Dynamic Mixed Dupoly
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On line Supplement: Proofs

Proof of Lemma.

Because in this subsection Microsoft is a monopolist, we set $y_L(t)=0$ for all t. Therefore, $y \equiv y_W - sy_L = y_W$. The following two assumptions on $\alpha_W(y_W)$ (see Section III) will be used in what follows. First, $\alpha_W(y_W)$ is increasing and strictly positive at all $y_W > -\infty$. Second, because Linux is not available, we assume that $\alpha_W(y)$ is concave for all $y \geq 0$. That is, we set y^0 in Assumption 4 (Section III) to be a negative number.

Consider Microsoft's constrained optimization program. The current value Hamiltonian is:[1]

$$H = \left(1 - \frac{p}{\alpha_W(y_W)}\right)p + m\left(1 - \frac{p}{\alpha_W(y_W)} - \delta y_W\right).$$

The first order conditions are:

$$\frac{\partial H}{\partial p} = 0 \Rightarrow p = \frac{1}{2}\left(\alpha_W(y_W) - m\right), \tag{A1}$$

$$mr - \frac{\partial H}{\partial y_W} = \dot{m} \Rightarrow \dot{m} = m(r+\delta) - \frac{p\alpha_W'(y_W)}{\alpha_W(y_W)^2}(p+m), \tag{A2}$$

and

$$\frac{\partial H}{\partial m} = \dot{y}_W \Rightarrow \dot{y}_W = 1 - \frac{p}{\alpha_W(y_W)} - \delta y_W. \tag{A3}$$

Using (A1), we may rewrite (A2) and (A3) as

$$\dot{m} = m(r+\delta) - \frac{\alpha_W'(y_W)\left(\alpha_W(y_W)^2 - m^2\right)}{4\alpha_W(y_W)^2} \tag{A4}$$

and

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[1] For notational simplicity, we omit time dependence of y_W, m (the Hamiltonian multiplier), and p.

$$\dot{y}_W = 1 - \frac{\left(\alpha_W\left(y_W\right) - m\right)}{2\alpha_W\left(y_W\right)} - \delta y_W . \tag{A5}$$

Now, (A4) and (A5) form a system of differential equations in m and y_W. Consider the following two equations

$$0 = m\left(r+\delta\right) - \frac{\alpha_W'\left(y_W\right)\left(\alpha_W\left(y_W\right)^2 - m^2\right)}{4\alpha_W\left(y_W\right)^2} \tag{A6}$$

and

$$0 = 1 - \frac{\left(\alpha_W\left(y_W\right) - m\right)}{2\alpha_W\left(y_W\right)} - \delta y_W . \tag{A7}$$

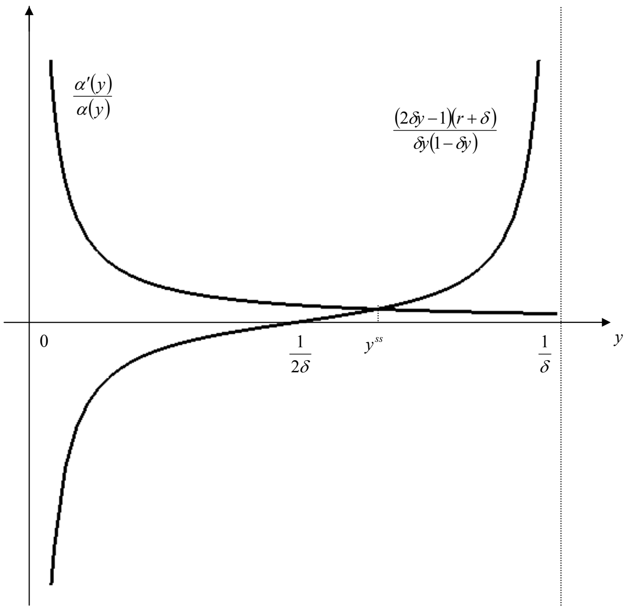
The steady states are all those pairs $\left\langle m, y_W \right\rangle$ that satisfy simultaneously equations (A6) and (A7). Solving (A6) and (A7) for m, equating the resulting expressions and simplifying yields:

$$\frac{\alpha_W'\left(y_W\right)}{\alpha_W\left(y_W\right)} = \frac{\left(2\delta y_W - 1\right)\left(r+\delta\right)}{\delta y_W\left(1-\delta y_W\right)} . \tag{A8}$$

Because $y_W\left(t\right) = \int_0^t q\left(\tau\right) e^{-\delta\left(t-\tau\right)} d\tau \geq 0$ for all t, we consider only $y_W\left(t\right) \geq 0$. In other words, there cannot be a steady state with $y_W\left(t\right) < 0$.

For $0 \leq y_W < \frac{1}{2\delta}$ the lhs of (A8) is positive, but the rhs is negative. Thus, there is no intersection in this range between the lhs and the rhs. However, at $y_W = \frac{1}{2\delta}$ the rhs is zero and from that point on it increases without bound until $y_W = \frac{1}{\delta}$. Because the lhs is positive and decreasing in this range, there is one (and only one) intersection point between $y_W = \frac{1}{2\delta}$ and $y_W = \frac{1}{\delta}$. Finally, for $y_W > \frac{1}{2\delta}$ the rhs of (A8) is negative while the lhs is positive, therefore there is no intersection here.

The above argument shows that the set of points y_W satisfying (A8) is a singleton. Let y_W^{ss} denote that point. As it can be seen in the graph below, $\frac{1}{2\delta} \leq y_W^{ss} \leq \frac{1}{\delta}$.



We turn now to the stability analysis of y_W^{ss}. To this end, we isolate *m* in (A6) and (A7) and investigate the shape of the resulting functions. Solving (A6) for *m*, we have

$$m_{\dot{m}=0}=\frac{-2\alpha_W(y_W)^2(r+\delta)\pm\alpha_W(y_W)\sqrt{4\alpha_W(y_W)^2(r+\delta)^2+(\alpha_W'(y_W))^2}}{\alpha_W'(y_W)}.$$

There are two cases, one for each sign of the square root. Consider first

$$m_{\dot{m}=0}=\frac{-2\alpha_W(y_W)^2(r+\delta)+\alpha_W(y_W)\sqrt{4\alpha_W(y_W)^2(r+\delta)^2+(\alpha_W'(y_W))^2}}{\alpha_W'(y_W)}. \qquad \text{(A9)}$$

Notice that (A9) is always greater than 0. In fact, (A9) is equal to 0 only at y_W such that $\alpha_W(y_W)=0$ but because $\alpha_W(y_W)$ is greater than zero for all $y_W>-\infty$, the expression is always strictly positive for $y_W\geq 0$.

We now show that $\lim_{y_W\to\infty} m_{\dot{m}=0}=0$. Because $\alpha_W(y_W)$ is an increasing and concave function of y_W for $y_W>0$, its first derivative $\alpha_W'(y_W)>0$ decreases with y_W. Thus, as $y_W\to\infty$, $\alpha_W'(y_W)$ approaches either to zero or a positive (finite) value.

