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ONLY AVAILABLE IN ELECTRONIC FORM

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Technical Supplement

PROPOSITION 1. (i) *Suppose that $i \in Z_k$, $j \in Z_m$, and $|Z_k| < |Z_m|$. Then, $\Pi_i^{\mathcal{Z}} > \Pi_j^{\mathcal{Z}}$.*

(ii) *Suppose that $j \in Z_k$ leaves the coalition, changing the coalition structure to $\mathcal{Z}' = \{Z_1, \dots, Z_k \backslash \{j\}, \{j\}, \dots, Z_l\}$. Then, the profit for members of the coalition $Z_k \backslash \{j\}$ decreases with respect to the profit realized in the coalition Z_k.*

(iii) *Let $\mathcal{Z}_1 = \{Z_1, \dots, Z_m\}$ and $\mathcal{Z}_2 = \{Z_1 \cup Z_2, Z_3, \dots, Z_m\}$ be the status quo position and the new position in which two of the coalitions merge, respectively. Then $\forall\, n, \alpha > 0$, we have $\Pi_{Z_1 \cup Z_2}^{\mathcal{Z}_2} \geq \Pi_{Z_1}^{\mathcal{Z}_1} + \Pi_{Z_2}^{\mathcal{Z}_1}$. Further, if $|Z_1| \geq |Z_2|$ then $\forall\, \alpha \geq 0$, $\Pi_{Z_1}^{\mathcal{Z}_2} \geq \Pi_{Z_1}^{\mathcal{Z}_1}$ and $\exists \alpha^*$ such that $\forall\, \alpha \leq \alpha^*$, $\Pi_{Z_2}^{\mathcal{Z}_1} \geq \Pi_{Z_2}^{\mathcal{Z}_2}$.*

To prove (i) in Proposition 1, we need the following very useful lemma, which states that smaller coalitions charge lower prices in equilibrium.

LEMMA 1. *Suppose that $i \in Z_k$, $j \in Z_m$, and $|Z_k| < |Z_m|$. Then, $p_i^* < p_j^*$.*

PROOF OF LEMMA 1. For each member, i, of a coalition Z_k in partition $\mathcal{Z}$,

$$\Pi_i^{\mathcal{Z}} = p_i D_i^{\mathcal{Z}} = p_i \left[A - (1+\alpha) p_i + \alpha \frac{|Z_k|}{n} p_i + \frac{\alpha}{n} \sum_{j \notin Z_k} p_j \right],$$

where $D_i^{\mathcal{Z}}$ denotes demand for a member, i, of coalition Z_k in partition $\mathcal{Z}$. The first-order condition for a coalition member can be written as $D_i^{\mathcal{Z}} + p_i[-(1+\alpha) + \alpha |Z_k|/n] = 0$, hence in optimality

$$D_i^{\mathcal{Z}} = p_i^* \left[(1+\alpha) - \alpha \frac{|Z_k|}{n} \right]. \tag{EC1}$$

Suppose that $p_i^* \geq p_j^*$ and $i \in Z_k$, $j \in Z_m$, and $|Z_k| < |Z_m|$. Then, because it follows from (1) that $D_i(p_1, \dots, p_n) - D_j(p_1, \dots, p_n) = (p_j - p_i)(1+\alpha)$, j faces larger demand than i, $D_j \geq D_i$. On the other hand, because $|Z_k| < |Z_m|$ and $p_i^* \geq p_j^*$, it follows from (EC1) that $D_i > D_j$, which is a contradiction. Therefore, $p_i^* < p_j^*$. □

PROOF OF PROPOSITION 1. (i) We show in Lemma 1 that $p_i^* < p_j^*$ when $i \in Z_k$, $j \in Z_m$, and $|Z_k| < |Z_m|$. Therefore,

$$\Pi_j(p_i^*, p_j^*, p_{-\{i,j\}}^*) < \Pi_j(p_j^*, p_j^*, p_{-\{i,j\}}^*) = \Pi_i(p_j^*, p_j^*, p_{-\{i,j\}}^*) < \Pi_i(p_i^*, p_j^*, p_{-\{i,j\}}^*),$$

where the first inequality follows from $p_i^* < p_j^*$, and the second one from the definition of the Nash equilibrium (NE).

(ii) Let us denote by $\mathcal{Z}$ the original coalition structure, by $\mathcal{Z}'$ the structure obtained when $j \in Z_k$ leaves the coalition, and by $s \in Z_k \backslash \{j\}$ an arbitrary coalition player. According to Proposition 1, j's profit in $\mathcal{Z}'$ is larger than the profit of the remaining coalition members,

$$\Pi_s^{\mathcal{Z}'} \leq \Pi_j^{\mathcal{Z}'}.$$

When all $|Z_k|$ members in $\mathcal{Z}$ select a price to maximize their profits, the total profit realized by all players is higher than the one realized when $|Z_k| - 1$ members select their price independently of the player j in $\mathcal{Z}'$,

$$\sum_{i \in Z_k} \Pi_i^{\mathcal{Z}} = |Z_k| \cdot \Pi_s^{\mathcal{Z}} \geq \sum_{i \in Z_k} \Pi_i^{\mathcal{Z}'} = (|Z_k| - 1) \Pi_s^{\mathcal{Z}'} + \Pi_j^{\mathcal{Z}'}.$$

Now, the above two inequalities together imply $\Pi_s^{\mathcal{Z}'} \leq \Pi_s^{\mathcal{Z}}$.

(iii) Suppose that a member of coalition $Z_i \in \mathcal{Z}_1$ charges price p_i, and a member of $Z_1 \cup Z_2 \in \mathcal{Z}_2$ charges p^*. The first-order condition before and after merger can be written as

$$\frac{d\Pi_{Z_i}^{\mathcal{Z}_1}}{dp_i} = D_{Z_i}^{\mathcal{Z}_1} + p_i\left[\frac{\alpha|Z_i|}{n} - (1+\alpha)\right] = 0 \tag{EC2}$$

$$+\frac{d\Pi_{Z_1\cup Z_2}^{\mathcal{Z}_2}}{dp^*} = D_{Z_1\cup Z_2}^{\mathcal{Z}_2} + p^*\left[\frac{\alpha|Z_1\cup Z_2|}{n} - (1+\alpha)\right] = 0. \tag{EC3}$$

Because $|Z_1| > |Z_2|$, we must have $p_1 > p_2$. If we assume $p^* = p_1$, then (EC3) is equivalent to

$$D_{Z_1}^{\mathcal{Z}_1} + (p_1 - p_2)|Z_2|\frac{\alpha}{n} - p_1\alpha + p_1\alpha\frac{|Z_1\cup Z_2|}{n} \geq 0.$$

It follows from concavity of $\Pi_{Z_1\cup Z_2}^{\mathcal{Z}_2}$ that $p_1 < p^*$, which implies the result. By continuity, we can show existence of α^*. □

Proposition 2. *Consider a basic coalition structure, $\mathcal{Z}_k^n$, with $k \geq 2$.*

(i) *The prices set by both coalition members and noncoalition members are higher with respect to the case in which all players act independently* (*i.e., no alliances are formed*).

(ii) *Prices of both coalition members and noncoalition members increase with the size of the coalition.*

(iii) *When $k > 1$, noncoalition members generate larger profits than retailers in a coalition.*

(iv) *Profits for coalition members increase with the size of the coalition.*

Proof of Proposition 2. Each player in the coalition charges the same price, p_C, whereas independent players charge $p_{k+1}, p_{k+2}, \ldots, p_n$. Then, the demand faced by a player, i, in a coalition C is

$$D_i^C(p_C, p_{k+1}, \ldots, p_n) = A - (1+\alpha)p_C + \frac{k\alpha}{n}p_C + \frac{\alpha}{n}\sum_{j\notin C} p_j.$$

If we denote $D(k) = 4n + 2\alpha(3n-k-1) + \alpha^2(n-k)(2n+k-2)/n$, then, in equilibrium, a coalition member selects price

$$p_C^* = A\frac{2n(\alpha+1)-\alpha}{D(k)}, \tag{EC4}$$

whereas each independent retailer charges the same price,

$$p_{\bar{C}}^* = A\frac{2n(\alpha+1)-k\alpha}{D(k)}, \tag{EC5}$$

hence $p_C^* > p_{\bar{C}}^*$.

(i) It follows from (2), (EC4), and (EC5):

$$p_C^* - p_0^* = \frac{A\alpha(k-1)[2n+\alpha(n+k-1)]}{D(k)} > 0, \qquad p_{\bar{C}}^* - p_0^* = \frac{A\alpha^2 k(k-1)}{D(k)} > 0.$$

(ii) First note that

$$\frac{\partial D(k)}{\partial k} = -2\alpha - \alpha^2\frac{n+2(k-1)}{n} \leq 0.$$

Thus, $D(k)$ is decreasing in k, hence p_c^* is increasing. In addition,

$$\frac{\partial p_{\bar{C}}^*}{\partial k} = A\frac{2\alpha^2(k-1) + \alpha^3\left(-2+4k-\frac{k^2}{n}\right)}{D(k)^2} \geq 0.$$

(iii) Suppose player i belongs to the coalition C, and player j does not. Observe that profits for retailer i and j, $\Pi_i(p_1, \ldots, p_n)$ and $\Pi_j(p_1, \ldots, p_n)$, are structurally similar except for p_i and p_j. Denote by $p_{-\{i,j\}} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)$. Then,

$$\Pi_i(p_i^*, p_j^*, p_{-\{i,j\}}^*) < \Pi_i(p_i^*, p_i^*, p_{-\{i,j\}}^*) = \Pi_j(p_i^*, p_i^*, p_{-\{i,j\}}^*) < \Pi_j(p_i^*, p_j^*, p_{-\{i,j\}}^*),$$

where the first inequality follows from $p_C^* > p_{\bar{C}}^*$, and the second one from the definition of the NE.

