	.0806	.216	.868	3.32	
	16	.0812	.207	.876	3.25	
	17	.0812	.207	.876	3.24	7
Color television	12	.0286	.280	.900	7.40	
	13	.0288	.298	.859	7.14	
	14	.0289	.296	.864	7.17	
	15	.0290	.285	.883	7.28	
	16	.0294	.268	.910	7.43	
	17	.0299	.244	.952	7.66	5
Air conditioner	32	4.80×10^{-5}	.274	.286	31.52	
	33	9.28×10^{-6}	.348	.241	30.30	
	34	5.96×10^{-6}	.370	.225	29.83	
	35	5.92×10^{-6}	.370	.225	29.83	31
Clothes dryer	25	4.96×10^{-5}	.430	.229	21.10	
	26	8.12×10^{-5}	.398	.241	21.37	
	27	8.63×10^{-5}	.387	.252	21.71	
	28	1.12×10^{-4}	.367	.264	22.06	
	29	1.48×10^{-4}	.345	.278	22.49	21
CT scanner	11	.00261	.622	.295	8.77	
	12	.00279	.608	.301	8.82	
	13	.00346	.567	.317	8.94	
	14	.00441	.520	.335	9.09	
	15	.00576	.467	.358	9.30	
	16	.00672	.435	.373	9.44	

	17	.00788	.400	.391	9.62	
	18	.00912	.367	.409	9.82	9
Compulsory school	10	.00471	.729	.830	6.87	
	11	.00816	.574	.967	7.31	
	12	.00768	.592	.950	7.25	
	13	.00925	.540	1.0	7.41	
	14	.00944	.533	1.0	7.44	7
Accelerated program	9	$6.05x10^{-4}$	.962	.633	7.66	
	10	$4.75x10^{-4}$	1.011	.608	7.58	
	11	$7.91x10^{-4}$	.917	.649	7.69	
	12	$7.42x10^{-4}$	.928	.644	7.68	7

[a] The data are annual, except CT scanners (semi-annual), compulsory school attendance (four-year data), and tetracycline (monthly).

Appendix 4

Descriptive Accuracy of the Bass (1969) Model and the Gamma/Shifted Gompertz Model

Product category	Bass model		Gamma/Shifted Gompertz model	
	Mean absolute deviation	Mean squared error	Mean absolute deviation	Mean squared error
Corn (1948)	7.37	73.91	4.45	29.06
Ultrasound	2.95	11.63	1.73	5.33
Foreign language	.66	.55	.52	.36
Mammography	1.70	3.93	1.07	2.12
Corn (1943)	3.35	14.15	3.09	12.28
Tetracycline	1.39	3.67	1.39	3.67
Color TV	1.75	5.33	1.70	4.71
Air conditioner	.15	.086	.17	.082
Clothes dryer	.21	.11	.13	.043
CT scanner	14.67	464.49	10.26	277.49
Compulsory school	.92	1.29	.82	1.09
Accelerated program	1.90	6.22	1.94	5.24
Overall mean	3.09	48.78	2.27	28.5

Appendix 5

Accuracy of the One-Step Ahead, Two-Step Ahead and Three-Step Ahead Forecasts Given By the Bass Model and the Gamma/Shifted Gompertz Model[a]

Product/Service	Forecasting Period	Bass model MAD1	Bass model MAD2	Bass model MAD3	Bass model MSE1	Bass model MSE2	Bass model MSE3	Gamma/shifted Gompertz model MAD1	Gamma/shifted Gompertz model MAD2	Gamma/shifted Gompertz model MAD3	Gamma/shifted Gompertz model MSE1	Gamma/shifted Gompertz model MSE2	Gamma/shifted Gompertz model MSE3
Air conditioners	1962-1964	.24	.50	.59	.15	.41	.53	.47	.64	.51	.33	.50	.33
Clothes dryers	1961-1964	.72	.82	.89	.56	.72	.85	.33	.43	.54	.15	.23	.35
Color TV	1975-1979	1.53	1.43	1.41	3.03	2.83	2.82	1.19	1.05	.99	1.88	1.53	1.44
Corn (1948)	1940-1942	6.92	7.83	8.03	53.84	65.38	68.03	2.85	2.93	3.83	15.01	13.54	18.56
Corn (1943)	1939-1941	5.03	4.27	3.25	25.74	23.03	15.33	12.62	13.64	15.93	328.25	299.71	386.48
Tetracyline	Nov 54-Feb 55	1.49	1.64	1.86	3.16	3.57	4.28	2.59	2.78	3.04	8.72	9.14	10.16
Ultrasound	1977-1978	6.02	8.25		37.27	69.59		.91	1.09		.84	1.19	
Mammography	1975-1978	2.09	1.65	2.16	6.79	5.10	7.19	1.35	1.75	1.27	3.31	4.29	2.64
CT scanning	1977-1981	31.41	35.37	38.68	1132.91	1388.09	1649.71	29.12	34.98	40.62	885.24	1331.18	1804.91
Foreign language	1963	2.31			5.34			1.60			2.54		
Accelerated program	1962-1963	1.89	1.59		5.23	5.05		1.89	1.58		3.82	3.15	
Compulsory school	1894-1902	.96	.94	1.00	1.32	1.18	1.37	.89	.84	.89	1.12	.97	1.13
Overall mean		5.05	5.85	6.43	106.28	142.27	194.46	4.65	5.61	7.51	104.27	151.40	247.33

[a] MAD1 denotes the mean absolute deviation of the one-step ahead forecasts and MSE1 denotes the mean squared error of the one-step ahead forecasts.