We will now show that $m_{n\&=0}$ approaches zero as $y_W \to \infty$. Let's consider first the case where $\alpha'_W(y_W)$ approaches zero as $y_W \to \infty$. Dividing the numerator and denominator of $m_{n\&=0}$ by $\alpha'_W(y_W)^2$ we obtain

$$m_{n\&=0} = \frac{-2\left(\frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\right)^2(r+\delta) + \frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\sqrt{4\left(\frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\right)^2(r+\delta)^2+1}}{\frac{1}{\alpha'_W(y_W)}}.$$

The denominator ($\frac{1}{\alpha'_W(y_W)}$) approaches infinity as $y_W \to \infty$. We now show that the numerator approaches a finite number as $y_W \to \infty$. Multiply and divide the numerator by $\left(\frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\right)^2$ to obtain

$$\left(\frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\right)^2\left(-2(r+\delta)+\sqrt{4(r+\delta)^2+\frac{1}{\left(\frac{\alpha_W(y_W)}{\alpha'_W(y_W)}\right)^2}}\right).$$

Rearranging we express the numerator as

$$\frac{-2(r+\delta)+\sqrt{4(r+\delta)^2+\left(\frac{\alpha'_W(y_W)}{\alpha_W(y_W)}\right)^2}}{\left(\frac{\alpha'_W(y_W)}{\alpha_W(y_W)}\right)^2}.$$

Taking limits as $y_W \to \infty$, we obtain $\frac{0}{0}$. Therefore, we apply l'Hôpital's Rule. Differentiating numerator and denominator and simplifying we have

$$\frac{1}{2\sqrt{4(r+\delta)^2+\left(\frac{\alpha'_W(y_W)}{\alpha_W(y_W)}\right)^2}}$$

which tends to $\frac{1}{4(r+\delta)}$ as $y_W \to \infty$. We conclude that $\lim_{y_W\to\infty} m_{n\&=0} = 0$.

Let's now consider the case where $\alpha'_W(y_W)$ approaches a finite (positive) constant when $y_W \to \infty$. Recall that

$$m_{n\&=0} = \frac{-2\alpha_W(y_W)^2(r+\delta)+\alpha_W(y_W)\sqrt{4\alpha_W(y_W)^2(r+\delta)^2+(\alpha'_W(y_W))^2}}{\alpha'_W(y_W)}$$

We will show that the numerator approaches zero as $y_W \to \infty$. Multiply and divide the numerator by $\alpha_W(y_W)^2$ to obtain

$$\alpha_W(y_W)^2\left(-2(r+\delta)+\sqrt{4(r+\delta)^2+\left(\tfrac{\alpha_W'(y_W)}{\alpha_W(y_W)}\right)^2}\right).$$

Rearranging we have

$$\frac{-2(r+\delta)+\sqrt{4(r+\delta)^2+\left(\tfrac{\alpha_W'(y_W)}{\alpha_W(y_W)}\right)^2}}{\tfrac{1}{\alpha_W(y_W)^2}}.$$

Taking limits as $y_W \to \infty$, we obtain $\frac{0}{0}$. Therefore, we apply l'Hôpital's Rule. Differentiating numerator and denominator and simplifying we have

$$\frac{\tfrac{\alpha_W'(y_W)^2}{\alpha_W(y_W)}-\alpha_W''(y_W)}{2\sqrt{4\alpha_W^2(y_W)(r+\delta)^2+\alpha_W'(y_W)^2}}$$

which tends to 0 as $y_W \to \infty$. Therefore,

$$\lim_{y_W\to\infty}\left(\alpha_W(y_W)^2\left(-2(r+\delta)+\sqrt{4(r+\delta)^2+\tfrac{\alpha_W'(y_W)}{\alpha_W(y_W)}}\right)\right)=0$$

and

$$\lim_{y_W\to\infty}\frac{-2\alpha_W(y_W)^2(r+\delta)+\alpha_W(y_W)\sqrt{4\alpha_W(y_W)^2(r+\delta)^2+(\alpha_W'(y_W))^2}}{\alpha_W'(y_W)}=0.$$

Let's now turn to equation (A7). Solving (A7) for *m*, we have

$$m_{\dot{y}_W=0}=\alpha_W(y_W)(2\delta y_W-1). \tag{A10}$$

Equation (A10) is positive for all $y_W \geq \frac{1}{2\delta}$ and negative otherwise. Taking the first and second derivative, we see that $m_{\dot{y}=0}$ is increasing for $y_W \geq \frac{1}{2\delta}$.

Our analysis immediately following equation (A8) shows that there is one steady state only. Our analysis of (A9) and (A10) shows that $m_{\dot{y}_W=0}$ cuts $m_{\dot{m}_W=0}$ from *below.*

To sketch the directions of movement compatible with equations (A4) and (A5), we first consider the locus $\langle m,y\rangle$ where $\dot{m}=0$. At a point $\langle m_A,y_A\rangle$ on the locus (A6), $\dot{m}=0$ is satisfied. At a point $\langle m_A+k,y_A\rangle$, $k>0$, above the locus, we have

$$(m_A+k)(r+\delta)-\frac{\alpha'_W(y_A)\left(\alpha_W(y_A)^2-(m_A+k)^2\right)}{4\alpha_W(y_A)^2}>0$$

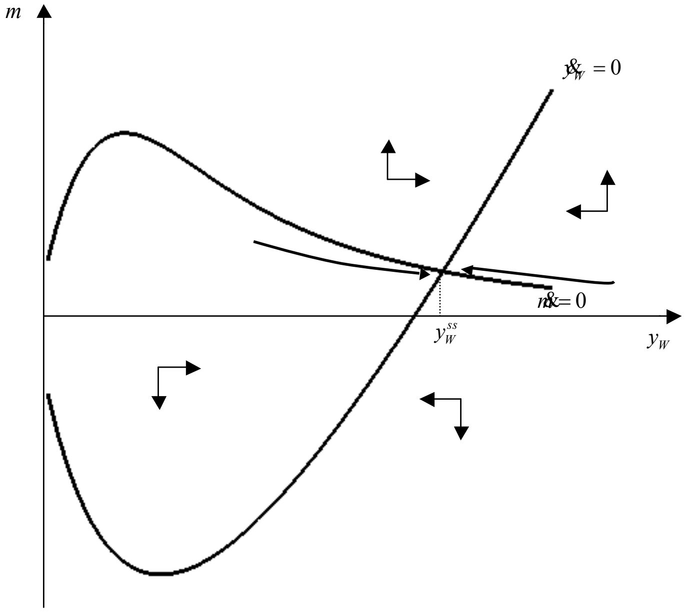
thus, $\dot{m}>0$. Similarly, we see that m is decreasing at points below the $\dot{m}=0$ locus.

Next, consider the points for which $\dot{y}_W=0$. At a point $\langle m_B, y_B\rangle$ on the locus (A7), $\dot{y}_W=0$ is satisfied. At a point $\langle m_B+k, y_B\rangle$, $k>0$, above the locus, we have

$$1-\frac{\left(\alpha_W(y_B)-(m_B+k)\right)}{2\alpha_W(y_B)}-\delta y_{\mathbf{B}}>0$$

thus, $\dot{y}_W>0$. Similarly, it is easy to see that y_W is decreasing at points below the $\dot{y}_W=0$ line.

