A Polyhedral Approach for the Staff Rostering Problem

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On-Line Appendix

Appendix

Proof of Theorem 4.2

Proof: We prove the theorem for an inequality of type (13) $y_{c-6,rr} + y_{crr} \leq x_{c-6,r}$. We recall that c corresponds to a Saturday and $c-6$ to the previous Sunday. As in the proof of Theorem 4.1 we consider a facet inducing inequality $\alpha y + \beta x \leq \alpha_0$ containing all the tight solutions for (13). Then we choose, for each week p, a cell $\bar{c}(p)$ different from $c-6$, and fix the coefficients $\beta_{\bar{c}(p)} = 0$.

First, let $c_1,\ c_2 \neq c-6$ be two cells and (y^1, x^1), (y^2, x^2) two tight solutions for (13) such that

- $x^1_{c_1 r} = 1$, $x^1_{c_2 r} = 0$, and $x^2_{c_1 r} = 0$, $x^2_{c_2 r} = 1$;
- $y^i_{c-6,rr} = y^i_{crr} = x^i_{c-6,r} = 0$ for $i \in \{1, 2\}$;
- (y^1, x^1) and (y^2, x^2) are equal for all other indices.

Then $\alpha y^1 + \beta x^1 = \alpha y^2 + \beta x^2 = \alpha_0$, so $\beta_{c_1} = \beta_{c_2}$ and, as c_1 and c_2 can be chosen equal to $\bar{c}(p)$ for a $p \in P$, it follows $\beta_{c_1} = \beta_{c_2} = 0$. Moreover, in the previous solutions it is possible that $y^1 = y^2 = 0$, and therefore $\alpha_0 = \beta x^1 = 0$.

Second, let $c' \neq c,\ c-6$; we can select a tight solution with $y_{c'rr} = 1$, $x_{c-6,r} = 0$, and all other y variables equal to zero, and then have $\alpha_{c'} = \alpha_0 = 0$.

Finally, consider a solution with $y_{c-6,rr} = x_{c-6,r} = 1$, then $\alpha_{c-6} + \beta_{c-6} = \alpha_0 = 0$, i.e., $\alpha_{c-6} = -\beta_{c-6}$. In a similar way we obtain $\alpha_c = -\beta_{c-6}$ (it is sufficient to consider a solution with $y_{crr} = x_{c-6,r} = 1$). Therefore, we have shown that the facet inducing inequality $\alpha y + \beta x \leq \alpha_0$ is equivalent to (13). The proof for inequalities of type (14) is similar. $\square$

Proof of Theorem 4.3

Proof: As in the proof of Theorem 4.1 we consider an inequality $\alpha\ y\ +\ \beta\ x\ \leq\ \alpha_0$ inducing a facet of $\mathcal{P}$ that contains all tight solutions of inequality (11); moreover, we choose indices $\bar{c}(p)$ different from c and $c+1$, so the coefficients $\beta_{\bar{c}(p)}$ can again be assumed to be zero. Now we determine all the other coefficients.

First, let $c_1, c_2 \notin \{c, c+1\}$ and consider two solutions for RRSS such that

- $x^1_{c_1 r} = 1$, $x^1_{c_2 r} = 0$, and $x^2_{c_1 r} = 0$, $x^2_{c_2 r} = 1$;
- $x^i_{cr} = 1$, $x^i_{c+1,r} = 0$ for $i \in \{1, 2\}$;
- (y^1, x^1) and (y^2, x^2) are equal for all other indices.

Since they are both tight for (11), then $\alpha y^1 + \beta x^1 = \alpha y^2 + \beta x^2 = \alpha_0$, and therefore $\beta_{c_1} = \beta_{c_2}$. As in Theorem 4.1, c_1 or c_2 may be chosen equal to one of the indices $\bar{c}(p)$, so $\beta_{c_1} = \beta_{c_2} = 0$.

Moreover, the solutions (y^1, x^1) and (y^2, x^2) can be chosen with $y^1 = y^2 = 0$, so that $\beta x^1 = \alpha_0$. As all the coefficients in vector β are equal to zero for the cells different from c and $c+1$, $x_{cr} = 1$, and $x_{c+1,r} = 0$, we also have $\beta_c = \alpha_0 = \mu$. The same applies if (y^1, x^1) is chosen such that $x_{cr} = 0$ and $x_{c+1,r} = 1$, deriving that $\beta_{c+1} = \alpha_0 = \mu$.

Second, let $c' \notin \{c,\ c+1,\ \ldots,\ c+5\}$; we can choose a tight solution with $y_{c'rr} = 1$, $x_{cr} = 1$, and $x_{c+1,r} = 0$, and all other y variables equal to zero; thus $\alpha_{c'} + \beta_c = \alpha_0$, i.e., $\alpha_{c'} = 0$.

Third, let $c' = c+1$; we can choose a tight solution with $y_{c+1,rr} = 1$ and all other y variables equal to zero. Then $x_{c+1,r} = x_{c+2,r} = 1$, and thus $\alpha_{c+1} + \beta_{c+1} = \alpha_0$, i.e., $\alpha_{c+1} = 0$ since $\beta_{c+1} = \alpha_0 = \mu$.

Fourth, let $c' \in \{c+2,\ \ldots,\ c+5\}$; we can choose a tight solution with $y_{c'rr} = 1$; since the other variables in the considered inequality must be equal to zero and we have determined that the other coefficients are equal to zero, we obtain $\alpha_{c'} = \alpha_0 = \mu$.

Finally, consider a tight solution with $y_{crr} = 1$ and $x_{cr} = x_{c+1,r} = 1$, then $\alpha y + \beta x = \alpha_c + \beta_c + \beta_{c+1} = \alpha_0$. As we have seen that $\beta_c = \beta_{c+1} = \alpha_0 = \mu$, then $\alpha_c + 2\mu = \mu$, i.e., $\alpha_c = -\mu$, and therefore we have shown that the facet inducing inequality $\alpha y + \beta x \leq \alpha_0$ is proportional to (11). $\square$

The proofs of validity and of facet inducing with respect to RRSS for the inequalities (15) is very similar to those for inequalities (11). In particular, in the proof of Theorem 4.3, one only has to pay attention to the coefficient of the y variable preceding the pair of x variables, applying the third step also in a backward sense. For instance, for the second inequality in (15) the steps of the proof must be "shifted" forward considering cell $c+1$ instead of cell c and so on. In order to find the coefficient α_c associated with the variables preceding the pair of x variables, we can choose a solution with $y_{crr} = 1$ and all other y variables equal to zero. Then $x_{cr} = x_{c+1,r} = 1$ and thus $\alpha_{cr} + \beta_{c+1,r} = \alpha_0$, i.e., $\alpha_{cr} = 0$ since in this case the first step establishes that $\beta_{c+1} = \beta_{c+2} = \alpha_0 = \mu$ and all other involved coefficients are equal to zero.