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Electronic Companion—"Capturing Flexible Heterogeneous Utility Curves: A Bayesian Spline Approach" by Jin Gyo Kim, Ulrich Menzefricke, and Fred M. Feinberg, *Management Science* 2007, 53(2) 340–354.

Online Appendix

This appendix presents the Markov chain Monte Carlo (MCMC) sampler utilized in the main paper. Details on prior literature, set up of the various spline and benchmark models, descriptions of data sets, and all empirical details are included in the main paper.

EC.1. Evaluation of the Posterior Distribution with an MCMC Sampler

Given l_m, the MCMC sampler is designed to estimate individual-specific spline functions with varying knot configuration. Let:

- $\mathbf{y}_h = (y_{h1}, \ldots, y_{hT_h})$, $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_H)'$,
- $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \ldots, \boldsymbol{\beta}_H')'$,
- $\mathbf{u}_h = (\mathbf{u}_{h1}, \ldots, \mathbf{u}_{hT_h})$ and $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_H)$,
- $(\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) = \{(q_{hm}, \theta_{hm}, \gamma_{hm}^{(q_{hm}, \theta_{hm})})\}_{m=1}^{M}$ and $(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma}) = \{(\mathbf{q}_1, \boldsymbol{\theta}_1, \boldsymbol{\gamma}_1), \ldots, (\mathbf{q}_H, \{\boldsymbol{\theta}_H, \boldsymbol{\gamma}_H)\}$,
- $\boldsymbol{\pi}_h^{(\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h)} = (\pi_{h1}^{(\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h)}, \ldots, \pi_{hT_h}^{(\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h)})'$ denote effects of $\{v_{hjt,m}\}$ for individual h depending on $\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h$, and π_{ht} as in the main paper, and
- $\boldsymbol{\pi}^{(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma})} = (\boldsymbol{\pi}_1^{(q_1, \theta_1, \gamma_1)}, \ldots, \boldsymbol{\pi}_H^{(q_H, \theta_H, \gamma_H)})'$.

Then, the full posterior distribution is

$$p(\mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta, \boldsymbol{\Sigma}_u, \mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma} \mid \mathbf{y}) \propto p(\mathbf{y} \mid \mathbf{u}) p(\mathbf{u} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}^{(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma})}) \times p(\boldsymbol{\beta} \mid \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) p(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma}) p(\boldsymbol{\mu}_\beta) p(\boldsymbol{\Sigma}_\beta) p(\boldsymbol{\Sigma}_u), \tag{EC1}$$

where

$$p(\mathbf{y} \mid \mathbf{u}) p(\mathbf{u} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}^{(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma})}) \propto \prod_{h=1}^{H} \prod_{t=1}^{T_h} \left(N_J\left(\mathbf{u}_{ht} \mid \mathbf{x}_{ht} \boldsymbol{\beta}_h + \boldsymbol{\pi}_{ht}^{(q_h, \theta_h, \gamma_h)}, \boldsymbol{\Sigma}_u\right) I(\mathbf{u}_{ht} \in A_{ht}) \right),$$

and $A_{ht} = A_{h1t} \times \cdots \times A_{hJt}$ is the sample space of $\mathbf{u}_{ht}$. Furthermore,

$$p(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma}) = \prod_{h=1}^{H} \prod_{m=1}^{M} p\left(\gamma_{hm}^{(q_{hm}, \theta_{hm})} \mid q_{hm}, \theta_{hm}\right) p(\theta_{hm} \mid q_{hm}) p(q_{hm}).$$

To evaluate (EC1), we use Markov chain Monte Carlo methods, sampling all unknown quantities in sequence.

EC.1.1. Sampling from $p(\mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\mu}_{\boldsymbol{\beta}}, \boldsymbol{\Sigma}_\beta, \boldsymbol{\Sigma}_u \mid \mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathbf{y})$

1. Sample $\mathbf{u}_{ht}$ from $p(\mathbf{u}_{ht} \mid \boldsymbol{\beta}_h, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}_{ht}^{(\mathbf{q}_h, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h)}, \mathbf{y}) = N_J(\mathbf{u}_{ht} \mid \mathbf{x}_{ht} \boldsymbol{\beta}_h + \pi_{ht}^{(\mathbf{q}_h, \{\boldsymbol{\theta}_h, \boldsymbol{\gamma}_h)}, \boldsymbol{\Sigma}_u) I(\mathbf{u}_{ht} \in A_{ht})$ for each individual h and choice occasion t. As J increases, rejection sampling becomes very inefficient. We therefore sample $\mathbf{u}_{ht}$ under the constraint $I(\mathbf{u}_{ht} \in A_{ht})$ by the multivariate slice sampling method (Neal 2003), which allows one to sample multiple quantities simultaneously with only one auxiliary random variable (see Figure 8 on p. 723 of Neal's paper). In order to implement the multivariate slice sampler, it is important to set the width of slices to be sufficiently large. We set the slice width to be 10.

