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Electronic Companion—"Managing a Single-Product Assemble-to-Order System with Technology Innovations" by Susan H. Xu and Zhaolin Li, *Management Science*, DOI 10.1287/mnsc.1070.0709.

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Appendix A. Lemma 1

LEMMA 1. (a) *$\Delta g_i(y)$ is a concave function if $a_i + b_i \geq 0$ and a convex function if $a_i + b_i \leq 0$.*

(b) *Let β_i be the largest value of y such that $\Delta g_i(y) = 0$, that is, β_i is the largest solution of*

$$\frac{b_i}{a_i + b_i} = \frac{E[\min(y, D)]}{y}, \tag{EC.A1}$$

where we let $\beta_i \equiv \infty$ if $\Delta g_i(y) \geq (\leq) 0$ for all $y \geq 0$. Then $\Delta g_i(y) \geq (\leq) 0$ if and only if $a_i \geq 0$ $(a_i < 0)$ and $y \leq \beta_i$.

(c) *If $a_i \geq 0$ $(a_i < 0)$, then $\Delta g_i(y) < 0$ (>0) and is decreasing (increasing) when $y > \beta_i$.*

PROOF.

(a) From (4.12), the second derivative of $\Delta g_i(y)$ satisfies

$$\frac{\partial^2 \Delta g_i(y)}{\partial y^2} = -(a_i + b_i) f(y), \tag{EC.A2}$$

which is concave (convex) in y if and only if $a_i + b_i \geq 0$ $(a_i + b_i \leq 0)$.

(b) If $a_i \geq 0$ and $b_i \leq 0$ $(a_i < 0$ and $b_i > 0)$, then $\Delta g_i(y) \geq 0$ $(\Delta g_i(y) \leq 0)$ for all y. Thus, $\beta_i = \infty$ and our claim holds true trivially. If $a_i \geq 0$ and $b_i \geq 0$ $(a_i < 0$ and $b_i < 0)$, then $\Delta g_i(y)$ is concave (convex), as shown in (a). Our claim follows immediately from the definition of β_i.

(c) This is a direct consequence of (a) and (b). □

Appendix B. An Example of Interval Partitioning Algorithm

EXAMPLE 1. We illustrate IPA using a 5-component system with the data showing in Table EC.1. Let demand D be an Erlang random variable with mean 4 and variance 8. Discount factor λ is set at 0.9. We assume $d_i = 1$, so the innovation time for each component is geometrically distributed with innovation rate given in the fifth column of Table EC.1. The salvage value of a Generation 2 (G2) technology is set at $c_i(2) \equiv 0$.

The result of IPA is reported in Table EC.2, which shows that $\{\beta_i\}$ partition $[0, \infty)$ into 4 subintervals. In each subinterval, we select the technology of each component using (4.15) or (4.16), and determine whether the resultant configuration is locally optimal by Step 3 of IPA. Here, we have a unique locally, and hence globally, optimal solution $\mathbf{k}^* = \{1, 1, 0, 0, 1\}$ and $y^* = 5.63$. IPA allows us to evaluate only 4 dominant configurations, rather than $2^5 = 32$ alternatives, to determine the optimal configuration and optimal inventory.

Appendix C. Proof of Theorem 1

(a) Consider two consecutive subintervals $[\beta_{(l)}, \beta_{(l+1)})$ and $[\beta_{(l+1)}, \beta_{(l+2)})$ and their corresponding dominant configurations $\mathbf{k}^{(l)}$ and $\mathbf{k}^{(l+1)}$, $l = 0, 1, \ldots, m-1$. Suppose $\beta_{(l+1)} < \infty$ and $\beta_{(l+1)} = \beta_i$. Consider $a_i \geq 0$ first. As suggested in Step 2 of Algorithm 1, Config-$\mathbf{k}^{(l)}$ and Config-$\mathbf{k}^{(l+1)}$ use the same technology