(iv) It is easy to evaluate that, for any coalition member i,

$$\frac{\partial \Pi_i}{\partial k}\bigg|_{(p_1^*,\ldots,p_n^*)} = \frac{\alpha^2 A p_C^*}{D(k)^2}[4n^2(k-1)+2\alpha n(5nk-4n-3k^2+k+1)+\alpha^2(n-k)(6nk-4n-3k+2)]. \quad \text{(EC6)}$$

Let us denote

$$G(k)=4n^2(k-1)+2\alpha n(5nk-4n-3k^2+k+1)+\alpha^2(n-k)(6nk-4n-3k+2).$$

Then,

$$G'(k)=4n^2+2\alpha n(5n-6k+1)+\alpha^2(6n^2-12nk+n+6k-2),$$

and

$$G''(k)=-6\alpha(2n+2\alpha n-\alpha)\le 0,$$

hence $G(k)$ is concave in k. Furthermore, $G(1)=2\alpha n(n-1)+\alpha^2(n-1)(2n-1)\ge 0$, and $G(n)=4n^2(n-1)+2\alpha n(2n^2-3n+1)\ge 0$, hence $G(k)\ge 0$ for any $k=1,\ldots,n$. Thus, the right-hand side (RHS) in (EC6) is nonnegative, and i's profit increases in k. □

Proposition 3. *For any $n\ge 4$ there is an $\alpha(n)$, defined by*

$$\alpha(n)=\frac{2n(n^2-4n+1)+2n\sqrt{(n^2-4n+1)^2+9(n^2-4n+3)}}{9(n-1)},$$

such that for $\alpha\le\alpha(n)$ a player realizes higher profit by staying independent and not joining the grand coalition, and for $\alpha\ge\alpha(n)$ a player realizes higher profit if he belongs to the grand coalition and does not defect. Furthermore, $\alpha(n)$ is an increasing and convex function of n.

Proof of Proposition 3. It is easy to evaluate that, for a noncoalition member,

$$p_{\bar{C}}^{\mathcal{I}_{n-1}^n}=A\frac{\alpha(n+1)+2n}{3\alpha^2(1-\frac{1}{n})+4n\alpha+4n}, \qquad D_{\bar{C}}^{\mathcal{I}_{n-1}^n}=A\frac{n(\alpha+2)(\alpha+1)-\alpha-\frac{\alpha^2}{n}}{3\alpha^2(1-\frac{1}{n})+4n\alpha+4n},$$

whereas for a coalition member,

$$p_{C}^{\mathcal{I}_{n-1}^n}=A\frac{\alpha(2n-1)+2n}{3\alpha^2(1-\frac{1}{n})+4n\alpha+4n}, \qquad D_{C}^{\mathcal{I}_{n-1}^n}=A\frac{2n(\alpha+1)+\alpha(2\alpha+1)-\frac{\alpha^2}{n}}{3\alpha^2(1-\frac{1}{n})+4n\alpha+4n}.$$

Therefore, the difference between his profits in these coalition structures can be written as

$$\Pi_{\bar{C}}^{\mathcal{I}_{n-1}^n}-\Pi_{C}^{\mathcal{I}_{n}^n}=A^2\frac{(n-1)[-9(n-1)\alpha^2+4n(n^2-4n+1)\alpha+4n^2(n-3)]}{4(3\alpha^2(1-\frac{1}{n})+4n\alpha+4n)^2}.$$

To determine when this difference is positive, let us first define a quadratic function $G(\alpha)=-9(n-1)\alpha^2+4n(n^2-4n+1)\alpha+4n^2(n-3)$. This is a concave function that satisfies $G(0)>0$. Thus, after determining its zeros, we can conclude that $G(\alpha)\ge 0$ for

$$\alpha\le\frac{2n(n^2-4n+1)+2n\sqrt{(n^2-4n+1)^2+9(n^2-4n+3)}}{9(n-1)}=\alpha(n), \quad \text{(EC7)}$$

or, in other words, $\Pi_{\bar{C}}^{\mathcal{I}_{n-1}^n}\ge\Pi_{C}^{\mathcal{I}_{n}^n}$ when (EC7) holds.

Now, the profit of the retailer who defects from the grand coalition can be expressed as a rational function of n,

$$A^2\frac{n[\alpha+n(\alpha+2)][n^2(\alpha+1)(\alpha+2)-n\alpha-\alpha^2]}{[3\alpha^2(n-1)+4n^2(\alpha+1)]^2}.$$

The lone retailer benefits from defection when

$$4n[\alpha+n(\alpha+2)][n^2(\alpha+1)(\alpha+2)-n\alpha-\alpha^2]-[3\alpha^2(n-1)+4n^2(\alpha+1)]^2\ge 0. \quad \text{(EC8)}$$

We know that (EC8) is satisfied as equality when $\alpha = 0$, and that it holds as strict inequality when $n \geq 4$ and α is sufficiently small. Thus, in order to show that a lone retailer does not benefit from a defection, we need to show that the left-hand side (LHS) of (EC8) is unimodal for $\alpha > 0$. As the LHS of (EC8) is a polynomial with degree 4, it can have at most four zeros. Note that, when $\alpha \to \pm\infty$, the LHS of (EC8) goes to $-\infty$. In addition, when $\alpha \to 0^-$, the LHS of (EC8) is positive. Thus, the LHS of (EC8) must intersect the x axis for some $\alpha \in (-\infty, 0)$, and there can only be one intersection with the x axis on $(0, +\infty)$.

To show that $\alpha(n)$ is an increasing convex function, note that

$$\alpha'(n) = 2n - 3 + \frac{2}{(n-1)^2} + \frac{2n^3 - 9n^2 + 14n - 5 + \frac{n-5}{(n-1)^3}}{\sqrt{n^4 - 6n^3 + 14n^2 - 10n - 6 - \frac{2(n-3)}{(n-1)^2}}}$$

$$\alpha''(n) = 2 - \frac{4}{(n-1)^3} + \frac{6n^2 - 18n + 14 - 2\frac{n-7}{(n-1)^4}}{\sqrt{n^4 - 6n^3 + 14n^2 - 10n - 6 - \frac{2(n-3)}{(n-1)^2}}} - \frac{\left(2n^3 - 9n^2 + 14n - 5 + \frac{n-5}{(n-1)^3}\right)^2}{\sqrt{n^4 - 6n^3 + 14n^2 - 10n - 6 - \frac{2(n-3)}{(n-1)^2}}^{\,3}}.$$

The second derivative of $\alpha(n)$ is minimized at $n = 2.35$, $\alpha''(2.35) = 1.627 > 0$. Thus, $\alpha(n)$ is convex, and $\alpha'(n)$ is increasing. Because $\alpha'(2) = 3$ and $\alpha'(n)$ is increasing, $\alpha'(n) \geq 0$ for $n \geq 2$, hence $\alpha(n)$ is increasing. □

Proposition 4. *For n large, every $\mathscr{Y} \notin$ the LCS is indirectly dominated by either the grand coalition or a basic coalition structure.*

Proof of Proposition 4. Recall that $\mathbf{Z}$ denotes the set of all possible coalition structures. For any $\mathbf{U} \subseteq \mathbf{Z}$, define a relation $\sim_{\mathbf{U}} \subseteq \mathbf{Z} \times \mathbf{Z}$ as follows:

$$(\mathscr{X}, \mathscr{Y}) \in \sim_{\mathbf{U}} \quad \text{if } \mathscr{Y} \in \mathbf{U} \Rightarrow \mathscr{X} \ll \mathscr{Y}, \quad \text{and} \quad \mathscr{Z} \in \mathbf{U} \quad \text{and} \quad \mathscr{Y} \ll \mathscr{Z} \Rightarrow \mathscr{X} \ll \mathscr{Z}.$$

Let $\mathbf{X} \subseteq \mathbf{Z}$, $\mathbf{X} = \{\text{grand coalition} + \text{set of all basic coalitions}\}$. Define $\lhd_{\mathbf{X}} \subseteq \mathbf{Z} \times \mathbf{X}$ as

$$(\mathscr{X}, \mathscr{Y}) \in \lhd_{\mathbf{X}} \quad \text{if } \mathscr{Y} \in \mathbf{X} \Rightarrow \mathscr{X} \ll \mathscr{Y} \quad \text{and} \quad \mathscr{Z} \in \mathbf{X} \quad \text{and} \quad \mathscr{Y} \ll \mathscr{Z} \Rightarrow \mathscr{X} \ll \mathscr{Z}.$$

Define $\diamond^{\mathbf{U}} = \sim_{\mathbf{U}} \cap \lhd_{\mathbf{X}} \subseteq \mathbf{Z} \times \mathbf{Z}$. We can conclude that $\diamond^{\mathbf{U}}$ is finite and transitive. Given any $\mathbf{Y} \subseteq \mathbf{Z}$, let $M(\mathbf{Y}, \diamond^{\mathbf{Y}}) = \{\mathscr{A} \in \mathbf{Y}\colon \nexists \mathscr{B} \in \mathbf{Y} \text{ such that } (\mathscr{A}, \mathscr{B}) \in \diamond^{\mathbf{Y}}\}$. We can show that M is nonempty. Define $\ell\colon 2^{\mathbf{Z}} \to 2^{\mathbf{Z}}$ as $\ell(\mathbf{A}) = M(\mathbf{A}, \diamond^{\mathbf{A}})$. We then have that there exists $\mathbf{W}$ such that $\ell(\mathbf{W}) = \mathbf{W}$, $\mathbf{W} \neq \varnothing$. This is true because Proposition 1 implies $\ell^{i+1}(\mathbf{A}) \subset \ell^{i}(\mathbf{A})\ \forall \mathbf{A}$, $\forall$large n, and we are guaranteed $\mathbf{W}$'s existence because $\mathbf{Z}$ is finite.

