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e-companion
ONLY AVAILABLE IN ELECTRONIC FORM

# Electronic Companion—"Asymmetric Consumer Learning and Inventory Competition" by Vishal Gaur and Young-Hoon Park, *Management Science* 2007, 53(2) 227–240.

## Appendix B. Proofs

Proof of Proposition 1. Given any time $t$, let $P^i_{st}$ denote the probability that customer $i$ will visit retailer $s$ at time $t$ or later and $\mu^i_{st}$ denotes the expected number of time periods till the next visit to retailer $s$. We show that given $f_s > 0$, (i) if $p^i_{st} > 0$, then $P^i_{st}$ is equal to 1 and $\mu^i_{st}$ is finite; (ii) $p^i_{st}$ does not go to 0 as $t$ increases. These two facts prove the required results.

*Step* 1. Consider a modified system wherein $p^i_{\bar{s}t} = 1$, i.e., the competitor of retailer $s$ offers a 100% service level. Model this system as a Markov chain with two states representing retailers $s$ and $\bar{s}$, respectively. In this chain, the probability that consumer $i$ ever visits retailer $s$ at time $t$ or later is given by

$$\hat{P}^i_{st} = \sum_{\tau=t}^{\infty}\left[\left(\frac{1}{1+p^i_{st}}\right)^{\tau-t}\frac{p^i_{st}}{1+p^i_{st}}\right] = 1, \tag{EC1}$$

and the expected number of time periods till the next visit to retailer $s$ is given by

$$\hat{\mu}^i_{st} = \sum_{\tau=t}^{\infty}\left[(\tau-t)\left(\frac{1}{1+p^i_{st}}\right)^{\tau-t}\frac{p^i_{st}}{1+p^i_{st}}\right] = \frac{1+p^i_{st}}{p^i_{st}}. \tag{EC2}$$

Here, we used the facts that $p^i_{st} > 0$, and that $p^i_{st}$ is not updated until the next visit of consumer $i$ to retailer $s$. From (EC1) and (EC2), it follows that state $s$ is positive recurrent in the Markov chain, i.e., $\hat{P}^i_{st} = 1$ and $\hat{\mu}^i_{st} < \infty$.

Now consider the case in which $p^i_{\bar{s}t} < 1$. The probability of visiting retailer $s$ in this case is always greater than the probability of visiting retailer $s$ in the above Markov chain, i.e.,

$$\eta^i_{st} = \frac{p^i_{st}}{p^i_{\bar{s}t} + p^i_{st}} > \frac{p^i_{st}}{1+p^i_{st}}. \tag{EC3}$$

Thus, it can be shown by induction over the number of time periods that $P^i_{st} \geq \hat{P}^i_{st}$ and $\mu^i_{st} \leq \hat{\mu}^i_{st}$. Combining with (EC1) and (EC2), we find that $p^i_{st} > 0$ implies that $P^i_{st}$ is equal to 1 and $\mu^i_{st}$ is finite.

*Step* 2. Now consider only the subsequence of time periods $\{t_k\}$ when consumer $i$ visits retailer $s$. Let $Y^i_{st_k}$ be 1 if the consumer's visit at time $t_k$ is satisfying, and 0 otherwise. We have

$$p^i_{s,\,t_k+1} = [(1-\theta^u)p^i_{st_k} + \theta^u]Y^i_{st_k} + (1-\theta^d)p^i_{st_k}(1-Y^i_{st_k}). \tag{EC4}$$

Define a Markov chain over this subsequence of time periods with two states, $Y^i_{st_k} = 0$ and 1. The transition probabilities of this embedded Markov chain are determined by $f_s$. Because $f_s$ is strictly positive, $Y^i_{st_k} = 1$ is a positive recurrent state. Therefore, (EC4) implies that $p^i_{st_k} > 0$ with probability 1 as $k$ tends to $\infty$. This further implies that $p^i_{st} > 0$ with probability 1 as $t$ tends to $\infty$ because $p^i_{st}$ is constant between successive visits to retailer $s$. On the other hand, if $f_s = 0$, then $p^i_{s,\,t_k+1} = (1-\theta^d)p^i_{st_k}$, so that $p^i_{st_k}$ goes to 0 with probability 1 as $t$ tends to $\infty$. □

Proof of Proposition 2. Let $X^{i\omega}_t$ be 1 if customer $i$ visits the retailer at time $t$ along the sample path $\omega$, and 0 otherwise. Let $Y^{i\omega}_t$ be 1 if the visit is satisfying, and 0 otherwise. Let $X^{\omega}_t$ be the vector of store visits across all consumers at time $t$, and $Y^{\omega}_t$ be the vector of outcomes of store visits across all consumers at time $t$. A state of the world $\omega$ is a sequence of pairs $(X^{\omega}_t, Y^{\omega}_t)$, $t = 1, \ldots, \infty$.