Table EC.1 Cost Parameters for Example 1

	G0 ($)		G1 ($)		
Component	Price $r_i(0)$	Cost $c_i(0)$	Price $r_i(1)$	Cost $c_i(1)$	Innov. prob. q_i
1	250	120	170	50	0.25
2	100	40	82	20	0.25
3	210	163	140	100	0.1
4	80	40	55	30	0.25
5	100	60	73	35	0.2

for each component except for component i, where the former uses Generation 0 (G0) and the latter uses Generation 1 (G1). Therefore,

$$[r(\mathbf{k}^{(l)})-c(\mathbf{k}^{(l)})]-[r(\mathbf{k}^{(l+1)})-c(\mathbf{k}^{(l+1)})]=a_i\geq 0,$$
$$[c(\mathbf{k}^{(l)})+H(\mathbf{k}^{(l)})]-[c(\mathbf{k}^{(l+1)})+H(\mathbf{k}^{(l+1)})]=b_i\geq 0,$$

where the last inequality holds since $\beta_i<\infty$. Thus, both the profit margin and overage cost of $\mathbf{k}^{(l)}$ decrease in l. Conversely, if $a_i<0$, Config-$\mathbf{k}^{(l)}$ uses G1 and Config-$\mathbf{k}^{(l+1)}$ uses G0 for component i, but both use the same technology for any other component. Hence, the LHSs of the above expressions are $-a_i>0$ and $-b_i>0$, where $-b_i>0$ since $\beta_i<\infty$ means $b_i<0$.

(b) If $a_i\geq(<)0$, then from (4.15) and (4.16), one sees that $k_i^{(l)}$ is increasing (decreasing) on l, $i\in E$. Consequently, $\mathbf{k}^{(l)}$ is increasing (decreasing) on l if $a_i\geq(<)0$ for all i. □

Appendix D. Lemma 2

Lemma 2 relies on the notion of modularity of a function $f(x,y)$, which is defined below.

Definition 2. A function $f(x,y)$ is called submodular (supermodular) if

$$f(x_2,y_2)-f(x_1,y_2)\leq(\geq)f(x_2,y_1)-f(x_1,y_1),\quad\text{for any } x_1\geq x_2 \text{ and } y_1\geq y_2, \tag{EC.D1}$$

or, equivalently, $f(x_2,y)-f(x_1,y)$ is increasing (decreasing) in y.

Lemma 2. (a) *Under Assumption* A1 (A2), $g_i(t_i,k_i,y)$ *is a submodular* (*supermodular*) *function of* t_i *and* y, *with* k_i *fixed. In other words,*

$$g_i(t_i,k_i,y)-g_i(t_i+1,k_i,y)\leq(\geq)g_i(t_i,k_i,y+1)-g_i(t_i+1,k_i,y+1),\quad i\in E. \tag{EC.D2}$$

(b) *Under Assumption* B1 (B2), $g_i(t_i,k_i,y)$ *is a submodular* (*supermodular*) *function of* t_i *and* k_i, *with* y *fixed. In other words,*

$$g_i(t_i,0,y)-g_i(t_i,1,y)\leq(\geq)g_i(t_i+1,0,y)-g_i(t_i+1,1,y). \tag{EC.D3}$$

(c) *Under Assumption* B1 (B2), $\beta_i(t_i)$ *is increasing* (*decreasing*) *in* t_i, $i\in E$.

Table EC.2 The Result of IPA for Example 1

				Dominant intervals and dominant configurations			
Component	a_i	b_i	β_i	$[0, 5.03)$	$[5.03, 5.52)$	$[5.52, 6.58)$	$[6.58, \infty)$
1	10	18.6	5.03	0	1	1	1
2	−2	4	∞	1	1	1	1
3	7.5	9.1	6.58	0	0	0	1
4	15	−2.5	∞	0	0	0	0
5	2	3.2	5.52	0	0	1	1
$r(\mathbf{k}^{(l)})-c(\mathbf{k}^{(l)})$ ($)				319.5	309.5	307.5	300
$c(\mathbf{k}^{(l)})+H(\mathbf{k}^{(l)})$ ($)				113.1	94.6	91.4	82.3
$y(\mathbf{k}^{(l)})$				5.27	5.57	5.63	5.80
Local optimum?				No	No	Yes	No
$g(\mathbf{k}^{(l)}, y(\mathbf{k}^{(l)}))$						$850.9	

Proof.

(a) Taking the partial derivative of the above function with respect to y, we obtain

$$\frac{\partial}{\partial y}(g_i(t_i,k_i,y)-g_i(t_i+1,k_i,y)) = [(r_i(t_i,k_i)-c_i(t_i,k_i))-(r_i(t_i+1,k_i)-c_i(t_i+1,k_i))]P(D\geq y)$$
$$-[(c_i(t_i,k_i)+H_i(t_i,k_i))-(c_i(t_i+1,k_i)+H_i(t_i+1,k_i))](1-P(D\geq y)).$$

If A1 (A2) holds, then the above expression is positive (negative), which implies that $g_i(t_i, k_i, y) - g_i(t_i+1, k_i, y)$ is increasing (decreasing) in y, or, equivalently, $g_i(t_i, k_i, y)$ is a submodular (supermodular) function of t_i and y, $i \in E$.