The following phase diagram summarizes the analysis:



Steady state y^{ss} is a saddle point. See Kamien and Schwartz, 1991, page 178, case (g). The steady state could also happen where the line $\dot{m}=0$ is increasing. Notice that $\dot{y}=0$ would again cut $\dot{m}=0$ from below. In this case, the steady state is also saddle point. See Kamien and Schwartz [1991], page 178, case (i). To see that the path leading to steady state y_W^{ss} is optimal, one can easily check Mangasarian’s sufficient conditions. (See Seierstad and Sydsæter [1997] Theorem 12, page 234.)

We now study the case in which the solution to (A6) is:

$$m_{\dot{m}=0} = \frac{\alpha_W\left(y_W\right)\left(-2\alpha_W\left(y_W\right)\left(r+\delta\right)-\sqrt{4\alpha_W\left(y_W\right)^2\left(r+\delta\right)^2+\left(\alpha_W'\left(y_W\right)\right)^2}\right)}{\alpha_W'\left(y_W\right)}. \tag{A11}$$

The steady states are the points where (A10) and (A11) intersect. For all $y_W \geq 0$, equation (A10) has value greater than or equal to $-\alpha_W\left(y_W\right)$. However, because $\alpha_W\left(y_W\right) > 0$ and $\alpha_W'\left(y_W\right) > 0$, (A11) is always less than $-\alpha_W\left(y_W\right)$. Therefore (A10) and (A11) never intersect. Therefore, there is no steady state in this case. □

Proof of Proposition 1.

(a) Differentiating (A8) implicitly with respect to *δ*, we get:

$$\frac{dy^{ss}(\delta)}{d\delta} = \frac{\alpha_W\left(y^{ss}\right)^2 y^{ss}\left(r\left(2\delta^2\left(y^{ss}\right)^2-2\delta y^{ss}+1\right)+\delta^2 y^{ss}\right)}{\delta\left(\left(1-\delta y^{ss}\right)^2\delta\left(y^{ss}\right)^2\left(\alpha_W''\left(y^{ss}\right)\alpha_W\left(y^{ss}\right)-\left(\alpha_W'\left(y^{ss}\right)\right)^2\right)+\left(r+\delta\right)\alpha_W\left(y^{ss}\right)^2\left(2\delta y^{ss}-1-2\left(\delta y^{ss}\right)^2\right)\right)} < 0$$

(b) Differentiate (A10) implicitly with respect to *r* to get

$$\frac{dy^{ss}(r)}{dr} = \frac{-\alpha_W\left(y^{ss}\right)^2 y^{ss}\left(\left(\delta y^{ss}-1\right)^2+\delta y^{ss}\left(\delta y^{ss}-1\right)\right)}{\left(1-\delta y^{ss}\right)^2\delta\left(y^{ss}\right)^2\left(\alpha_W''\left(y^{ss}\right)\alpha_W\left(y^{ss}\right)-\left(\alpha_W'\left(y^{ss}\right)\right)^2\right)+\left(r+\delta\right)\alpha_W\left(y^{ss}\right)^2\left(2\delta y^{ss}-1-2\left(\delta y^{ss}\right)^2\right)}.$$

The expression above is negative if $\left(\delta y^{ss}-1\right)^2+\delta y^{ss}\left(\delta y^{ss}-1\right) < 0$ for all $\frac{1}{2\delta} < y^{ss} < \frac{1}{\delta}$. To see that this is the case notice that $\left(\delta y^{ss}-1\right)^2+\delta y^{ss}\left(\delta y^{ss}-1\right)$ is zero at $\frac{1}{2\delta}$ and $\frac{1}{\delta}$. Moreover, first derivative ($4\delta y^2-3\delta$) is negative for $\frac{1}{2\delta} < y^{ss} < \frac{3}{4\delta}$ and positive for $\frac{3}{4\delta} < y^{ss} < \frac{1}{\delta}$. Finally, the second derivative ($8\delta y$) is always positive. Therefore, $\left(\delta y^{ss}-1\right)^2+\delta y^{ss}\left(\delta y^{ss}-1\right)$ is convex. We conclude that $\left(\delta y^{ss}-1\right)^2+\delta y^{ss}\left(\delta y^{ss}-1\right)$ is negative for all $\frac{1}{2\delta} < y^{ss} < \frac{1}{\delta}$.

(c) As $\delta \to 0^+$, we have $\frac{1}{2\delta} \to \infty$. As shown above, $\frac{1}{2\delta} < y^{ss}$. Therefore, we have that $y^{ss} \to \infty$.

(d) As $\delta \to \infty$, we have $\frac{1}{2\delta} \to 0$ and $\frac{1}{\delta} \to 0$. Because $\frac{1}{2\delta} < y^{ss} < \frac{1}{\delta}$, we have that $y^{ss} \to 0$.

(e) When $r \to \infty$, $\frac{(2\delta y^{ss}-1)(r+\delta)}{\delta y^{ss}(1-\delta y^{ss})} \to \infty$ as long as $\frac{1}{\delta} > y^{ss} > \frac{1}{2\delta}$. For $\frac{\alpha_W'(y^{ss})}{\alpha_W(y^{ss})} = \frac{(2\delta y^{ss}-1)(r+\delta)}{\delta y^{ss}(1-\delta y^{ss})}$ to hold when r is large, y^{ss} must be sufficiently close to $\frac{1}{2\delta}$. In the limit, $y^{ss} = \frac{1}{2\delta}$. □

Proof of Proposition 2. The following three properties of $\beta(y) \equiv \alpha_W(y) - \alpha_L(y)$ will be used in what follows. First, $\beta(y)$ is increasing for all y:

$$\frac{d\beta(y)}{dy} = \underbrace{\frac{d\alpha_W(y)}{dy}}_{+} - s\underbrace{\frac{d\alpha_L(y)}{dy}}_{-} > 0 .$$

Second,

$$\beta(y(0)) \geq 0 .$$

Finally, for $y > y^0$

$$\frac{d^2\beta(y)}{dy^2} = \underbrace{\frac{d^2\alpha_W(y)}{dy^2}}_{-} - s\underbrace{\frac{d^2\alpha_L(y)}{dy^2}}_{+} \leq 0 .$$

Thus, $\beta(y)$ is concave for $y > y^0$.