Now we can show that if $\mathscr{X} \in \mathbf{Z} \backslash \mathbf{W}$, there is $\mathscr{Y} \in \mathbf{W}$ such that $(\mathscr{X}, \mathscr{Y}) \in \sim_{\mathbf{W}}$ and that $\mathbf{W} \subset M(\mathbf{Z}, \diamond^{\mathbf{W}})$. Using Tarski (1955) we can show that if $f\colon 2^{\mathbf{Z}} \to 2^{\mathbf{Z}}$ is defined as $f(\mathbf{X}) = \{\mathscr{Z} \in \mathbf{Z}$ such that $\forall \mathscr{V}$, S, such that $\mathscr{Z} \rightharpoonup_S \mathscr{V}$, $\exists \mathscr{B} \in \mathbf{X}$, where $\mathscr{V} = \mathscr{B}$ or $\mathscr{V} \ll \mathscr{B}$, and $\mathscr{Z} \not\prec_S \mathscr{B}\}$. We note that for large n there is an α such that $\mathbf{V} \subseteq f(\mathbf{V})$ and thus $\mathbf{V} \subseteq LCS$. This concludes the proof. □

Proposition 5. *When $n = 3$, the only stable coalition structure is the alliance of all players, $\mathscr{Z}_3^3$.*

Proof of Proposition 5. It follows from Proposition 2 that $\Pi^{\mathscr{Z}_1^3} < \Pi_C^{\mathscr{Z}_2^3} < \Pi_{\bar{C}}^{\mathscr{Z}_2^3}$, and $\Pi_C^{\mathscr{Z}_2^3} < \Pi_C^{\mathscr{Z}_3^3}$. We need to find the relationship between the profit an independent player makes in $\mathscr{Z}_2^3$ and the profit he can make in the grand coalition. When a player is independent, while the other two players form a coalition, his price at optimality and the corresponding demand are given by

$$p_{\bar{C}}^{\mathscr{Z}_2^3} = A\frac{6 + 4\alpha}{12 + 12\alpha + 2\alpha^2}, \qquad D_{\bar{C}}^{\mathscr{Z}_2^3} = A\frac{6 + 8\alpha + \frac{8}{3}\alpha^2}{12 + 12\alpha + 2\alpha^2}.$$

Therefore,

$$\Pi_C^{\mathscr{Z}_3^3} - \Pi_{\bar{C}}^{\mathscr{Z}_2^3} = A^2\frac{\alpha^4 + \frac{4}{3}\alpha^3}{(12 + 12\alpha + 2\alpha^2)^2} \geq 0,$$

hence the profit in the grand coalition dominates the profit he can make in any other coalition structure,

$$\Pi^{\mathscr{Z}_1^3} < \Pi_C^{\mathscr{Z}_2^3} < \Pi_{\bar{C}}^{\mathscr{Z}_2^3} < \Pi_C^{\mathscr{Z}_3^3},$$

or if we write it in terms of players' preferences,

$$\mathcal{Z}_1^3 \prec_i \mathcal{Z}_2^3 \prec_i \mathcal{Z}_3^3 \quad \forall i \in N. \tag{EC9}$$

1. First, we show that the grand coalition belongs to the LCS. Consider a deviation by an arbitrary coalition $S \subset N$, where S can consist of either 1 or 2 players, which leads to a coalition structure $\mathcal{Z}_S^3$. Any such deviation can be followed by another deviation of all three players, $\mathcal{Z}_3^3 \rightharpoonup_S \mathcal{Z}_S^3 \rightharpoonup_{1,2,3} \mathcal{Z}_3^3$. Clearly, it follows from (EC9) that $\mathcal{Z}_S^3 \prec_S \mathcal{Z}_3^3$ for any $S \subset N$, while at the same time $\mathcal{Z}_3^3 \not\prec_S \mathcal{Z}_3^3$. In other words, if $\mathcal{Z} = \mathcal{Z}_3^3$, $\mathcal{V} = \mathcal{Z}_S^3$, and $\mathcal{B} = \mathcal{Z}_3^3$, then $\mathcal{Z}_S^3 = \mathcal{V} \ll \mathcal{B} = \mathcal{Z}_3^3$, and $\mathcal{Z}_3^3 = \mathcal{Z} \not\prec_S \mathcal{B} = \mathcal{Z}_3^3$. Thus, any possible deviation from the grand coalition is deterred.

2. Now, assume that the current status quo is $\mathcal{Z}_1^3$, and consider a deviation by all three players, in which they form the grand coalition, $\mathcal{Z}_1^3 \rightharpoonup_{1,2,3} \mathcal{Z}_3^3$. It follows from (EC9) that $\mathcal{Z}_S^3 \prec_S \mathcal{Z}_3^3$ for any $S \subset N$, hence we cannot find a coalition structure that can be obtained by a deviation from $\mathcal{Z}_3^3$ which strictly dominates $\mathcal{Z}_3^3$ for any subset of players. That is, if $\mathcal{Z} = \mathcal{Z}_1^3$, $\mathcal{V} = \mathcal{Z}_3^3$, and $S = \{1,2,3\}$, then clearly $\mathcal{Z} \rightharpoonup_S \mathcal{V}$, but we cannot find a coalition structure $\mathcal{B}$ such that $\mathcal{Z}_3^3 = \mathcal{V} \ll \mathcal{B}$. In addition, (EC9) implies that $\mathcal{Z}_1^3 \prec_{\{1,2,3\}} \mathcal{Z}_3^3$, hence if $\mathcal{V} = \mathcal{B}$, then $\mathcal{Z} \prec_S \mathcal{B}$. Therefore, when $\mathcal{Z} = \mathcal{Z}_1^3$, $\mathcal{V} = \mathcal{Z}_3^3$, and $S = \{1,2,3\}$, we cannot find a coalition structure $\mathcal{B}$, where $\mathcal{V} = \mathcal{B}$ or $\mathcal{V} \ll \mathcal{B}$, such that $\mathcal{Z} \not\prec_S \mathcal{B}$, hence $\mathcal{Z}_1^3$ cannot be stable. Similar analysis can be used to show that $\mathcal{Z}_2^3$ cannot be stable. □

Proposition 6. *When $n = 4$, the coalition structure with no alliances, $\mathcal{Z}_1^4$, and a coalition structure with an alliance of two players, $\mathcal{Z}_2^4$, are never stable.*

(i) *If $\alpha > 1.864$, the only stable coalition structure is the alliance of all players, $\mathcal{Z}_4^4$.*

(ii) *If $1.2516 < \alpha \le 1.864$, the coalition structure $\mathcal{Z}_3^4$, where an alliance of three players is formed, and the grand coalition, $\mathcal{Z}_4^4$, are stable.*

(iii) *If $\alpha \le 1.2516$, the coalition structure $\mathcal{Z}_{2,2}^4$, where two alliances of two players are formed, the coalition structure $\mathcal{Z}_3^4$, and the grand coalition $\mathcal{Z}_4^4$ are all stable.*

Proof of Proposition 6. It follows from Proposition 2 that

$$\Pi^{\mathcal{Z}_1^4} < \Pi_C^{\mathcal{Z}_2^4} < \Pi_{\bar{C}}^{\mathcal{Z}_2^4}, \qquad \Pi_C^{\mathcal{Z}_2^4} < \Pi_C^{\mathcal{Z}_3^4} < \Pi_{\bar{C}}^{\mathcal{Z}_3^4}, \qquad \text{and} \qquad \Pi_C^{\mathcal{Z}_3^4} < \Pi_C^{\mathcal{Z}_4^4}.$$

We need to find the relationship between $\Pi_{\bar{C}}^{\mathcal{Z}_2^4}$ and $\Pi_C^{\mathcal{Z}_3^4}$, the relationship between $\Pi_{\bar{C}}^{\mathcal{Z}_3^4}$ and $\Pi_C^{\mathcal{Z}_4^4}$, and the relationship between the profit realized in $\mathcal{Z}_{2,2}^4$ and the one realized in other coalition structures.

The optimal prices charged by the players when a coalition of three retailers is formed and demands that they face are given by

$$p_C^{\mathcal{Z}_3^4} = A\frac{8+7\alpha}{16+16\alpha+\frac{9}{4}\alpha^2}, \qquad p_{\bar{C}}^{\mathcal{Z}_3^4} = A\frac{8+5\alpha}{16+16\alpha+\frac{9}{4}\alpha^2},$$

$$D_C^{\mathcal{Z}_3^4} = A\frac{8+9\alpha+\frac{7}{4}\alpha^2}{16+16\alpha+\frac{9}{4}\alpha^2}, \qquad D_{\bar{C}}^{\mathcal{Z}_3^4} = A\frac{8+11\alpha+\frac{15}{4}\alpha^2}{16+16\alpha+\frac{9}{4}\alpha^2}.$$

It is, then, easy to verify that

$$\Pi_C^{\mathcal{Z}_4^4} - \Pi_{\bar{C}}^{\mathcal{Z}_3^4} = A^2\frac{\alpha^2\left(\frac{81}{16}\alpha^2-3\alpha-12\right)}{4\left(16+16\alpha+\frac{9}{4}\alpha^2\right)^2},$$

which is positive for $\alpha \ge 1.864$. Next, for one 2-player coalition, the optimal prices charged in optimality and corresponding demands are given by

$$p_C^{\mathcal{Z}_2^4} = A\frac{8+7\alpha}{16+18\alpha+4\alpha^2}, \qquad p_{\bar{C}}^{\mathcal{Z}_2^4} = A\frac{8+6\alpha}{16+18\alpha+4\alpha^2},$$

$$D_C^{\mathcal{Z}_2^4} = A\frac{8+11\alpha+\frac{7}{2}\alpha^2}{16+18\alpha+4\alpha^2}, \qquad D_{\bar{C}}^{\mathcal{Z}_2^4} = A\frac{8+12\alpha+\frac{9}{2}\alpha^2}{16+18\alpha+4\alpha^2}.$$

