INVENTORY SHARING AND RATIONING IN DECENTRALIZED DEALER NETWORKS

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Technical Appendices

A Summary of Notation

$\hat{h}$ = holding cost per unit per unit time.

π = backorder cost for each unit backordered.

$\hat{\pi}$ = delay cost for each unit of time a demand is backordered.

h_1^- = cost for each dealer's own-customer demand filled through inventory sharing from another dealer, referred to as the cost of sharing to the requesting dealer.

h_2^- = penalty cost to a dealer for not filling an inventory sharing request, referred to as the incentive for sharing to the sharing dealer.

S_i = base-stock level for dealer i.

K_i = rationing level for dealer i.

λ_i = arrival rate of own end-customer demands to dealer i.

$\lambda = \lambda_1 + \lambda_2$ = the total end-customer demand rate for the system.

$\beta_i = \frac{\lambda_i}{\lambda_i + \lambda_j}$ = the probability that a demand arrival to the system is a dealer i demand, $i, j = 1, 2$.

$p(x; y)$ = the probability that a Poisson random variable with mean of y takes value x.

$b(\alpha; y; z)$ = the binomial probability that there are z successes in y trials with the probability of success on each trial equal to α.

B Proof of Theorem 1

Recall that a dealer's inventory level is determined by the inventory position at the two dealers, the sequence of end-customer demand arrivals, as well as how replenishments are allocated between the dealers. Equation (3) tells us the inventory level at any time $t + \tau$ given that the replenishment allocation policy, as defined in Section 4.2, and the rationing policy, as defined in Section 3, are followed. Notice that equation (3) is also how we would write the inventory level at the dealer at any time $t + \tau$, if the *on-hand inventory levels* at the two dealers were S_i and S_j, respectively, at time t and the rationing policy is followed subsequently. Under this condition, there would be no replenishment arrivals during $(t, t + \tau)$, and hence no replenishment allocation decisions need to be made during $(t, t + \tau)$. Therefore, it is trivial to show that if equation (3) holds true immediately after the arrival of a replenishment, it holds true until the arrival of the next replenishment, since there are no replenishment allocation decisions to be made between two consecutive replenishment arrivals. However, when a replenishment arrives, equation (3) may or may not hold, depending on how the replenishment is allocated. Therefore, to prove this theorem, we prove that if equation (3) holds true before a replenishment arrives, it holds true after the replenishment arrives, given that this arriving replenishment is allocated according to the replenishment allocation policy defined in Section 4.2.

Let an end-customer demand occur at dealer i at some random time t. Due to the one-for-one replenishment policy, there will be a replenishment arriving at dealer i at time $t+\tau$. We assume that there is no end-customer demand arrival at time $t+\tau$[2]. Choose a positive $\epsilon \to 0$ such that $D(t+\tau-\epsilon, t+\tau] = 0$ and $D(t-\epsilon, t] = 1$ (this is equivalent to $D(t-\epsilon, t) = 0$ as we assume there is a demand arrival at dealer i at time t), where $D(t_1, t_2]$ represents the end-customer demand at the system (dealer 1 and dealer 2) during time interval $(t_1, t_2]$. We next show that if equation (3) holds for time $t+\tau-\epsilon$ (equation (12) below), it will hold for time $t+\tau$ (equation (3)), if the replenishment is allocated according to the replenishment allocation policy. At time $t+\tau-\epsilon$, assume the following holds:

$$IL_i(t+\tau-\epsilon) = S_i - D_i(t-\epsilon, t+\tau-\epsilon] + [D_i(t-\epsilon, t_{kj}^{\epsilon}] - S_i]^+ - [D_j(t-\epsilon, t_{ki}^{\epsilon}] - S_j]^+, \qquad (12)$$

where t_{kj}^{ϵ} is the time at which the $(S_i+S_j-K_j)^{th}$ demand arrives at the system during the time interval $(t-\epsilon, t+\tau-\epsilon]$. If less than $(S_i+S_j-K_j)$ demands arrive during the time interval, then $t_{kj}^{\epsilon} = t+\tau-\epsilon$.

Since there is a demand arrival at dealer i at time t, and since no demands arrive during $(t-\epsilon, t)$ and $(t+\tau-\epsilon, t+\tau]$, we can write the following:

$$D_i(t, t+\tau] = D_i(t-\epsilon, t+\tau-\epsilon] - 1. \qquad (13)$$

Similarly,

$$D_i(t, t_{kj}] = D_i(t-\epsilon, t_{kj}^{\epsilon}] + D_i(t_{kj}^{\epsilon}, t_{kj}] - 1, \qquad (14)$$

$$D_j(t, t_{ki}] = D_j(t-\epsilon, t_{ki}^{\epsilon}] + D_j(t_{ki}^{\epsilon}, t_{ki}]. \qquad (15)$$

Notice that there is only one demand arrival in the time interval $(t_{kj}^{\epsilon}, t_{kj}]$. This is the demand at time t_{kj}. By definition, if there are at least $S_i+S_j-K_j$ demand arrivals during $(t, t+\tau]$, then the demand at time t_{kj} is the $S_i+S_j-K_j^{th}$ demand arrival during $(t, t+\tau]$. Therefore,

$$D_i(t_{kj}^{\epsilon}, t_{kj}] = \begin{cases} 1 & \text{if the } S_i+S_j-K_j^{th} \text{ demand arrival during } (t, t+\tau] \text{ is a demand at dealer } i, \\ 0 & \text{if the } S_i+S_j-K_j^{th} \text{ demand arrival during } (t, t+\tau] \text{ is a demand at dealer } j. \end{cases}$$

If there are less than $S_i+S_j-K_j$ demand arrivals during $(t, t+\tau]$, then $D_i(t_{kj}^{\epsilon}, t_{kj}] = D_i(t+\tau-\epsilon, t+\tau] = 0$.

Similarly, $D(t_{ki}^{\epsilon}, t_{ki}]$ refers to the end-customer demand arrival at time t_{ki}.

Next we prove the theorem based on three cases:

[2]Without loss of generality, if a demand arrival coincides with the replenishment arrival at time $t+\tau$, we assume that we implement the replenishment allocation policy prior to observing the demand.

1. $[D_i(t-\epsilon, t^{\epsilon}_{kj}] - S_i]^+ = 0$ and $[D_j(t-\epsilon, t^{\epsilon}_{ki}] - S_j]^+ > 0$.

   Under this case,

$$IL_i(t+\tau-\epsilon) = S_i - D_i(t-\epsilon, t+\tau-\epsilon] - [D_j(t-\epsilon, t^{\epsilon}_{ki}] - S_j].$$

   Based on equation (14), $D_i(t, t_{kj}] \leq D_i(t-\epsilon, t^{\epsilon}_{kj}]$. Thus, $[D_i(t-\epsilon, t^{\epsilon}_{kj}] - S_i]^+ = 0$ leads to $[D_i(t, t_{kj}] - S_i]^+ = 0$. Similarly, based on equation (15), $D_j(t, t_{ki}] \geq D_j(t-\epsilon, t^{\epsilon}_{ki}]$, $[D_j(t-\epsilon, t^{\epsilon}_{ki}] - S_j]^+ > 0$ leads to $[D_j(t, t_{ki}] - S_j]^+ > 0$. Therefore, to prove that equation (3) holds, we need to show that

$$\begin{aligned} IL_i(t+\tau) &= S_i - D_i(t, t+\tau] + [D_i(t, t_{kj}] - S_i]^+ - [D_j(t, t_{ki}] - S_j]^+ \\ &= S_i - D_i(t-\epsilon, t+\tau-\epsilon] + 1 - [D_j(t-\epsilon, t^{\epsilon}_{ki}] + D_j(t^{\epsilon}_{ki}, t_{ki}] - S_j]. \\ &= IL_i(t+\tau-\epsilon) + 1 - D_j(t^{\epsilon}_{ki}, t_{ki}]. \end{aligned} \tag{16}$$

   Recall that $D(t^{\epsilon}_{ki}, t_{ki}]$ represents the demand arrival to the system at time t_{ki}, or the $S_i + S_j - K_i^{th}$ demand arrivals during $(t, t+\tau]$ if there are at least $S_i + S_j - K_i$ total demands during $(t, t+\tau]$. Notice that

$$IL_i(t+\tau) = \begin{cases} IL_i(t+\tau-\epsilon) + 1 & \text{if the replenishment part is kept at dealer } i, \\ IL_i(t+\tau-\epsilon) & \text{if the replenishment part is allocated to dealer } j. \end{cases}$$

   Therefore, the third bullet of the replenishment allocation policy described in Section 4.2 ensures that equation (16) holds.