Microsoft's problem is

$$\max_{p(t)} \int_0^\infty e^{-rt} q(t) p(t) dt$$

subject to

$$\begin{aligned}
&\dot{y} = q(t) - s(1-q(t)) - \delta y(t) \\
&p(t) = (1-q(t))\beta(y(t)) \\
&\beta(y(0)) \geq 0 \\
&p(t) \geq 0 .
\end{aligned}$$

The current value Hamiltonian is:[2]

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[2] For notational simplicity, we omit time dependence of *y*, *m* (the Hamiltonian multiplier), and *p*.

$$H = \left(1 - \frac{p}{\beta(y)}\right)p + m\left(\left(1 - \frac{p}{\beta(y)}\right)(1+s) - s - \delta y\right).$$

The first order conditions are

$$\frac{\partial H}{\partial p} = 0 \Rightarrow p = \frac{1}{2}\left(\beta(y) - m(1+s)\right), \tag{B1}$$

$$mr - \frac{\partial H}{\partial y} = \dot{m} \Rightarrow \dot{m} = mr - \frac{p\beta'(y)(p + m(1+s)) - m\delta\beta(y)^2}{\beta(y)^2}, \tag{B2}$$

and

$$\frac{\partial H}{\partial m} = \dot{y} \Rightarrow \dot{y} = \left(1 - \frac{p}{\beta(y)}\right)(1+s) - s - \delta y. \tag{B3}$$

Using (B1), we may rewrite (B2) and (B3) as

$$\dot{m} = mr - \frac{\beta'(y)\left(\beta(y)^2 - m^2(1+s)^2\right) - 4m\delta\beta(y)^2}{4\beta(y)^2}, \tag{B4}$$

and

$$\dot{y} = \frac{\beta(y)(1 - s - 2\delta y) + m(1+s)^2}{2\beta(y)}. \tag{B5}$$

Now, (B4) and (B5) form a system of differential equations in *m* and *y*. Consider the following two equations

$$0 = mr - \frac{\beta'(y)\left(\beta(y)^2 - m^2(1+s)^2\right) - 4m\delta\beta(y)^2}{4\beta(y)^2}, \tag{B6}$$

and

$$0 = \beta(y)(1 - s - 2\delta y) + m(1+s)^2. \tag{B7}$$

The steady states are all those pairs $\langle m, y \rangle$ that satisfy simultaneously equations (B6) and (B7). Solving (B6) for *m*, we have

$$m_{\dot{m}=0}=\frac{\beta(y)\left(-2(r+\delta)\beta(y)\pm\sqrt{4(r+\delta)^2\beta(y)^2+(\beta'(y))^2(1+s)^2}\right)}{\beta'(y)(1+s)^2}. \tag{B8}$$

Solving (B7) for *m*, we have

$$m_{\dot{y}=0}=\frac{\beta(y)(s-1+2\delta y)}{(1+s)^2}. \tag{B9}$$

Equating (B8) and (B9) and simplifying yields two characterizations for the steady states. First, there is one set of steady states that satisfy $\beta(y)=0$. As mentioned above, there is only one value satisfying the condition, y^0. If y^0 was ever reached, Microsoft would leave the market because when $y=y^0$ Windows is as valuable as Linux and the only possible way to compete is by setting $p(t)=0$ and earning zero profit forever after.

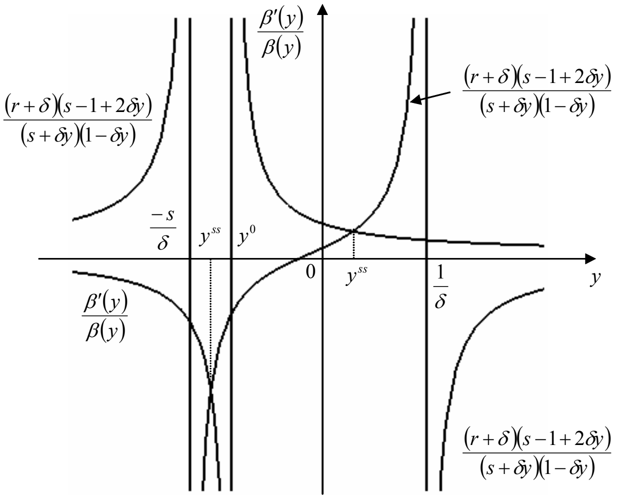
The second set of steady states are all those y that satisfy

$$\frac{\beta'(y)}{\beta(y)}=\frac{(r+\delta)(s-1+2\delta y)}{(s+\delta y)(1-\delta y)}. \tag{B10}$$

(Notice that this is a generalization of equation (A8), the characterization of the steady state for the monopoly case.) The lhs of (B10) has a unique vertical asymptote at y^0. Further, $\frac{\beta'(y)}{\beta(y)}$ is decreasing for $y>y^0$. The rhs has vertical asymptotes at $\frac{-s}{\delta}$ and $\frac{1}{\delta}$. Further, $\frac{(r+\delta)(s-1+2\delta y)}{(s+\delta y)(1-\delta y)}$ is greater than zero for $y\le\frac{-s}{\delta}$, lower than zero for $\frac{-s}{\delta}<y<\frac{1-s}{2\delta}$, greater than zero for $\frac{1-s}{2\delta}<y\le\frac{1}{\delta}$, and lower than zero for $y>\frac{1}{\delta}$.

We study the three possible cases determined by the values that y^0 may take in relation to the vertical asymptotes $\frac{-s}{\delta}$ and $\frac{1}{\delta}$. For each case, we consider only the steady states to the right of y^0 because to the left of y^0 Windows is better off exiting.

Case 1: $\frac{-s}{\delta}<y^0<\frac{1}{\delta}$. The rhs and lhs of (B10) have the following shape:



Inspecting the graph, we see that there is one and only one y to the right of y^0 satisfying equation (B10). Thus, in this case there is one steady state only. (The y that satisfies (B10) that is to the left of y^0 cannot be a valid steady state equilibrium because Windows would not be selling anything in this case).

To draw the phase diagram, we study the shape of (B8) and (B9). (B9) equals zero at y^0. After this point, it can be either positive or negative, but as y increases it will eventually become increasing, then positive ($y > \frac{1-s}{2\delta}$) and finally concave. (B8) equals zero at y^0. After this point, it is strictly positive. (B8) has positive derivative at $y = y^0$ and it tends to zero as y approaches infinity (this follows replicating the argument we presented in the case of the monopoly).

To check that $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$ intersect to the right of y^0, we evaluate the slopes of both curves at $y = y^0$:

$$\left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)}{(1+s)}$$

$$\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)(s-1+2\delta y^0)}{(1+s)^2}.$$

Because $y^0 < \frac{1}{2\delta}$, we have that $\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} < \left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0}$. Therefore $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$ intersect to the right of y^0.

Our analysis immediately following equation (A10) shows that there is one steady state only. Our analysis of (A8) and (A9) shows that $m_{\dot{y}=0}$ cuts $m_{\dot{m}=0}$ from *below*.

