Agent Competition Double Auction Mechanism

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ON LINE APPENDIX

Online Appendix[9]

B Analysis of Incentives

We show in this Appendix that the seller competition mechanism is (*ex post*) individually-rational and strategy-proof under Properties 2 and 3. At the end of this section, we conclude that as long as we formulate the social welfare according to a quasi-linear formulation, for example, using formulation $\mathcal{P}$ for the simple exchange environment, each agent will only face a "take-it-or-leave-it" situation, and the mechanism is both (*ex post*) individually-rational and strategy-proof. Therefore, the quasi-linearity assumption on the agents' preferences is not required.[10]

Let us first discuss the individually-rational issue. We use $\tilde{J}$ to denote the set of sellers after the elimination phase and let $\hat{p}_-(k) = p_-(k)(I, \tilde{J})$.

Theorem B.1 *The seller competition mechanism is (ex post) individually-rational.*

Proof: Note that if seller k trades, his bid g_k must be no more than $p_+(k)$ because otherwise he would be eliminated. Since his transaction price is $p_+(k)$, which is no less than g_k, the price he is willing to sell, his utility from participation is non-negative. Because all the non-trading sellers have utility zero, the mechanism is (ex post) individually-rational for all the sellers.

If buyer l trades, she is involved in the efficient allocation of agent set $I \cup \tilde{J}$. That is, her bid f_l must be no less than $\hat{p}_-(l)$. Since her transaction price is $\hat{p}_-(l)$, which is no more than f_l, the price she is willing to pay, her utility from participation is non-negative. Because all the non-trading buyers have utility zero, the mechanism is (ex post) individually-rational for all the buyers.

Therefore, the mechanism is (ex post) individually-rational. ■

To show that the mechanism is strategy-proof, we first show that truth-revelation is a dominant strategy for all the sellers. This is accomplished by the following lemmas.

Lemma B.2 $p_+(k) \leq p_-(k)$ *for seller* $k \in J$.

Proof: Consider an exchange environment with one more seller who is identical to k. By Property 2, we have $V_k + V_{-k} \leq V + V$. Thus, $V_k - V \leq V - V_{-k}$. Since $p_+(k)$ and $p_-(k)$ are the suprema of the

[9] We only prove the results for the seller competition mechanism in the appendix, and it is understood that all theorems about the seller competition mechanism also hold, *mutatis mutandis*, for the buyer competition mechanism.

[10] In fact, we can derive other individually-rational and strategy-proof mechanisms by using non-quasi-linear formulations. The proofs will hold as long as the objective function of the new social welfare formulation is an *additive* measure over functions, each of which is a function of an agent's bid.

bid prices such that $V_k - V > 0$ and $V - V_{-k} > 0$, respectively, we have $p_+(k) \leq p_-(k)$. ■

The following lemma implies that if seller $k \in J' \subset J$ bids g_k, which is lower than $p_-(k)(I, J)$ (he is involved in every efficient allocation for agent set $I \cup J$), seller k is involved in every efficient allocation for agent set $I \cup J'$.

Lemma B.3 *For $k \in J' \subset J$, $V - V_{-k} \leq V_{-J \setminus J'} - V_{-(J \setminus J') \cup \{k\}}$.*

Proof: It suffices to prove the result for the case $|J'| = |J| - 1$. Suppose l is the unique element in $J \setminus J'$. From Property 2, we have $V + V_{-\{k,l\}} \leq V_{-k} + V_{-l}$, i.e., $V - V_{-k} \leq V_{-l} - V_{-\{k,l\}}$. Note if $g_k < p_-(k)$, we have $V > V_{-k}$. Thus, $V_{-l} > V_{-\{k,l\}}$, which means that seller k is involved in every efficient allocation for agent set $I \cup J'$. ■

Lemma B.4 *If seller k bids lower than $p_+(k)$, he trades his bundle.*

Proof: From Lemma B.2, we know if seller k bids lower than $p_+(k)$, his bid is also lower than $p_-(k)$. Note that after the elimination phase, the agent set becomes $I \bigcup \tilde{J}$ where $\tilde{J} \subseteq J$ and $k \in \tilde{J}$. By Lemma B.3, seller k wins in every efficient allocation for agent set $I \bigcup \tilde{J}$. ■