2. Sample a correlation matrix $\boldsymbol{\Sigma}_u = \{\sigma_{ij}\}$ from $p(\boldsymbol{\Sigma}_u \mid \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\pi}^{(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\gamma})}) = IW_J(\overline{m}, \bar{\mathbf{C}})$ by using the slice sampler for off-diagonal elements in sequence given the constraints $\sigma_{ij} = 1$ and $|\sigma_{ij}| < 1$, where $\overline{m} = m + \sum_{h=1}^{H} T_h$ and

$$\bar{\mathbf{C}} = \mathbf{C} + \sum_{h=1}^{H}\sum_{t=1}^{T_h}\big(\mathbf{u}_{ht} - \mathbf{x}_{ht}\boldsymbol{\beta}_h - \boldsymbol{\pi}_{ht}^{(\mathbf{q_h},\boldsymbol{\theta}_\mathbf{h},\boldsymbol{\gamma}_\mathbf{h})}\big)\big(\mathbf{u}_{ht} - \mathbf{x}_{ht}\boldsymbol{\beta}_h - \boldsymbol{\pi}_{ht}^{(\mathbf{q_h},\boldsymbol{\theta}_\mathbf{h},\boldsymbol{\gamma}_\mathbf{h})}\big)'.$$

3. Sample $\boldsymbol{\beta}_h$ from $p(\boldsymbol{\beta}_h \mid \mathbf{u}_h, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}_{ht}^{(\mathbf{q}_h,\boldsymbol{\theta}_h,\boldsymbol{\gamma}_h)}, \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) = N(\bar{\boldsymbol{\mu}}_\beta, \bar{\boldsymbol{\Sigma}}_\beta)$ for each household h, where $\bar{\boldsymbol{\mu}}_\beta = \bar{\boldsymbol{\Sigma}}_\beta\{\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta + \sum_{t=1}^{T_h}\mathbf{x}'_{ht}\boldsymbol{\Sigma}_u^{-1}(\mathbf{u}_{ht} - \boldsymbol{\pi}_{ht}^{(\mathbf{q}_h,\boldsymbol{\theta}_h,\boldsymbol{\gamma}_h)})\}$ and $\bar{\boldsymbol{\Sigma}}_\beta = (\boldsymbol{\Sigma}_\beta^{-1} + \sum_{t=1}^{T_h}\mathbf{x}'_{ht}\boldsymbol{\Sigma}_u^{-1}\mathbf{x}_{ht})^{-1}$.

4. Sample $\boldsymbol{\mu}_\beta$ from $p(\boldsymbol{\mu}_\beta \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}_\beta) = N(\bar{\boldsymbol{\mu}}_\beta, \bar{\boldsymbol{\Sigma}}_\beta)$, a multivariate normal density with $\bar{\boldsymbol{\mu}}_\beta = \bar{\boldsymbol{\Sigma}}_\beta(\mathbf{C}_\beta^{-1}\mathbf{m}_\beta + \boldsymbol{\Sigma}_\beta^{-1}\sum_{h=1}^{H}\boldsymbol{\beta}_h)$ and $\bar{\boldsymbol{\Sigma}}_\beta = (\mathbf{C}_\beta^{-1} + H\boldsymbol{\Sigma}_\beta^{-1})^{-1}$.

5. Sample $\boldsymbol{\Sigma}_\beta$ from $p(\boldsymbol{\Sigma}_\beta \mid \boldsymbol{\beta}, \boldsymbol{\mu}_\beta) = IW(\bar{r}_\beta, \overline{\mathbf{W}}_\beta)$, an inverse Wishart density with $\bar{r}_\beta = r_\beta + H$ and $\overline{\mathbf{W}}_\beta = \mathbf{W}_\beta + \sum_{h=1}^{H}(\boldsymbol{\beta}_h - \boldsymbol{\mu}_\beta)(\boldsymbol{\beta}_h - \boldsymbol{\mu}_\beta)'$.