To sketch the directions of movement compatible with equations (B4) and (B5), we first consider the locus $\langle m, y\rangle$ where $\dot{m}=0$. At a point $\langle m_A, y_A\rangle$ on the locus (B6), $\dot{m}=0$ is satisfied. At a point $\langle m_A+k, y_A\rangle$, $k>0$, above the locus, we have

$$(m_A+k)r-\frac{\beta'(y_A)\left(\beta(y_A)^2-(m_A+k)^2(1+s)^2\right)-4(m_A+k)\delta\beta(y_A)^2}{4\beta(y_A)^2}>0$$

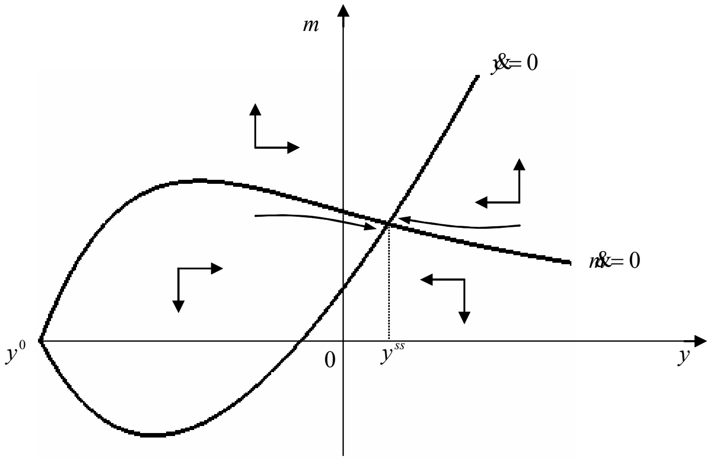
thus, $\dot{m}>0$. Similarly, we see that m is decreasing at points below the $\dot{m}=0$ locus.

Next, consider the points for which $\dot{y}=0$. At a point $\langle m_B, y_B\rangle$ on the locus (B7), $\dot{y}=0$ is satisfied. At a point $\langle m_B+k, y_B\rangle$, $k>0$, *above* the locus, we have

$$\beta(y_B)(1-s-2\delta y_B)+(m_B+k)(1+s)^2>0$$

thus, $\dot{y}>0$. Similarly, it is easy to see that y is decreasing at points below the $\dot{y}=0$ line.

The following phase diagram summarizes the analysis:



Steady state y^{ss} is a saddle point and y^0 is unstable. (See Kamien and Schwartz [1991], page 178, cases (g) and (d), respectively. The steady state could also happen where the line $\dot{m}=0$ is increasing. Notice that $\dot{y}=0$ would again cut $\dot{m}=0$ from below. In this case, the steady state is also saddle point. See Kamien and Schwartz [1991], page 178, case (i).

Finally, to see that the path leading to steady state y^{ss} is optimal (in the *catching up* sense), we check the next sufficient condition

$$\liminf_{T\to\infty}\left(J_T\left(p(\cdot)\right)-J_T\left(\tilde{p}(\cdot)\right)\right)\geq 0,$$

where $J_T\left(p(\cdot)\right)$ is the time T-truncation of the objective functional, $\tilde{p}(\cdot)$ is every feasible control path, and $p(\cdot)$ is the path that we have found leads to y^{ss}. (See Dockner, Jørgensen, Van Long, and Sorger [2000], pages 61 to 65.)

Because $y \geq y^0$ for all y and $m(T)\geq 0$ for sufficiently large *T*, the previous condition is implied by

$$\liminf_{T\to\infty} e^{-rT} m(T)\left(y(T)-y^0\right)\leq 0.$$[3]

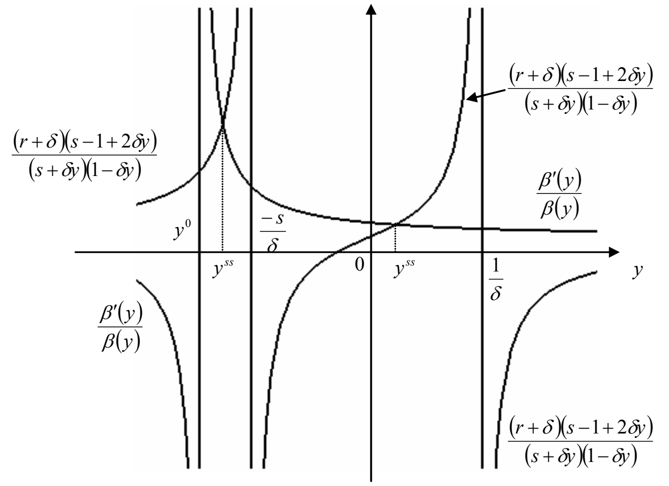
Now, the above analysis shows that $y(t)$ and $m(t)$ converge to *finite* limits (y^{ss} and its correspondent value for the costate m), as a consequence as *T* grows large $e^{-rT} m(T)\left(y(T)-y^0\right)$ becomes smaller and smaller. In fact,

$$\liminf_{T\to\infty} e^{-rT} m(T)\left(y(T)-y^0\right)=0.$$

Therefore, the path characterized by the necessary conditions above is in fact optimal.

Finally, we need to consider the negative root of (B8). It is easy to see that in this case, (B8) and (B9) do not intersect to the right of y^0.

Case 2: $y^0 < \frac{-s}{\delta}$. The rhs and lhs of (B10) look as follows:



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[3] See Dockner, Jørgensen, Van Long, and Sorger (2000), page 65.

As the diagram shows, two steady states are obtained, one below $\frac{-s}{\delta}$ and another between $\frac{1-s}{2\delta}$ and $\frac{1}{\delta}$.

To draw the phase diagram, we study the shape of (B8) and (B9). For the case of (B8) with positive root, the phase diagram is exactly the same as in the previous case, so y^{ss} is a saddle point.

Solving for the negative sign of the root, the other y^{ss} is obtained. To construct the phase diagram, we study the shapes of $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$. $m_{\dot{y}=0}$ is as before. $m_{\dot{m}=0}$ equals zero at y^0 and, contrary to the previous case, it is always negative. (B8) has negative derivative at $y = y^0$ and it tends to minus infinity as y approaches infinity.

To check that $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$ intersect to the right of y^0, we evaluate the slopes of both curves at $y = y^0$:

$$\left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0} = \frac{-\beta'(y^0)}{(1+s)}$$

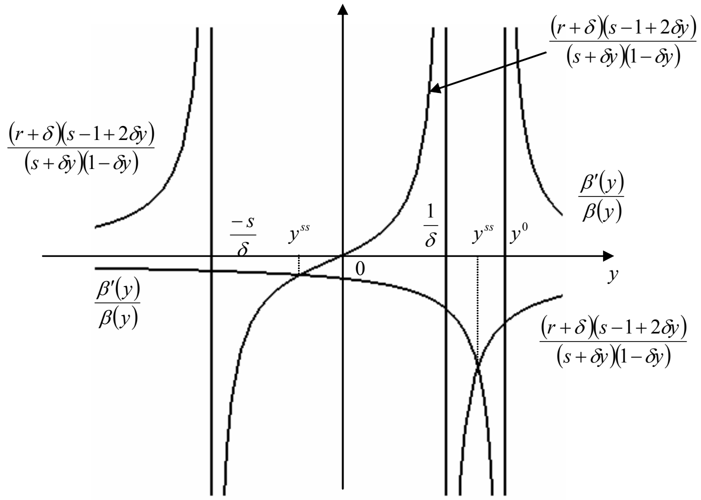
$$\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)(s-1+2\delta y^0)}{(1+s)^2}.$$

At $y^0 = \frac{-s}{\delta}$ we have $\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} = \frac{-\beta'(y^0)}{(1+s)} = \left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0}$.