(b) We have

$$g_i(t_i,0,y)-g_i(t_i,1,y) = a_i(t_i)E[\min(y,D)] - b_i(t_i)(y-E[\min(y,D)]),$$

which is increasing (decreasing) in t_i, $i \in E$, if Assumption B1 (B2) holds.

(c) Recall that $\beta_i(t_i)$ is the largest solution of (4.13). Under B1 (B2), the RHS of (4.13), $b_i(t_i)/(a_i(t_i)+b_i(t_i))$, is decreasing (increasing) in t_i. Furthermore, $E[\min(y,D)]/y$ is decreasing in y since

$$\frac{d}{dy}\left(\frac{E[\min(y,D)]}{y}\right) = \frac{d}{dy}\left(\frac{\int_0^y P(D\geq u)\,du}{y}\right) = \frac{P(D\geq y)y-\int_0^y P(D\geq u)\,du}{y^2}$$
$$\leq \frac{P(D\geq y)y-P(D\geq y)\int_0^y du}{y^2} = 0. \tag{EC.D4}$$

Therefore, $\beta_i(t_i)$ is increasing (decreasing) in t_i under B1 (B2). □

Appendix E. Proof of Theorem 2

(a) Let 1_i be the vector whose i-th element is 1 and others 0. It is sufficient to show $\mathbf{k}^*(\mathbf{t}) \geq \mathbf{k}^*(\mathbf{t}+1_i)$. For notational simplicity, denote the optimal solution pair in state $\mathbf{t}$ by $(\mathbf{k}^*, y^*)$ and the optimal solution pair in state $\mathbf{t}+1_i$ by $(\bar{\mathbf{k}}, \bar{y})$. Recall that $\mathbf{k}^*$ must be one of the dominant configurations $\mathbf{k}^{(l)}(\mathbf{t})$, $l = 0, 1, \ldots, m$. Without loss of generality, we assume $\beta_l(t_l)$ is increasing in l so that $\beta_{(l)}(t_l) = \beta_l(t_l)$. Since $a_i \geq 0$, $i \in E$, by Theorem 1(1), a dominant configuration in state $\mathbf{t}$ takes the form $\mathbf{k}^{(l)}(\mathbf{t}) = (\underbrace{1,1,\ldots,1}_{\text{first } l \text{ elements}}, 0,0,\ldots,0)$, $l = 0,1,\ldots,m$. Let $\mathbf{k}^* = \mathbf{k}^{(l^*)}(\mathbf{t})$, i.e., $\mathbf{k}^*$ uses G1 technologies for components $1,2,\ldots,l^*$, and G0 technologies for components $l^*+1,\ldots,m$. We establish our result via contradiction. Suppose $\bar{\mathbf{k}} \nleq \mathbf{k}^*$. This implies that there exists $l > l^*$, such that $k_l^* = 0$ but $\bar{k}_l = 1$ (i.e., component l uses G0 in state $\mathbf{t}$ but G1 in state $\mathbf{t}+1_i$). Since the optimal order quantity must belong to its dominant interval,

$$\begin{cases} \beta_{l^*}(t_{l^*}) \leq y^* < \beta_{l^*+1}(t_{l^*+1}) \leq \beta_l(t_l) \leq \bar{y} & \text{if } l \neq i, \\ \beta_{l^*}(t_{l^*}) \leq y^* < \beta_{l^*+1}(t_{l^*+1}) \leq \beta_l(t_l) = \beta_i(t_i) \leq \beta_i(t_i+1) \leq \bar{y}, & \text{if } l = i. \end{cases}$$

In either case, $y^* < \bar{y}$. We need to consider two cases.