EC.1.2. Sampling from $p(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\gamma} \mid \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta, \boldsymbol{\Sigma}_u, \mathbf{y})$

For each household and each spline, we have:

$$p\big(q_{hm}, \theta_{hm}, \gamma_{hm}^{(q_{hm},\theta_{hm})} \mid \mathbf{u}_h, \boldsymbol{\beta}_h, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}_h^{(\mathbf{q}_h,\boldsymbol{\theta}_h,\boldsymbol{\gamma}_h)}\big) \propto p\big(\mathbf{u}_h \mid \boldsymbol{\beta}_h, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}_h^{((\mathbf{q}_h,\boldsymbol{\theta}_h,\boldsymbol{\gamma}_h)}\big)p\big(\gamma_{hm}^{(q_{hm},\theta_{hm})} \mid q_{hm}, \theta_{hm}\big) \times p(\theta_{hm} \mid q_{hm})p(q_{hm}).$$

It is necessary to allow the dimension of $(q_{hm}, \theta_{hm}, \gamma_{hm})$ to change across iterations. So, as discussed previously, we use the reversible jump MCMC method (Green 1995), which is designed to move around a countable union of subspaces $\Phi = \bigcup_{i=0}^{\infty}\Phi_i$ by making random transitions between Φ_i and Φ_k. To implement it, we consider four possible transitions: (a) a birth step (addition of a knot), (b) a death step (deletion of a knot), (c) movement of a knot, and (d) update of γ_{hm} without changes in q_{hm} and θ_{hm}. Thus, the set of possible moves is $w \in \{U, M, 0, 1, 2, \ldots\}$, where U means an update of γ_{hm} without changes in q_{hm}, θ_{hm}, M means a random movement of a knot, and $m = 0, 1, 2, \ldots$ refers to increasing the number of interior knots from $q_{hm} = m$ to $q_h = m + 1$ or decreasing from $q_{hm} = m + 1$ to $q_{hm} = m$.

We let the probabilities for these four possible transitions be

$$\varsigma_{q_{hm}} = a\min\left(1, \frac{p(q_{hm}+1)}{p(q_{hm})}\right) \quad \text{for a birth step,} \tag{EC2}$$

$$\tau_{q_{hm}} = a\min\left(1, \frac{p(q_{hm}-1)}{p(q_{hm})}\right) \quad \text{for a death step,} \tag{EC3}$$

$$\upsilon_{q_{hm}} = b(1 - \varsigma_{q_{hm}} - \tau_{q_{hm}}) \quad \text{for a move step,} \quad \text{and} \tag{EC4}$$

$$\xi_{q_{hm}} = 1 - \varsigma_{q_{hm}} - \tau_{q_{hm}} - \upsilon_{q_{hm}} \quad \text{for an update step.} \tag{EC5}$$

The positive constant a should be as large as possible subject to $\varsigma_{q_h} + \tau_{q_h} \le 0.9$ for all $q_{hm} = 0, 1, \ldots, Q_m$. Note that this has the practical consequence of bounding the move and update step probabilities below by $1 - 2a$. It is easy to find the constant a by examining $(\varsigma_{q_h} + \tau_{q_h})$ over all values of $q_{hm} \le Q_m$. Note that a should fall in the interval $[0, 0.5]$; otherwise, if $a > 0.5$, the sum of the probabilities ς_{q_h} and τ_{q_h} could be greater than one for some values of q_{hm}. We take $a = 0.45$, but other values are also valid. Because q_{hm} should be in the interval $[0, Q_m]$, we use $\tau_0 = \upsilon_0 = 0$ and $\varsigma_{Q_m} = \upsilon_{Q_m} = 0$. Note that the ratio $p(q_{hm} \pm 1)/p(q_{hm})$ is not affected by this truncation because $p(q_{hm})I(q_{hm} \le Q_m) = (1/\sum_{i=0}^{Q_m}\text{Po}(q_{im} \mid \lambda_m)) \cdot \text{Po}(q_{hm} \mid \lambda_m)$. In addition, we let $b = 0.5$, but other nonnegative values from the interval $(0, 1)$ are also valid.