At $y^0 < \frac{-s}{\delta}$ we have $\left|\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0}\right| > \left|\left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0}\right|$.

Therefore $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$ intersect to the right of y^0. Because the Hamiltonian multiplier on the path leading to this steady state is negative, the path cannot be optimal.

Case 3: $y^0 > \frac{1}{\delta}$. The rhs and lhs of (B10) have the following shape:



The graph shows that there is *no* steady state to the right side of y^0.

We turn to the analysis of the phase diagram. $m_{\dot{y}=0}$ equals zero at y^0. After this point, it is strictly positive and increasing. It ends up becoming concave and as y grows, it approaches either infinity or a strictly positive constant. $m_{\dot{m}=0}$ also equals zero at y^0. To the right of y^0 it is strictly positive and approaches zero as $y \to \infty$.

To check that there is no intersection point between $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$, we compare the derivatives at $y = y^0$:

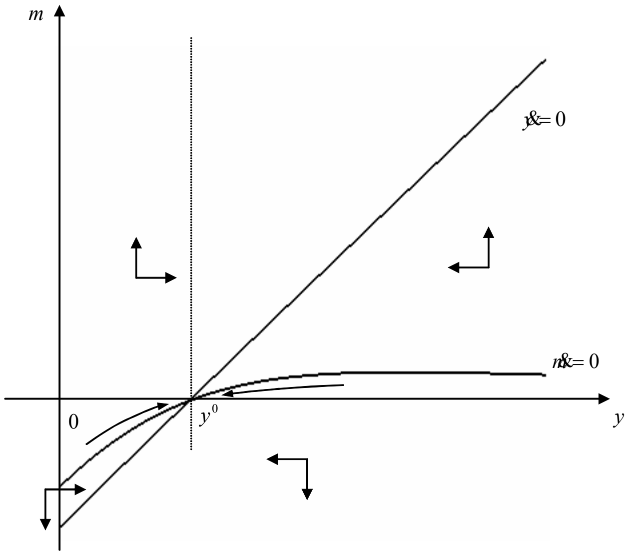
$$\left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)}{(1+s)},$$
$$\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)(s-1+2\delta y^0)}{(1+s)^2}.$$

At $y^0 = \frac{1}{\delta}$ we have $\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} = \frac{\beta'(y^0)}{(1+s)} = \left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0}$.

At $y^0 > \frac{1}{\delta}$ we have $\left.\frac{dm_{\dot{y}=0}}{dy}\right|_{y=y^0} > \left.\frac{dm_{\dot{m}=0}}{dy}\right|_{y=y^0}$.

This shows that $m_{\dot{y}=0}$ and $m_{\dot{m}=0}$ do not intersect to the right of y^0.

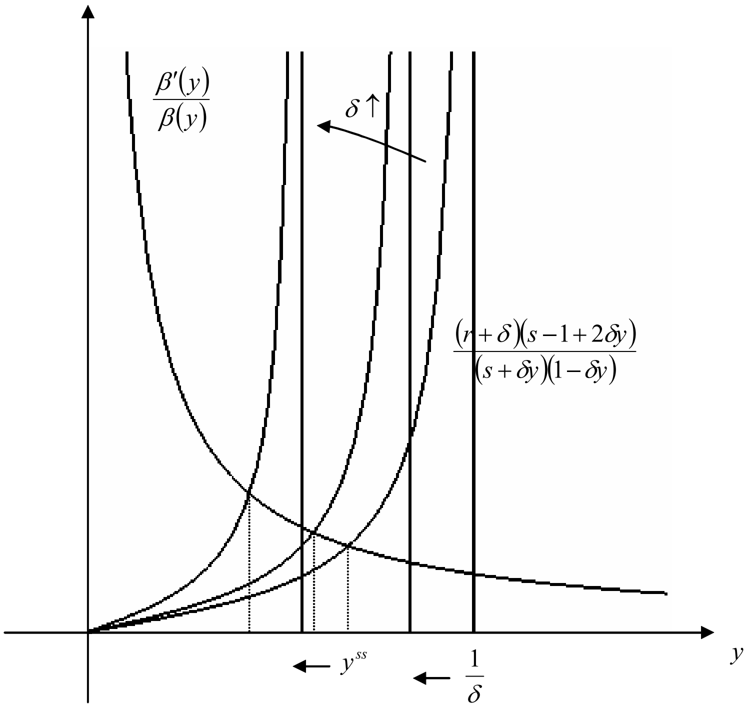
The phase diagram is



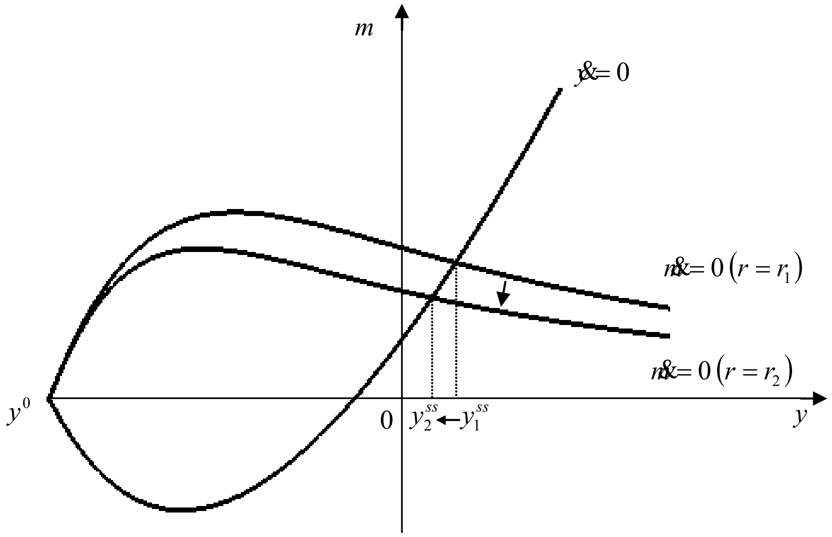
It is easy to see that y^0 is a saddle point. See Kamien and Schwartz, 1991, page 178, case (i).

Proof of Proposition 3.