Case 1. $k_i^* \leq \bar{k}_i$: We prove the solution pair $(\bar{\mathbf{k}}, \bar{y})$ outperforms the presumed optimal solution pair $(\mathbf{k}^*, y^*)$ in state $\mathbf{t}$, resulting in a contradiction. We have

$$\begin{aligned} g(\mathbf{t},\mathbf{k}^*,y^*)-g(\mathbf{t}+1_i,\bar{\mathbf{k}},\bar{y}) &\leq g(\mathbf{t},\mathbf{k}^*,y^*)-g(\mathbf{t}+1_i,\mathbf{k}^*,y^*) \\ &= g_i(t_i,k_i^*,y^*)-g_i(t_i+1,k_i^*,y^*) \\ &\leq g_i(t_i,k_i^*,\bar{y})-g_i(t_i+1,k_i^*,\bar{y}) \\ &\leq g_i(t_i,\bar{k}_i,\bar{y})-g_i(t_i+1,\bar{k}_i,\bar{y}) \\ &= g(\mathbf{t},\bar{\mathbf{k}},\bar{y})-g(\mathbf{t}+1_i,\bar{\mathbf{k}},\bar{y}). \end{aligned} \tag{EC.E1}$$

Here, the first inequality follows, because $(\bar{\mathbf{k}}, \bar{y})$ is the optimal solution pair in state $\mathbf{t}+1_i$. s inequality is due to Lemma 2(a) under A1. The third inequality uses $k_i^* \leq \bar{k}_i$ and Lemma 2(b) under B1. However, (EC.E1) yields

$$g(\mathbf{t},\mathbf{k}^*,y^*) \leq g(\mathbf{t},\bar{\mathbf{k}},\bar{y}),$$

which contradicts to the optimality of $(\mathbf{k}^*, y^*)$ in state $\mathbf{t}$.

Case 2. $k_i^* > \bar{k}_i$: This means $k_i^* = 1$ and $\bar{k}_i = 0$, that is, component i, which uses G1 in state $\mathbf{t}$, is upgraded to G0 in state $\mathbf{t}+1_i$. Since $\mathbf{k}^* = \mathbf{k}^{(l^*)}(\mathbf{t})$, $i \le l^*$. From Lemma 2(c), $\beta_i(t_i) \le \beta_i(t_i+1)$ under B1. In addition, since $k_i^* = 1$ and $\bar{k}_i = 0$, and y^* and $\bar{y}$ belong to their respective dominant intervals, we have $\beta_i(t_i) \le y^* < \bar{y} < \beta_i(t_i+1)$. Let $\bar{l}$ be the largest integer such that $\beta_{\bar{l}}(t_{\bar{l}}) \le \bar{y} < \beta_i(t_i+1)$, $\bar{l} \ge l > l^* \ge i$. This, however, implies that $\bar{\mathbf{k}}$ must take the form

$$\bar{\mathbf{k}} = (\underbrace{1, 1, \ldots, 1,}_{\text{first } i-1 \text{ elements}} 0, \underbrace{1, 1, \ldots, 1,}_{i+1 \text{ to } \bar{l} \text{ elements}} 0, 0, \ldots, 0).$$

Let us define

$$\mathbf{k}' = \bar{\mathbf{k}} + 1_i \qquad \text{and} \qquad \mathbf{k}'' = \mathbf{k}^* - 1_i. \tag{EC.E2}$$

Then, by our assumption, the solution pair $(\mathbf{k}^*, y^*)$ outperforms the solution pair $(\mathbf{k}', \bar{y})$ in state $\mathbf{t}$, that is,

$$g(\mathbf{t}, \mathbf{k}^*, y^*) \ge g(\mathbf{t}, \mathbf{k}', \bar{y}), \tag{EC.E3}$$

or equivalently,

$$g_i(t_i, 1, y^*) - g_i(t_i, 1, \bar{y}) \ge \sum_{j \ne i} g_j(t_j, k_j', \bar{y}) - \sum_{j \ne i} g_j(t_j, k_j^*, y^*). \tag{EC.E4}$$

On the other hand, the solution pair $(\bar{\mathbf{k}}, \bar{y})$ outperforms the solution pair $(\mathbf{k}'', y^*)$ in state $\mathbf{t}+1_i$:

$$g(\mathbf{t}+1_i, \bar{\mathbf{k}}, \bar{y}) \ge g(\mathbf{t}+1_i, \mathbf{k}'', y^*), \tag{EC.E5}$$

or equivalently,

$$\sum_{j \ne i} g_j(t_j, \bar{k}_j, \bar{y}) - \sum_{j \ne i} g_j(t_j, k_j'', y^*) \ge g_i(t_i+1, 0, y^*) - g_i(t_i+1, 0, \bar{y}). \tag{EC.E6}$$

