Option Pricing with Downward Sloping Demand Curves: The Case of Supply Chain Options*

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Appendix

Lemma A.1 *If the manufacturer does not provide options, the optimal quantity of inventory the retailer orders at time 0 is given by:*

$$I^* = \begin{cases} \frac{a_H e_H - S_0}{2\delta e_H} & \textit{if} \quad 0 < S_0 \le (a_H - a_L)e_H \\ \frac{A_0 - S_0}{2\delta B_0} & \textit{if} \quad (a_H - a_L)e_H \le S_0 \le A_0 \\ 0 & \textit{if} \quad S_0 > A_0. \end{cases}$$

Proof

In period 1 the retailer solves the problem:

$$R_1(q^*|I, a) = Max_{0 \le q \le I} q(a - \delta q)$$

The optimal solution is:

$$q^* = \begin{cases} \frac{a}{2\delta} & \text{if} \quad \frac{a}{2\delta} < I \\ I & \text{if} \quad \frac{a}{2\delta} \ge I \end{cases}$$

Now consider the retailer's problem in period 0. We have:

$$R_0(I) = -IS_0 + e_L R_1(q_L^*|I) + e_H R_1(q_H^*|I).$$

There are two cases that need to be considered.

1. $0 \le I \le \frac{a_L}{2\delta}$
2. $\frac{a_L}{2\delta} \le I \le \frac{a_H}{2\delta}$

For case 1 we have

$$R_0(I_1^*) = Max_{0 \le I \le \frac{a_L}{2\delta}} \{-IS_0 + e_L I(a_L - \delta I) + e_H I(a_H - \delta I)\}.$$

The optimal solution is

$$I_1^* = \begin{cases} \frac{A_0 - S_0}{2\delta B_0} & \text{if} \quad (a_H - a_L)e_H \le S_0 \le A_0 \\ \frac{a_L}{2\delta} & \text{if} \quad S_0 < (a_H - a_L)e_H \end{cases}$$

For the second case, we have:

$$R_0(I_2^*) = Max_{\frac{a_L}{2\delta} \le I \le \frac{a_H}{2\delta}} \{-IS_0 + e_L \frac{a_L}{2\delta}(a_L - \delta\frac{a_L}{2\delta} + e_H I(a_H - \delta I)\}.$$

The optimal solution is

$$I_2^* = \begin{cases} \frac{a_H e_H - S_0}{2\delta e_H} & \text{if} \quad 0 < S_0 < (a_H - a_L)e_H \\ \frac{a_L}{2\delta} & \text{if} \quad S_0 \geq (a_H - a_L)e_H \end{cases}$$

The result then follows.

Proof of Proposition 1

Given the results of Lemma A.1, the manufacturer's profit as a function of S_0 is

$$M_0(S_0) = \begin{cases} M_1(S_0) & \text{if } \ 0 < S_0 \leq (a_H - a_L)e_H \\ M_2(S_0) & \text{if } \ (a_H - a_L)e_H < S_0 \leq A_0 \\ 0 & \text{if } \ S_0 > A_0 \end{cases},$$

where

$$M_1(S_0) = \frac{e_H a_H - S_0}{2\delta e_H}(S_0 - K_0)$$

and

$$M_2(S_0) = \frac{A_0 - S_0}{2\delta B_0}(S_0 - K_0).$$

Therefore, the maximum value of $M_0(S_0)$ is given by

$$M_0^* = M_0(S_0^*) = \max\{M_1^*, M_2^*\},$$

where

$$M_1^* = M_1(S_1^*) = Max_{0 \leq S_0 \leq (a_H - a_L)e_H} M_1(S_0) \tag{A.1}$$

and

$$M_2^* = M_2(S_2^*) = Max_{(a_H - a_L)e_H \leq S_0 \leq A_0} M_2(S_0). \tag{A.2}$$

The maximizing S_0 value for the problem in (A.1) is

$$S_1^* = \begin{cases} \frac{e_H a_H + K_0}{2} & \text{if} \quad K_0 \leq \overline{K} \\ (a_H - a_L)e_H & \text{if} \quad K_0 > \overline{K} \end{cases},$$

where the value of $\overline{K}$ is determined by solving the inequality $\frac{e_H a_H + K_0}{2} \leq (a_H - a_L)e_H$ in K_0, i.e.,

$$\overline{K} = a_H e_H - 2a_L e_H = (a_H - a_L)e_H - a_L e_H.$$

In this case the manufacturer's profits are:

$$M_1^* = \begin{cases} \frac{(a_H e_H - K_0)^2}{8\delta e_H} & \text{if} \quad K_0 \leq \overline{K} \\ \frac{[(a_H - a_L)e_H - K_0]a_L}{2\delta} & \text{if} \quad K_0 > \overline{K}. \end{cases}$$

