Managing Inventory and Supply Performance in Assembly Systems with Random Supply Capacity and Demand

Online Supplement / Electronic Companion Pages

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Proofs of Results

Proof of Proposition 1 Follows from Theorem 5.1 of Glasserman and Yao (1995) because the shortfall recursion belongs to their case (A1). To see this, observe that the shortfall vector may be written as $Y_{n+1} = A(n) \bigotimes Y_n \bigoplus B(n)$, where $\bigotimes$ corresponds to $+$, $\bigoplus$ corresponds to max, and $A(n)$ and $B(n)$ satisfy,

$$A(n) = \begin{vmatrix} -\infty & \xi_n - s^1 & \xi_n - s^2 \\ -\infty & \xi_n - \eta_n^1 & -\infty \\ -\infty & -\infty & \xi_n - \eta_n^2 \end{vmatrix}, \; B(n) = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}.$$

To complete the proof, we need to verify the following conditions: (1) $E[(\|A(0)\|_v)^+] < \infty$, (2) $\gamma_{v+} < 0$, and (3) $E[(\|B(0)\|_v)^+] < \infty$, where $\|\cdot\|_v$ and γ_{v+} are defined as in Glasserman and Yao (1995). Because it is easy to see that both $E[(\|A(0)\|_v)^+] < \infty$ and $E[(\|B(0)\|_v)^+] < \infty$ hold, we only show that $\gamma_{v+} < 0$.

By Lemma 5A of Glasserman and Yao (1995), $\lim_{n\to\infty} n^{-1}\|P(m,n)\|_v = \gamma_{v+}$ with probability one, where $P(m,n) = A(n) \bigotimes \cdots \bigotimes A(m)$, $m < n$. In our case

$$P(m,n) = \begin{vmatrix} -\infty & \xi_n - s^1 + \sum_{k=m}^{n-1}(\xi_k - \eta_k^1) & \xi_n - s^2 + \sum_{k=m}^{n-1}(\xi_k - \eta_k^2) \\ -\infty & \sum_{k=m}^{n}(\xi_k - \eta_k^1) & -\infty \\ -\infty & -\infty & \sum_{k=m}^{n}(\xi_k - \eta_k^2) \end{vmatrix},$$

so that

$$\|P(m,n)\|_v = \max \begin{cases} \xi_n - s^i + \sum_{k=m}^{n-1}(\xi_k - \eta_k^i), i = 1,2, \\ \sum_{k=m}^{n}(\xi_k - \eta_k^i), i = 1,2. \end{cases} \tag{8}$$

From the ergodicity condition and the law of large numbers, we know that, with probability one,

$$\lim_{n\to\infty} \frac{1}{n}(\xi_n - s^i + \sum_{k=m}^{n-1}(\xi_k - \eta_k^i)) = E[\xi] - E[\eta^i] - s^i, \text{ and,}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m}^{n}(\xi_k - \eta_k^i) = E[\xi] - E[\eta^i], \text{ for } i = 1,2.$$

Consider the intersection of the probability spaces on which the above limits hold. Then, on this space, whose probability is one, for any $\epsilon > 0$, when n is big enough, we have:

$$\frac{1}{n}(\xi_n - s^i + \sum_{k=m}^{n-1}(\xi_k - \eta_k^i)) \le E[\xi] - E[\eta^i] - s^i + \epsilon, \quad \text{and,}$$

$$\frac{1}{n}\sum_{k=m}^{n}(\xi_k - \eta_k^i) \le E[\xi] - E[\eta^i] + \epsilon, \quad \text{for } i = 1, 2.$$

