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Online Appendix

Proof of Equation 2. The cycle time is defined as $\tau = \min(T^1_{\Delta_H}, \ldots, T^N_{\Delta_H}, T_{Q_H})$; the event "$\tau > t$" is equivalent to each retailer having at most $\Delta_H - 1$ units of demand; and the total demand from all retailers being less than or equal to $Q_H - 1$ in $[0, t]$. Because the demand at any retailer is Poisson with rate λt, we have the following expression considering all possible demand realizations in the interval $[0, t]$:

$$F(t) = 1 - \Pr(\tau > t) = 1 - \Pr\left(D_1(t) \le \Delta_H - 1, \ldots, D_N(t) \le \Delta_H - 1, \sum_{i=1}^{N} D_i(t) \le Q_H - 1\right)$$
$$= 1 - \sum_{\substack{d_1, d_2, \ldots, d_N \le \Delta_H - 1 \\ d_1 + d_2 + \cdots + d_N \le Q_H - 1}} \prod_{i=1}^{N} p(d_i; \lambda t) = 1 - \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \cdots \sum_{d_N=0}^{U_N} \prod_{i=1}^{N} p(d_i; \lambda t),$$

with $U_1 = \min(\Delta_H - 1, Q_H - 1)$ and $U_i = \min(\Delta_H - 1, Q_H - 1 - d_1 - \cdots - d_{i-1})$ for $i = 2, \ldots, N$.

Note that the upper limits, U_i, for the variables, d_i, make sure that the total demand from all retailers is less than Q_H and demand at each retailer is less than Δ_H.

Proof of Lemma 1. Define:

$$V_i(Q_H, \Delta_H, t) = \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \cdots \sum_{d_N=0}^{U_N} \frac{dp(d_i; \lambda t)}{dt} \prod_{\forall j \ne i} p(d_j; \lambda t). \tag{EC1}$$

Using $dp(d_i; \lambda t)/dt = \lambda(p(d_i - 1; \lambda t) - p(d_i; \lambda t))$ and (EC1), with some effort, it can be shown that:

$$V_i(Q_H, \Delta_H, t) = V_N(Q_H, \Delta_H, t) = -\lambda \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \cdots \sum_{d_{N-1}=0}^{U_{N-1}} (p(U_N; \lambda t)) \prod_{\forall j \ne N} p(d_j; \lambda t), \quad i = 1, \ldots, N. \tag{EC2}$$

Next, the probability density function of the cycle time, τ, can be written as:

$$f(t) = \frac{dF(t)}{dt} = -\sum_{i=1}^{N} V_i(Q_H, \Delta_H, t).$$

Using (EC2), $f(t)$ reduces to:

$$f(t) = -NV_N(Q_H, \Delta_H, t) = N\lambda \sum_{d_1=0}^{U_1} \sum_{d_2=0}^{U_2} \cdots \sum_{d_{N-1}=0}^{U_{N-1}} (p(U_N; \lambda t)) \prod_{\forall j \ne N} p(d_j; \lambda t). \tag{EC3}$$

The proof is now complete. □

Proof of Lemma 2. Without loss of generality, assume that the retailer of interest is retailer 1. The event "$D_1(\tau) \ge n$" is equivalent to the event that the nth demand arrives at retailer 1 before the replenishment has been triggered. Considering all the possible combinations of demand from other retailers (as long as the total is less than or equal to $Q_H - 1$), we have:

$$\Pr(D_1(\tau) \geq n) = \int_{t=0}^{\infty} \sum_{d_2=0}^{U_2} \cdots \sum_{d_N=0}^{U_N} \prod_{i=2}^{N} p(d_i;\, \lambda t) \frac{e^{-\lambda t} \lambda (\lambda t)^{(n-1)}}{(n-1)!}\, dt, \tag{EC4}$$

where, for $i = 2, \ldots, N$: $U_i = \min(\Delta_H - 1, Q_H - 1 - d_1 - \cdots - d_{i-1})$ and $d_1 = n$.