Therefore,

$$\Pi_C^{\mathcal{Z}_3^4} - \Pi_{\bar{C}}^{\mathcal{Z}_2^4} = A^2\frac{\alpha^3\left(59\frac{5}{16}\alpha^4+505\frac{1}{4}\alpha^3+1312\alpha^2+1376\alpha+512\right)}{\left(16+16\alpha+\frac{9}{4}\alpha^2\right)^2(16+18\alpha+4\alpha^2)^2} \ge 0,$$

hence

$$\Pi_{\bar{C}}^{\mathcal{L}_2^4} \le \Pi_C^{\mathcal{L}_3^4}.$$

Finally, for two 2-player coalitions,

$$p_C^{\mathcal{L}_{2,2}^4} = A\frac{2}{4+\alpha}, \qquad D_C^{\mathcal{L}_{2,2}^4} = A\frac{2+\alpha}{4+\alpha}.$$

hence

$$\Pi_C^{\mathcal{L}_4^4} - \Pi_C^{\mathcal{L}_{2,2}^4} = A^2\frac{\alpha^2}{4(\alpha+4)^2} \ge 0,$$

$$\Pi_{\bar{C}}^{\mathcal{L}_3^4} - \Pi_C^{\mathcal{L}_{2,2}^4} = A^2\frac{\alpha^2\left(\frac{69}{8}\alpha^3+\frac{283}{4}\alpha^2+164\alpha+112\right)}{(\alpha+4)^2\left(16+16\alpha+\frac{9}{4}\alpha^2\right)^2} \ge 0,$$

$$\Pi_C^{\mathcal{L}_3^4} - \Pi_C^{\mathcal{L}_{2,2}^4} = A^2\frac{\alpha^2\left(\frac{17}{8}\alpha^3+\frac{43}{4}\alpha^2-4\alpha-16\right)}{(\alpha+4)^2\left(16+16\alpha+\frac{9}{4}\alpha^2\right)^2} \ge 0 \quad \text{for } \alpha \ge 1.2516,$$

$$\Pi_C^{\mathcal{L}_{2,2}^4} - \Pi_{\bar{C}}^{\mathcal{L}_2^4} = A^2\frac{\alpha^2(5\alpha^3+28\alpha^2+40\alpha+16)}{(\alpha+4)^2(16+18\alpha+4\alpha^2)^2} \ge 0.$$

Therefore, for $\alpha \le 1.2516$,

$$\Pi^{\mathcal{L}_1^4} < \Pi_C^{\mathcal{L}_2^4} < \Pi_{\bar{C}}^{\mathcal{L}_2^4} < \Pi_C^{\mathcal{L}_3^4} < \Pi_C^{\mathcal{L}_{2,2}^4} < \Pi_C^{\mathcal{L}_4^4} < \Pi_{\bar{C}}^{\mathcal{L}_3^4},$$

and

$$\mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_3^4, \qquad \mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_4^4, \qquad \mathcal{L}_3^4 \prec_C \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_4^4 \prec_{\bar{C}} \mathcal{L}_3^4 \quad \forall\, i \in N, \tag{EC10}$$

for $1.2516 < \alpha \le 1.864$,

$$\Pi^{\mathcal{L}_1^4} < \Pi_C^{\mathcal{L}_2^4} < \Pi_{\bar{C}}^{\mathcal{L}_2^4} < \Pi_C^{\mathcal{L}_{2,2}^4} < \Pi_C^{\mathcal{L}_3^4} < \Pi_C^{\mathcal{L}_4^4} < \Pi_{\bar{C}}^{\mathcal{L}_3^4},$$

and

$$\mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_3^4, \qquad \mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_4^4 \quad \forall\, i \in N, \qquad \mathcal{L}_3^4 \prec_C \mathcal{L}_4^4 \prec_{\bar{C}} \mathcal{L}_3^4, \tag{EC11}$$

and for $\alpha > 1.864$,

$$\Pi^{\mathcal{L}_1^4} < \Pi_C^{\mathcal{L}_2^4} < \Pi_{\bar{C}}^{\mathcal{L}_2^4} < \Pi_C^{\mathcal{L}_{2,2}^4} < \Pi_C^{\mathcal{L}_3^4} < \Pi_{\bar{C}}^{\mathcal{L}_3^4} < \Pi_C^{\mathcal{L}_4^4},$$

and

$$\mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_3^4 \prec_i \mathcal{L}_4^4 \quad \forall\, i \in N. \tag{EC12}$$

(i) Let us first suppose $\alpha > 1.864$. Then, it follows from (EC9) that each player realizes the highest profit in the grand coalition. Therefore, an argument similar to the one used in the proof of Proposition 5 may be used to show that the grand coalition is the only stable coalition structure.

(ii) Next, assume that $1.2516 < \alpha \le 1.864$.

(a) As before, assume that the status quo is the grand coalition, $\mathcal{L}_4^4$. Because (EC8) implies $\mathcal{L}_1^4 \prec_i \mathcal{L}_2^4 \prec_i \mathcal{L}_{2,2}^4 \prec_i \mathcal{L}_4^4$ for all i, for any coalition structure $\mathcal{V} \ne \mathcal{L}_3^4$, obtained when some coalition S deviates from $\mathcal{L}_4^4$, letting $\mathcal{B} = \mathcal{L} = \mathcal{L}_4^4$ satisfies $\mathcal{V} \ll \mathcal{B} = \mathcal{L}_4^4$ and $\mathcal{L} = \mathcal{L}_4^4 \not\prec_S \mathcal{B} = \mathcal{L}_4^4$. Because $\mathcal{L}_3^4 \prec_C \mathcal{L}_4^4$, a similar argument can be used when any three players deviate from the grand coalition to form a 3-player coalition. If a single player, say player 4, decides to deviate, that is, $S = \{4\}$ and $\mathcal{V} = \mathcal{L}_3^4$, consider a following sequence of deviations:

$$\mathcal{L}_4^4 \rightharpoonup_4 \{(123), 4\} = \mathcal{L}_3^4 \rightharpoonup_3 \{(12), 3, 4\} = \mathcal{L}_2^4 \rightharpoonup_{1,2,3,4} \mathcal{L}_4^4. \tag{EC13}$$

Then, $\mathcal{B} = \mathcal{L} = \mathcal{L}_4^4$, $\mathcal{L} = \mathcal{L}_4^4 \not\prec_S \mathcal{B} = \mathcal{L}_4^4$, whereas $\mathcal{L}_3^4 = \mathcal{V} \ll \mathcal{B} = \mathcal{L}_4^4$ because of (EC13) and because $\mathcal{L}_3^4 \prec_3 \mathcal{L}_4^4$ and $\mathcal{L}_2^4 \prec_{\{1,2,3,4\}} \mathcal{L}_4^4$, which follows from (EC8). Thus, $\mathcal{L}_4^4$ belongs to the LCS.

(b) Now, suppose that the current status quo is $\mathcal{Z}_1^4$, and assume that all four players deviate and form the grand coalition. It follows from (EC8) that $\mathcal{Z}_j^4 \prec_i \mathcal{Z}_4^4$ for any $i \in N$ and $j \neq 3, 4$, and that $\mathcal{Z}_3^4 \prec_C \mathcal{Z}_4^4$. Thus, we can obtain $\mathcal{Z}_4^4 \prec_S \mathcal{B}$, wherein $\mathcal{Z}_4^4 \rightharpoonup_S \mathcal{B}$, only when S contains a single player and $\mathcal{B} = \mathcal{Z}_3^4$. If a single player, say player 4, decides to deviate, $\mathcal{Z}_1^4 \rightharpoonup_{1,2,3,4} \mathcal{Z}_4^4 \rightharpoonup_4 \{(123), 4\} = \mathcal{Z}_3^4$, then $\mathcal{Z}_4^4 \prec_4 \mathcal{Z}_3^4$, but $\mathcal{Z}_1^4 \prec_{1,2,3,4} \mathcal{Z}_3^4$. Therefore, if $\mathcal{Z} = \mathcal{Z}_1^4$, $\mathcal{V} = \mathcal{Z}_4^4$ and $S = \{1, 2, 3, 4\}$, then clearly $\mathcal{Z} \rightharpoonup_S \mathcal{V}$, but we cannot find a coalition structure $\mathcal{B}$, where $\mathcal{V} = \mathcal{B} = \mathcal{Z}_4^4$ or $\mathcal{Z}_4^4 = \mathcal{V} \ll \mathcal{B}$, such that $\mathcal{Z} = \mathcal{Z}_1^4 \not\prec_S \mathcal{B}$, hence $\mathcal{Z}_1^4$ cannot be stable.

(c) Next, suppose that the current status quo is $\mathcal{Z}_2^4$, and assume that all four players deviate and form the grand coalition, $\mathcal{Z}_2^4 = \{(12), 3, 4\} \rightharpoonup_{1,2,3,4} \mathcal{Z}_4^4$. Because it follows from (EC8) that $\mathcal{Z}_2^4 \prec_{1,2,3,4} \mathcal{Z}_3^4$, a similar argument as the one used in (b) can be used to show that $\mathcal{Z}_2^4$ cannot be stable.

(d) Assume that the current coalition structure is $\mathcal{Z}_{2,2}^4$, and assume that all four players deviate and form the grand coalition, $\mathcal{Z}_{2,2}^4 \rightharpoonup_{1,2,3,4} \mathcal{Z}_4^4$. Because it follows from (EC8) that $\mathcal{Z}_{2,2}^4 \prec_{1,2,3,4} \mathcal{Z}_3^4$, a similar argument as the one used in (b) can be used to show that $\mathcal{Z}_{2,2}^4$ cannot be stable.