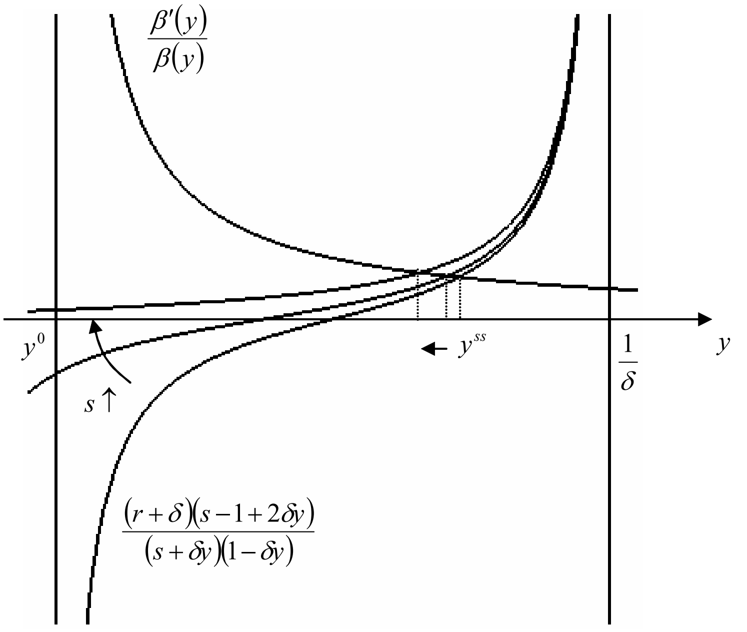
(*a*) Consider equation (B10). The lhs does not change with δ while the rhs moves left. As a consequence, the steady-state moves down



(*b*) Equation (B9) does not depend on r. Equation (B8) moves down as r increases. Looking at the phase diagram ($r_2 > r_1$), it is immediate that y^{ss} decreases with an increase in r.



Proof of Proposition 4. Consider equation (B10). The lhs does not change with s while the rhs approaches $\frac{r+\delta}{1-\delta y}$. Therefore, as $s \to \infty$, the steady state approaches y that solves $\frac{\beta'(y)}{\beta(y)} = \frac{r+\delta}{1-\delta y}$ which, of course, is strictly larger than y^0. Graphically,



It is interesting to notice that when δ is small, it is never optimal for Microsoft to "milk" its initial advantage ($\beta(y(0)) > 0$) by setting high prices in the short term and at some future point leave the market. To understand why, notice first that the myopic profit maximizing price in period t is $p = \frac{1}{2}\beta(y(t))$. Clearly, if Microsoft was determined to milk its short term advantage, it would eventually set prices at this level. In particular, this would be Microsoft's optimal choice in the last period before exiting the market. The following argument shows that when δ is small and $r < \infty$, Microsoft will *never* set $p = \frac{1}{2}\beta(y(t))$: The derivative of instantaneous profit evaluated at $p = \frac{1}{2}\beta(y(t))$ is 0. Thus, lowering price a tiny bit has virtually no effect on instantaneous profit. However, in the last period before Windows exits, it must happen that $\beta(y(t)) \approx 0$. But then, because $q = 1 - \frac{p}{\beta(y(t))}$, we have $\left|\frac{dq}{dp}\right| \approx \infty$. Now, recall that $\dot{y} = q(t) - s(1-q(t)) - \delta y$. Thus, if δ is small, $\left|\frac{d\dot{y}}{dp}\right| >> 0$ and Microsoft is better off by not setting $p = \frac{1}{2}\beta(y(t))$. Lowering price a tiny bit in the "last period," has a huge effect on next period's perceived value of Windows and almost no effect on the present period profit level. Thus, if $r < \infty$ it is optimal for Windows to set a lower price than the myopic optimum and stay active at least one more period.

Proof of Proposition 5. Suppose first that for some technological trajectories ($\alpha_W(\cdot)$ and $\alpha_L(\cdot)$) $TS_W^{Monopoly} > TS^{Duopoly}$. For simplicity, assume in this example that $\delta = 0$ and $\lim_{y\to\infty} \alpha_W(y) = \overline{\alpha}_W < \infty$ and $\lim_{y\to-\infty} \alpha_L(y) = \overline{\alpha}_L < \infty$. In this case, monopoly price at the steady state is $p^{ss} = \frac{\overline{\alpha}_W}{2}$ and steady state monopoly surplus is $TS_W^{Monopoly} = \frac{3}{8}\overline{\alpha}_W$. Let $r \to 0$. $TS_W^{Monopoly}$ does not change. However, the duopoly steady state $y^{ss} \to \infty$. As a consequence, $\alpha_W(y^{ss}) \to \overline{\alpha}_W$ and $\alpha_L(y^{ss}) \to 0$. Furthermore, because $p^{ss} = \beta(y^{ss})\frac{1}{s+1}$, we have that $p^{ss} \to \overline{\alpha}_W \frac{1}{s+1}$. Finally, in the steady state, the demand function for Windows is the same regardless of market structure (the vertical intercept is $\overline{\alpha}_W$ both for monopoly and duopoly) but Windows is sold at a lower price in a duopoly ($\frac{\overline{\alpha}_W}{s+1} < \frac{\overline{\alpha}_W}{2}$), therefore we have that $TS_W^{Monopoly} < TS^{Duopoly}$.

Let's now consider the case in which for some technological trajectories ($\alpha_W(\cdot)$ and $\alpha_L(\cdot)$) we have that $TS^{Duopoly} > TS_W^{Monopoly}$. Consider now a new technological trajectory $\alpha_W^*(y)$ that coincides with $\alpha_W(y)$ everywhere up to a point $y^* > y^{ss}$. From that point on, $\alpha_W^*(y)$ grows linearly (with slope $\left.\frac{d\alpha_W(y)}{dy}\right|_{y=y^*}$). Clearly, if δ is sufficiently close to zero, with this new technological trajectory, $TS_W^{Monopoly} > TS^{Duopoly}$.

Proof of Proposition 6. Because $\rho\mu$ 'would be buyers' pirate Windows, demand for Windows at price *p* is

$$q = 1 - \frac{p}{\beta} - \rho\mu .$$

Notice that $\dot{y}_W = q + \rho$ and $\dot{y}_L = 1 - q - \rho$. Therefore,

$$\dot{y} = q + \rho - s(1 - q - \rho).$$

Microsoft's problem is:

$$\max_{p(t)} \int_0^\infty e^{-rt} q(t) p(t)\,dt$$

subject to

$$\dot{y} = q(t) + \rho - s(1 - q(t) - \rho) - \delta y$$

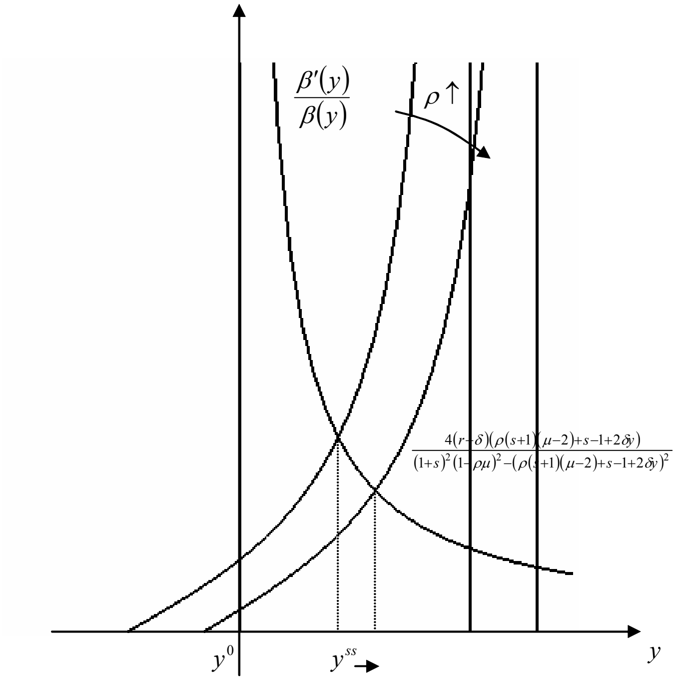
$$q(t) = 1 - \frac{p(t)}{\beta(y)} - \rho\mu$$