   On the other hand, if there are less than $S_i + S_j - K_i$ total demand arrivals during $(t, t+\tau]$, $t^{\epsilon}_{ki} = t+\tau-\epsilon$ and $t_{ki} = t+\tau$. Since $D_j(t+\tau-\epsilon, t+\tau] = 0$, equation (16) implies that $IL_i(t+\tau) = IL_i(t+\tau-\epsilon) + 1$. Therefore, the replenishment allocation policy described in Section 4.2, which specifies that the replenishment part should be kept at dealer i under this situation, ensures that equation (16) holds.

2. $[D_i(t-\epsilon, t^{\epsilon}_{kj}] - S_i]^+ = 0$ and $[D_j(t-\epsilon, t^{\epsilon}_{ki}] - S_j]^+ = 0$.

   Under this case,

$$IL_i(t+\tau-\epsilon) = S_i - D_i(t-\epsilon, t+\tau-\epsilon].$$

   As we show in case 1, $[D_i(t-\epsilon, t^{\epsilon}_{kj}] - S_i]^+ = 0$ leads to $[D_i(t, t_{kj}] - S_i]^+ = 0$. Therefore, in order to prove that equation (3) holds, we need to show that

$$\begin{aligned} IL_i(t+\tau) &= S_i - D_i(t, t+\tau] + [D_i(t, t_{kj}] - S_i]^+ - [D_j(t, t_{ki}] - S_j]^+ \\ &= S_i - D_i(t-\epsilon, t+\tau-\epsilon] + 1 - [D_j(t-\epsilon, t^{\epsilon}_{ki}] + D_j(t^{\epsilon}_{ki}, t_{ki}] - S_j]^+ \\ &= \begin{cases} IL_i(t+\tau-\epsilon) & \text{if } D_j(t-\epsilon, t^{\epsilon}_{ki}] = S_j \text{ and } D_j(t^{\epsilon}_{ki}, t_{ki}] = 1, \\ IL_i(t+\tau-\epsilon) + 1 & \text{otherwise,} \end{cases} \end{aligned} \tag{17}$$

in which the last equality is based on the fact that $D_j(t-\epsilon, t_{ki}^{\epsilon}] \leq D_j(t, t_{ki}] \leq D_j(t-\epsilon, t_{ki}^{\epsilon}]+1$, which directly follows from equation (15).

Notice that the condition of $D_j(t-\epsilon, t_{ki}^{\epsilon}] = S_j$ and $D_j(t_{ki}^{\epsilon}, t_{ki}] = 1$ implies that, during $(t, t+\tau]$, there are at least $S_i + S_j - K_i$ total end-customer arrivals, *and* there are exactly S_j end customers arrivals at dealer j during $(t-\epsilon, t_{ki}^{\epsilon}]$, *and* the $S_i + S_j - K_i^{th}$ end-customer is a dealer j customer. This is equivalent to the condition that exactly S_j+1 end customers arrive at dealer j during $(t, t_{ki}]$ *and* the $S_i+S_j-K_i^{th}$ end-customer is a dealer j customer. According to the third bullet of the replenishment allocation policy defined in Section 4.2, under this condition, the replenishment should be allocated to dealer j. Therefore, equation (17) holds true for this case. Similarly, if the condition of $D_j(t-\epsilon, t_{ki}^{\epsilon}] = S_j$ and $D_j(t_{ki}^{\epsilon}, t_{ki}] = 1$ is not satisfied, according to the replenishment allocation policy, the replenishment should be kept at dealer i. Therefore, equation (17) also holds true for this case.

3. $[D_i(t-\epsilon, t_{kj}^{\epsilon}] - S_i]^+ > 0$ and $[D_j(t-\epsilon, t_{ki}^{\epsilon}] - S_j]^+ = 0$.

   Under this case,

$$IL_i(t+\tau-\epsilon) = S_i - D_i(t-\epsilon, t+\tau-\epsilon] + [D_i(t-\epsilon, t_{kj}^{\epsilon}] - S_i]. \tag{18}$$

Based on equation (14), $[D_i(t-\epsilon, t_{kj}^{\epsilon}]-S_i]^+ > 0$ leads to $D_i(t, t_{kj}]-S_i \geq 0$. Based on equation (15), $D_j(t-\epsilon, t_{ki}^{\epsilon}] \leq D_j(t, t_{ki}] \leq D_j(t-\epsilon, t_{ki}^{\epsilon}]+1$. Therefore, as $[D_j(t-\epsilon, t_{ki}^{\epsilon}] - S_j]^+ = 0$, $[D_j(t, t_{ki}] - S_j]^+$ will take the value of either 1 or 0. Since it is impossible for a dealer to both receive transshipments from other dealers (i.e., he has stocked out) and share with other dealers during a lead time, i.e., the last two terms in equation (3) cannot both take positive values at the same time, it can be easily shown using contradiction that under this case, $[D_j(t, t_{ki}] - S_j]^+$ can only take the value of 0. Hence, given equation (18) we want to prove that

$$\begin{aligned} IL_i(t+\tau) &= S_i - D_i(t, t+\tau] + [D_i(t, t_{kj}] - S_i]^+ - [D_j(t, t_{ki}] - S_j]^+ \\ &= S_i - D_i(t-\epsilon, t+\tau-\epsilon] + 1 + [D_i(t-\epsilon, t_{kj}^{\epsilon}] + D_i(t_{kj}^{\epsilon}, t_{kj}] - 1 - S_i] \\ &= IL_i(t+\tau-\epsilon) + D_i(t_{kj}^{\epsilon}, t_{kj}]. \end{aligned} \tag{19}$$

Recall that $D(t_{kj}^{\epsilon}, t_{kj}]$ represents the demand arrival to the system at time t_{kj}. Specifically, if, during $(t, t+\tau]$, there are at least $S_i + S_j - K_j$ total end-customer arrivals, *and* if at least S_i of the first $S_i + S_j - K_j$ demands are dealer i customers ($[D_i(t-\epsilon, t_{kj}^{\epsilon}] - S_i]^+ > 0$), *and* if the $S_i + S_j - K_j^{th}$ end-customer is a dealer j customer ($D_i(t_{kj}^{\epsilon}, t_{kj}] = 0$), according to the second bullet of the replenishment allocation policy in Section 4.2, the replenishment should be allocated to dealer j. Therefore, equation (19) holds true under these conditions.

If, during $(t, t+\tau]$, however, there are less than $S_i + S_j - K_j$ total end-customer arrivals, *and* if at least S_i of these demands are dealer i customers, then $D_i(t^\epsilon_{kj}, t_{kj}] = D_i(t+\tau-\epsilon, t+\tau] = 0$. According to the first bullet of the replenishment allocation policy in Section 4.2, the replenishment should be allocated to dealer j. Therefore, equation (19) holds true under these conditions.

Since $[D_i(t-\epsilon, t^\epsilon_{kj}] - S_i]^+$ and $[D_j(t-\epsilon, t^\epsilon_{ki}] - S_j]^+$ cannot both take positive values at the same time, we have considered all possible cases and proved under all cases, equation (3) holds true. ∎

C Derivation of the Steady-State Probabilities

According to the discussion in Section 4.2, the system steady-state probability, p_{mn}, can be calculated as the probability that, given that the system starts from state (S_i, S_j) at any time t and the threshold rationing policy is used to fill demands, the system reaches state (m, n) at time $t + \tau$. Thus, in order to calculate p_{mn}, we need to know only the sequence of demand arrivals during the lead time. Based on this concept, we detail the derivation of the steady-state probabilities for regions A and B of the state space as shown in Figure 1.

1. Region A: States $\{(m, n) | m \leq 0, 0 < n \leq K_2\}$.

$$p_{mn} = p(S_1+S_2-m-n;\lambda\tau)\left[\sum_{x=S_1}^{S_1+S_2-K_2} b(\beta_1; S_1+S_2-K_2; x)b(\beta_1; K_2-m-n; -m) + \sum_{x=max(0,S_1-K_2+n)}^{S_1-1} b(\beta_1; S_1+S_2-K_2; x)b(\beta_1; K_2-m-n; S_1-m-x)\right]. \quad (20)$$

   For the system to reach state (m, n) in this region at the end of a lead time, there must be at least $S_1 + S_2 - K_2$ total end-customer demands arrivals during the lead time, τ, with at least S_1 demand arrivals at dealer 1. Since $m \leq 0$, starting from state (S_1, S_2), there can be two paths through which the system can reach the state (m, n) at the end of a lead time: a path with inventory sharing from dealer 2 to dealer 1 during the lead time; or a path with no inventory sharing between the two dealers during the lead time.