From (8), we conclude that, with probability one, $n^{-1}\|P(m,n)\|_v \le \max\{E[\xi] - E[\eta^1] - s^1, E[\xi] - E[\eta^2] - s^2, E[\xi] - E[\eta^1], E[\xi] - E[\eta^2]\} + \epsilon$, when n is big enough. Hence, because $s^1, s^2 \ge 0$, $\gamma_{v+} = \lim_{n\to\infty} n^{-1}\|P(m,n)\|_v \le \max\{E[\xi] - E[\eta^1], E[\xi] - E[\eta^2]\} < 0$, where the last inequality is based on the condition that $E[\xi] < E[\eta^i]$ for $i = 1, 2$. Therefore, condition (2), stated earlier in the proof, is satisfied, which completes the proof. ■

Proof of Proposition 2. Note that $Y^0 = \max_i\{0, Y^i + \xi - s^i\}$ is a convex function because each argument of the max operation is a linear function, and the maxima of convex functions is convex. Hence, $s^0 - Y^0$ is a concave function. By Proposition 3.1 of Wets 1989, the feasible region $A(s^0, s^1, s^2)$ is a closed convex set. ■

Proof of Proposition 3. We only prove result (i), because (ii) follows using similar arguments. Because the monotonicity in (i) is trivial, we only need to show that $s^0(s^1, s^2)$ is convex in (s^1, s^2). Let $s^1 = \theta s_x^1 + (1-\theta)s_y^1$ and $s^2 = \theta s_x^2 + (1-\theta)s_y^2$. If $s_x^0 = s^0(s_x^1, s_x^2)$ and $s_y^0 = s^0(s_y^1, s_y^2)$, then $\vec{s}_x \equiv (s_x^0, s_x^1, s_x^2) \in A$ and $\vec{s}_y \equiv (s_y^0, s_y^1, s_y^2) \in A$, where A is the feasible region for the two-echelon assembly system problem. Because A is a convex set, we know that for any $0 \le \theta \le 1$, $\theta\vec{s}_x + (1-\theta)\vec{s}_y \in A$, so $\theta s_x^0 + (1-\theta)s_y^0$ is a feasible s^0 value corresponding to s^1 and s^2. By definition, $s^0(\theta s_x^1 + (1-\theta)s_y^1, \theta s_x^2 + (1-\theta)s_y^2)$ is the minimal feasible value of s^0 corresponding to s^1 and s^2. Consequently,

$$s^0(\theta s_x^1 + (1-\theta)s_y^1, \theta s_x^2 + (1-\theta)s_y^2) \le \theta s_x^0 + (1-\theta)s_y^0.$$

That is, $s^0(\theta s_x^1 + (1-\theta)s_y^1, \theta s_x^2 + (1-\theta)s_y^2) \le \theta s^0(s_x^1, s_x^2) + (1-\theta)s^0(s_y^1, s_y^2)$, proving that $s^0(s^1, s^2)$ is convex in (s^1, s^2). ■

Proof of Proposition 5. By contradiction. Suppose $s^{1*} > s^{2*}$ instead, then there exists $\epsilon > 0$ such that $s^{1*} - \epsilon \ge s^{2*}$.

Because $\eta^1 \ge_{st} \eta^2$, by Proposition 1, $Y^1 \le_{st} Y^2$ (Stoyan 1983). Therefore, there exist $\hat{Y}^1$ and $\hat{Y}^2$ defined on the same probability space such that $\hat{Y}^1 \overset{st}{=} Y^1$, $\hat{Y}^2 \overset{st}{=} Y^2$ and $\hat{Y}^1 \le \hat{Y}^2$ a.s. (Stoyan 1983). Combining the above three statements, we have

$$\xi - s^{1*} + \hat{Y}^1 \;\; \le \;\; \xi - (s^{1*} - \epsilon) + \hat{Y}^1 \;\le\; \xi - s^{2*} + \hat{Y}^2 \; a.\, s. \tag{9}$$