Similarly, the maximizing S_0 value for the problem in (A.2) is

$$S_2^* = \begin{cases} \frac{A_0+K_0}{2} & \text{if} \quad K_0 \geq \underline{K} \\ (a_H - a_L)e_H & \text{if} \quad K_0 < \underline{K}, \end{cases},$$

where the value of $\underline{K}$ is determined by solving the inequality $\frac{A_0+K_0}{2} \geq (a_H - a_L)e_H$ in K_0, i.e.,

$$\underline{K} = a_H e_H - 2a_L e_H - a_L e_L = (a_H - a_L)e_H - a_L e_H(1+\rho),$$

where $\rho = e_L/e_H$.

In this case the manufacturer's profits are:

$$M_2^* = \begin{cases} \frac{(A_0-K_0)^2}{8\delta B_0} & \text{if} \quad K_0 \geq \underline{K} \\ \frac{[(a_H-a_L)e_H-K_0]a_L}{2\delta} & \text{if} \quad K_0 < \underline{K} \end{cases}$$

Since $\underline{K} < \overline{K}$, the solution can be summarized as follows

$$S_0^* = \begin{cases} \frac{a_H e_H + K_0}{2} & \text{if} \quad K_0 \leq \underline{K} \\ \frac{A_0+K_0}{2} & \text{if} \quad K_0 \geq \overline{K} \\ \frac{e_H a_H + K_0}{2} & \text{if} \quad \underline{K} < K_0 < \overline{K} \text{ and } M_1^* > M_2^* \\ \frac{A_0+K_0}{2} & \text{if} \quad \underline{K} < K_0 < \overline{K} \text{ and } M_1^* \leq M_2^* \end{cases} \tag{A.3}$$

The solution can be simplified. Substituting the values of M_1^* and M_2^* for the case where $\underline{K} < K_0 < \overline{K}$, the relationship $M_1^* - M_2^* \leq 0$ can be written as the following quadratic inequality in K_0

$$K_0^2 + 2(a_H - a_L)e_H K_0 - 2a_H a_L e_H^2 - a_L^2 e_H e_L + a_H^2 e_H^2 \leq 0.$$

The two roots of this quadratic function in K_0 are

$$\begin{aligned} k_1 &= (a_H - a_L)e_H - a_L e_H\sqrt{1+\rho} \\ k_2 &= (a_H - a_L)e_H + a_L e_H\sqrt{1+\rho} \end{aligned}$$

and the inequality is valid for $k_1 \leq K_0 \leq k_2$. In addition, in this case it must be true that $\underline{K} < K_0 < \overline{K}$. It is easy to see from the definitions of $\underline{K}$ and $\overline{K}$ that $k_2 > \overline{K}$ and $\underline{K} < k_1 < \overline{K}$. Summarizing the above relationships, it follows that, in the case where $\underline{K} < K_0 < \overline{K}$, $M_1^* - M_2^* \leq 0$ is true if and only if $k_1 < K_0 < \overline{K}$. Based on this, equation (A.3) can simplified as follows.

$$S_0^* = \begin{cases} \frac{a_H e_H - K_0}{2} & \text{if} \quad K_0 \leq k_1 \\ \frac{A_0+K_0}{2} & \text{if} \quad K_0 > k_1 \end{cases} \tag{A.4}$$

Using the transformations (11) - (14), we find

$$\frac{a_H e_H - K_0}{2} = S_{det} - \frac{e_L}{2}\left(\frac{A_0}{B_0} - \frac{\sigma_A}{\sqrt{\rho}}\right).$$

Further, the condition $K_0 \le k_1$ can be reexpressed as an equivalent condition involving the volatility of the demand curve. In particular, substituting $a_H = \frac{A_0}{B_0} + \sqrt{\rho}\sigma_A$ and $a_L = \frac{A_0}{B_0} - \frac{\sigma_A}{\sqrt{\rho}}$ into the expression for k_1, we find that $K_0 \le k_1$ if and only if

$$\sigma_A \ge \eta \equiv \frac{\sqrt{\rho}}{1+\rho+\sqrt{1+\rho}}\left[(1+\rho)\frac{K_0}{B_0} + \sqrt{1+\rho}\frac{A_0}{B_0}\right].$$

Therefore equation (15) for the optimal wholesale price is established.

To complete the proof, it remains to determine the retailer's policy in each of the two cases.