Similarly, for the outbound quantity, the expression will be the same as in (EC4) except that the total can be equal to Q_H now (instead of $Q_H - 1$). Thus, we have the following:

$$\Pr(Z_1(\tau) \geq n) = \int_{t=0}^{\infty} \sum_{d_2=0}^{U_2} \cdots \sum_{d_N=0}^{U_N} \prod_{i=2}^{N} p(d_i;\, \lambda t) \frac{e^{-\lambda t} \lambda (\lambda t)^{(n-1)}}{(n-1)!}\, dt, \tag{EC5}$$

where, for $i = 2, \ldots, N$: $U_i = \min(\Delta_H - 1, Q_H - 1 - d_1 - \cdots - d_{i-1})$ and $d_1 = n - 1$.

Therefore, when $Q_H > \Delta_H + (N-1) \cdot (\Delta_H - 1)$, we have:

$$\Pr(D_1(\tau) \geq n) = \int_{t=0}^{\infty} \sum_{d_2=0}^{\Delta_H - 1} \cdots \sum_{d_N=0}^{\Delta_H - 1} \prod_{i=2}^{N} p(d_i;\, \lambda t) \frac{e^{-\lambda t} \lambda (\lambda t)^{(n-1)}}{(n-1)!}\, dt = \int_{t=0}^{\infty} \lambda p(n-1;\, \lambda t) [1 - P(\Delta_H;\, \lambda t)]^{(N-1)}\, dt.$$

The same expression holds for the outbound quantity. $\square$

Proof of Lemma 4. The above result is easily obtained using the conditional expectation formula and the relation between the inbound quantity (Z_0) and τ

$$E[Z_0] = E_\tau[E[Z_0 \mid \tau]] = E_\tau[N\lambda\tau] = N\lambda E[\tau].$$

Therefore,

$$E[\tau] = \frac{E[Z_0]}{N\lambda}. \quad \square \tag{EC6}$$

Proof of Lemma 5. We know that the expected cycle time for given Q_H, Δ_H, and N can be written as:

$$\begin{aligned} E[\tau] &= \int_{t=0}^{\infty} \Pr(\tau \geq t)\, dt = \int_{t=0}^{\infty} \sum_{d_1=0}^{U_1} \cdots \sum_{d_N=0}^{U_N} \prod_{i=1}^{N} p(d_i;\, \lambda t)\, dt = \sum_{d_1=0}^{U_1} \cdots \sum_{d_N=0}^{U_N} \frac{1}{d_1! \cdots d_N!} \int_{t=0}^{\infty} e^{-N\lambda t} (\lambda t)^{D_0}\, dt \\ &= \sum_{d_1=0}^{U_1} \cdots \sum_{d_N=0}^{U_N} \frac{D_0!}{N\lambda (N)^{D_0} d_1! \cdots d_N!} = \sum_{d_1=0}^{U_1} \cdots \sum_{d_N=0}^{U_N} \frac{h(d_1, d_2, \ldots, d_N)}{N\lambda}, \end{aligned} \tag{EC7}$$

where

$$D_0 = \sum_{i=1}^{N} d_i, \qquad h(d_1, d_2, \ldots, d_N) = \frac{D_0!}{N^{D_0} * \prod_{i=1}^{N} d_i!}.$$

Using (EC7), we can write the following recursive relation of the expected cycle time with respect to Q_H for $N-1$ retailers:

$$\begin{aligned} &E[\tau \mid Q_H + 1 - n, \Delta_H, N-1] - E[\tau \mid Q_H - n, \Delta_H, N-1] \\ &\quad = \frac{1}{(N-1)\lambda} \sum_{\substack{d_1, \ldots, d_{N-1} \leq \Delta_H - 1 \\ D_0 = Q_H - n}} \frac{D_0!}{(N-1)^{D_0} d_1! \cdots d_{N-1}!} \\ &\quad = \frac{(Q_H - n)!}{\lambda (N-1)^{(Q_H + 1 - n)}} \sum_{\substack{d_1, \ldots, d_{N-1} \leq \Delta_H - 1 \\ D_0 = Q_H - n}} \frac{1}{d_1! \cdots d_{N-1}!}. \end{aligned} \tag{EC8}$$