However, our construction of $\mathbf{k}'$ and $\mathbf{k}''$ in (EC.E2) implies that the RHS of (EC.E4) is equal to the LHS of (EC.E6), and thus we have, from (EC.E4) and (EC.E6),

$$\begin{aligned} g_i(t_i, 1, y^*) - g_i(t_i, 1, \bar{y}) &\ge g_i(t_i+1, 0, y^*) - g_i(t_i+1, 0, \bar{y}) \\ &\ge g_i(t_i, 0, y^*) - g_i(t_i, 0, \bar{y}), \end{aligned} \tag{EC.E7}$$

where the last inequality is due to Lemma 2(a). Rearranging the above inequality, we obtain

$$g_i(t_i, 0, y^*) - g_i(t_i, 1, y^*) \le g_i(t_i, 0, \bar{y}) - g_i(t_i, 1, \bar{y}).$$

However, this contradicts Lemma 1(c), which states that $g_i(t_i, 0, y) - g_i(t_i, 1, y) < 0$ and is decreasing in y for $y > \beta_i(t_i)$. Therefore, we must have $\bar{\mathbf{k}} \le \mathbf{k}^*$.

(b) First note that, from (4.8), we can write

$$g(\mathbf{t}, \mathbf{k}, y) - g(\mathbf{t}+1_i, \mathbf{k}, y) = g_i(t_i, k_i, y) - g_i(t_i+1, k_i, y).$$

Then Lemma 2(a) means that $g(\mathbf{t}, \mathbf{k}, y)$ is a submodular function of t_i and y. Hence $y(\mathbf{t}, \mathbf{k})$ is decreasing in t_i for any fixed $\mathbf{k}$ and t_l, $l \ne i$. This result means that it is sufficient to show $y' \equiv y(\mathbf{t}, \bar{\mathbf{k}}) \le y^*$, since $y' \ge \bar{y} \equiv y(\mathbf{t}+1_i, \bar{\mathbf{k}})$. Suppose, on the contrary, $y' > y^*$. Note that $\beta_{l^*}(t_{l^*}) < y^*$. Recall that both functions $g(\mathbf{t}, \mathbf{k}^*, y)$ and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ are concave functions of y. Since $g(\mathbf{t}, \mathbf{k}^*, y)$ reaches the maximum at y^* and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ reaches the maximum at y', $y^* < y'$, $g(\mathbf{t}, \mathbf{k}^*, y)$ is decreasing in y and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ is increasing in y in the interval $[y^*, y')$. This implies that the difference function

$$g(\mathbf{t}, \bar{\mathbf{k}}, y) - g(\mathbf{t}, \mathbf{k}^*, y) = \sum_{\bar{k}_j < k_j^*,\, j \le l^*} [g_j(t_j, 0, y) - g_j(t_j, 1, y)]$$

is increasing in y in the interval $[y^*, y')$, with $y^* \ge \beta_{l^*}(t_{l^*})$. However, by Lemma 1(c), each term on the RHS of the above expression, $g_j(t_j, 0, y) - g_j(t_j, 1, y)$, is decreasing in y for $y > \beta(t_{l^*}) \ge \beta(t_j)$, $j \le l^*$, resulting in a contradiction. Hence we must have $\bar{y} \le y' \le y^*$. $\square$

Appendix F. Proof of Theorem 3

We truncate $W^\tau(\mathbf{t},\mathbf{j},x)$ to $W^\tau_n(\mathbf{t},\mathbf{j},x)$, where $W^\tau_n(\mathbf{t},\mathbf{j},x)$ is the n-period counterpart of $W^\tau(\mathbf{t},\mathbf{j},x)$. The truncated optimality equation of (5.3) is

$$W^\tau_n(\mathbf{t},\mathbf{j},x)=\max_{y\ge 0}\left\{G^\tau(\mathbf{t},\mathbf{j},x,y)+\lambda\sum_{\mathbf{t}'}\sum_{\mathbf{M}}p^{\mathbf{M}}(\mathbf{t}'\mid\mathbf{t})EW^\tau_{n-1}(\mathbf{t}',\mathbf{k}(\mathbf{t})+\mathbf{M},y-Z^\tau(\mathbf{t},y))\right\}. \tag{EC.F1}$$