Theorem B.5 *Each seller has a (weakly) dominant strategy to bid truthfully.*

Proof: Note that $p_+(k)$ is determined by agent set $I \cup J \setminus \{k\}$; thus, it is independent of the bid price g_k. Also, if seller k trades, his revenue is $p_+(k)$. Thus, if seller k's valuation is higher than $p_+(k)$, he prefers not to trade, which can be achieved by bidding his valuation. If seller k's valuation is lower than $p_+(k)$, he prefers to trade at price $p_+(k)$, which can also be achieved by bidding his valuation. If seller k's valuation is equal to $p_+(k)$, he is indifferent between trading and not trading. Thus, it is a (weakly) dominant strategy for all the sellers to bid truthfully. ■

Now we show that truth-revelation is a dominant strategy for all the buyers. The result builds on the following lemmas. The next lemma implies that if buyer k bids an amount f_k, which is higher than $p_-(k)(I, J')$ for $J' \subset J$, buyer k is involved in every efficient allocation for agent set $I \cup J$.

Lemma B.6 *Given a subset $J' \subset J$ and a buyer $k \in I$, $V - V_{-k} \geq V_{-J \setminus J'} - V_{-(J \setminus J') \cup \{k\}}$.*

Proof: It suffices to prove the result for the case $|J'| = |J| - 1$. Suppose l is the unique element in $J \setminus J'$. From Property 3, we have $V + V_{-\{k,l\}} \geq V_{-k} + V_{-l}$, or $V - V_{-k} \geq V_{-l} - V_{-\{k,l\}}$. Note that if buyer k bids an amount f_k, which is higher than $p_-(k)(I, J')$, $V_{-l} > V_{-\{k,l\}}$. Thus, $V > V_{-l}$ and, buyer k is involved in every efficient allocation for agent set $I \cup J$. ■

Lemma B.7 *If buyer $k \in I$ bids an amount f_k, which is lower than $p_-(k)$, then she does not trade in any efficient allocation of agent set $I \cup J'$, where $J' \subset J$.*

Proof: It suffices to prove the result for the case $|J'| = |J| - 1$. Suppose l is the unique element in $J \backslash J'$. We have $f_k < p_-(k)$ and $V = V_{-k}$. From Property 3, we know $V_{-\{k,l\}} = V_{-l}$ as we always have $V_{-\{k,l\}} \leq V_{-l}$. Now suppose buyer k bids $p_-(k) - \epsilon > f_k$ instead of f_k for some small $\epsilon > 0$. Then we still have $V = V_{-k}$ and $V_{-\{k,l\}} = V_{-l}$ for the new bid. This means that the value of V_{-l} remains the same, since $V_{-\{k,l\}}$ is independent of k's bid. Thus, buyer k does not trade in any efficient allocation for agent set $I \cup J'$, otherwise V_{-l} increases as buyer k increases her bid to $p_-(k) - \epsilon$. ■

Lemma B.8 *$p_+(l)$ for $l \in J$ does not increase if one buyer lowers her bid.*

Proof: Since we define $p_+(l)$ as the supremum of seller l's price to make $V_l > V$, it suffices to show that $V_l - V$ does not increase when some buyer k lowers her bid. Let buyer k lower her bid continuously. Both V_l and V either decrease or remain the same when f_k decreases. If V decreases as f_k decreases, buyer k is involved in every efficient allocation for agent set $I \cup J$. Due to Lemma B.6, if there is one more seller who bids the same price as seller l's price, buyer k still trades; thus, V_l decreases at the same rate, and $V_l - V$ does not increase when some buyer k lowers her bid. ■