Note that here $D_0 = d_1 + \cdots + d_{N-1}$. Equation (EC8) yields:

$$\sum_{\substack{d_1,\ldots,d_{N-1}\le\Delta_H-1\\ D_0=Q_H-n}} \frac{1}{d_1!\cdots d_{N-1}!} = \frac{\lambda(N-1)^{(Q_H+1-n)}(E[\tau \mid Q_H+1-n,\Delta_H,N-1]-E[\tau\mid Q_H-n,\Delta_H,N-1])}{(Q_H-n)!}, \tag{EC9}$$

and we know from (EC5) that

$$\Pr(Z_1(\tau)\ge n) = \int_{t=0}^{\infty}\sum_{d_2=0}^{U_2}\cdots\sum_{d_N=0}^{U_N}\prod_{i=2}^{N} p(d_i;\lambda t)\frac{e^{-\lambda t}\lambda(\lambda t)^{(n-1)}}{(n-1)!}\,dt = \sum_{d_1=n-1}^{n-1}\sum_{d_2=0}^{U_2}\cdots\sum_{d_N=0}^{U_N}\frac{h(d_1,d_2,\ldots,d_N)}{N}.$$

Therefore, when $Q_H > \Delta_H$ we can write the following recursive relation of $\Pr(Z\ge n)$ with respect to Q_H as follows:

$$\begin{aligned} &P(Z\ge n\mid Q_H,\Delta_H,N) - P(Z\ge n\mid Q_H-1,\Delta_H,N)\\ &\quad= \frac{1}{N}\sum_{\substack{d_2,\ldots,d_N\le\Delta_H-1\\ D_0=Q_H-n}}\frac{(D_0+n-1)!}{(N)^{(D_0+n-1)}(n-1)!d_2!\cdots d_N!}\\ &\quad= \left(\frac{1}{N}\right)^{Q_H}\frac{(Q_H-1)!}{(n-1)!}\sum_{\substack{d_2,\ldots,d_N\le\Delta_H-1\\ D_0=Q_H-n}}\frac{1}{d_2!\cdots d_N!},\qquad D_0=d_2+\cdots+d_N. \end{aligned} \tag{EC10}$$

Finally, substituting (EC9) in (EC10) yields:

$$\begin{aligned} &P(Z\ge n\mid Q_H,\Delta_H,N) - P(Z\ge n\mid Q_H-1,\Delta_H,N)\\ &\quad= \left(\frac{1}{N}\right)^{Q_H}\frac{(Q_H-1)!}{(n-1)!}\sum_{\substack{d_2,\ldots,d_N\le\Delta_H-1\\ D_0=Q_H-n}}\frac{1}{d_2!\cdots d_N!}\\ &\quad= \left(\frac{1}{N}\right)^{Q_H}\frac{(Q_H-1)!}{(n-1)!}\,\frac{\lambda(N-1)^{Q_H+1-n}(E[\tau\mid Q_H+1-n,\Delta_H,N-1]-E[\tau\mid Q_H-n,\Delta_H,N-1])}{(Q_H-n)!}\\ &\quad= \lambda\binom{Q_H-1}{n-1} = \left(\frac{1}{N}\right)^{n-1}\left(1-\frac{1}{N}\right)^{Q_H+1-n}(E[\tau\mid Q_H+1-n,\Delta_H,N-1]-E[\tau\mid Q_H-n,\Delta_H,N-1]). \end{aligned}$$