EC.1.2.1 Update Step. In the update step, only $\gamma_{hm} = \gamma_{hm}^{(q_{hm},\theta_{hm})}$ is updated, q_{hm} and θ_{hm} being unchanged. Let the scalar $\tilde{u}_{hjtm} = u_{hjt} - \mathbf{x}'_{hjt}\boldsymbol{\beta}_h - \sum_{i=1,\, i\neq m}^{M} f_i^h(v_{hjt,\,i})$, let $\tilde{\mathbf{u}}_{htm} = (\tilde{u}_{h1tm}, \ldots, \tilde{u}_{hJtm})'$, a

J-dimensional vector, and let $\tilde{\mathbf{u}}_{hm} = (\tilde{\mathbf{u}}_{h1m}, \ldots, \tilde{\mathbf{u}}_{hT_hm})$. Furthermore, let the $(l_m + q_{hm})$-dimensional vector $\mathbf{z}_{hjtm}$ collect the coefficients of γ_{hm}, that is,

$$\mathbf{z}_{hjtm} = \begin{pmatrix} (v_{hjtm} - s_{hm0})_+^1 \\ (v_{hjtm} - s_{hm0})_+^2 \\ \vdots \\ (v_{hjtm} - s_{hm0})_+^{l_m} \\ (v_{hjtm} - s_{hm1})_+^{l_m} \\ \vdots \\ (v_{hjtm} - s_{hmq_{hm}})_+^{l_m} \end{pmatrix},$$

and let $\mathbf{z}_{htm} = (\mathbf{z}_{h1tm}, \ldots, \mathbf{z}_{hJtm})'$ be a $J \times (l_m + q_{hm})$ matrix. Thus, $\tilde{\mathbf{u}}_{htm} = \mathbf{z}_{htm}\boldsymbol{\gamma}_{hm}^{(q_{hm},\,\theta_{hm})} + \boldsymbol{\varepsilon}_{ht}$. We then have

$$p(\gamma_{hm} \mid \tilde{\mathbf{u}}_{hm}, \boldsymbol{\Sigma}_u, \mathbf{q}_h, \boldsymbol{\theta}_h) = \frac{1}{c_{hm}} p(\tilde{\mathbf{u}}_{hm} \mid \gamma_{hm}, \boldsymbol{\Sigma}_u) p(\gamma_{hm} \mid q_h, \theta_{hm}),$$

with normalizing constant

$$c_{hm} = c(q_{hm}, \theta_{hm}) = \frac{\exp\{-\frac{1}{2}[\sum_{t=1}^{T_h} \tilde{\mathbf{u}}'_{htm} \boldsymbol{\Sigma}_u^{-1} \tilde{\mathbf{u}}_{htm} + (l_m + q_{hm})\frac{a_m^2}{b_m} + \mathbf{g}'_{hm}\mathbf{G}_{hm}^{-1}\mathbf{g}_{hm}]\}}{(2\pi)^{JT_h/2} |\boldsymbol{\Sigma}_u|^{T_h/2} \left|\mathbf{I}_{l_m+q_{hm}} + b_m \sum_{t=1}^{T_h} \mathbf{z}'_{htm}\boldsymbol{\Sigma}_u^{-1}\mathbf{z}_{htm}\right|^{1/2}}, \tag{EC6}$$

where $\mathbf{g}_{hm} = \mathbf{G}_{hm}\{(a_m/b_m)\mathbf{1}_{l_m+q_{hm}} + \sum_{t=1}^{T_h} \mathbf{z}'_{htm}\boldsymbol{\Sigma}_u^{-1}\tilde{\mathbf{u}}_{htm}\}$, $\mathbf{G}_{hm} = \{(1/b_m)\mathbf{I}_{l_m+q_{hm}} + \sum_{t=1}^{T_h} \mathbf{z}'_{htm}\boldsymbol{\Sigma}_u^{-1}\mathbf{z}_{htm}\}^{-1}$, $\mathbf{1}_q$ is a vector of q ones, and the prior parameters a_m and b_m are defined as in the main paper, that is, the conditional posterior distribution for γ_{hm} given $(\tilde{\mathbf{u}}_{htm}, \boldsymbol{\Sigma}_u, \boldsymbol{\pi}_h^{(\mathbf{q}_h,\,\boldsymbol{\theta}_h,\,\gamma_h)}, \mathbf{q}_h, \boldsymbol{\theta}_h)$ is

$$\boldsymbol{\gamma}_{hm} \sim N_{l_m+q_{hm}}(\mathbf{g}_{hm}, \mathbf{G}_{hm}). \tag{EC7}$$