Let $p^i_{st}(p^i_{s1}, \omega)$ be consumer $i$'s estimate of the fill rate at retailer $s$ at the start of period $t$, as a function of consumer $i$'s initial estimate of the fill rate, $p^i_{s1}$ and the sample path $\omega$. Let $F^i_{st}$ denote the distribution function of $p^i_{st}$. We wish to show that there exists a random variable $p_s$ with distribution function $F_s$ such that $F^i_{st}(x) \to F_s(x)$ for every $x$ where $F_s$ is continuous. To prove this, we show that $F^i_{st}(x)$ is a Cauchy sequence in $[0, 1]$, i.e., for all $\epsilon > 0$, there exists $T$ such that $|F^i_{s,t+\tau}(x) - F^i_{st}(x)| < \epsilon$ for all $t \geq T$, for all $\tau$.

We only need to consider the subsequence of time periods $\{t_k\}$ when consumer $i$ visits retailer $s$. From Proposition 1, this subsequence is infinite. Thus, the subscript $k$ is suppressed for convenience. The superscript $i$ is also ignored to simplify the notation. The updating rule (4) gives the stochastic recursion,

$$\begin{aligned} p_{s,t+1} &= [p_{st}(1-\theta^u) + \theta^u]Y_{st} + p_{st}(1-\theta^d)(1-Y_{st}) \\ &= \theta^u Y_{st} + [(\theta^d - \theta^u)Y_{st} + (1-\theta^d)]p_{st}. \end{aligned}$$

Expanding this for $p_{s,t+\tau}$, we get

$$p_{s,t+\tau} = p_{s,\tau+1}u(t,\tau) + v(t,\tau), \tag{EC5}$$

where $p_{s,\tau+1}$ is consumer $i$'s estimate of the service level at the start of period $\tau+1$, and

$$u(t,\tau) \equiv \prod_{k=1}^{t-1}[(\theta^d - \theta^u)Y_{s,t+\tau-k} + (1-\theta^d)],$$

$$v(t,\tau) \equiv \sum_{i=1}^{t-1}\prod_{k=1}^{i-1}[(\theta^d - \theta^u)Y_{s,t+\tau-k} + (1-\theta^d)]\theta^u Y_{s,t+\tau-i}.$$

Thus,

$$\begin{aligned} F_{s,t+\tau}(x) - F_{st}(x) &= \Pr[p_{s,\tau+1}u(t,\tau) + v(t,\tau) \leq x] - \Pr[p_{s1}u(t,0) + v(t,0) \leq x] \\ &\leq \Pr[v(t,\tau) \leq x] - \Pr[p_{s1}u(t,0) + v(t,0) \leq x], \end{aligned} \tag{EC6}$$

where the inequality follows because $p_{s,\tau+1}u(t,\tau) > 0$. Consider the second term on the right-hand side of (EC6). Let $\delta_{\max} = \max\{1-\theta^u, 1-\theta^d\}$. Because $[(\theta^d - \theta^u)Y_{st} + (1-\theta^d)]$ is equal to $(1-\theta^u)$ if $Y_{st} = 1$ and $(1-\theta^d)$ otherwise, we have that $u(t,0) \leq \delta^{t-1}_{\max}$. Thus,

$$\begin{aligned} F_{s,t+\tau}(x) - F_{st}(x) &\leq \Pr[v(t,\tau) \leq x] - \Pr[p_{s1}\delta^{t-1}_{\max} + v(t,0) \leq x] \\ &\leq \Pr[x - \delta^{t-1}_{\max} \leq v(t,0) \leq x]. \end{aligned}$$

Here, $v(t,\tau)$ can be replaced by $v(t,0)$ in the second inequality because the retailer maintains constant service level and $Y^i_{st}$ are iid random variables. Because $\Pr[x - \delta^{t-1}_{\max} \leq v(t,0) \leq x] \to 0$ for all $t$ sufficiently large, we obtain the required result. Thus, there exists a random variable $p_s$ such that $p^i_{st} \xrightarrow{D} p_s$ as $t \to \infty$. From symmetry, the limiting distribution is identical for all consumers.