   If there is inventory sharing from dealer 2 to dealer 1 during the lead time, it must occur during the first $S_1 + S_2 - K_2$ end-customer demands to the system during the lead time, since dealer 2's inventory level will be below his rationing level (K_2) after that point. This concept is reflected in the expression for p_{mn} and can be best demonstrated by a graph. In Figure 7, line *abcde* represents the first $S_1 + S_2 - K_2$ end-customer demand arrivals during a lead time. Different points on the line represent different demand splits between dealer 1 and dealer 2 during the first $S_1 + S_2 - K_2$ demands.

- The first item in the bracket in equation (20) represents the case in which there is inventory sharing from dealer 2 to dealer 1 during the lead time. Therefore, there are at least S_1 demand arrivals at dealer 1 during the first $S_1 + S_2 - K_2$ demands to the system (part ab of line $abcd$), i.e., dealer 1 stocks out while dealer 2's on-hand inventory is still above his rationing level K_2. Hence, dealer 2 will share with dealer 1 during the first $S_1 + S_2 - K_2$ demands for any demand arrival at dealer 1 after dealer 1 stocks out. Thus, immediately after the $(S_1+S_2-K_2)^{th}$ demand, the system is at state $(0, K_2)$. After that, there can be no more sharing in the system, which implies that there are exactly $-m$ dealer 1 demands in the following $K_2 - m - n$ demands in order to reach state (m, n).

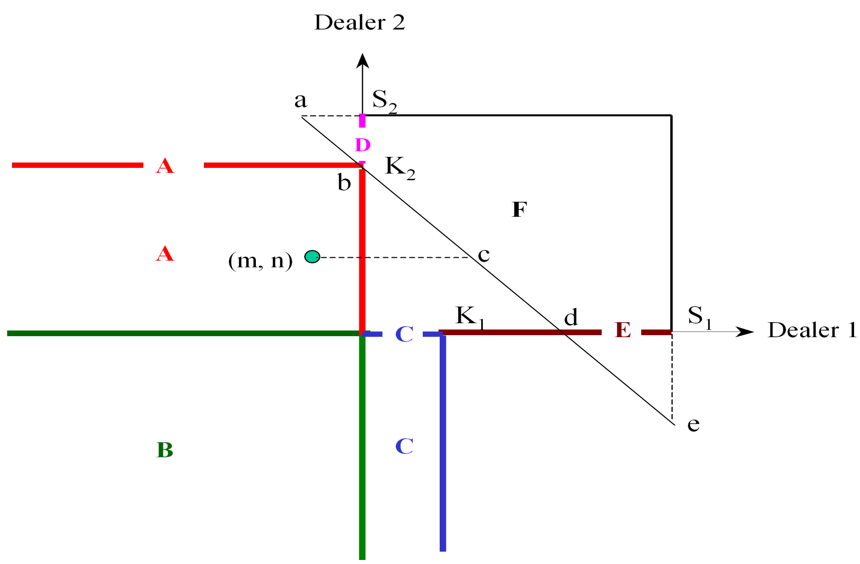

Figure 7: Illustration of the Calculation of the Steady-State Distribution for States in Region A

- The second term represents the case in which there is no sharing between the dealers (part bc of line $abcd$) during the lead time. In this case, the number of dealer 1 demands during the first $S_1 + S_2 - K_2$ end-customer demands, x, is less than S_1, but great than $max(0, S_1-(S_2-n)$ to ensure that it is possible to reach the state of (m, n) at the end of the lead time. The $max(\ )$ in the second term deals with the case in which $S_1 - K_2 + n$ is negative. There will not be any inventory sharing after the $(S_1 + S_2 - K_2)^{th}$ demand either (since dealer 2 is already below his rationing level). Therefore, in order to reach state (m, n), out of the following $K_2 - m - n$ demands to the system, there are exactly $S_1 - m - x$ dealer 1 demands.

2. Region B: States $\{(m, n)|m \leq 0, n \leq 0\}$. The steady-state probabilities in region B depend

on the relative values of K_1 and K_2.

- $K_2 \geq K_1$:

$$\begin{aligned}
p_{mn} &= \Bigg\{ \sum_{x=S_1}^{S_1+S_2-K_2} b(\beta_1; S_1+S_2-K_2; x) b(\beta_1; K_2-m-n; -m) \\
&+ \sum_{x=S_1-K_1}^{S_1-1} b(\beta_1; S_1+S_2-K_2; x) b(\beta_1; K_2-m-n; S_1-m-x) \\
&+ \Bigg[\sum_{x=max(S_1-K_2,0)}^{S_1-K_1-1} b(\beta_1; S_1+S_2-K_2; x) \\
&\times \sum_{x'=S_1-K_1-x+1}^{min(K_2-K_1,S_1-x-m)} b(\beta_1; K_2-K_1; x') b(\beta_1; K_1-m-n; S_1-m-x-x') \Bigg] \\
&+ \sum_{x=0}^{S_1-K_1-1} b(\beta_1; S_1+S_2-K_1; x) b(\beta_1; K_1-m-n; K_1-m) \Bigg\} p(S_1+S_2-m-n; \lambda\tau)
\end{aligned} \tag{21}$$

- $K_2 < K_1$:

$$\begin{aligned}
& p_{mn} \\
= \; & p(S_1+S_2-m-n; \lambda\tau) \\
& \times \Bigg\{ \sum_{x=0}^{S_1-K_1} b(\beta_1; S_1+S_2-K_1; x) b(\beta_1; K_1-m-n; K_1-m) \\
& + \sum_{x=S_1-K_1+1}^{S_1-K_1+K_2} b(\beta_1; S_1+S_2-K_1; x) b(\beta_1; K_1-m-n; S_1-m-x) \\
& + \Bigg[\sum_{x=S_1-K_1+K_2+1}^{min(S_1,S_1+S_2-K_1)} b(\beta_1; S_1+S_2-K_1; x) \\
& \times \sum_{x'=max(0,S_1-K_2-x+n})^{S_1-x-1} b(\beta_1; K_1-K_2; x') b(\beta_1; K_2-m-n; S_1-m-x-x') \Bigg] \\
& + \sum_{x=S_1+1}^{S_1+S_2-K_2} b(\beta_1; S_1+S_2-K_2; x) b(\beta_1; K_2-m-n; -m) \Bigg\}.
\end{aligned}$$

We focus on our derivation of the case for $K_2 \geq K_1$ as follows, and the derivation of the case for $K_2 < K_1$ can be conducted analogously.

Notice that in Region B, both dealers have stocked out. Therefore, starting from state (S_1, S_2) at the beginning of a lead time, there are three possible paths through which the system can arrive at the state (m, n) at the end of the lead time: dealer 1 stocks out first and dealer 2 shares with dealer 1, dealer 2 stocks out first and dealer 1 shares with dealer 2, or no inventory sharing during the lead time according to the dealers' (S, K) policy. Specifically, since inventory sharing from dealer i to dealer j can only occur during the first $S_i + S_j - K_i$ end-customer demands during the lead time, whether or not there is sharing from dealer i to dealer j depends on the demand split between the two dealers during the first $S_i + S_j - K_i$ end-customer demands. Again, this can be best understood using a graph. Taking $K_2 \geq K_1$ as an example (see Figure 8), as in Figure 7, line $abcde$ (or line $a'b'c'd'e'$) in this figure represents the first $S_1 + S_2 - K_2$ (or $S_1 + S_2 - K_1$) end-customer demands. Different points on line $abcde$ (or line $a'b'c'd'e'$) represent different demand splits between dealer 1 and dealer 2 during the first $S_1 + S_2 - K_2$ (or $S_1 + S_2 - K_1$) end-customer demands. Thus, sharing from dealer 2 to dealer 1 would only happen if the demand split between the two dealers in the first $S_1 + S_2 - K_2$ demands falls on part ab of line $abcde$ (i.e., when there are at least S_1 demand arrivals at dealer 1 during the first $S_1 + S_2 - K_2$ end-customer demands). Similarly, sharing from dealer 1 to dealer 2 could only happen if the demand split between the two dealers in the first $S_1 + S_2 - K_1$ demands fall on part $c'e'$ of line $a'b'c'd'e'$ (i.e., when there are at least S_2 demand arrivals at dealer 2 during the first $S_1 + S_2 - K_1$ end-customer demands). Since $K_2 \geq K_1$, we focus on analyzing the first $S_1 + S_2 - K_2$ and the following $K_2 - K_1$ demands (giving a total demand of $S_1 + S_2 - K_1$). After the $(S_1 + S_2 - K_1)^{th}$ demand, there can be no more sharing between the dealers as both dealers' inventory levels will be below their rationing levels.