Now let us consider the set of remaining sellers in the limit as buyer k's bid approaches infinity. This limiting set is well defined by Lemma B.8. Since there are only finitely many possible remaining seller sets, if buyer k bids high enough, the set of remaining sellers is the limiting set. We use $\tilde{J}_k$ to denote this limiting remaining seller set and use $\tilde{p}_-(k)$ to denote $p_-(k)(I, \tilde{J}_k)$. We have the following lemma:

Lemma B.9 *The buyer k who bids higher than $\tilde{p}_-(k)$ trades the bundle at price $\tilde{p}_-(k)$.*

Proof: We know if buyer k bids higher than $\tilde{p}_-(k)$, she is involved in every efficient allocation for agent set $I \cup \tilde{J}_k$. From Lemma B.6, we know that buyer k is involved in every efficient allocation for $V(I, J)$ and $V_l(I, J)$ for any seller $l \in J$. Thus, $V_l - V$ and $p_+(l)(I, J)$ remain the same, as long as buyer k bids higher than $\tilde{p}_-(k)$. We claim that the set of remaining sellers is $\tilde{J}_k$ because the set of remaining sellers is $\tilde{J}_k$ if buyer k bids high enough. Since buyer k bids higher than $\tilde{p}_-(k)$, she is involved in every efficient allocation for agent set $I \cup \tilde{J}_k$. Therefore, she trades at price $\tilde{p}_-(k)$. ■

Corollary B.10 *The remaining seller set is $\tilde{J}_k$ as long as buyer k bids higher than $\tilde{p}_-(k)$. Furthermore, $p_+(j)(I,J)$ remains the same for all $j \in \tilde{J}_k$.*

Theorem B.11 *Each buyer has a (weakly) dominant strategy to bid truthfully.*

Proof: We have already shown that if buyer k bids higher than $\tilde{p}_-(k)$, she trades at price $\tilde{p}_-(k)$. Suppose she bids lower than $\tilde{p}_-(k)$, then the set of remaining sellers $\tilde{J}$ becomes a subset of $\tilde{J}_k$ by Lemma B.8. Furthermore, she does not trade in any efficient allocation for agent set $I \cup J'$ since she bids lower than $\tilde{p}_-(k)$ by Lemma B.7. Thus, buyer k does not trade if she bids lower than $\tilde{p}_-(k)$. Now if she bids $\tilde{p}_-(k)$, the set of remaining sellers $\tilde{J}$ is a subset of $\tilde{J}_k$, and she may either not trade or trade at price $\tilde{p}_-(k)$. Thus, if her valuation of the item is lower than $\tilde{p}_-(k)$, she prefers not to trade, which can be achieved by bidding her valuation. If her valuation is higher than $\tilde{p}_-(k)$, she prefers to trade at price $\tilde{p}_-(k)$, which can also be achieved by bidding her valuation. If her valuation is equal to $\tilde{p}_-(k)$, she is indifferent between trading and not trading. Thus, it is a (weakly) dominant strategy for all the buyers to bid truthfully. ■

C Budget Analysis

We have shown that the seller competition mechanism is (*ex post*) individually-rational and strategy-proof under Properties 2 and 3. We show in this Appendix that the seller competition mechanism is (weakly) budget-balanced with two more additional conditions, Property 1 and a quasi-linear social welfare function.

Since $\tilde{J} \subseteq J$, $p_+(j)(I,J) \leq \tilde{p}_+(j)$ by Lemma B.3, that is, no seller is eliminated if the auction starts with agent set $I \cup \tilde{J}$. Note that the trading prices for the buyers remain the same, while the trading prices of the sellers increase from $p_+(j)(I,J)$s to $\tilde{p}_+(j)$s, and the auctioneer's revenue does not increase. Thus, it suffices to prove the budget balance result only for the special case $J = \tilde{J}$. Note that in such an environment, the allocation determined by our mechanism is efficient.

The following lemma implies that if buyer $k \in I' \subset I$ bids f_k, which is higher than $p_-(k)(I,J)$ (she is involved in every efficient allocation for agent set $I \cup J$), buyer k is involved in every efficient allocation for agent set $I' \cup J$.