The expectation of $p_s$ is now directly obtained from the updating rule (4) or from the expansion (EC5) because convergence in distribution implies convergence in expectation. □

Proof of Lemma 1. We have

$$\frac{dv_s}{dQ_s} = \frac{\mathrm{E}[p_{\bar{s}}]}{(\mathrm{E}[p_s] + \mathrm{E}[p_{\bar{s}}])^2}\frac{\theta}{[f_s(\theta-1)+1]^2}\frac{1-G(Q_s)}{N\omega} > 0.$$

Thus, $v_s$ is increasing in $Q_s$ for given $Q_{\bar{s}}$. Further, with some algebraic manipulation, the second derivative of $v_s$ with respect to $Q_s$ can be written as

$$\frac{d^2v_s}{dQ_s^2} = -\frac{\mathrm{E}[p_{\bar{s}}]}{(\mathrm{E}[p_s] + \mathrm{E}[p_{\bar{s}}])^2}\frac{\theta}{[f_s(\theta-1)+1]^2}\left[\frac{g(Q_s)}{N\omega} + \left(\frac{1-G(Q_s)}{N\omega}\right)^2 \frac{2\{\theta - (1-\theta)\mathrm{E}[p_{\bar{s}}]\}}{\mathrm{E}[p_{\bar{s}}] + f_s\{\theta - (1-\theta)\mathrm{E}[p_{\bar{s}}]\}}\right].$$

All the terms in the above expression are positive, with the exception of $\theta - (1-\theta)\mathrm{E}[p_{\bar{s}}]$, which is negative if $\mathrm{E}[p_{\bar{s}}] > \theta/(1-\theta)$. However, $\theta \geq 0.5$ implies that $\theta/(1-\theta) \geq 1$. Thus, $\theta - (1-\theta)\mathrm{E}[p_{\bar{s}}]$ is nonnegative because $\mathrm{E}[p_{\bar{s}}] \leq 1$ by definition. Therefore, $v_s$ is concave in $Q_s$. □

Proof of Lemma 2. From the profit function (7), we have

$$\frac{d^2\mathrm{E}[\pi_s]}{dQ_s^2} = v_s''h(Q_s) + v_s h''(Q_s) + 2h'(Q_s)v_s'.$$

For $Q_s \in (Q_s^M, \bar{Q}_s)$, we have $h(Q_s) > 0$, $h'(Q_s) < 0$, and $h''(Q_s) < 0$. By Lemma 1, we further have $v_s > 0$, $v_s' > 0$, and $v_s'' < 0$. Therefore, $d^2\mathrm{E}[\pi_s]/dQ_s^2 < 0$ for $Q_s \in (Q_s^M, \bar{Q}_s)$. □

Proof of Lemma 3. Applying the Implicit Function Theorem to the first order condition (8), we get

$$\frac{dQ_s^S}{dQ_{\bar{s}}} = -\frac{dF_s/dQ_{\bar{s}}}{dF_s/dQ_s^S}.$$

It can easily be seen that $dF_s/dQ_{\bar{s}} > 0$. Further, $dF_s/dQ_s^S < 0$ from the concavity of $\mathrm{E}[\pi_s]$ for all $Q_s \in [Q_s^M, \bar{Q}_s]$. Thus, $dQ_s^S/dQ_{\bar{s}} > 0$. □

Proof of Proposition 4. *Existence*: The strategy spaces of the retailers are nonempty, compact, convex subsets of the real line and each retailer's response function is continuous and strictly concave in the inventory level. Therefore, from Debreu (1952), the result follows.

*Uniqueness*: We need to show that the reaction curves of the two retailers intersect at most once, so that there is at most one fixed point and the equilibrium is unique. Equivalently, we show that there is at most one point that satisfies the first-order conditions of both retailers.

The first-order conditions of the two retailers can be rewritten as

$$v_{\bar{s}} = -\frac{h_s'(Q_s)}{h_s(Q_s)\phi(Q_s)} \quad \text{for } s = 1, 2,$$

where $\phi(Q_s) \equiv \mathrm{E}[p_s]\cdot(1-G(Q_s))/(f_s^2\theta N\omega)$. Suppose that the solution to these simultaneous equations is not unique, and there exist two distinct equilibria, $(Q_1, Q_2)$ and $(Q_1', Q_2')$. Assume, without loss of generality, that $Q_1' > Q_1$. This implies that $Q_2' > Q_2$ because $dQ_2/dQ_1 > 0$ by Lemma 3.