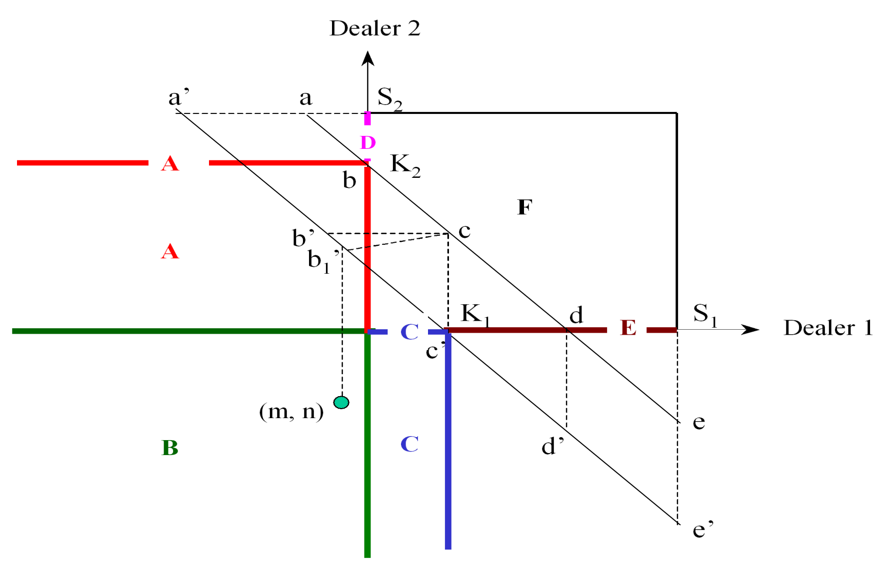

Figure 8: Illustration of the Calculation of the Steady-State Distribution for States in Region B ($K_2 \geq K_1$)

The three possible ways to reach state (m, n) in region B correspond to the different terms of equation (21).

- If there are at least S_1 demand arrivals at dealer 1 during the first $S_1 + S_2 - K_2$ demands (part *ab* on line *abcde* in Figure 8), then dealer 2 will share with dealer 1 during the first $S_1 + S_2 - K_2$ demands for any demand arrival at dealer 1 after dealer 1 stocks out. Thus, immediately after the $(S_1 + S_2 - K_2)^{th}$ demand, the system is at state $(0, K_2)$. After that, there can be no more sharing in the system, which implies that there are exactly $-m$ dealer 1 demands in the following $K_2 - m - n$ demands in order to reach state (m, n). This is represented by the first term in equation (21).
- If the number of dealer 1 demands during the first $S_1 + S_2 - K_2$ end-customer demands, x, is between $S_1 - K_1$ and $S_1 - 1$ (part *bc* on line *abcde*), then there is no inventory sharing during the first $S_1 + S_2 - K_2$ demands (since neither dealer has stocked out) and there will not be any sharing regardless of what the next $K_2 - m - n$ demands are (since both dealers are already at or below their rationing levels). Therefore, in order to reach state (m, n), out of the following $K_2 - m - n$ demands to the system, there are exactly $S_1 - m - x$ dealer 1 demands. This is represented by the second term in equation (21).
- If the number of dealer 1 demands during the first $S_1 + S_2 - K_2$ demands, x, is between $S_1 - K_2$ (or 0 if $S_1 < K_2$) and $S_1 - K_1 - 1$ (part *cd* on line *abcde*) [3], then there is no sharing from dealer 2 to dealer 1 during the first $S_1 + S_2 - K_2$ demands (since neither dealer has stocked out). However, depending on the demand split between dealer 1 and dealer 2 for the next $K_2 - K_1$ demands, inventory sharing from dealer 1 to dealer 2 may or may not happen. Specifically, if, the next $K_2 - K_1$ demands put the system on part $b'c'$ of line $a'b'c'd'e'$ (or part b'_1c' if point b'_1 lies within $b'c'$ [4]) immediately after the $(S_1 + S_2 - K_1)^{th}$ demand to the system, then there is no inventory sharing from dealer 1 to dealer 2, and also no sharing in the last $K_1 - m - n$ end-customer demands. This is represented by the third term (in the square brackets) of equation (21).

  If, however, the next $K_2 - K_1$ demands put the system on part $c'd'$ of line $a'b'c'd'e'$ in Figure 8 immediately after the $(S_1 + S_2 - K_1)^{th}$ demand to the system, then there will be sharing from dealer 1 to dealer 2, which is captured in the last term of equation (21), described in the next bullet.
- If there are at least S_2 demand arrivals at dealer 2 during the first $S_1 + S_2 - K_1$ end-customer demands (part $c'e'$ on line $a'b'c'd'e'$), then dealer 1 will share with dealer 2 during the first $S_1 + S_2 - K_1$ end-customer demands for any demand arrival at dealer

[3] If $K_1 = K_2$, then this third term equals to zero.

[4] b'_1 is determined by the point (m, n). Notice that the system cannot fall on $b'b'_1$ in order to reach state (m, n). This is captured by *min*() in the third term of equation (21).

2 after dealer 2 stocks out. This includes both the case in which there is no inventory sharing during the first $S_1+S_2-K_2$ end-customer demands to the system and then there is inventory sharing in the following K_2-K_1 demands (part *cd* of line*abcde* leading to part *c'd'* of line a'b'c'd'e'), and the case in which there is inventory sharing during the first $S_1+S_2-K_2$ end-customer demands (part *de* of line *abcde* leading to part *c'e'* of line $a'b'c'd'e'$). Thus, immediately after the $(S_1+S_2-K_1)^{th}$ demand, the system will be at the state of $(K_1, 0)$. After that, there will be no more sharing in the system (since neither dealer will be above his rationing level), which implies that there are exactly K_1-m dealer 1 demands in the last K_1-m-n demands to reach state (m, n). This is represented by the last term of equation (21). ∎

D Proof of Lemma 1

For the full-sharing case, to show that $C_1(S_1, S_2)$ and

$C_2(S_1, S_2)$ have decreasing differences (in order) in S_1 and S_2, we need to prove equation (9). We will focus our proof of equation (9) for $i=1$. The proof for $i=2$ can be obtained analogously. We obtain the cross difference of C_1 with regard to S_1 and S_2 based on the performance measures of the full-sharing game we developed in Section 5.1. We first obtain the cross difference for each performance measure for dealer 1:

1. EI_1:

$$\begin{aligned}
& (EI_1(S_1+1,S_2+1)-EI_1(S_1,S_2+1))-(EI_1(S_1+1,S_2)-EI_1(S_1,S_2)) \\
= & \sum_{m=1}^{S_1+1}\left[\sum_{n=1}^{S_2+1} p(S_1+S_2-m-n+2;\lambda\tau)b(\beta_1;S_1+S_2-m-n+2;S_1-m+1)\right. \\
& \left. -\sum_{n=1}^{S_2} p(S_1+S_2-m-n+1;\lambda\tau)b(\beta_1;S_1+S_2-m-n+1;S_1-m+1)\right] \\
& +\sum_{y=0}^{S_1} p(S_1+S_2-y+1;\lambda\tau)\sum_{x=0}^{S_1-m} b(\beta_1;S_1+S_2-m+1;x) \\
& -\sum_{y=0}^{S_1} p(S_1+S_2-y;\lambda\tau)\sum_{x=0}^{S_1-m} b(\beta_1;S_1+S_2-m;x) \\
= & \; p(S_1+S_2+1;\lambda\tau)\sum_{x=0}^{S_1} b(\beta_1;S_1+S_2+1;x)
\end{aligned}$$

2. EB_1:

$$\begin{aligned}
& (EB_1(S_1+1,S_2+1)-EB_1(S_1,S_2+1))-(EB_1(S_1+1,S_2)-EB_1(S_1,S_2)) \\
= & \; \beta_1\, p(S_1+S_2+1;\lambda\tau)
\end{aligned}$$