It follows that,

$$\begin{aligned}
& P[s^{0*} - \max\{0, Y^1 + \xi - (s^* - \epsilon), Y^2 + \xi - s^{2*}\} \ge \xi] \\
\overset{\hat{Y}^i \overset{st}{=} Y^i}{\Longrightarrow} \;& P[s^{0*} - \max\{0, \hat{Y}^1 + \xi - (s^{1*} - \epsilon), \hat{Y}^2 + \xi - s^{2*}\} \ge \xi] \\
\overset{(9)}{=} \;& P[s^{0*} - \max\{0, \hat{Y}^1 + \xi - s^{1*}, \hat{Y}^2 + \xi - s^{2*}\} \ge \xi] \\
\overset{\hat{Y}^i \overset{st}{=} Y^i}{\Longrightarrow} \;& P[s^{0*} - \max\{0, Y^1 + \xi - s^{1*}, Y^2 + \xi - s^{2*}\} \ge \xi] \;\ge\; \alpha.
\end{aligned}$$

Thus we find a feasible solution $(s^{0*}, s^{1*} - \epsilon, s^{2*})$ for the problem with a lower objective function value than the optimal, which yields a contradiction. ■

Proof of Proposition 9. By contradiction. We only show that s^{1*} decreases with c^1. The other claims can be proved similarly.

Let the optimal base-stock level at cost (c^1, c^2, c^0) be (s^{1*}, s^{2*}, s^{3*}). Suppose s^{1*} strictly increases in c^1. Then, when c^1 increases to $c^1 + \varepsilon$ with $\varepsilon > 0$, the optimal base-stock level can be written as $(s^{1*} + \delta^1, s^{2*} + \delta^2, s^{0*} + \delta^0)$, where $\delta^1 > 0$. It follows that

$$c^0(s^{0*} + \delta^0) + (c^1 + \varepsilon)(s^{1*} + \delta^1) + c^2(s^{2*} + \delta^2) < c^0 s^{0*} + (c^1 + \varepsilon)s^{1*} + c^2 s^{2*},$$

which implies

$$c^0\delta^0 + (c^1 + \varepsilon)\delta^1 + c^2\delta^2 < 0. \tag{10}$$

Because (s^{1*}, s^{2*}, s^{3*}) is the optimal base-stock level for (c^1, c^2, c^0), we have

$$c^0 s^{0*} + c^1 s^{1*} + c^2 s^{2*} \leq c^0(s^{0*} + \delta^0) + c^1(s^{1*} + \delta^1) + c^2(s^{2*} + \delta^2),$$

which implies $0 \leq c^0\delta^0 + c^1\delta^1 + c^2\delta^2$. And this contradicts (10) because $\varepsilon\delta^1 > 0$. ■

Proof of Proposition 10. By Proposition 3, $s^0(s^1, s^2)$ is convex in (s^1, s^2), hence the objective function of the assembly system problem is convex in (s^1, s^2). Because (s^1, s^2) is monotone in π, we know that the objective function is quasi-convex in π because a convex function of a monotone function is quasi-convex. ■

Proof of Lemma 11. The proof is based on induction on n.

Because the lead times are not zero, we need to consider the pipeline inventory. Denote T_n^{Ai} (T_n^{Bi}) as the pipeline inventory of node i in system A (system B) at the beginning of period n. Denote R_n^{Ai} (R_n^{Bi}) as the shipped quantity of node i in system A (system B) at the beginning of period n. Based on the sequence events described in §1.1, we see that the pipeline inventory and the delivery quantity of an item have the following relationship:

$$\begin{aligned} T_n^i &= T_{n-1}^i + R_{n-1}^i - R_{n-l^i-1} & (11)\\ &= R_{n-1}^i + \cdots + R_{n-l^i} \stackrel{def}{=} \mathcal{R}^i_{[n-l^i,n-1]}. & (12) \end{aligned}$$

where l denotes the norminal lead time of a generic node i and $\mathcal{R}^i_{[n-l^i,n-1]}$ is defined as the total delivery of node i between period $n-l$ and $n-1$ inclusive. And at the beginning of period n, due to the base-stock policy, the following equation holds in both systems:

$$I_n^i + T_n^i + Y_n^i - Y_n^j = s^i, \tag{13}$$

where j is the index of direct successor of node i.