Note that $h_s'(Q_s)$ is negative and decreasing in $Q_s$, and $h_s(Q_s)$ and $\phi(Q_s)$ are both positive and decreasing in $Q_s$. Thus, $-h_s'(Q_s)/[h_s(Q_s)\phi(Q_s)]$ is positive and increasing in $Q_s$. Therefore, we have

$$-\frac{h_s'(Q_s')}{h_s(Q_s')\phi(Q_s')} > -\frac{h_s'(Q_s)}{h_s(Q_s)\phi(Q_s)} \quad \text{for } s = 1, 2.$$

Adding the inequalities for $s = 1$ and 2 gives

$$v_1(Q_1', Q_2') + v_2(Q_1', Q_2') > v_1(Q_1, Q_2) + v_2(Q_1, Q_2).$$

But this is an impossibility because $v_1(Q_1, Q_2) + v_2(Q_1, Q_2) = 1$ for all $(Q_1, Q_2)$. Therefore, it must be that $(Q_1, Q_2) = (Q_1', Q_2')$ and there is at most one Nash equilibrium. □

Proof of Proposition 5. $Q_1$ and $Q_2$ are implicit functions of $\theta$ defined by the first-order conditions of the two retailers given in (8). The derivative of (8) with respect to $\theta$ gives

$$\frac{\partial F_s}{\partial\theta} + \frac{dF_s}{dQ_s}\frac{dQ_s}{d\theta} + \frac{dF_s}{dQ_{\bar{s}}}\frac{dQ_{\bar{s}}}{d\theta} = 0 \quad \text{for } s = 1, 2.$$

By solving these simultaneous equations, the derivative of $Q_s$ with respect to $\theta$ is obtained as

$$\frac{dQ_s}{d\theta} = -\frac{\dfrac{dF_{\bar{s}}}{dQ_{\bar{s}}}\dfrac{\partial F_s}{\partial\theta} - \dfrac{dF_s}{dQ_{\bar{s}}}\dfrac{\partial F_{\bar{s}}}{\partial\theta}}{\dfrac{dF_1}{dQ_1}\dfrac{dF_2}{dQ_2} - \dfrac{dF_1}{dQ_2}\dfrac{dF_2}{dQ_1}}. \tag{EC7}$$

Recall that

$$\frac{dF_s}{dQ_s} < 0 \quad \text{and} \quad \frac{dF_s}{dQ_{\bar{s}}} > 0. \tag{EC8}$$

Additional inequalities are established by the following lemmas:

LEMMA 4.
$$\left.\frac{\partial F_s}{\partial \theta}\right|_{F_s=0} < 0.$$

PROOF. Differentiating condition (8) with respect to $\theta$ and simplifying, we get
$$\left.\frac{\partial F_s}{\partial \theta}\right|_{F_s=0} = h_s(Q_s)\frac{v_s^2 v_{\bar{s}}}{f_s\theta^2[f_s(\theta-1)+1]}\frac{1-G(Q_s)}{N\omega}\left[\mathrm{E}[p_s]\left(1-\theta-\frac{1}{f_s}\right)+\mathrm{E}[p_{\bar{s}}]\left(-1-\theta+\frac{1}{f_{\bar{s}}}\right)\right]. \tag{EC9}$$
Here,
$$\begin{aligned}\mathrm{E}[p_s]\left(1-\theta-\frac{1}{f_s}\right)+\mathrm{E}[p_{\bar{s}}]\left(-1-\theta+\frac{1}{f_{\bar{s}}}\right) &= \frac{\theta f_s}{f_s(\theta-1)+1}\frac{f_s-1-f_s\theta}{f_s}+\frac{\theta f_{\bar{s}}}{f_{\bar{s}}(\theta-1)+1}\frac{1-f_{\bar{s}}-f_{\bar{s}}\theta}{f_{\bar{s}}}\\ &= -\theta+\frac{\theta(1-f_{\bar{s}}-f_{\bar{s}}\theta)}{f_{\bar{s}}(\theta-1)+1}\\ &= \frac{-2f_{\bar{s}}\theta^2}{f_{\bar{s}}(\theta-1)+1}\\ &< 0.\end{aligned}$$
Because all other terms in (EC9) are nonnegative, the result follows. □