In the case when $K_0 \le k_1$, we have that $S_0^* = \frac{a_H e_H + K_0}{2} < (a_H - a_L)e_H$, and the results of the above lemma imply that

$$I^* = \frac{a_H e_H - S_0^*}{2\delta e_H} = \frac{a_H e_H - K_0}{4\delta e_H}.$$

Furthermore, it is easy to see that $I^* < \frac{a_H}{2\delta}$, therefore, $q_H^* = I^*$. On the other hand,

$$I^* - \frac{a_L}{2\delta} = \frac{a_H e_H - 2a_L e_H - K_0}{4\delta e_H} = \frac{\overline{K} - K_0}{4\delta e_H} > 0,$$

since $K_0 \le k_1 < \overline{K}$. Thus, $I^* > \frac{a_L}{2\delta}$ and $q_L^* = \frac{a_L}{2\delta}$.

Finally, in the case when $K_0 > k_1$, it follows that

$$(a_H - a_L)e_H < S_0^* = \frac{A_0 + K_0}{2} < A_0,$$

and the results of the above lemma imply that

$$I^* = \frac{A_0 - S_0^*}{2\delta B_0} = \frac{A_0 - K_0}{4\delta B_0}.$$

Furthermore,

$$I^* - \frac{a_L}{2\delta} = \frac{A_0 - 2a_L B_0 - K_0}{4\delta B_0} == \frac{a_H e_H - 2a_L e_H - a_L e_L}{4\delta B_0} = \frac{\underline{K} - K_0}{4\delta B_0} < 0,$$

since $K_0 > k_1 > \underline{K}$. Hence $I^* < \frac{a_L}{2\delta} < \frac{a_H}{2\delta}$, and $q_L^* = q_H^* = I^*$. Equations (17), (18), (19) follow from the substitutions (11) to (14). This completes the proof.

Proof of Lemma 1

Let

$$\begin{aligned} R_1^{(1)}(q) &= R_1(q,0) = q(q - \delta q) \\ R_1^{(2)}(v) &= R_1(I,v) = (I+v)(a - \delta(I+v)) - Xv. \end{aligned}$$

Then, since no options will be exercised when inventory is available, we have:

$$\max_{0\le q\le I, 0\le v\le U} R_1(q,v) = \max\{\max_{0\le q\le I} R_1^{(1)}(q), \max_{0\le v\le U} R_1^{(2)}(v)\}.$$

We also have

$$\frac{dR_1^{(1)}}{dq}(I) = a - 2\delta I$$
$$\frac{dR_1^{(2)}}{dv}(0) = a - 2\delta I - X$$

First, assume $\frac{dR_1^{(1)}}{dq}(I) < 0$. This implies that $R_1^{(1)}(q)$ is maximized for $q = q^* = \frac{a}{2\delta} < I$. Further, for this case, $\frac{dR_1^{(2)}}{dv}(0) < 0$. This and the concavity of $R_1^{(2)}(v)$ imply that $\frac{dR_1^{(2)}}{dv}(v) < 0$ for all $v > 0$. Hence $Max_{0 \le v \le U}[R_1^{(2)}(v)] = R_1^{(2)}(0)$. This implies that:

$$R_1^{(2)}(v) \le R_1^{(2)}(0) = R_1(I,0) = R^{(1)}(I) \le R_1^{(1)}(q^*)$$

which means that $q = q^*, v = 0$ is optimal in this case.

Second, consider the case where $I < \frac{a}{2\delta}$. In this case, $\frac{dR_1^{(1)}}{dq}(I) > 0$. Hence:

$$R_1^{(1)}(q) \le R_1^{(1)}(I) = R_1^{(2)}(0) \le R_1^{(2)}(v^*)$$

where v^* is the value maximizing $R_1^{(2)}(v)$. Specifically, using the first order conditions for $R_1^{(2)}(v)$ we obtain:

$$v^* = \begin{cases} 0 & \text{if} \quad I + U > \frac{a-X}{2\delta} \\ \frac{a-X}{2\delta} - I & \text{if} \quad \frac{a-X}{2\delta} > I \end{cases}$$

Proof of Lemma 2

First, consider the regions

1. $I < \frac{a_L - X}{2\delta}$ and $U > \frac{a_H - X}{2\delta} - I$.
2. $\frac{a_L - X}{2\delta} < I < \frac{a_H - X}{2\delta}$ and $U > \frac{a_H - X}{2\delta} - I$.
3. $I > \frac{a_H - X}{2\delta}$ and $U > 0$.

In each of these regions, regardless of which state occurs in the future, the maximum number of options that can be exercised, U, is never attained. Hence, if $C_0 > 0$, then clearly U can be reduced and the retailer can obtain savings. Hence the optimal solution for the retailer will never lie in these regions.