When a constraint is imposed on the splines, for example, monotonicity, then $\boldsymbol{\gamma}_{hm}$ must be sampled from $N_{l_m+q_{hm}}(\mathbf{g}_{hm}, \mathbf{G}_{hm}) I(\boldsymbol{\gamma}_{hm} \in B)$, where B denotes the space of $\boldsymbol{\gamma}_{hm}$ under the imposed constraint. It is thus necessary to multiply the normalizing constant in (EC6) by a correction factor involving $\phi(q_{hm}, \theta_{hm}) = \int_{\gamma_{hm}\in B} p(\boldsymbol{\gamma}_{hm} \mid \mathbf{g}_{hm}, \mathbf{G}_{hm})\, d\gamma_{hm}$ so that the normalizing constant is

$$c_{hm}^* = c^*(q_{hm}, \theta_{hm}) = \frac{1}{\phi(q_{hm}, \theta_{hm})} c(q_{hm}, \theta_{hm}).$$

In this case, we can use the single variable slice sampler (cf., Neal 2003) to sample $\boldsymbol{\gamma}_{hm}$.

EC.1.2.2. Birth, Death, and Move Steps. In the birth, death, and move steps, the transition probabilities in (EC2) to (EC4) yield a proposed new value for $\bar{q}_{hm} = q_{hm} + r$, where $r = -1, 0, 1$. Given $\bar{q}_{hm}$, we must then propose a new value for θ_{hm}, $\bar{\theta}_{hm}$. To add a new knot or move/delete one of the knots currently present, we have a set of candidate knots, $\mathcal{D}_{hm} = \{D_{hm1}, \ldots, D_{hmQ_m}\}$, which can be, for example, a set of prespecified grid points or a subset of $\{v_{hjt,\,m}\}$ for all j and t. Then, for each iteration, a newly proposed value of $\bar{\theta}_{hm}$, given $\bar{q}_{hm}$, is generated as follows:

1. Birth step: Add a knot uniformly chosen from one of the $Q_m - q_{hm}$ candidate grid points from $\mathcal{D}_{hm}$.
2. Death step: Delete a knot uniformly chosen from the q_{hm} knots currently present.
3. Move step: Choose one of the currently present q_h knots uniformly and change its location to a value uniformly chosen from the currently nonpresent knots.

Given the newly proposed knot configuration $(\bar{q}_{hm}, \bar{\theta}_{hm})$, we must finally propose a value for the spline coefficients for each household and each spline, $\bar{\boldsymbol{\gamma}}_{hm} = \bar{\boldsymbol{\gamma}}_{hm}^{(\bar{q}_{hm},\,\bar{\theta}_{hm})}$, where we drop the superscript for expository convenience. Our proposal distribution for these spline coefficients, $\bar{\gamma}_{hm}$, is their conditional posterior distribution, that is, $\bar{\boldsymbol{\gamma}}_{hm} \sim N_{l_m+\bar{q}_{hm}}(\bar{\mathbf{g}}_{hm}, \bar{\mathbf{G}}_{hm})$, where $\bar{\mathbf{g}}_{hm}$ and $\bar{\mathbf{G}}_{hm}$ are defined just before (EC7). Note that we have changed the notation from $\mathbf{g}_{hm}$ and $\mathbf{G}_{hm}$ in (EC7) to $\bar{\mathbf{g}}_{hm}$ and $\bar{\mathbf{G}}_{hm}$ to make explicit that these quantities are computed for the newly proposed knot configuration $(\bar{q}_{hm}, \bar{\theta}_{hm})$.

Given the proposal $(\bar{q}_{hm}, \bar{\theta}_{hm}, \bar{\boldsymbol{\gamma}}_{hm}^{(\bar{q}_{hm}, \bar{\theta}_{hm})})$ generated above, we must now decide whether or not to accept it. Using the notation in Green (1995), the acceptance probability for each move type is

$$\alpha = \min\{(1, (\text{likelihood ratio}) \times (\text{prior ratio}) \times (\text{proposal ratio}) \times \text{Jacobian})\}.$$

The Jacobian is needed for the steps in which the dimension is changing: It does not matter for the move step, but it does for the birth and death steps. In both these steps, the Jacobian term is one because the new knot configuration, $\bar{\mathbf{q}}_h$ and $\boldsymbol{\theta}_h^{(\bar{\mathbf{q}}_h)}$, and the spline coefficients, $\bar{\boldsymbol{\gamma}}_{hm}^{(\bar{q}_{hm}, \bar{\theta}_{hm})}$, are generated independently of the previous values.