(e) Suppose now that the current coalition structure is $\mathcal{Z}_3^4$. It follows from (EC8) that $\mathcal{Z}_1^4 \prec_i \mathcal{Z}_2^4 \prec_i \mathcal{Z}_{2,2}^4 \prec_i \mathcal{Z}_3^4$ for all i, hence for any coalition structure $\mathcal{V} \neq \mathcal{Z}_4^4$, obtained when some coalition S deviates from $\mathcal{Z}_3^4$, letting $\mathcal{B} = \mathcal{Z} = \mathcal{Z}_3^4$ satisfies $\mathcal{V} \ll \mathcal{B} = \mathcal{Z}_3^4$ and $\mathcal{Z} = \mathcal{Z}_3^4 \not\prec_S \mathcal{B} = \mathcal{Z}_3^4$. If $S = \{1, 2, 3, 4\}$ and $\mathcal{V} = \mathcal{Z}_4^4$, consider a following deviation by a player, 4,

$$\mathcal{Z}_3^4 = \{(123), 4\} \rightharpoonup_{1,2,3,4} \mathcal{Z}_4^4 \rightharpoonup_4 \{(123), 4\} = \mathcal{Z}_3^4. \tag{EC14}$$

Then, $\mathcal{B} = \mathcal{Z} = \mathcal{Z}_3^4$, $\mathcal{Z} = \mathcal{Z}_3^4 \not\prec_S \mathcal{B} = \mathcal{Z}_3^4$, while $\mathcal{Z}_4^4 = \mathcal{V} \ll \mathcal{B} = \mathcal{Z}_3^4$ because of (EC14) and because $\mathcal{Z}_4^4 \prec_4 \{(123), 4\} = \mathcal{Z}_3^4$, which follows from (EC8). Thus, $\mathcal{Z}_3^4$ belongs to the LCS.

(iii) Lastly, suppose that $\alpha \leq 1.2516$. To show that $\mathcal{Z}_1^4$ and $\mathcal{Z}_2^4$ cannot be stable, one can use arguments similar to those in (b) and (c) of (ii) in this proof. In addition, showing stability of $\mathcal{Z}_4^4$ can be done in a similar way as in (a) of (ii) in this proof.

(a) The analysis for the case $\mathcal{Z}_3^4$ is similar to that in (e) of (ii) of this proof, except for the case $\mathcal{V} = \mathcal{Z}_2^2$. Suppose first $\mathcal{Z} = \mathcal{Z}_3^1 = \{(123), 4\}$, $S = \{1, 2\}$, $\mathcal{V} = \mathcal{Z}_2^2 = \{(12), (34)\}$, and consider the following:

$$\mathcal{Z}_3^4 = \{(123), 4\} \rightharpoonup_{1,2} \{(12), (34)\} = \mathcal{Z}_{2,2}^4 \rightharpoonup_4 \{(12), 3, 4\} = \mathcal{Z}_2^4 \rightharpoonup_{1,2,3} \{(123), 4\} = \mathcal{Z}_3^4. \tag{EC15}$$

Then, $\mathcal{B} = \mathcal{Z} = \mathcal{Z}_3^4$, $\mathcal{Z} = \mathcal{Z}_3^4 \not\prec_S \mathcal{B} = \mathcal{Z}_3^4$, while $\mathcal{Z}_{2,2}^4 = \mathcal{V} \ll \mathcal{B} = \mathcal{Z}_3^4$ because of (EC15) and because $\mathcal{Z}_{2,2}^4 \prec_4 \mathcal{Z}_3^4$ and $\mathcal{Z}_2^4 \prec_{\{1,2,3\}} \mathcal{Z}_3^4$, which follows from (EC7). A similar argument can be used when the deviating coalition contains the independent player, $S = \{3, 4\}$, and $\mathcal{V} = \mathcal{Z}_2^2 = \{(12), (34)\}$.

(b) Suppose that the current coalition structure is $\mathcal{Z}_{2,2}^4$, and that $\mathcal{Z}_{2,2}^4$ belongs to the LCS. Because (EC7) implies $\mathcal{Z}_1^4 \prec_i \mathcal{Z}_2^4 \prec_i \mathcal{Z}_{2,2}^4$ for all i, for any coalition structure $\mathcal{V}$ that consists of all players acting independently, or contains a single 2-player coalition, obtained when some coalition S deviates from $\mathcal{Z}_{2,2}^4$, letting $\mathcal{B} = \mathcal{Z} = \mathcal{Z}_{2,2}^4$ satisfies $\mathcal{V} \ll \mathcal{B} = \mathcal{Z}_{2,2}^4$ and $\mathcal{Z} = \mathcal{Z}_{2,2}^4 \not\prec_S \mathcal{B} = \mathcal{Z}_{2,2}^4$. Next, suppose $S = \{1, 2, 3\}$ and $\mathcal{V} = \{(123), 4\} = \mathcal{Z}_3^4$. Letting $\mathcal{V} = \mathcal{B} = \mathcal{Z}_3^4$ satisfies $\mathcal{Z} = \mathcal{Z}_{2,2}^4 \not\prec_S \mathcal{B} = \mathcal{Z}_3^4$ because $\mathcal{Z}_3^4 \prec_C \mathcal{Z}_{2,2}^4$. If $S = \{1, 2, 3, 4\}$ and $\mathcal{V} = \{(1234)\} = \mathcal{Z}_4^4$, consider the following sequence of deviations:

$$\mathcal{Z}_{2,2}^4 = \{(12), (34)\} \rightharpoonup_{1,2,3,4} \mathcal{Z}_4^4 \rightharpoonup_4 \{(123), 4\} = \mathcal{Z}_3^4. \tag{EC16}$$

Now, $\mathcal{B} = \{(123), 4\} = \mathcal{Z}_3^4$, $\mathcal{Z}_4^4 = \mathcal{V} \ll \mathcal{B} = \mathcal{Z}_3^4$ because of (EC16) and because $\mathcal{Z}_4^4 \prec_{\bar{C}} \mathcal{Z}_3^4$, while $\mathcal{Z} = \mathcal{Z}_{2,2}^4 \not\prec_{\{1,2,3,4\}} \mathcal{B} = \mathcal{Z}_3^4$ because $\mathcal{Z}_3^4 \prec_C \mathcal{Z}_{2,2}^4$. Thus, $\mathcal{Z}_{2,2}^4$ belongs to the LCS. □

Theorem 2. (i) *The grand coalition is always stable.*

(ii) *For* $n \geq 3$, *there is an* $\alpha(n)$, *defined by* (3), *such that any coalition structure of the form* $\mathcal{Z}_{n-1}^n$ *that contains an* $(n-1)$*-members coalition, is stable for* $\alpha \leq \alpha(n)$.

(iii) *For large* n *there are values* α^1 *and* α^2, $\alpha^1 < \alpha^2 < \alpha(n)$, *such that*: (a) *when* $\alpha < \alpha^2$, *the outcome* $\mathcal{Z}_{n-2,1,1}^n$ *is stable*; (b) *when* $\alpha < \alpha^1$, *the outcome* $\mathcal{Z}_{n-3,1,1,1}^n$ *is stable*.

In order to prove Theorem 2, we first need the following lemmas.

Lemma 2. 1. *For any* $k \leq (n-1)/2$, *noncoalition members in the coalition structure* $\mathcal{Z}_k^n$ *realize lower profit than the coalition members in the coalition structure* $\mathcal{Z}_{n-1}^n$.

2. *For any* $k \leq (n-1)/2$, *noncoalition members in the coalition structure* $\mathcal{Z}_k^n$ *realize lower profit than the members of the grand coalition*.

Proof of Lemma 2. 1. Recall that the profit for a coalition member in the basic coalition structure $\mathcal{Z}_k^n$ increases with the size of the coalition. When $n \le 4$, $(n-1)/2 \le 3/2$, hence the condition $k \le (n-1)/2$ implies $k = 1$, which corresponds to the coalition structure where all players act independently. Thus, the statement of Lemma 2 for $n \le 4$ follows from Proposition 2. If we show that the statement holds for $k = (n-1)/2$ when $n \ge 5$ odd, the proof is complete.

By using expressions (EC4) and (EC5) from the proof of Proposition 2, one can show, albeit after tedious calculation, that

$$\begin{aligned}\Pi_C^{\mathcal{Z}_{n-1}^n} - \Pi_{\bar{C}}^{\mathcal{Z}_{(n-1)/2}^n} = K\Bigg[& 8\alpha^2 n^2(n^2-4n+5) + 2\alpha^3 n(47n^3 - 119n + 93n + 3) \\ & + \alpha^4\left(42\frac{1}{4}n^4 + 130n^3 - 915\frac{1}{2}n^2 + 896n - 80\frac{3}{4}\right) \\ & + \alpha^5\left(26\frac{1}{2}n^4 - 60n^3 - 42\frac{1}{2}n^2 + 202\frac{1}{2}n - 106 - \frac{9}{2n}\right) \\ & + \alpha^6\left(\frac{25}{4}n^4 + \frac{9}{4}n^3 - 119\frac{7}{16}n^2 + 143n - 54\frac{3}{8} - \frac{125}{4n} + \frac{217}{16n^2}\right) \\ & + \frac{\alpha^7}{16}\left(100n^3 - 424n^2 + 581n + 488 - \frac{166}{n} + \frac{8}{n^2} + \frac{61}{n^3}\right)\Bigg], \qquad \text{(EC17)}\end{aligned}$$

where K is positive for $\alpha > 0$. It is easy to evaluate that the RHS of (EC17) is positive for $n \ge 5$ and $\alpha > 0$, hence the first statement of the proposition follows.

2. Follows from Part 1 because the coalition members in $\mathcal{Z}_{n-1}^n$ always realize lower profit than the members of the grand coalition. □

Next, denote by $\mathcal{Z}_{n-k-1,k,1} = \{(12\cdots n-k-1), (n-k\cdots n-1), n\}$ a coalition structure where players are divided into three coalitions, one containing $n-k-1$ players, the other containing k players, and one having a single member.