Lemma C.1 *For buyer $k \in I' \subset I$, $V - V_{-k} \leq V_{-I \setminus I'} - V_{-(I \setminus I') \cup \{k\}}$.*

Proof: Similar to the proof of Lemma B.3, except that we apply Property 1 instead of Property 2. ■

Lemma C.2 *Suppose buyer k bids $f_k > p_-(k)(I,J)$, and she lowers her bid to $p_-(k)(I,J)+\epsilon_k$ ($\epsilon_k > 0$), then i) the set of remaining sellers and $p_+(j)(I,J)$ for seller $j \in J$ do not change; ii) for buyer i with $f_i > p_-(i)(I,J)$, $p_-(i)(I,J)$ does not change; iii) the original efficient allocation is still efficient.*

Proof: i) Since $J = \tilde{J}$, this is just a restatement of Corollary B.10.

To prove ii), note that since $f_k > \tilde{p}_-(k)$, buyer k is involved in every efficient allocation for agent set $I \cup J = I \cup \tilde{J}$. By Lemma C.1, she is also involved in every efficient allocation for agent set $I \cup J = I \cup \tilde{J} \backslash \{i\}$ for $i \in I$ and $i \neq k$. As buyer k lowers her bid, both V and V_{-i} change by the same amount, $p_-(i)(I,J)$ of buyer i, who bids higher than original $p_-(i)(I,J)$, remains unchanged.

For iii), since bidding higher than $p_-(k)(I,J)$ guarantees that buyer k is involved in every efficient allocation for agent set $I \cup J$, the decrease of social welfare is determined by the amount of bid decrease. Thus, iii) follows. ■

Lemma C.3 *Suppose seller k bids g_k, which is less than $\tilde{p}_-(k)$, and he raises his bid to $\tilde{p}_-(k) - \epsilon_k$ ($\epsilon_k > 0$). Then i) for trading seller j ($\neq k$), $p_+(j)(I,J)$ does not change; ii) the original efficient allocation is still efficient.*

Proof: i) If seller k changes his bid to $p_+(k)(I,J) - \epsilon_k$, V decreases, and V_j decreases by at most the same amount for any other seller $j \in J$. Thus, $p_+(j)(I,J)$ for seller $j \in J$ does not decrease, and no seller is eliminated after this change.

For ii), since bidding lower than $p_+(k)(I,J)$ guarantees that seller k is involved in every efficient allocation, the social welfare decreases according to the amount changed in the bid. Thus, ii) follows. ■

We denote the trading buyer set as $\hat{I}$ and the trading seller set as $\hat{J}$. We now prove the main theorem for budget balance. Recall that we have $J = \tilde{J}$.

Theorem C.4 *The seller competition mechanism is (weakly) budget-balanced.*

Proof: Suppose we let all trading buyers who bid higher than $p_-(k)(I,J)$ lower their bids to $p_-(k)(I,J)+\epsilon_k$ (for some $\epsilon_k > 0$) one by one. Then by Lemma C.2, the original allocation remains efficient and $p_+(j)(I,J)$ remains the same for seller $j \in J$. After that, by Lemma C.3, we can let all of the trading sellers who bid lower than $p_+(k)(I,J)$ raise their bids to $p_+(k)(I,J) - \epsilon_k$ (for some

$\epsilon_k > 0$) one by one and still keep the original allocation efficient. Since the original allocation is still efficient, it is better than making no transaction. Under quasi-linear social welfare function, we have $\sum_{i \in \hat{I}}(p_-(i)(I,J) + \epsilon_i) - \sum_{j \in \hat{J}}(p_+(j)(I,J) - \epsilon_j) - C(Q) \geq 0$, where $C(Q)$ is the transaction costs to execute the allocation Q. Since the ϵ_ks are arbitrary positive numbers and $J = \tilde{J}$, $\sum_{i \in \hat{I}} \tilde{p}_-(i) \geq \sum_{j \in \hat{J}} p_+(j)(I,J) + C(Q)$, i.e., the total cash inflow is no less than the total cash outflow. Thus, the seller competition mechanism is (weakly) budget-balanced. ■