3. PNS_1:

$$\begin{aligned} & (PNS_1(S_1+1,S_2+1) - PNS_1(S_1,S_2+1)) - (PNS_1(S_1+1,S_2) - PNS_1(S_1,S_2)) \\ = \; & -p(S_1+S_2+1;\lambda\tau) + p(S_1+S_2;\lambda\tau) \end{aligned}$$

4. PS_1:

$$\begin{aligned} & (PS_1(S_1+1,S_2+1) - PS_1(S_1,S_2+1)) - (PS_1(S_1+1,S_2) - PS_1(S_1,S_2)) \\ = \; & p(S_1+S_2+1;\lambda\tau) \sum_{x=S_1+1}^{S_1+S_2+1} b(\beta_1;S_1+S_2+1;x) \\ & -p(S_1+S_2;\lambda\tau) \sum_{x=S_1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) \end{aligned}$$

Now, combining all terms together, we obtain the cross difference of dealer 1's cost, C_1, with respect to S_1 and S_2, as follows:

$$\begin{aligned} & (C_1(S_1+1,S_2+1) - C_1(S_1,S_2+1)) - (C_1(S_1+1,S_2) - C_1(S_1,S_2)) \\ = \; & p(S_1+S_2+1;\lambda\tau) \times \left[\hat{h} \sum_{x=0}^{S_1} b(\beta_1;S_1+S_2+1;x) \right. \\ & \left. + h_1^- \lambda_1 \sum_{x=S_1+1}^{S_1+S_2+1} b(\beta_1;S_1+S_2+1;x) + (\beta_1\hat{\pi} - h_2^-\lambda_2 - \pi\lambda_1) \right] \\ & + p(S_1+S_2;\lambda\tau) \left[h_2^-\lambda_2 + \pi\lambda_1 - h_1^-\lambda_1 \sum_{x=S_1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) \right] \\ \geq \; & p(S_1+S_2+1;\lambda\tau) \left[min(\hat{h}, h_1^-\lambda_1) + (\beta_1\hat{\pi} - h_2^-\lambda_2 - \pi\lambda_1) \right] \\ & + p(S_1+S_2;\lambda\tau) \left[h_2^-\lambda_2 + \pi\lambda_1 - h_1^-\lambda_1 \sum_{x=S_1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) \right] \end{aligned}$$

Therefore, if $h_1^-\lambda_1 \leq h_2^-\lambda_2 + \pi\lambda_1 \leq \beta_1\hat{\pi} + min(\hat{h}, h_1^-\lambda_1)$, then C_1 has decreasing differences (in order) in S_1 and S_2. Similarly, if $h_1^-\lambda_2 \leq h_2^-\lambda_1 + \pi\lambda_2 \leq \beta_2\hat{\pi} + min(\hat{h}, h_1^-\lambda_2)$, then $C_2(S_1,S_2)$ has decreasing differences (in order) in S_1 and S_2. ∎

E Proof of Theorem 2

There are several conditions required for a game to be supermodular (Milgrom and Roberts (1990)):

1. Each player's strategies must be ordered. Since supermodularity implies that the players' strategies are "strategic complements", we adopt the following ordering convention, which requires that the ordering of the two dealers' base-stock levels, S_i, be opposite. Specifically, we use the natural ordering for one dealer, say dealer 1. Therefore, a higher strategy (in

order) for dealer 1, implies a greater S_1 (in number). For dealer 2, we use a reverse ordering (reverse of the natural ordering). Therefore, a higher strategy (in order) for dealer 2, implies a lower S_2 (in number). A similar ordering convention was adopted for the reorder point, R, in the (Q, R) game considered in Cachon (2001). For more details on ordering, see Cachon (2001). Therefore, "strategic complements" implies that a greater S_1 (higher in order) leads to a lower S_2 (higher in order). In other words, if dealer 1 stocks more (a higher base-stock level at dealer 1), then dealer 2 stocks less (a lower base-stock level at dealer 2).

2. The strategy sets for dealer 1 and dealer 2 are complete lattices. The strategy spaces are lattices because they are single dimensional and bounded. Although the real strategy space for each dealer is infinite, for any real system, we would only stock a finite number of parts in inventory. Therefore, it is reasonable to assume that each dealer's strategy space is the set of integers between $[0, M]$, in which M is an extremely large positive integer. Thus, the strategy spaces are bounded. The dealer's strategy space is complete because any finite lattice is complete.

3. The payoff function $-C_i$ is order upper semi-continuous in S_i (for fixed S_j) and order continuous in S_j for fixed S_i (Milgrom and Roberts, 1990). Since each dealer's strategy space is finite and compact, it is easy to confirm order continuity and order upper semi-continuity of the cost functions.

4. Each dealer's payoff function $(-C_i)$ has an upper bound because the strategy set is finite and bounded, and therefore, a minimum cost exists.

5. Each dealer's payoff function $(-C_i)$ is supermodular in S_i, which is trivial to show in the one dimensional case (i.e., one decision variable for each dealer).

6. Each dealer's payoff function $(-C_i)$ has increasing differences in S_1 and S_2, or each dealer's cost function has decreasing differences in S_1 and S_2, which was shown in Lemma 1.

Therefore, the full-sharing inventory game is supermodular and the existence of a pure strategy Nash equilibrium follows immediately from the first corollary of Theorem 5 in Milgrom and Roberts (1990). ∎

F Proof of Theorem 3

For the full-sharing game, to show that C_i is convex in S_i for a given S_j, we want to show that

$$C_1(S_1 + 1, S_2) - C_1(S_1, S_2) \geq C_1(S_1, S_2) - C_1(S_1 - 1, S_2) \tag{22}$$

and

$$C_2(S_1, S_2 + 1) - C_2(S_1, S_2) \geq C_2(S_1, S_2) - C_2(S_1, S_2 - 1). \tag{23}$$

We will focus on proving equation (22). Equation (23) can be proven analogously. We first calculate the second difference of each term in dealer 1's cost function according to performance measures of the full-sharing game we developed in Section 5.1.

1. EI_1: We can write

$$\begin{aligned} & EI_1(S_1+1,S_2) - EI_1(S_1,S_2) \\ = & \sum_{n=1}^{S_2}\sum_{m=1}^{S_1+1} p(S_1+S_2-m-n+1;\lambda\tau)b(\beta_1;S_1+S_2-m-n+1;S_1-m+1) \\ & +\sum_{m=0}^{S_1} p(S_1+S_2-m;\lambda\tau)\sum_{x=0}^{S_1-m} b(\beta_1;S_1+S_2-m;x), \end{aligned}$$

and

$$\begin{aligned} & (EI_1(S_1+1,S_2) - EI_1(S_1,S_2)) - (EI_1(S_1,S_2) - EI_1(S_1-1,S_2)) \\ = & \sum_{z=S_1}^{S_1+S_2-1} p(z;\lambda\tau)b(\beta_1;z;S_1) + p(S_1+S_2;\lambda\tau)\sum_{x=0}^{S_1} b(\beta_1;S_1+S_2;x). \end{aligned}$$

2. EB_1: We can write

$$EB_1(S_1+1,S_2) - EB_1(S_1,S_2) = -\beta_1 \sum_{z=S_1+S_2+1}^{\infty} p(z;\lambda\tau),$$

and

$$\begin{aligned} & (EB_1(S_1+1,S_2) - EB_1(S_1,S_2)) - (EB_1(S_1,S_2) - EB_1(S_1-1,S_2)) \\ = & \beta_1 p(S_1+S_2;\lambda\tau). \end{aligned}$$

3. PNS_1: We can write

$$PNS_1(S_1+1,S_2) - PNS_1(S_1,S_2) = -p(S_1+S_2;\lambda\tau),$$

and

$$\begin{aligned} & (PNS_1(S_1+1,S_2) - PNS_1(S_1,S_2)) - (PNS_1(S_1,S_2) - PNS_1(S_1-1,S_2)) \\ = & -p(S_1+S_2;\lambda\tau) + p(S_1+S_2-1;\lambda\tau). \end{aligned}$$

4. PS_1: We can write

$$\begin{aligned} & PS_1(S_1+1,S_2) - PS_1(S_1,S_2) \\ = & p(S_1+S_2;\lambda\tau)\sum_{x=S_1+1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) - \sum_{z=S_1}^{S_1+S_2-1} p(z;\lambda\tau)b(\beta_1;z;S_1), \end{aligned}$$

and

$$
\begin{aligned}
& (PS_1(S_1+1,S_2) - PS_1(S_1,S_2)) - (PS_1(S_1,S_2) - PS_1(S_1-1,S_2)) \\
= \; & p(S_1+S_2;\lambda\tau) \sum_{x=S_1+1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) \\
& -p(S_1+S_2-1;\lambda\tau) \sum_{x=S_1}^{S_1+S_2-1} b(\beta_1;S_1+S_2-1;x) \\
& - \sum_{z=S_1}^{S_1+S_2-1} p(z;\lambda\tau) b(\beta_1;z;S_1) + \sum_{z=S_1-1}^{S_1+S_2-2} p(z;\lambda\tau) b(\beta_1;z;S_1-1).
\end{aligned}
$$