Lemma 3. 1. *In any coalition structure $\mathcal{Z}_{n-k-1,k,1}^n$ that consists of three sets—one set has only one member—a member of the largest coalition realizes lower profit than a coalition member in the coalition structure $\mathcal{Z}_{n-1}^n$.*

2. *In any coalition structure $\mathcal{Z}_{n-k-1,k,1}^n$ that consists of three sets—one set has only one member—a member of the largest coalition realizes lower profit than a member of the grand coalition $\mathcal{Z}_n^n$.*

Proof of Lemma 3. Recall that Proposition 1 states that in any coalition structure, members of larger coalitions realize lower profit than the members of smaller coalitions. Therefore, in a coalition structure that consists of three sets—one set has only one member—the members of the largest coalition realize the lowest profit, and it attains its highest value if, for n odd, $k = (n-1)/2$. Thus, if we prove the statement for the coalition structure $\mathcal{Z}_{(n-1)/2,(n-1)/2,1}^n$, the statement of the lemma follows. Note that each member of a coalition $C_{(n-1)/2}$ selects the same price and realizes the same profit, which maximizes the profit of $(n-1)/2$ members. Clearly, this profit is smaller than the one realized by a coalition member in the coalition structure $\mathcal{Z}_{n-1}^n$, wherein the price is selected as to maximize the profit of all $n-1$ members. Therefore, the first statement of the lemma holds. The second statement follows immediately from the first. □

Denote by $\mathcal{Z}_{n-i,i} = \{(12\cdots n-i), (n-i+1\cdots n)\}$ a coalition structure where n players are divided into two coalitions, one containing i players, C_i, the other containing the remaining $n-i$ players, C_{n-i}.

Lemma 4. *When the set of all players is divided into two coalitions, a player realizes the highest profit if he belongs to a one-member coalition.*

Proof of Lemma 4. If we denote $D(i, n-i) = 4n^2(\alpha+1) + 3i\alpha^2(n-i)$, then the profit for a player who belongs to the coalition C_i can be written as $\Pi_{C_i}(n, i, \alpha) = A^2 n[2n + \alpha(n+i)][n^2(\alpha+1)(\alpha+2) - i\alpha(n + i\alpha)]/D(i, n-i)^2$. Although i is a discrete variable, let us suppose for a moment that it is continuous, and consider partial derivatives of $\Pi_{C_i}(n, i, \alpha)$ with respect to i. The second partial derivative of $\Pi_{C_i}(n, i, \alpha)$ with respect to i corresponds to

$$\begin{aligned}\frac{\partial^2 \Pi_{C_i}(n, i, \alpha)}{\partial i^2} = \frac{A^2\alpha^2 n}{D(i, n-i)^3} & [24n^2 i\alpha^4 + 3\alpha^3(3n^3 + 22n^2 i - 9ni^2 - 4i^3) \\ & + 2\alpha^2 n(23n^2 - 18ni - 9i^2) + 4n(16n^2 - 18ni + 9i^2)].\end{aligned}$$

One can evaluate that, for any $n \geq 2$ and $\alpha > 0$, the RHS of (EC18) is positive, hence $\Pi_{C_i}(n, i, \alpha)$ is convex in i. It follows from Theorem 1 that, whenever the independent player in $\mathcal{Z}_{n-1}^n$ realizes higher profit than he can generate in the grand coalition, each member of coalition C_{n-1} realizes a lower profit than the independent player, $\Pi_{\bar{C}}^{\mathcal{Z}_{n-1}^n} > \Pi_{C}^{\mathcal{Z}_{n-1}^n}$. Because $\Pi_{C_i}(n, i, \alpha)$ is convex in i, it further implies that $\Pi_{\bar{C}}^{\mathcal{Z}_{n-1}^n} > \Pi_{C_i}^{\mathcal{Z}_{n-i,i}^n}$ for all $i = 2, \ldots, n-1$. □

Proof of Theorem 2. (i) Let us suppose that the grand coalition, $\mathcal{Z}_n^n$, belongs to the LCS. Observe that, for $k \in \{1, \ldots, n-1\}$, any deviation from $\mathcal{Z}_n^n$ has the form $\mathcal{Z}_n^n \rightharpoonup_{n-k+1,\ldots,n} \mathcal{Z}_{n-k,k}^n$. Suppose that $k \geq n/2$, and consider the following sequence of deviations: $\mathcal{Z}_n^n \rightharpoonup_{n-k+1,\ldots,n} \mathcal{Z}_{n-k,k}^n \rightharpoonup_n \mathcal{Z}_{n-k,k-1,1}^n \cdots \rightharpoonup_{n-k+1} \mathcal{Z}_{n-k}^n \rightharpoonup_{1,\ldots,n} \mathcal{Z}_n^n$, where $\mathcal{Z}_{n-k}^n = \{(12\cdots n-k), 1, \ldots, 1\}$. Let $\mathcal{Z} = \mathcal{B} = \mathcal{Z}_n^n$, $\mathcal{V} = \mathcal{Z}_{n-k,k}^n$, and $S = \{n-k+1, \ldots, n\}$. Now, it is true that $\mathcal{Z}_{n-k,k}^n = \mathcal{V} \ll \mathcal{B} = \mathcal{Z}_n^n$, because:

(a) $\mathcal{Z}_{n-k,k}^n \prec_n \mathcal{Z}_n^n$ follows from the fact that the grand coalition realizes the highest total profit, and that a member of the larger coalition in $\mathcal{Z}_{n-k,k}^n$ realizes lower profit than a member of the smaller coalition;

(b) $\mathcal{Z}_{n-k,k-j,1,\ldots,1}^n \prec_{n-j-1} \mathcal{Z}_{n-k}^n$, for all $j = 0, \ldots, k-1$ follows from Lemma 3 and Proposition 1;

(c) $\mathcal{Z}_{n-k}^n \prec_{1,\ldots,n} \mathcal{Z}_n^n$ follows from Lemma 2, the fact that $n-k \leq n/2$ for $k \geq n/2$, and the fact that the profit for a coalition member in $\mathcal{Z}_j^n$ increases with the size of the coalition.

In addition, $\mathcal{Z}_n^n = \mathcal{Z} \not\prec_S \mathcal{B} = \mathcal{Z}_n^n$. Note that, when $k < n/2$, the analysis still holds after replacing the deviation of players $n-k+1, \ldots, n$ by a deviation of players $1, \ldots, n-k$. Thus, the grand coalition always belongs to the LCS.

(ii) Proof of (ii) is similar to that of (i) and is therefore omitted. It uses Proposition 1 and Lemmas 2, 3, and 4.

(iii) Consider the coalition structure $\mathcal{Z}_{n-2,1,1}^n$ (and $\mathcal{Z}_{n-3,1,1,1}^n$), where $n-2$ $(n-3)$ players form a coalition, and the remaining two (three) players, say i and j (i, j, and k) remain independent. Using exactly the same analysis as in Proposition 3, we can show that there exists $\alpha^1 < \alpha^2 < \alpha(n)$ such that:

I. $\Pi_i^{\mathcal{Z}_{n-2,1,1}^n} > \Pi_i^{\mathcal{Z}_n^n}$ when $\alpha < \alpha^2$;
II. $\Pi_i^{\mathcal{Z}_{n-3,1,1,1}^n} > \Pi_i^{\mathcal{Z}_n^n}$ when $\alpha < \alpha^1$; and
III. $\Pi_i^{\mathcal{Z}_{n-3,1,1,1}^n} < \Pi_i^{\mathcal{Z}_{n-2,1,1}^n}$ when $\alpha > \alpha^1$.

Now consider all possible outcomes that are obtained by a single defection from $\mathcal{Z}_{n-2,1,1}^n$. They are: (1) $\mathcal{Z}_{n-2,1,1}^n \rightharpoonup \mathcal{Z}_n^n$; (2) $\mathcal{Z}_{n-2,1,1}^n \rightharpoonup \mathcal{Z}_{n-k,k}^n$; (3) $\mathcal{Z}_{n-2,1,1}^n \rightharpoonup \mathcal{Z}_{n-k-l,k,l}^n$; and (4) $\mathcal{Z}_{n-2,1,1}^n \rightharpoonup \mathcal{Z}_{n-k-l-1,k,l,1}^n$. Note that (2) and (3) can be further divided into subcases depending on whether either i or j are involved in the defection.

It is easy to show that (1) is deterred by considering the sequence $\mathcal{Z}_{n-2,1,1}^n \rightharpoonup \mathcal{Z}_n^n \rightharpoonup \mathcal{Z}_{n-2,1,1}^n$. We then show, exactly imitating the proof of Proposition 4, that there exists $\psi > \alpha^1$ such that for $\alpha < \min\{\psi, \alpha^2\}$ coalition outcome with exactly two coalitions that are not in the LCS is indirectly dominated by a basic coalition structure. This shows that (2) is deterred. It can be shown that (3) and (4) are deterred using chains as in (i). This proves part (iii)(a). The proof of (iii)(b) is similar and uses Parts II and III above. □

Proposition 7. *For $n < 4$, the grand coalition is the unique stable outcome.*

Proof of Proposition 7. When there are no coalitions, i.e., players operate independently, the expected profit can be written as

$$\pi(p_i, q_i) = E[(p_i - c)(D_i(p_i) + \varepsilon) - \{(p_i - c)[D_i(p_i) + \varepsilon - q]^+ + (c+h)[q - (D_i(p_i) + \varepsilon)]^+\}],$$

where

$$D_i(p_i) = A - (1+\alpha)p_i + \frac{\alpha}{n}\sum_{j=1}^{n} p_j.$$

Instead of working with the purchasing quantity q_i, we use the transformation $\lambda_i = q_i - D_i(p_i)$. Thus,

$$\pi(p_i, \lambda_i) = (p_i - c)(D_i(p_i) + \mu) - \left\{(p_i - c)\int_{\lambda_i}^{\infty}(u - \lambda_i) f(u)\, du + (c+h)\int_{-\infty}^{\lambda_i}(\lambda_i - u) f(u)\, du\right\}.$$

Let us denote

$$I(x) = \int_x^{\infty}(u - x) f(x)\, dx.$$

The equilibrium prices and quantity are given by

$$p^* = \frac{A + \left(1 + \frac{n-1}{n}\alpha\right)c + \mu - I(\lambda^*)}{2 + \frac{n-1}{n}\alpha}, \qquad F(\lambda^*) = \frac{p^* - c}{p^* + h},$$

where F and f are the cumulative distribution and density of $N(\mu, \sigma)$, respectively.