And when $Q_H \le \Delta_H$, the hybrid policy reduces to policy EB (echelon-based), and the probability of the outbound quantity being greater than $n=\min(\Delta_H, Q_H)$, is just equal to the outbound quantity being equal to Q_H. And it is given by:

$$P(Z=Q_H) = \binom{Q_H}{Q_H}\left(\frac{1}{N}\right)^{Q_H}\left(1-\frac{1}{N}\right)^{Q_H-Q_H} = \left(\frac{1}{N}\right)^{Q_H}. \quad \square$$

Proof of Proposition 1. The first term in (11) is the warehouse ordering cost per unit time. A fixed cost of K_0 independent of the order size is incurred at the warehouse each time an order is placed, so the expected ordering cost per unit time at the warehouse is (Ross 1993, p. 318):

$$E[\text{Ordering Cost}] = \frac{K_0}{E[\tau]}.$$

Because the demand during lead time is Poisson with rate $\lambda * LT$, we obtain the following by conditioning on IP:

$$E[IL^+] = \sum_{j=(\max(S_H-Q_H+1,\,S_H-\Delta_H+1))^+}^{S_H}\ \sum_{l=0}^{j}(j-l)p(l;\lambda*LT)\Pr\{IP=j\} = E[(IP-D(LT))^+], \tag{EC11}$$

$$
\begin{aligned}
E[IL^-] = & \sum_{j=\max(S_H-Q_H+1,\, S_H-\Delta_H+1)}^{-1} \sum_{l=0}^{\infty} (l-j)p(l;\, \lambda * LT)\Pr\{IP=j\} \\
& + \sum_{j=(\max(S_H-Q_H+1,\, S_H-\Delta_H+1))^+}^{S_H} \sum_{l=j+1}^{\infty} (l-j)p(l;\, \lambda * LT)\Pr\{IP=j\} \\
= & E[(D(LT)-IP)^+]. \qquad \text{(EC12)}
\end{aligned}
$$

Using (EC11) and (EC12), one can express the expected backorders in terms of the expected on-hand inventory as follows:

$$
E[IL^-] = E[IL^+] + E[D(LT)] - E[IP] = E[IL^+] + E[D(LT)] - \sum_{j=\max(S_H-Q_H+1,\, S_H-\Delta_H+1)}^{S_H} j\Pr(IP=j). \qquad \text{(EC13)}
$$

Consequently, using (EC13), the sum of the expected holding and shortage costs at any retailer will be:

$$
\begin{aligned}
E[\text{Holding} + \text{Shortage Cost at a retailer}] &= hE[IL^+] + \pi E[IL^-] \\
&= (h+\pi)E[IL^+] + \pi(E[D(LT)] - E[IP]).
\end{aligned}
$$

The last two terms of (11) are the expected cost of transportation penalty costs due to inbound and outbound quantities exceeding the capacity limits, respectively. □

Proof of Proposition 2. The second order difference with respect to S_H is always greater than zero as follows:

$$
\Delta^2_{S_H} CR = N(h+\pi) \sum_{j=\max(1,\, S_H-Q_H+1,\, S_H-\Delta_H+1)}^{S_H} \Pr(IP=j)p(j;\, \lambda LT) > 0.
$$

Therefore, the following first order difference can be used to find the optimal S_H given Δ_H and Q_H:

$$
\Delta_{S_H} CR = N\left\{(h+\pi)\left(\sum_{j=\max(1,\, S_H-Q_H+1,\, S_H-\Delta_H+1)}^{S_H} \Pr(IP=j)(1-P(j;\, \lambda * LT))\right) - \pi\right\}. \quad \square
$$

Proof of Lemma 7. Recall that (8) states $P(IP=j \mid D(\tau)=n) = 1/(n+1)$ for $j = S_{EB}-n,\, S_{EB}-n+1, \ldots, S_{EB}$. Because the quantity demanded from any retailer is the random disaggregation of the total demand, it is binomially distributed

$$
\Pr(D(\tau)=n) = \binom{Q_{EB}-1}{n}\left(\frac{1}{N}\right)^n\left(1-\frac{1}{N}\right)^{Q_{EB}-1-n}, \quad 0 \le n \le Q_{EB}-1. \qquad \text{(EC14)}
$$