$$\beta(y(0)) \geq 0$$

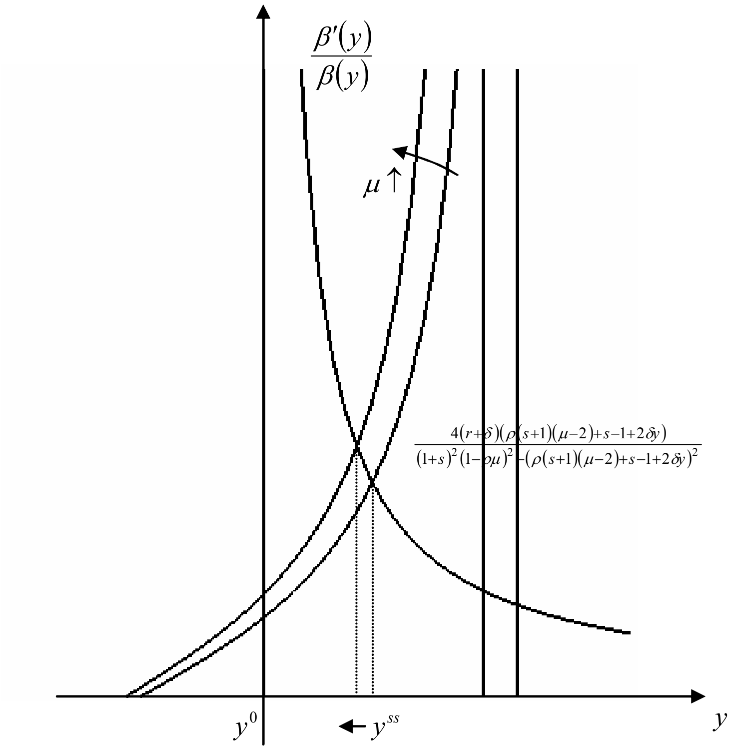
$$p(t) \geq 0$$

Solving the program, the unique saddle point steady state is characterized by:

$$\frac{\beta'\left(y^{ss}\right)}{\beta\left(y^{ss}\right)}=\frac{4(r+\delta)(\rho(s+1)(\mu-2)+s-1+2\delta y)}{(1+s)^2(1-\rho\mu)^2-(\rho(s+1)(\mu-2)+s-1+2\delta y)^2}.$$



An increase in piracy ρ has two effects. On the one hand, if $\mu < 1$, some people that would have chosen Linux, now use Windows because they get the OS for free. So, there is an increase in period market share of size $\rho(1-\mu)$. On the other hand, if $\mu > 0$, some people who would have bought Windows now get Windows for free. This does not affect instantaneous market share, but shrinks demand and Microsoft is induced to set lower prices and, as a consequence, the steady state market share differential also increases.



Finally, $\frac{dy^{ss}}{d\mu}$ is negative. The larger the piracy by ‘would be buyers’ the less the increase in instantaneous market share (as compared to a situation where piracy comes from individuals who would have used Linux).

Proof of Proposition 7. Recall that absent strategic commitment to Linux, demand for Windows is

$$q = 1 - \frac{p}{\beta(y)}.$$

But because ε 'would-be-buyers' of Windows are committed to Linux instead, demand for Windows is

$$q = 1 - \frac{p}{\beta(y)} - \varepsilon \qquad \text{or} \qquad p = (1 - q - \varepsilon)\beta(y).$$

The equation of motion of y is as in the benchmark model. Microsoft's problem is

$$\max_{p(t)} \int_0^\infty e^{-rt} q(t) p(t) dt$$

subject to

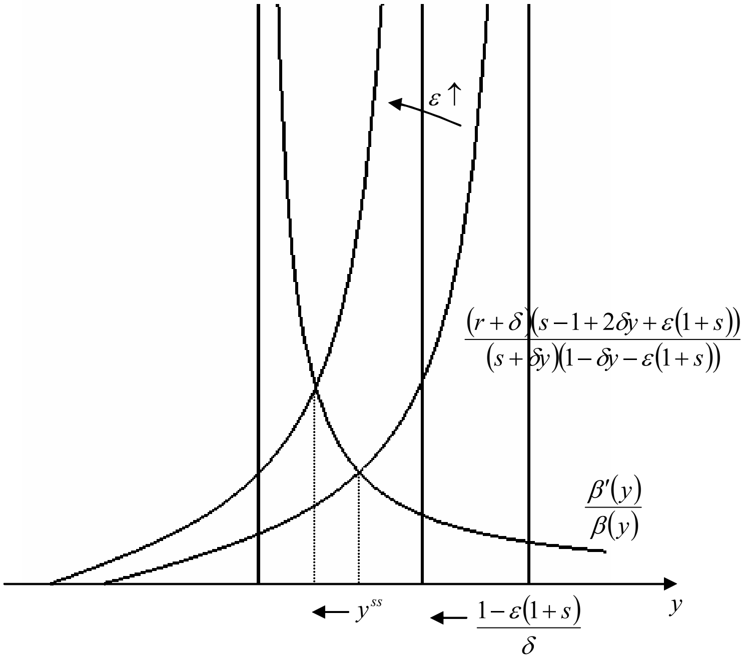
$$\dot{y} = q(t) - s(1 - q(t)) - \delta y$$
$$q(t) = 1 - \frac{p(t)}{\beta(y(t))} - \varepsilon$$
$$\beta(y(0)) > 0$$
$$p(t) \geq 0$$

Solving for the unique saddle point steady state y^{ss}, we obtain

$$\frac{\beta'(y^{ss})}{\beta(y^{ss})} = \frac{(r + \delta)(s - 1 + 2\delta y + \varepsilon(1 + s))}{(s + \delta y)(1 - \delta y - \varepsilon(1 + s))}.$$

Strategic commitment to Linux by a portion ε of Windows' 'would-be-buyers' has two effects: helps build market share of Linux, thus increasing its value vis a vis Windows, and it also forces Microsoft to lower its prices for Windows.

The following plot shows that if $\varepsilon > 0$, if s is sufficiently large, Microsoft is pushed out of the market (because as s grows, $\frac{1-\varepsilon(1+s)}{\delta}$ eventually becomes smaller than y^0). Equivalently, for given s, if ε is sufficiently large, Microsoft is pushed out of the market.


$\varepsilon\uparrow$
$\frac{(r+\delta)(s-1+2\delta y+\varepsilon(1+s))}{(s+\delta y)(1-\delta y-\varepsilon(1+s))}$
$\frac{\beta'(y)}{\beta(y)}$
$\leftarrow y^{ss}$
$\leftarrow \frac{1-\varepsilon(1+s)}{\delta}$
y

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