Now, combining all the terms together, we obtain the second difference of dealer 1's cost, C_1, with regard to S_1, as follows:

$$
\begin{aligned}
& (C_1(S_1+1,S_2) - C_1(S_1,S_2)) - (C_1(S_1,S_2) - C_1(S_1-1,S_2)) \\
= \; & (\hat{h} - h_1^- \lambda_1) \sum_{z=S_1}^{S_1+S_2-1} p(z;\lambda\tau) b(\beta_1;z;S_1) \\
& +p(S_1+S_2;\lambda\tau) \times \left[\hat{h} \sum_{x=0}^{S_1} b(\beta_1;S_1+S_2;x) + h_1^- \lambda_1 \sum_{x=S_1+1}^{S_1+S_2} b(\beta_1;S_1+S_2;x) \right. \\
& \left. +(\beta_1 \hat{\pi} - h_2^- \lambda_2 - \pi \lambda_1) \right] \\
& + \left[h_2^- \lambda_2 + \pi \lambda_1 - h_1^- \lambda_1 \sum_{x=S_1}^{S_1+S_2-1} b(\beta_1;S_1+S_2-1;x) \right] p(S_1+S_2-1;\lambda\tau) \\
& +h_1^- \lambda_1 \sum_{z=S_1-1}^{S_1+S_2-2} p(z;\lambda\tau) b(\beta_1;z;S_1-1) \\
\geq \; & (\hat{h} - h_1^- \lambda_1) \sum_{z=S_1}^{S_1+S_2-1} p(z;\lambda\tau) b(\beta_1;z;S_1) \\
& +p(S_1+S_2;\lambda\tau) \left[min(\hat{h}, h_1^- \lambda_1) + (\beta_1 \hat{\pi} - h_2^- \lambda_2 - \pi \lambda_1) \right] \\
& +p(S_1+S_2-1;\lambda\tau) \left[h_2^- \lambda_2 + \pi \lambda_1 - h_1^- \lambda_1 \sum_{x=S_1}^{S_1+S_2-1} b(\beta_1;S_1+S_2-1;x) \right] \\
& +h_1^- \lambda_1 \sum_{z=S_1-1}^{S_1+S_2-2} p(z;\lambda\tau) b(\beta_1;z;S_1-1)
\end{aligned} \tag{24}
$$

Therefore, if $\hat{h} \geq h_1^- \lambda_1$ and $h_1^- \lambda_1 \leq h_2^- \lambda_2 + \pi \lambda_1 \leq \beta_1 \hat{\pi} + min(\hat{h}, h_1^- \lambda_1)$, then C_1 is convex with regard to S_1, given a fixed S_2. Similarly, we can prove that C_2 is convex with regard to S_2 if $\hat{h} \geq h_1^- \lambda_2$ and $h_1^- \lambda_2 \leq h_2^- \lambda_1 + \pi \lambda_2 \leq \beta_2 \hat{\pi} + min(\hat{h}, h_1^- \lambda_2)$. ∎

G Proof of Lemma 2

We prove this lemma by first obtaining a closed-form expression for the cross partial difference of C_i with respect to S_i and S_j, denoted $\Delta_{S_i,S_j}C_i$, where

$$\Delta_{S_i,S_j}C_i \triangleq C_i(S_i{+}1,Q_i,S_j{+}1,Q_j)-C_i(S_i,Q_i,S_j{+}1,Q_j)-C_i(S_i{+}1,Q_i,S_j,Q_j)+C_i(S_i,Q_i,S_j,Q_j).$$

In this proof, we prove the lemma for $i = 1$ and $j = 2$. The case for $i = 2$ and $j = 1$ can be proved analogously and is thus omitted here.

We obtain the expression of $\Delta_{S_i,S_j}C_i$ by using a sample path approach, i.e., we evaluate the cross partial difference of dealer 1's cost with respect to S_1 and S_2 on all possible lead-time demand sample paths and weight each sample path by the associated probability that such a sample path will happen. Starting from state (S_1, S_2), let (m, n) be the state reached by the system at the end of a lead time. The sample paths can be divided into three cases: (1) Those sample paths on which no sharing from either dealer takes place; (2) Sample paths on which dealer 1 shares with dealer 2; and (3) Sample paths on which dealer 2 shares with dealer 1. These sample paths can be further divided based on the region to which (m, n) belongs (i.e., either region A, B, C, D, E, or region F as shown in Figures 1 and 8).

Case 1:

Consider sample paths on which, beginning from state (S_1, S_2), the system reaches state (m, n) at the end of a lead time through no inventory sharing between the two dealers. On these sample paths, increasing S_1 by 1 (while keeping Q_1 constant) would lead the system to reach the state $(m + 1, n)$ at the end of a lead time (since no inventory sharing is involved on these sample paths, increasing the stocking level, without changing the sharing level, leads to an increase in dealer 1's inventory level by 1). Similarly, increasing S_2 by 1 will result in the system reaching the state $(m, n + 1)$ at the end of a lead time, while simultaneously increasing S_1 and S_2 by 1 results in the system reaching state $(m + 1, n + 1)$at the end of a lead time. Thus on these sample paths we can write the cross partial difference of dealer 1's cost with respect to S_1 and S_2 as

$$c_1(m + 1, n + 1) - c_1(m, n + 1) - c_1(m + 1, n) + c_1(m, n), \tag{25}$$

where $c_1(m, n)$ is dealer 1's cost per unit time when dealer 1's inventory level is m and dealer 2's inventory level is n, respectively.

For the subset of these sample paths which end in region F (positive inventory levels for both dealers, i.e., $m > 0$ and $n > 0$, see figure 8), equation (25) is equal to

$$(m + 1)\hat{h} - m\hat{h} - (m + 1)\hat{h} + m\hat{h} = 0.$$

Similarly, it can be shown that the cross partial differences also equal to zero for sample paths ending in all other regions (i.e., region A, B, C, D, and region E) when no sharing is involved.

Case 2:

Now consider sample paths on which dealer 1 shares with dealer 2 during the lead time. Hence these sample paths would end in regions B, C, or E of Figure 8.

Consider the subset of these sample paths ending in region E, in which dealer 2 stocks out and dealer 1 is above his rationing level, i.e., $K_1 < m \leq S_1$ and $n = 0$. On these sample paths, adding one unit to either S_1 or S_2 would result in the system reaching state $(m+1, n)$ at the end of a lead time. Notice that adding one unit to S_2 reduces the number of sharing requests from dealer 2 to dealer 1 by one, and hence results in dealer 1's inventory level increasing by one (rather than dealer 2's inventory level increasing by one since in this region, all sharing requests are filled by dealer 1). Similarly, simultaneously adding one unit to S_1 and S_2 results in the system reaching state $(m+2, n)$. Thus, on these sample paths we have the cross partial difference of dealer 1's cost with respect to S_1 and S_2 as

$$\begin{aligned} & c_1(m+2,0) - c_1(m+1,0) - c_1(m+1,0) + c_1(m,0) \\ = \; & (m+2)\hat{h} - (m+1)\hat{h} - (m+1)\hat{h} + m\hat{h} \\ = \; & 0. \end{aligned}$$