Concentrating first on the pair $(q_{hm}, \theta_{hm}^{(q_{hm})})$ and a birth step, the prior ratio is

$$\frac{p(q_{hm}+1)}{p(q_{hm})} \frac{1/\binom{Q_m}{q_{hm}+1}}{1/\binom{Q_m}{q_{hm}}} = \frac{p(q_{hm}+1)}{p(q_{hm})} \frac{q_{hm}+1}{Q_m - q_{hm}},$$

and the proposal ratio is

$$\frac{P(\text{death} \mid q_{hm}+1)\frac{1}{q_{hm}+1}}{P(\text{birth} \mid q_{hm})\frac{1}{Q_m - q_{hm}}} = \frac{p(q_{hm})}{p(q_{hm}+1)} \frac{Q_m - q_{hm}}{q_{hm}+1},$$

where we have used the fact that $P(\text{death} \mid q_{hm}+1) = a\min\{1, p(q_{hm})/p(q_{hm}+1)\}$ and $P(\text{birth} \mid q_{hm}) = a\min\{1, p(q_{hm}+1)/p(q_{hm})\}$. Combining prior and proposal ratios for the pair $(q_{hm}, \theta_{hm}^{(q_{hm})})$, we find the product to be one. For a death step the ratio is the inverse, that is, one, and for a move step, it is also one.

Let us now concentrate on the likelihood ratio, and on the prior ratio and proposal ratio contributed by the spline coefficients. Because the proposal distribution for the spline coefficient vector is its conditional posterior distribution as given in (EC7), the product of likelihood ratio, prior ratio, and proposal ratio is just the corresponding ratio of the normalizing constants in (EC6), $c(\bar{q}_{hm}, \bar{\theta}_{hm})/c(q_{hm}, \theta_{hm})$, where $c(\bar{q}_{hm}, \bar{\theta}_{hm})$ indicates that the normalizing constant is computed for the newly proposed knot configuration.

The acceptance probability is thus

$$\alpha = \min\left\{1, \frac{c(\bar{q}_{hm}, \bar{\theta}_{hm})}{c(q_{hm}, \theta_{hm})}\right\}, \tag{EC8}$$

with

$$\frac{c(\bar{q}_{hm}, \bar{\theta}_{hm})}{c(q_{hm}, \theta_{hm})} = \frac{\left|\mathbf{I}_{l_m+q_{hm}} + b_m \sum_{t=1}^{T_h} \mathbf{z}'_{htm} \boldsymbol{\Sigma}_u^{-1} \mathbf{z}_{htm}\right|^{1/2}}{\left|\mathbf{I}_{l_m+\bar{q}_{hm}} + b_m \sum_{t=1}^{T_h} \bar{\mathbf{z}}'_{htm} \boldsymbol{\Sigma}_u^{-1} \bar{\mathbf{z}}_{htm}\right|^{1/2}} \times \exp\left\{\frac{1}{2}\left[(q_{hm} - \bar{q}_{hm})\frac{a_m^2}{b_m} + \mathbf{g}'_{hm}\mathbf{G}_{hm}^{-1}\mathbf{g}_{hm} - \bar{\mathbf{g}}'_{hm}\bar{\mathbf{G}}_{hm}^{-1}\bar{\mathbf{g}}_{hm}\right]\right\},$$

where the term $(q_{hm} - \bar{q}_{hm})$ equals $1, 0$, and -1 for a death, move, and birth step, respectively.

When there is a constraint on the spline coefficient vector such that $\boldsymbol{\gamma}_{hm} \in B$, each of the normalizing constants in (EC8) must be multiplied by its appropriate correction factor involving $\phi(\bullet)$ as discussed before and we must sample the newly proposed value for the spline coefficient vector, $\bar{\boldsymbol{\gamma}}_{hm}$, from this region. We found rejection sampling to be very inefficient for this purpose, and so used a slice sampler instead.

References

See references list in the main paper.

Neal, R. M. 2003. Slice sampling (with discussion). *Ann. Statist.* **31** 705–767.