LEMMA 5.
$$\frac{dF_1}{dQ_1}\frac{dF_2}{dQ_2}-\frac{dF_1}{dQ_2}\frac{dF_2}{dQ_1}>0. \tag{EC10}$$

PROOF. From (8), note that
$$\begin{aligned}\frac{dF_s}{dQ_s} &= h_s''(Q_s)v_s+2h_s'(Q_s)v_s'+h_s(Q_s)v_s'',\\ \frac{dF_s}{dQ_{\bar{s}}} &= h_s'(Q_s)\frac{dv_s}{dQ_{\bar{s}}}+h_s(Q_s)\frac{d^2v_s}{dQ_s dQ_{\bar{s}}}.\end{aligned}$$
Simplifying (EC10) using the fact that $dv_{\bar{s}}/dQ_s = -v_s'$, we get
$$\begin{aligned}\frac{dF_1}{dQ_1}\frac{dF_2}{dQ_2}-\frac{dF_1}{dQ_2}\frac{dF_2}{dQ_1} &= \left(h_1''(Q_1)v_1\frac{dF_2}{dQ_2}+2h_1'(Q_1)h_2''(Q_2)v_1'v_2+h_1(Q_1)h_2''(Q_2)v_1''v_2\right)\\ &\quad+(4h_1'(Q_1)h_2'(Q_2)v_1'v_2'-h_1'(Q_1)h_2'(Q_2)v_1'v_2')\\ &\quad+\left(2h_1(Q_1)h_2'(Q_2)v_1''v_2'+h_1(Q_1)h_2'(Q_2)v_1'\frac{d^2v_1}{dQ_1dQ_2}\right)\\ &\quad+\left(2h_1'(Q_1)h_2(Q_2)v_1'v_2''+h_1'(Q_1)h_2(Q_2)v_2'\frac{d^2v_2}{dQ_1dQ_2}\right)\\ &\quad+\left(h_1(Q_1)h_2(Q_2)v_1''v_2''-h_1(Q_1)h_2(Q_2)\frac{d^2v_1}{dQ_1dQ_2}\frac{d^2v_2}{dQ_1dQ_2}\right).\end{aligned}$$
Denote the terms in the five sets of brackets as $A$, $B$, $C$, $D$, $E$, respectively. $A$ is positive because $h_s(Q_s)$ and $v_s$ are positive and concave, $h_s'(Q_s)<0$, $v_s'>0$ and $dF_s/dQ_s<0$. $B$ is positive because $h_s'(Q_s)<0$ and $v_s'>0$.

The following additional facts are useful to analyze $C$, $D$, and $E$.
$$\begin{aligned}v_s' &= v_s v_{\bar{s}}\phi_s\\ v_s'' &= (v_{\bar{s}}-v_s)v_s'\phi_s+v_s v_{\bar{s}}\phi_s'\\ \frac{d^2v_s}{dQ_s dQ_{\bar{s}}} &= (v_s-v_{\bar{s}})v_{\bar{s}}'\phi_s=(v_s-v_{\bar{s}})v_s'\phi_{\bar{s}},\end{aligned}$$

where $\phi_s = \mathrm{E}[p_s] \cdot (1 - G(Q_s))/(f_s^2 \theta N \omega)$ is positive and decreasing in $Q_s$. Thus, $C$ gives

$$\left(2v_1''v_2' + v_1' \frac{d^2 v_1}{dQ_1 dQ_2}\right) h_1(Q_1) h_2'(Q_2) = \left(v_1''v_2' + v_1 v_2 v_2' \frac{d\phi_1}{dQ_1}\right) h_1(Q_1) h_2'(Q_2) > 0.$$

$D$ is analogous to $C$. Thus, it can be shown that $D > 0$. $E$ gives

$$\begin{aligned} v_1''v_2'' - \frac{d^2 v_1}{dQ_1 dQ_2} \frac{d^2 v_2}{dQ_1 dQ_2} &= v_1''v_2'' - \phi_1 \phi_2 v_1' v_2' (v_1 - v_2)(v_2 - v_1) \\ &= v_1''v_2'' + \phi_1 \phi_2 v_1' v_2' (v_1 - v_2)^2 \\ &> 0. \end{aligned}$$

This proves the required inequality. □

Applying (EC8) and Lemmas 4 and 5 to (EC7), it follows that $dQ_s/d\theta < 0$ for $s = 1, 2$. □

## Reference

Debreu, D. 1952. A social equilibrium existence theorem. *Proc. Natl. Acad. Sci. USA* **38** 886–893.