When all players form the grand coalition, each faces a demand $A - p + \varepsilon$. In this case, calculating the optimal (p^*, λ^*) yields

$$p^* = \frac{A + c + \mu - I(\lambda^*)}{2}, \qquad F(\lambda^*) = \frac{p^* - c}{p^* + h}.$$

Finally, when $n-1$ players form an alliance (which corresponds to the structure Z_n^{n-1}), denote the price set by the coalition at equilibrium as p_C^*, and the price set by the lone nonmember as $p_{\bar{C}}^*$. Then, we have

$$p_C^* = \frac{(A+\mu)[2n + (2n-1)\alpha] + c\left[3\frac{n-1}{n}\alpha^2 + (2n+1)\alpha + 2n\right] - \alpha I(\lambda_{\bar{C}}^*) - 2[n + (n-1)\alpha] I(\lambda_C^*)}{4n + 4n\alpha + 3\frac{n-1}{n}\alpha^2},$$

$$F(\lambda_C^*) = \frac{p_C^* - c}{p_C^* + h},$$

$$p_{\bar{C}}^* = \frac{(A+\mu)[2n + (n+1)\alpha] + c\left[3\frac{n-1}{n}\alpha^2 + (3n-1)\alpha + 2n\right] - 2(n+\alpha) I(\lambda_{\bar{C}}^*) - (n-1)\alpha I(\lambda_C^*)}{4n + 4n\alpha + 3\frac{n-1}{n}\alpha^2},$$

$$F(\lambda_{\bar{C}}^*) = \frac{p_{\bar{C}}^* - c}{p_{\bar{C}}^* + h}.$$

The remaining analysis follows the steps similar to those in §3, after substituting $n = 3$. □

Proposition 8. *Let us denote by p^* the price charged by players in the structure $\mathscr{Z}_1^n$, and suppose that the parameters are such that $p^* > h + 2c$. Then,*

(i) *There exists α^* such that the grand coalition is uniquely stable when $\alpha \geq \alpha^*$.*

(ii) *Let $n \geq 4$. There exist μ^*, σ_1^*, σ_2^*, and β^*, where $\sigma_1^* < \sigma_2^*$ and $\beta^* < \alpha^*$, such that*

(a) *$\mu \geq \mu^*$, $\sigma \leq \sigma_1^*$, and $\alpha < \alpha^*$ implies that $\mathscr{Z}_{n-1}^n$ is also stable.*

(b) *$\mu \geq \mu^*$, $\sigma \leq \sigma_1^*$, and $\alpha < \beta^*$ implies that $\mathscr{Z}_{n-2,1,1}^n$ is also stable.*

(c) *$\mu \geq \mu^*$ and $\sigma_2^* > \sigma > \sigma_1^*$ implies that, for any α such that $\beta^* < \alpha < \alpha^*$, $\mathscr{Z}_{n-1}^n$ and $\mathscr{Z}_n^n$ are the only stable outcomes.*

(d) *$\mu \geq \mu^*$ and $\sigma > \sigma_2^*$ implies that, for any $\alpha > 0$, $\mathscr{Z}_n^n$ is uniquely stable.*

Proof of Proposition 8. We first observe that the price charged by the grand coalition, $\mathscr{Z}_n^n$, is larger than in $\mathscr{Z}_1^n$. In fact, a result similar to Proposition 2 holds in this setting.

Now, in any structure, if player i sets (p_i, λ_i), his profit is given by

$$\Pi_i(s_i, \lambda_i) = s_i(A + \mu - c - s_i) - (h+c)\int_{-\infty}^{\lambda_i} (\lambda_i - u) f(u)\, du - s_i \int_{\lambda_i}^{\infty} (u - \lambda_i) f(u)\, du + \alpha s_i\left(-s_i + \frac{1}{n}\sum_{j=1}^{n} s_j\right),$$

where $s_i = p_i - c$. We write $\Pi_i^{\mathscr{Z}}(s_i, \lambda_i) = B_i(s_i, \lambda_i, \mathscr{Z}) + L_i(s_i, \lambda_i, \mathscr{Z})$, where

$$B_i(s_i, \lambda_i) = s_i(A + \mu - c - s_i) - (h+c)\int_{-\infty}^{\lambda_i} (\lambda_i - u) f(u)\, du - s_i \int_{\lambda_i}^{\infty} (u - \lambda_i) f(u)\, du$$

$$L_i(s_i, \lambda_i) = \alpha s_i\left(-s_i + \frac{1}{n}\sum_{j=1}^{n} s_j\right).$$

This expression is in free form. In a particular coalition structure $\mathscr{Z} = \{Z_1, \ldots, Z_k\}$, players in coalition Z_k will set the same price and inventory levels. Thus, even though formally there is a dependence on $\mathscr{Z}$, we drop this from the expressions. The regularity conditions imply that for a fixed vector $(\lambda_1, \ldots, \lambda_k)$, B_i and L_i are unimodal in $(s_1, \ldots, s_k)$ and vice versa. Clearly, they are differentiable and hence continuous in the variables as well. The total profit of all players under any coalitional structure equals $\Pi^{\mathcal{Z}}_{\text{Total}} = \sum_{i=1}^n \Pi_i(s_i) = \sum_{i=1}^n B_i(s_i) + \sum_{i=1}^n L_i(s_i)$. Note that

$$\sum_{i=1}^n L(s_i) = \alpha\left(-\sum_{i=1}^n s_i^2 + \frac{1}{n}\sum_{j=1}^n\sum_{k=1}^n s_j s_k\right) = \alpha\left(-\frac{n-1}{n}\sum_{i=1}^n s_i^2 + \frac{1}{n}\sum_{j\neq k}^n 2s_j s_k\right) = -\frac{\alpha}{n}\sum_{j\neq k}(s_j - s_k)^2 \le 0$$

and

$$\frac{\partial \sum_{j=1}^n B(s_j)}{\partial s_i} = \frac{\partial B(s_i)}{\partial s_i} = A + \mu - c - 2s_i - \int_{\lambda_i}^\infty (u-\lambda_i) f(u)\,du$$
$$- (h+c)\frac{\partial \int_{-\infty}^{\lambda_i}(\lambda_i - u) f(u)\,du}{\partial s_i} - s_i \frac{\partial \int_{\lambda_i}^\infty (u-\lambda_i) f(u)\,du}{\partial s_i}.$$

Therefore, in any equilibrium outcome $\lambda_i = F^{-1}(s_i/(s_i + c + h))$ implies that

$$\frac{\partial \sum_{j=1}^n B(s_j)}{\partial s_i} = \frac{\partial B(s_i)}{\partial s_i} = A + \mu - c - 2s_i - \int_{\lambda_i}^\infty (u-\lambda_i) f(u)\,du$$

and $\Pi^{\mathcal{Z}}_{\text{Total}} = \sum_{i=1}^n B_i(s_i) + \sum_{i=1}^n L_i(s_i)$ achieves its maximum at

$$s_i^* = \frac{A + \mu - c - \int_{\lambda_i^*}^\infty (u - \lambda_i^*) f(u)\,du}{2}, \qquad \lambda_i^* = F^{-1}\left(\frac{s_i^*}{s_i^* + c + h}\right)$$

for all $i = 1, \ldots, n$ when $\mathcal{Z} = N$, the grand coalition.

Now, when $p^* > h + 2c$, we can show using continuity that there exists $\alpha^* > 0$ such that for $\alpha > \alpha^*$, $\Pi_1^{\mathcal{Z}^n_{n-1}} = B_1(s_1^*, \lambda_1^*, \mathcal{Z}^n_{n-1}) + L_1(s_1^*, \lambda_1^*, \mathcal{Z}^n_{n-1}) < \Pi^N_{\text{Total}}/n$. This immediately implies that the grand coalition is uniquely stable. This proves (i).

To prove (ii), note that $\Pi_i(s_i, \lambda_i)$ is continuous with respect to μ, σ and α. Thus, we can show that there are μ^* and σ_1^* such that, for $\mu < \mu^*$, $\sigma < \sigma_1^*$, and $\alpha < \alpha^*$, Lemmas 2, 3, and 4 hold. This proves (a) and (b). To prove (c) and (d), note that $\Pi_i^{\mathcal{Z}^{n-1}_n}(s_i, \lambda_i)$ and $\Pi_i^{\mathcal{Z}^n_n}(s_i, \lambda_i)$ are both decreasing in σ for all i. Thus, using the logic in Proposition 4, we can show that

- there is $\sigma_A^* > \sigma_1^*$ such that, for all $\sigma > \sigma_A^*$, only basic coalitions (including $\mathcal{Z}^n_n$) indirectly dominate all other outcomes;
- there is $\sigma_B^* > \sigma_1^*$ such that, for all $\sigma > \sigma_B^*$, the lone player in $\mathcal{Z}^n_{n-2,1,1}$ makes less than any player in $\mathcal{Z}^n_{n-1,1}$;
- for any α and μ, there is $\sigma_C^* > \sigma_1^*$ such that $\mathcal{Z}^n_n$ is the only indirectly dominating outcome.