Note that both systems face the same demand stream and both nodes 2 and 3 have identical suppliers in systems A and B. Therefore, as in §3, we can write down the delivery quantities and the shortfalls of nodes 2 and 3 in systems A and B as:

$$\begin{aligned} R_n^{Ai} &= \min\{Y_n^{Ai} + \xi_n, \eta_n^i\}, & (14)\\ R_n^{Bi} &= \min\{Y_n^{Bi} + \xi_n, \eta_n^i\}, & (15)\\ Y_{n+1}^{Ai} &= [Y_n^{Ai} + \xi_n - \eta_n^i]^+, & (16)\\ Y_{n+1}^{Bi} &= [Y_n^{Bi} + \xi_n - \eta_n^i]^+, \text{ for } i = 2, 3. & (17) \end{aligned}$$

We first characterize the ordering policy of node 1 in system B. Recall that node 1 of system B has a perfect supplier. Therefore, there is no shortfall of node 1 if a *regular* base-stock policy is followed. However, we introduce a *modified* base-stock policy for the ordering activities of node 1 in system B. According to this policy, in period n, an order of node 1 is placed to raise its inventory position back to S^1 subject to the constraint that its on-hand inventory level in any period is no higher than the minimum on-hand inventory level of nodes 2 and 3.

One should note the following with regard to the ordering policy of node 1:

1. This additional constraint, which *modifies* the base-stock policy, does not affect the service level of the end-product because the assembly of the end-product depends on the availability of all components.
2. This additional constraint ensures that no unnecessary inventory of component 1 is kept compared to the *regular* base-stock policy.
3. Under this policy, there might be shortfalls of node 1, however this shortfall is caused by the lower inventory level of node 2 or 3.
4. Under this policy, the assembly activity of the end-product in a period will not depend on the inventory level of nodes 2 and 3 because both are no less than the inventory level of node 1.

Clearly, to bring the inventory position of node 1 back to the base-stock level, one should order $Y_n^{B1} + \xi_n$. Note that R_n^{B1} will only affect the inventory level of node 1 starting from period $n + L^1 + 1$ onwards. And the inventory level at the beginning of period $n+L^1+1$ is $T_n^{B1} + I_n^{B1} + R_n^{B1}$, assuming no assembly activity of the end-product happens between periods n and $n+L^1$. And under the assumption that there is no assembly activity of the end-product between periods n and $n+L^1$, the inventory level of node i at the beginning of period $n+L^1+1$ is $I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]}$. Hence R_n^{B1} must satisfy $T_n^{B1} + I_n^{B1} + R_n^{B1} \leq I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]}$. Therefore, the actual ordering quantity of node 1 in system B under the modified base-stock policy is given by

$$R_n^{B1} = \min\{Y_n^{B1} + \xi_n, I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]} - I_n^{B1} - T_n^{B1}\}. \tag{18}$$

Note that the assumption that there is no assembly activity of the end-product between periods n and $n + L^1$ is only used to faciliate the development of (18). Because the assembly of the end-product consumes the same amount of nodes 1, 2 and 3, even where there are assembly activities of the end-product during those periods, (18) still holds.

Now we are ready to prove that the end-product shortfalls are the same in both systems with induction. Assume that $Y_k^{Ai} = Y_k^{Bi}$ for $k \leq n$ and all i, $R_k^{A1} = R_k^{B1}$, $R_k^{A0} = R_k^{A0}$ for $k \leq n-1$. We now show that $Y_k^{Ai} = Y_k^{Bi}$ holds for $k = n+1$ and all i, and $R_k^{A1} = R_k^{B1}$, $R_k^{A0} = R_k^{A0}$ holds for $k = n$.