Using (EC14) to remove the condition on "n," we obtain:

$$
\begin{aligned}
\Pr(IP=j) &= \sum_{n=S-j}^{Q_{EB}-1} \Pr(IP=j \mid D(\tau)=n)\Pr(D(\tau)=n) \\
&= \sum_{n=S-j}^{Q_{EB}-1} \frac{1}{n+1}\binom{Q_{EB}-1}{n}\left(\frac{1}{N}\right)^n\left(1-\frac{1}{N}\right)^{Q_{EB}-1-n}. \qquad \text{(EC15)}
\end{aligned}
$$

Simplifying (EC15), we obtain the expression in (15). □

$B(1, S_{TB}-1, T)$ and the First-Order Difference w.r.t. S_{TB} Under Policy TB (Time-Based)
These expressions can also be found in Hadley and Whitin (1963):

$$\begin{aligned} B(1, S_{TB}-1, T) = \frac{1}{T}\bigg\{ &\frac{\lambda}{2}[(LT+T)^2 P(S_{TB}-1; \lambda(LT+T)) - LT^2 P(S_{TB}-1; \lambda * LT)] \\ &+ \frac{S_{TB}(S_{TB}+1)}{2\lambda}[P(S_{TB}+1; \lambda(LT+T)) - P(S_{TB}+1; \lambda * LT)] \\ &- S_{TB}[(LT+T)P(S_{TB}; \lambda(LT+T)) - LTP(S_{TB}; \lambda * LT)]\bigg\}, \end{aligned}$$

and

$$\begin{aligned} \Delta_{S_{TB}} CR = CR|_{S_{TB}} - CR|_{S_{TB}-1} = h &+ \frac{S_{TB}}{\lambda T}(h+\pi)[P(S_{TB}; \lambda(LT+T)) - P(S_{TB}; \lambda * LT)] \\ &- \frac{h+\pi}{T}[(LT+T)P(S_{TB}; \lambda(LT+T)) - LT_i P(S_{TB}; \lambda * LT)] \\ &- \frac{(h+\pi)S_{TB}}{\lambda T}[p(S_{TB}; \lambda(LT+T)) - p(S_{TB}; \lambda * LT)]. \end{aligned} \tag{EC16}$$

For a given T, the optimal value of S_{TB} will be the largest integer where the expression in (EC16) is nonpositive. To find the optimal T, we search over possible values of T.

Variance of Demand Rate at the Supplier
Denote by X the random variable for the supplier's demand rate, Y the number of orders received by the supplier per time unit, and Z_0 the quantity demanded in each order. It is easy to see that $E[X] = N\lambda$ under all the policies. Next we derive the variance of the demand rate.

For the noncoordinated model, we use Equation (4) in Svoronos and Zipkin (1988) for the variance of the number of orders received by the supplier during time L_w:

$$\mathrm{Var}(D_w) = \frac{\lambda L_W N}{Q_{NC}} + \frac{N}{Q_{NC}^2}\sum_{k=1}^{Q_{NC}-1}\frac{[1-\exp(-\alpha_k \lambda L_W)\cos(\beta_k \lambda L_W)]}{\alpha_k},$$

where $\alpha_k = 1-\cos(2\pi k/Q_{NC})$ and $\beta_k = \sin(2\pi k/Q_{NC})$.

Then the demand rate variance can be derived as follows:

$$\mathrm{Var}(X) = \lim_{L_w\to\infty}(Q_{NC}^2 \mathrm{Var}(D_W)/L_w) = N\lambda.$$

For policy TB, cycle time is a constant T. So the demand received by the supplier every cycle is Poisson with rate $N\lambda T$, which is also its variance. Therefore, the demand rate variance is $N\lambda$.