The proof follows if one chooses $\sigma_2^* = \max\{\sigma_A^*, \sigma_B^*, \sigma_C^*\}$. □

Proposition 9. (i) *When $n = 3$, the grand coalition is the unique absorbing state of the equilibrium process of coalition formation* (*EPCF*).

(ii) *When $n = 4$, the grand coalition is the unique absorbing state if $\alpha > 1.864$; if $\alpha \le 1.864$, then any coalition structure with an alliance of three players is also absorbing.*

Proof of Proposition 9. To prove (ii), let us first define p as follows:

$$\{(123), 4\} \rightharpoonup_3 \{(12), 3, 4\}, \qquad \{(234), 1\} \rightharpoonup_4 \{(23), 1, 4\},$$
$$\{(134), 2\} \rightharpoonup_1 \{(34), 1, 2\}, \qquad \{(124), 3\} \rightharpoonup_2 \{(14), 2, 3\},$$

and $p(\mathcal{Z}, \mathcal{Z}^4_4) = 1$ for all remaining coalition structures $\mathcal{Z}$. We want to show that this is an EPCF with absorbing state at the grand coalition. It is easy to verify that players always benefit by deviating from any coalition structure that is not in the LCS. This is also true for $\mathcal{Z}^4_{2,2}$: Because $v(\mathcal{Z}^4_4, p) = \Pi_C^{\mathcal{Z}^4_4} + \delta v(\mathcal{Z}^4_4, p)$, it follows that $v(\mathcal{Z}^4_4, p) = \Pi_C^{\mathcal{Z}^4_4}/(1-\delta)$, while $v(\mathcal{Z}^4_{2,2}, p) = \Pi_C^{\mathcal{Z}^4_{2,2}} + \delta v(\mathcal{Z}^4_4, p)$. Thus, $v(\mathcal{Z}^4_4, p) - v(\mathcal{Z}^4_{2,2}, p) = \Pi_C^{\mathcal{Z}^4_4} - \Pi_C^{\mathcal{Z}^4_{2,2}} > 0$. Lastly, we need to show that player i has an incentive to deviate from $\{(ijk), l\}$ to $\{(jk), i, l\}$, which is equivalent to $v_i(\{(ijk), l\}, p) \le \Pi_{\bar{C}}^{\mathcal{Z}^4_2} + \delta v_i(\mathcal{Z}^4_4, p)$. This is true if $\Pi_C^{\mathcal{Z}^4_3} + \delta\Pi_{\bar{C}}^{\mathcal{Z}^4_2} + \delta^2/(1-\delta)\Pi_C^{\mathcal{Z}^4_4} \le \Pi_{\bar{C}}^{\mathcal{Z}^4_2} + \delta/(1-\delta)\Pi_C^{\mathcal{Z}^4_4}$, which is satisfied for $\delta > (\Pi_C^{\mathcal{Z}^4_3} - \Pi_{\bar{C}}^{\mathcal{Z}^4_2})/(\Pi_C^{\mathcal{Z}^4_4} - \Pi_{\bar{C}}^{\mathcal{Z}^4_2})$. It is easy to evaluate that the RHS of this inequality never exceeds 0.25.

Next, we want to construct a process of coalition formation (PCF) which is an EPCF with absorbing state at $\{(123), 4\}$. For $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k \neq i$, let us define PCF as follows:

$$\{(1234)\} \rightharpoonup_4 \{(123), 4\}, \quad \{(123), 4\} \rightharpoonup \{(123), 4\}, \quad \{(ij4), k\} \rightharpoonup_4 \{(ij), k, 4\}, \quad \{(ij), k, 4\} \rightharpoonup_{1,2,3} \{(123), 4\},$$
$$\{(i4), j, k\} \rightharpoonup_4 \{1, 2, 3, 4\}, \quad \{(ij), (k4)\} \rightharpoonup_4 \{(ij), k, 4\}, \quad \{1, 2, 3, 4\} \rightharpoonup_{1,2,3} \{(123), 4\}.$$

Again, it is easy to verify that players always benefit by deviating from any coalition structure which is not in the LCS, and that player 4 benefits when he deviates from the grand coalition. Next, consider a coalition structure wherein player 4 is a member of a 3-player coalition, say $\{(124), 3\}$. We need to show that player 4 has an incentive to deviate from $\{(124), 3\}$, which is equivalent to $v_4(\{(124), 3\}, p) \leq \Pi_{\bar{C}}^{\mathscr{Z}_2^4} + \delta v_4(\{(123), 4\}, p)$. Thus, we must have $\Pi_C^{\mathscr{Z}_3^4} + \delta \Pi_{\bar{C}}^{\mathscr{Z}_2^4} + \delta^2/(1-\delta)\Pi_{\bar{C}}^{\mathscr{Z}_3^4} \leq \Pi_{\bar{C}}^{\mathscr{Z}_2^4} + \delta/(1-\delta)\Pi_{\bar{C}}^{\mathscr{Z}_3^4}$, which is satisfied for $\delta > (\Pi_C^{\mathscr{Z}_3^4} - \Pi_{\bar{C}}^{\mathscr{Z}_2^4})/(\Pi_{\bar{C}}^{\mathscr{Z}_3^4} - \Pi_{\bar{C}}^{\mathscr{Z}_2^4})$. It is easy to evaluate that the RHS of this inequality never exceeds 0.25 when $\alpha \leq 1.864$. Similarly, player 4 has an incentive to deviate from $\{(12), (34)\}$ if $\delta > (\Pi_C^{\mathscr{Z}_{2,2}^4} - \Pi_{\bar{C}}^{\mathscr{Z}_2^4})/(\Pi_{\bar{C}}^{\mathscr{Z}_3^4} - \Pi_{\bar{C}}^{\mathscr{Z}_2^4})$. It is easy to evaluate that the RHS of this inequality never exceeds 0.18 when $\alpha \leq 1.864$. In a similar way, it can be shown that the remaining coalition structures of the form $\mathscr{Z}_3^4$ are absorbing. □

Theorem 3. (i) *In the model with deterministic demand, for large n, $\exists \alpha^{\diamond} > 0$ and $0 < \Delta_1^* < \Delta_2^* < 1$ such that $N \notin EP(n, \alpha)$ and $\mathscr{Z}_k^n \in EP(n, \alpha)$ hold whenever*

$$\alpha < \alpha^{\diamond} \quad \text{and} \quad \delta > \Delta_1^*, \quad \text{or} \quad \alpha \geq \alpha^{\diamond} \quad \text{and} \quad \Delta_1^* < \delta < \Delta_2^*.$$

(ii) *For the model with uncertain demand, let us denote by p^* the price charged by players in the structure $\mathscr{Z}_1^n$. If $p^* > h + 2c$, then there exists α^* such that the grand coalition is the unique absorbing state of the EPCF when $\alpha > \alpha^*$.*

Proof of Theorem 3. (i) Let $|\mathbf{Z}| = t$, and let m and M be minimum and maximum single period payoffs to any player, respectively. Choose $\Delta_1^* \in (0, 1)$ such that for any two states $\mathscr{X}, \mathscr{Y}$, and any player i with $u_i(\mathscr{X}) > u_i(\mathscr{Y})$, we have

$$u_i(\mathscr{X}) > [1 - (\Delta_1^*)^t]M + (\Delta_1^*)^t u_i(\mathscr{Y}), \qquad \text{and} \qquad u_i(\mathscr{X})(\Delta_1^*)^t + [1 - (\Delta_1^*)^t]m > u_i(\mathscr{Y}).$$

Consider any $\delta \in (\Delta_1^*, 1)$ and fix an arbitrary absorbing deterministic EPCF with absorbing states $\mathbf{X} \subseteq \mathbf{Z}$. Let $\mathscr{Z} \in \mathbf{Z}$, and $\mathscr{Y} \in F_S(\mathscr{Z})$ for some fixed coalition S. Let $\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_r$ be the subsequent PCF path, where $\mathscr{Y}_1 = \mathscr{Z}$, and $\mathscr{Y}_r = \mathscr{Y}$. Then, there is an S_i such that $\mathscr{Y}_{i+1} \in F_{S_i}(\mathscr{Y}_i)$ and $v_i(\mathscr{Y}_{i+1}) \geq v_i(\mathscr{Y}_i)$. It is now possible to chose $\alpha^{\diamond}$ and n such that for all $\alpha < \alpha^{\diamond}$ and $\delta > \Delta_1^*$, a basic coalition is in the LCS and

$$v_j(\mathscr{Y}_i) \geq \Pi^N/n \quad \text{for } j \in S_i. \tag{EC18}$$

When $\alpha \geq \alpha^{\diamond}$, by continuity there is a Δ_2^* such that for $\delta \in (\Delta_1^*, \Delta_2^*)$ the above holds (note that, when $\delta \to 1$, (EC18) is not true). Now, for this range of δ and n, we can also show $v_i(\mathscr{Y}) \geq v_i(\mathscr{Y}_j)$ and $u_i(\mathscr{Y}) \geq u_i(\mathscr{Z})$. Define $f\colon \mathbf{Z} \to \mathbf{Z}$ as $f(\mathscr{Z}) = \{\mathscr{X} \in \mathbf{Z}\colon \forall S$ and $\mathscr{Y} \in F_S(\mathscr{X})\ \exists \mathscr{Z} \in \mathbf{Z}$ such that $\mathscr{Y} = \mathscr{Z}$ or $\mathscr{Y} \ll \mathscr{Z}$, and $u_i(\mathscr{X}) \geq u_i(\mathscr{Z})$ for some $i \in S\}$. It is possible to show $\mathscr{Z} \subseteq f(\mathscr{Z})$. Choose δ and n as above so that $\mathscr{Z} \subseteq f(\mathscr{Z}) \subset LCS \cap \{$basic coalition structure$\}$, where the second inclusion follows from Proposition 4 and (EC18).

(ii) The result follows directly from Proposition 8, because $EP(n)$ is a subset of the LCS. □

Reference

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