From (12), we have

$$\begin{aligned} T_{n-1}^{Bi} &= \mathcal{R}^{Bi}_{[n-(L^1+L^i+2),n-2]} = R^{Bi}_{n-1-(L^1+L^i+1)} + \cdots + R^{Bi}_{n-L^i-2} + \mathcal{R}^{Ai}_{[n-(L^i+1),n-2]} \\ &= R^{Bi}_{n-1-(L^1+L^i+1)} + \cdots + R^{Bi}_{n-L^i-2} + T^{Ai}_{n-1} \end{aligned} \tag{19}$$

We now examine the relationship between I_n^{Ai} and $I_n^{Bi}$and the relationship between R_n^{A1} and R_n^{B1}.

Due to the base-stock policy, for $i = 2,\ 3$,

$$
\begin{aligned}
I_n^{Ai} + T_n^{Ai} + Y_n^{Ai} - Y_n^{A1} &= S^i & (20)\\
I_n^{Bi} + T_n^{Bi} + Y_n^{Bi} - Y_n^{B0} &= S^1 + S^i & (21)
\end{aligned}
$$

Because $Y_n^{Ai} = Y_n^{Bi}$ and (19), we have:

$$I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]} = I_n^{Ai} + S^1 + Y_n^{B0} - Y_n^{A1}. \tag{22}$$

R_n^{A1} depends on the available inventories of items 2 and 3, therefore,

$$R_n^{A1} = \min_{i=2,3}\{Y_n^{A1} + \xi_n, I_n^{Ai}\}. \tag{23}$$

Hence R_n^{B1} must satisfy $T_n^{B1} + I_n^{B1} + R_n^{B1} \leq I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]}$ for $i = 2, 3$ to satisfy the constraint of the modified base-stock policy. Herefore,

$$
\begin{aligned}
R_n^{B1} &= \min\{Y_n^{B1} + \xi_n, I_n^{Bi} + \mathcal{R}^{Bi}_{[n-(L^1+L^i+1),n-(L^i+1)]} - I_n^{B1} - T_n^{B1}\}\\
&\overset{(22)}{=} \min\{Y_n^{B1} + \xi_n, I_n^{Ai} + S^1 + Y_n^{B0} - Y_n^{A1} - I_n^{B1} - T_n^{B1}\}\\
&= \min\{Y_n^{B1} + \xi_n, I_n^{Ai} + S^1 + Y_n^{B0} - Y_n^{B1} - I_n^{B1} - T_n^{B1}\}\\
&= \min\{Y_n^{B1} + \xi_n, I_n^{Ai}\}, & (24)
\end{aligned}
$$

where the second-last equality is based on the induction assumption that $Y_n^{A1} = Y_n^{B1}$ and the last equality is based on the fact that $S^1 + Y_n^{B0} - Y_n^{B1} - I_n^{B1} - T_n^{B1} = 0$.

From (14), (15), (16) and (17), for $i = 2,\ 3$, we have

$$R_n^{Ai} = R_n^{Bi},\ Y_{n+1}^{Ai} = Y_{n+1}^{Bi} \tag{25}$$

based on the induction assumption. Comparing (23) and (24), we see that

$$R_n^{A1} = R_n^{B1} \tag{26}$$

because $Y_n^{A1} = Y_n^{B1}$. Therefore, combining (26) with the induction assumption yields

$$Y_{n+1}^{A1} = Y_{n+1}^{B1}. \tag{27}$$

Because $Y_k^{A0} = Y_k^{B0}$ for $k \leq n$ and $I_k^{A1} = I_k^{B1}$, $R_k^{A1} = R_k^{B1}$ for $k \leq n-1$, at the beginning of period n, the inventory levels of node 1 are the same in both systems A and B (this is based on the fact that for both systems the delivery quantities for node 1 are the same up to period n). Note that in system B, because the inventory level of node 1 is no higher than the inventory level of node 2 or 3 under the modified base-stock policy, the assembly of the end-product depends on the inventory level of node 1 only as in system A. Because both systems face the same demand stream,

$$R_n^{A0} = R_n^{B0}. \tag{28}$$

Hence

$$Y_{n+1}^{A0} = Y_{n+1}^{B0}. \tag{29}$$

Combining (25), (26), (27), (28), (29), we complete the induction step. ■