For policy EB, each order at the supplier has a constant size Q_{EB}, and the cycle time has Erlang distribution, with a mean of $Q_{EB}/(N\lambda)$ and a variance of $Q_{EB}/(N\lambda)^2$. According to Deurmeyer and Schwarz (1981), the demand rate variance at the supplier is again $N\lambda$.

Finally, we use the following relation to derive the supplier's demand rate variance. Because this relation assumes independence between Y and Z, our derivation below is an approximation.

$$\mathrm{Var}(X) = \mathrm{Var}(Y)E^2(Z_0) + E(Y)\mathrm{Var}(Z_0). \tag{EC17}$$

From Ross (1993, p. 317) the asymptotic mean and variance for Y is $1/\mu$ and σ^2/μ^3, respectively, in the steady state where μ and σ^2 are the mean and variance of the cycle time in our problem. To compute μ and σ^2, we use the following relations:

$$\mu = E[\tau] = \frac{E[Z_0]}{N\lambda}, \tag{EC18}$$

and

$$\begin{aligned} E[\tau^2] &= \int_{t=0}^{\infty} t^2 f(t)\,dt = \sum_{d_1=0}^{U_1}\cdots\sum_{d_{N-1}=0}^{U_{N-1}}\sum_{d_N=U_N}^{U_N}\frac{1}{N\lambda N^{D_0} d_1!\cdots d_N!}\int_{t=0}^{\infty} e^{-N\lambda t}(N\lambda t)^{(D_0+2)}\,dt \\ &= \sum_{d_1=0}^{U_1}\cdots\sum_{d_{N-1}=0}^{U_{N-1}}\sum_{d_N=U_N}^{U_N}\frac{(D_0+2)!}{(N\lambda)^2 N^{D_0} d_1!\cdots d_N!} \end{aligned}$$

Table EC.1 Variance and Mean of X for All Policies

	Hybrid policy	Policy IB	Policy EB	Policy TB	Noncoordinated model
E(X)	$N\lambda$	$N\lambda$	$N\lambda$	$N\lambda$	$N\lambda$
Var(X)	$N\lambda\left(1+2\frac{\mathrm{Var}(Z_0)}{E(Z_0)}\right)$	$N\lambda\left(1+2\frac{\mathrm{Var}(Z_0)}{E(Z_0)}\right)$	$N\lambda$	$N\lambda$	$N\lambda$

$$\begin{aligned}
&= \sum_{d_1=0}^{U_1}\cdots\sum_{d_{N-1}=0}^{U_{N-1}}\sum_{d_N=U_N}^{U_N}\frac{(D_0+2)(D_0+1)h(d_1,d_2,\ldots,d_N)}{(N\lambda)^2}\\
&= \frac{1}{(N\lambda)^2}\sum_{k=(\Delta-1)\wedge(Q-1)}^{Q-1}(k+2)(k+1)P(D_0=k) = \frac{1}{(N\lambda)^2}\sum_{n=\Delta\wedge Q}^{Q}n(n+1)P(Z_0=n)\\
&= \frac{1}{(N\lambda)^2}(E[Z_0^2]+E[Z_0]). \qquad \text{(EC19)}
\end{aligned}$$

Using (EC18) and (EC19), we get the variance of the cycle time for the hybrid policy and policy IB as follows:

$$\sigma^2=\mathrm{Var}[\tau]=E[\tau^2]-E^2[\tau]=\frac{1}{(N\lambda)^2}(E[Z_0^2]+E[Z_0]-E^2[Z_0])=\frac{1}{(N\lambda)^2}(\mathrm{Var}[Z_0]+E[Z_0]). \qquad \text{(EC20)}$$

Equations (EC17), (EC18), and (EC20) yield the following for the demand rate variance at the supplier for the hybrid policy and policy IB as follows:

$$\mathrm{Var}(X)=N\lambda\left(1+2\frac{\mathrm{Var}(Z_0)}{E(Z_0)}\right).$$

Table EC.1 summarizes the variance and mean of X for all policies.

References

See references list in the main paper.

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