Now consider the subset of these sample paths ending in regions B or C, in which dealer 2 stocks out and dealer 1 is at or below his rationing level, i.e., $m \leq K_1$ and $n \leq 0$ (and hence the system passes through the state $(K_1, 0)$, point c' in Figure 8, before reaching (m, n)). On these sample paths, adding one unit to S_1, without changing Q_1, results in system reaching state $(m+1, n)$, because dealer 1 will satisfy the same number of sharing requests. However, on these sample paths, the impact of adding one unit to S_2 depends on the type of the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival. This is because, now, inventory sharing from dealer 1 to dealer 2 can happen during the first $S_1 + S_2 - K_1 + 1$ demand arrivals instead of only the first $S_1 + S_2 - K_1$ demand arrivals. In fact, any sharing request from dealer 2 to dealer 1 will be satisfied by dealer 1 during the first $S_1 + S_2 - K_1 + 1$ demand arrivals. Thus, if the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival on this sample path is a dealer 2 demand, then adding one unit to S_2 results in the sharing request generated by the $S_1 + S_2 - K_1 + 1^{th}$ demand being satisfied by dealer 1, resulting in an increase in dealer 2's inventory level by 1. Hence, the system reaches state $(m, n+1)$ if the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival on these sample paths is a dealer 2 demand. On the other hand, if the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival on these sample paths is a dealer 1 demand, adding one unit to S_2 decreases the number of sharing requests satisfied by dealer 1 by one unit and hence increases dealer 1's inventory level by 1 (since all sharing requests from dealer 2 to dealer 1 during the first $S_1 + S_2 - K_1 + 1$

demand arrivals would have been filled by dealer 1). Hence the system reaches state $(m+1, n)$ if the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival on this sample path is a dealer 1 demand. Note that the probability of the $S_1 + S_2 - K_1 + 1^{th}$ demand arrival on these sample paths is a dealer 1 demand is β_1. Similarly, the impact of adding one unit to S_1 and S_2 simultaneously is determined by the type of the $S_1+1+S_2+1-(K_1+1)^{th}$ demand arrival, i.e., the $S_1+S_2-K_1+1^{th}$ demand arrival. Hence, on these sample paths we can write the cross partial difference of dealer 1's cost with respect to S_1 and S_2 as

$$\begin{aligned}
& (1-\beta_1)\{c_1(m+1, n+1) - c_1(m+1, n) - c_1(m, n+1) + c_1(m, n)\} \\
& +\beta_1\{c_1(m+2, n) - c_1(m+1, n) - c_1(m+1, n) + c_1(m, n)\} \\
= \;& \beta_1\{c_1(m+2, n) - c_1(m+1, n) - c_1(m+1, n) + c_1(m, n)\},
\end{aligned}$$

where the equation is obtained from the fact that $c_1(m+1, n+1) - c_1(m+1, n) - c_1(m, n+1) + c_1(m, n)$ is always zero (as shown in case 1). It is also easy to confirm that, on these sample paths, the cross partial difference is zero for all values of (m, n), except when $m = 0, -1$ (recall $n \leq 0$ in these regions). When $m = 0$ and $n \leq 0$, the cross partial difference becomes

$$\begin{aligned}
& \beta_1\{c_1(2, n) - c_1(1, n) - c_1(1, n) + c_1(0, n)\} \\
= \;& \beta_1\{(2\hat{h} + h_2^- \lambda_2) - (\hat{h} + h_2^- \lambda_2) - (\hat{h} + h_2^- \lambda_2) + (h_2^- \lambda_2 + \pi\lambda_1)\} \\
= \;& \beta_1 \pi \lambda_1
\end{aligned}$$

Similarly, when $m = -1$ and $n \leq 0$, the cross partial difference becomes

$$\begin{aligned}
& \beta_1\{c_1(1, n) - c_1(0, n) - c_1(0, n) + c_1(-1, n)\} \\
= \;& \beta_1\{(\hat{h} + h_2^- \lambda_2) - (\pi\lambda_1 + h_2^- \lambda_2) - (\pi\lambda_1 + h_2^- \lambda_2) + (\hat{\pi} + h_2^- \lambda_2 + \pi\lambda_1)\} \\
= \;& \beta_1(\hat{\pi} + \hat{h} - \pi\lambda_1).
\end{aligned}$$

It is obvious that $\Delta_{S_1,S_2} C_1$ depends only on the sample paths for which the cross partial difference do not equal to zero. Also note that the probability of sample paths where dealer 1 shares with dealer 2 and the system reaches state $(0, n)$ $(n \leq 0)$, with the $S_1 + S_2 - K_1 + 1^{th}$ arrival being a dealer 1 demand, is $\beta_1 \sum_{D=S_1+S_2}^{\infty} p(D; \lambda\tau) \sum_{x=0}^{S_1-K_1-1} b(\beta_1; S_1+S_2-K_1; x) b(\beta_1; D-(S_1+S_2-K_1)-1; K_1-1)$. Similarly, the probability of sample paths where dealer 1 shares with dealer 2 and the system reaches state $(-1, n)$, with the $S_1 + S_2 - K_1 + 1^{th}$ arrival being a dealer 1 demand, is $\beta_1 \sum_{D=S_1+S_2+1}^{\infty} p(D; \lambda\tau) \sum_{x=0}^{S_1-K_1-1} b(\beta_1; S_1 + S_2 - K_1; x) b(\beta_1; D - (S_1 + S_2 - K_1) - 1; K_1)$. These probabilities will be used as the weight for the corresponding sample path for the calculation of the $\Delta_{S_1,S_2} C_1$ shown at the end of the proof.

Case 3:

We now consider sample paths where dealer 2 shares with dealer 1 during the lead time. These sample paths will end in regions A, B or D of Figure 8.

Consider the subset of these sample paths ending in region D, in which dealer 1 stocks out and dealer 2 is above his rationing level, i.e., $m = 0$ and $K_2 < n \leq S_2$. Using reasoning similar to that used in case 2, it is easy to show that the cross partial differences of dealer 1's cost with respect to S_1 and S_2 are zero on these sample paths.

Now consider the subset of these sample paths ending in region A or B, regions in which dealer 1 stocks out and dealer 2 is at or below his rationing level, i.e., $m \leq 0$ and $n \leq K_2$ (and hence the system passes through the state $(0, K_2)$ before reaching (m, n)). Using an analysis similar to that used in case 2, we can write the cross partial difference of dealer 1's cost with respect to S_1 and S_2 as

$$\begin{aligned} & \beta_1\{c_1(m+1,n+1) - c_1(m+1,n) - c_1(m,n+1) + c_1(m,n)\} \\ + \quad & (1-\beta_1)\{c_1(m,n+2) - c_1(m,n+1) - c_1(m,n+1) + c_1(m,n)\} \end{aligned}$$

It is easy to confirm that the above expression is zero for all values of (m, n), except when $n = 0, -1$ (recall $m \leq 0$ for these regions). When $n = 0$ and $m \leq 0$, the cross partial difference becomes

$$\begin{aligned} & (1-\beta_1)\{c_1(m,2) - c_1(m,1) - c_1(m,1) + c_1(m,0)\} \\ = \quad & (1-\beta_1)\{(-m\hat{\pi} + \pi\lambda_1) - (-m\hat{\pi} + \pi\lambda_1) - (-m\hat{\pi} + \pi\lambda_1) + (h_2^- \lambda_2 - m\hat{\pi} + \pi\lambda_1)\} \\ = \quad & (1-\beta_1)h_2^- \lambda_2 \end{aligned}$$

Similarly, when $n = -1$ and $m \leq 0$, the cross partial difference becomes

$$\begin{aligned} & (1-\beta_1)\{c_1(m,1) - c_1(m,0) - c_1(m,0) + c_1(m,-1)\} \\ = \quad & \beta_1\{(-m\hat{\pi} + \pi\lambda_1) - (h_2^- \lambda_2 - m\hat{\pi} + \pi\lambda_1) - (h_2^- \lambda_2 - m\hat{\pi} + \pi\lambda_1) + (h_2^- \lambda_2 - m\hat{\pi} + \pi\lambda_1)\} \\ = \quad & -(1-\beta_1)h_2^- \lambda_2 \end{aligned}$$

Also note that the probability of sample paths where dealer 2 shares with dealer 1 and the system reaches state $(m, 0)$, with the $S_1 + S_2 - K_2 + 1$th arrival being a dealer 2 demand, is $(1-\beta_1) \sum_{D=S_1+S_2}^{\infty} p(D; \lambda\tau) \cdot \sum_{x=S_1+1}^{S_1+S_2-K_2} b(\beta_1; S_1 + S_2 - K_2; x) b(\beta_2; D - (S_1 + S_2 - K_2) - 1; K_2 - 1)$. Similarly, the probability of sample paths where dealer 2 shares with dealer 1 and the system reaches state $(m, -1)$, with the $S_1 + S_2 - K_2 + 1$th arrival being a dealer 2 demand, is $(1 - \beta_1) \sum_{D=S_1+S_2+1}^{\infty} p(D; \lambda\tau) \sum_{x=S_1+1}^{S_1+S_2-K_2} b(\beta_1; S_1 + S_2 - K_2; x) b(\beta_2; D - (S_1 + S_2 - K_2) - 1; K_2)$.