Second, we show that if $U > 0$ and $I < \frac{(a_L - X)}{2\delta}$, then (I, U) cannot be optimal.

If $X > a_L$, then $a_L - X < 0$ and $I > (a_L - X)/2\delta$. Now consider $X \le a_L$. Take (I, U) such that $U > 0$ and $I \le \frac{a_L - X}{2\delta}$. Then $v_L^*, v_H^* > 0$, for all $U > 0$. Now

$$R_0(I,U) = -S_0 I - C_0 U + e_L(I + v_L^*)(a_L - \delta(I + v_L^*) - X e_L v_L^* + e_H(I + v_H^*)(a_H - \delta(I + v_H^*) - X e_H v_H^*$$

Let $I' = I + \epsilon$, $U' = U - \epsilon$, where $0 < \epsilon < Min[v_L^*, v_H^*]$. Then $v_L' = v_L^* - \epsilon < U - \epsilon = U'$ and $v_H' = v_H^* - \epsilon < U - \epsilon = U'$ are feasible, but perhaps not optimal, exercise policies. Also, $I' + v_L' = I + v_L^*$, and $I' + v_H' = I + v_H^*$. Then:

$$R_0(I', U') \geq -S_0 I' - C_0 U' + e_L(I + v_L^*)(a_L - \delta(I + v_L^*)) - Xe_L(v_L^* - \epsilon) + e_H(I + v_H^*)(a_H - \delta(I + v_H^*)) - Xe_H(v_H^* - \epsilon)$$

Hence, $R_0(I', U') - R_0(I, U) \geq (C_0 - S_0 + XB_0)\epsilon > 0$. Therefore, (I, U) is not optimal.

The only region that remains when $U > 0$ is the region where $I > \frac{a_L - X}{2\delta}$ and $U \leq \frac{a_H - X}{2\delta} - I$. This completes the proof.

Proof of Proposition 5

(i) First, consider the case where $\sigma_A^2 < \eta^2$ and $\widetilde{K}_L \leq K_1 \leq \widetilde{K}_H$. Then, computing the difference between the manufacturer's profit with and without options, leads, upon simplification to:

$$M^* - M_0 = \frac{e_H}{8\delta e_L B_0}[a_L e_L - a_H e_L - K_0 + e_H K_1 + K_1 e_L]^2 > 0$$

which shows that, for this case, the manufacturer is better off with options.

(ii) Now consider the case where $\sigma_A^2 \geq \eta^2$ and $\widetilde{K}_L \leq K_1 \leq \widetilde{K}_H$. Then, computing the difference between the manufacturer's profit with and without options, leads, upon simplification to:

$$M^* - M_0 = \frac{N(K_1)}{8\delta e_H e_L}$$

where

$$N(K_1) = a_0 K_1^2 + b_0 K_1 + c_0$$

and

$$\begin{aligned} a_0 &= e_H^2 B_0 \\ b_0 &= -2e_H^2[K_0 + (a_H - a_L)e_L] \\ c_0 &= (e_H - e_L)K_0^2 + e_H e_L^2 a_L^2 + 2K_0 e_H e_L (a_H - a_L) \end{aligned}$$

Recall that the volatility range in (32) is equivalent to the range of costs in (31). Furthermore, since in the case we examine $\sigma_A > \eta$ and from (16) $\eta \geq \sqrt{\rho}\frac{K_0}{B_0}$, it follows that $\sigma > \sqrt{\rho}\frac{K_0}{B_0}$ and the upper bound in (31) becomes $\bar{k}_1(\sigma_A) = \frac{K_0}{e_H}$.

In addition,

$$N(\bar{k}_1) = e_H e_L^2 a_L^2$$

and

$$N'(\bar{k}_1) = 2e_H e_L \left[K0 - (a_H - a_L)e_H\right],$$

which, using the substitutions (11)-(14), becomes

$$N'(\bar{k}_1) = 2\frac{B_0^2\sqrt{\rho}\left(\sqrt{\rho}K_0 - \sigma B_0\right)}{(1+\rho)^2}$$

As we have seen above, $\sigma > \sqrt{\rho}\frac{K_0}{B_0}$, thus $N'(\bar{k}_1) < 0$. Therefore, $N(K_1)$ is a convex quadratic function which is positive and decreasing at the upper bound of the range of K_1. It is thus positive in the entire range $\underline{k}_1(\sigma_A) \leq K_1 \leq \overline{k}_1(\sigma_A)$, for any $\sigma_A > \eta$. This completes the proof of the proposition.