Taking the expressions for $\Delta_{S_1,S_2} C_1$ obtained in cases 1, 2, and 3, multiplying by the probability of the associated sample path, and adding up, we have

$$\Delta_{S_1,S_2} C_1 = \beta_1 \pi \lambda_1 \sum_{D=S_1+S_2}^{\infty} p(D; \lambda\tau) \cdot \sum_{x=0}^{S_1-K_1-1} b(\beta_1; S_1 + S_2 - K_1; x) b(\beta_1; D - (S_1 + S_2 - K_1) - 1; K_1 - 1)$$

$$+ \beta_1\{\hat{\pi} + \hat{h} - \pi\lambda_1\} \sum_{D=S_1+S_2+1}^{\infty} p(D;\lambda\tau) \sum_{x=0}^{S_1-K_1-1} b(\beta_1; S_1+S_2-K_1; x) b(\beta_1; D-(S_1+S_2-K_1)-1; K_1)$$

$$+ (1-\beta_1)h_2^- \lambda_2 \sum_{D=S_1+S_2}^{\infty} p(D;\lambda\tau) \cdot \sum_{x=S_1+1}^{S_1+S_2-K_2} b(\beta_1; S_1+S_2-K_2; x) b(\beta_2; D-(S_1+S_2-K_2)-1; K_2-1)$$

$$- (1-\beta_1)h_2^- \lambda_2 \sum_{D=S_1+S_2+1}^{\infty} p(D;\lambda\tau) \sum_{x=S_1+1}^{S_1+S_2-K_2} b(\beta_1; S_1+S_2-K_2; x) b(\beta_2; D-(S_1+S_2-K_2)-1; K_2)$$

Thus, if $\pi\lambda_1 \leq \hat{\pi} + \hat{h}$ and $h_2^- = 0$, then $\Delta_{S_1,S_2} C_1 \geq 0$. ∎

H Proof of Theorem 4

Theorem 4 follows directly from Lemma 2 using the outline of Theorem 2. ∎

I Proof of Lemma 3

We prove this lemma using a sample path approach, i.e. we show that $\Delta_{Q_i,Q_j} C_i \triangleq C_i(S_i, Q_i+1, S_j, Q_j+1) - C_i(S_i, Q_i, S_j, Q_j+1) - C_i(S_i, Q_i+1, S_j, Q_j) + C_i(S_i, Q_i, S_j, Q_j)$ is zero on all possible lead-time demand sample paths. Starting from the state (S_1, S_2), let (m, n) be the state reached by the system at the end of a lead time. The sample paths can be divided into three cases: (1) sample paths on which neither dealer stocks out, (2) sample paths on which dealer j stocks out before dealer i, and (3) sample paths on which dealer i stocks out before dealer j. We evaluate each of these three cases below.

1. During the lead time, neither dealer i nor dealer j stocks out. In this case, there is no sharing between the dealers. Since changing Q_i (or Q_j) has an impact on the dealers' costs only if this change leads to a change in the amount of sharing between the dealers, equation (11) automatically holds.

2. During the lead time, dealer j stocks out before dealer i. Define $\Delta_{Q_i} C_i \triangleq C_i(S_i, Q_i+1, S_j, Q_j) - C_i(S_i, Q_i, S_j, Q_j)$. Equation (11) states that $\Delta_{Q_i} C_i$ is not affected by a change in dealer j's sharing level, Q_j (or rationing level, K_j). Recall from Section 4.3 that, in order for sharing to happen during a lead time, either dealer i stocks out and dealer j shares with dealer i, or dealer j stocks out and dealer i shares with dealer j, but not both. Therefore, in this case, it is not possible to have sharing from dealer j to dealer i (since dealer j stocks out before dealer i). Since a change in Q_j has an impact on the dealers' costs only if this change leads to a change in the amount of sharing from dealer j to dealer i, a change in Q_j (or K_j) will not affect $\Delta_{Q_i} C_i$, i.e., will not affect the change in dealer i's cost resulting from an increase in Q_i, in this case. Thus, equation (11) holds for this case.

3. During the lead time, dealer i stocks out before dealer j. Define $\Delta_{Q_j} C_i \stackrel{\triangle}{=} C_i(S_i, Q_i, S_j, Q_j + 1) - C_i(S_i, Q_i, S_j, Q_j)$. Equation (11) indicates that $\Delta_{Q_j} C_i$ is not affected by a change in dealer i's sharing level, Q_i (or K_i). The reasoning is the same as in the previous case.

Since, on all sample paths, $\Delta_{Q_i, Q_j} C_i = 0$, the result holds. ∎

J Proof of Theorem 5

Theorem 5 follows directly from Lemma 3.

K Proof of Theorem 6

Using a sample path similar to that used in Lemma 2, we obtain the expression of $\Delta_{S_i, Q_j} C_i$ as follows:

$$\begin{aligned}\Delta_{S_i,Q_i} C_i &\stackrel{\triangle}{=} C_i(S_i+1, Q_i+1, S_j, Q_j) - C_i(S_i, Q_i+1, S_j, Q_j) - C_i(S_i+1, Q_i, S_j, Q_j) + C_i(S_i, Q_i, S_j, Q_j) \\ &= -(\hat{h} + \hat{\pi} - \pi\lambda_i) \sum_{D=S_i+S_j+1}^{\infty} p(D; \lambda\tau) \sum_{x=0}^{Q_i} b(\beta_i; S_j + Q_i; x)(1-\beta_i) b(\beta_i; D - (S_j + Q_i) - 1; S_i - Q_i)\end{aligned}$$

Theorem 6 immediately follows from the above expression. ∎

L Proof of Theorem 7

Using a sample path similar to that used in Lemma 2, we obtain the expression of $\Delta_{S_i, Q_j} C_i$ as follows:

$$\begin{aligned}\Delta_{S_i,Q_j} C_i &\stackrel{\triangle}{=} C_i(S_i+1, Q_i, S_j, Q_j+1) - C_i(S_i, Q_i, S_j, Q_j+1) - C_i(S_i+1, Q_i, S_j, Q_j) + C_i(S_i, Q_j, S_j, Q_i) \\ &= (\pi\lambda_i - h_1^- \lambda_i) p(S_i + Q_j; \lambda\tau) \sum_{x=S_i}^{S_i+Q_j} b(\beta_i; S_i + Q_j; x) \\ &+ (h_1^- \lambda_i - \pi\lambda_i + \beta_i \hat{\pi}) p(S_i + Q_j + 1; \lambda\tau) \sum_{x=S_i+1}^{S_i+Q_j} b(\beta_i; S_i + Q_j; x) \\ &+ \beta_i (h_1^- \lambda_i - \pi\lambda_i + \hat{\pi}) p(S_i + Q_j + 1; \lambda\tau) b(\beta_i; S_i + Q_j; S_i) \\ &- h_2^- \lambda_j \beta_i (1-\beta_i) \sum_{D=S_i+S_j}^{\infty} p(D; \lambda\tau) \sum_{x=S_i+1}^{S_i+Q_j} b(\beta_i; S_i + Q_j; x) b(\beta_j; D - (S_i + Q_j) - 2; S_j - Q_j - 2) \\ &+ h_2^- \lambda_j \beta_i (1-\beta_i) \sum_{D=S_i+S_j+1}^{\infty} p(D; \lambda\tau) \sum_{x=S_i+1}^{S_i+Q_j} b(\beta_i; S_i + Q_j; x) b(\beta_j; D - (S_i + Q_j) - 2; S_j - Q_j - 1) \\ &+ \hat{\pi} \beta_i (1-\beta_i) \sum_{D=S_i+Q_j+2}^{\infty} p(D; \lambda\tau) b(\beta_i; S_i + Q_j; S_i) \sum_{x=0}^{S_j-Q_j-3} b(\beta_j; D - (S_i + Q_j) - 2; x)\end{aligned}$$

$$+ (\hat{\pi} - h_2^- \lambda_j)\beta_i(1-\beta_i) \sum_{D=S_i+S_j}^{\infty} p(D;\lambda\tau) b(\beta_i; S_i+Q_j; S_i) b(\beta_j; D-(S_i+Q_j)-2; S_j-Q_j-2)$$

$$+ \hat{\pi}\beta_i(1-\beta_i) \sum_{D=S_i+S_j+1}^{\infty} p(D;\lambda\tau) b(\beta_i; S_i+Q_j; S_i) \sum_{x=K_j-1}^{D-(S_i+Q_j)-2} b(\beta_j; D-(S_i+Q_j)-2; x). \quad (26)$$

Theorem 7 immediately follows from the above expression. ∎