Using induction, we shall prove both (a) and (b) of Theorem 3 hold for any n. When $n=1$, the optimality equation becomes

$$\begin{aligned}W^\tau_1(\mathbf{t},\mathbf{j},x)&=\max_{y\ge 0}\{G^\tau(\mathbf{t},\mathbf{j},x,y)\}\\&=\max_{y\ge 0}\left\{[r^\tau(\mathbf{t})+H^\tau(\mathbf{t})]E[Z^\tau(\mathbf{t},y)]-[c^\tau(\mathbf{t})+H^\tau(\mathbf{t})]y-\sum_{j_i=k_i(\mathbf{t})}\Delta_i(t_i,j_i)(x-y)^+\right\}.\end{aligned} \tag{EC.F2}$$

From Assumption A0 and (5.2), $r^\tau(\mathbf{t})\ge c^\tau(\mathbf{t})\ge \bar{s}^\tau(\mathbf{t})$, which implies $r^\tau(\mathbf{t})+H^\tau(\mathbf{t})=r^\tau(\mathbf{t})+h^\tau(\mathbf{t})-\bar{s}^\tau(\mathbf{t})>0$. In addition, with positive salvage loss, $\Delta_i(t_i,j_i)>0$. Therefore, each term in (EC.F2), and consequently $G^\tau(\mathbf{t},\mathbf{j},x,y)$, is concave in y for fixed x. From (EC.F2), we find that the first derivative of $G^\tau(\mathbf{t},\mathbf{j},x,y)$ with respect to y satisfies

$$\frac{\partial G^\tau(\mathbf{t},\mathbf{j},x,y)}{\partial y}=\begin{cases}[r^\tau(\mathbf{t})+H^\tau(\mathbf{t})][1-F_{\mathbf{t},\mathbf{k}(\mathbf{t})}(y)]-[c^\tau(\mathbf{t})+H^\tau(\mathbf{t})], & \text{if } y\ge x,\\[2ex] [r^\tau(\mathbf{t})+H^\tau(\mathbf{t})][1-F_{\mathbf{t},\mathbf{k}(\mathbf{t})}(y)]-\left[c^\tau(\mathbf{t})+H^\tau(\mathbf{t})-\displaystyle\sum_{j_i=k_i(\mathbf{t})}\Delta_i(t_i,j_i)\right], & \text{if } y<x.\end{cases} \tag{EC.F3}$$

Setting each of the above expressions to zero, we obtain two base-stock levels of the newsvender-type:

$$F_{\mathbf{t},\mathbf{k}(\mathbf{t})}(y^\tau_1(\mathbf{t}))=\frac{r^\tau(\mathbf{t})-c^\tau(\mathbf{t})}{r^\tau(\mathbf{t})+H^\tau(\mathbf{t})},\qquad F_{\mathbf{t},\mathbf{k}(\mathbf{t})}(\bar{y}^\tau_1(\mathbf{t},\mathbf{j}))=\frac{r^\tau(\mathbf{t})-c^\tau(\mathbf{t})+\sum_{j_i=k_i(\mathbf{t})}\Delta_i(t_i,j_i)}{r^\tau(\mathbf{t})+H^\tau(\mathbf{t})}. \tag{EC.F4}$$

Note that both $y^\tau_1(\mathbf{t})$ and $\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$ are well defined and satisfy $y^\tau_1(\mathbf{t})\le\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$, since $\sum_{j_i=k_i(\mathbf{t})}\Delta_i(t_i,j_i)\ge 0$. Clearly, if $x\le y^\tau_1(\mathbf{t})$, then $y^\tau_1(\mathbf{t})$ maximizes $G^\tau(\mathbf{t},\mathbf{j},x,y)$. If $x\ge\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$, then $\bar{y}^\tau_2(\mathbf{t})$ maximizes $G^\tau(\mathbf{t},\mathbf{j},x,y)$. If $y^\tau_1(\mathbf{t})<x<\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$ and $y<x$, which implies $y<\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$, then the second expression of (EC.F3) is positive. On the other hand, if $y^\tau_1(\mathbf{t})<x<\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$ and $y>x$, which implies $y>y^\tau_1(\mathbf{t})$, then the first expression of (EC.F3) is negative. Therefore, if $y^\tau_1(\mathbf{t})<x<\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$, then $G^\tau(\mathbf{t},\mathbf{j},x,y)$ is maximized at x. This proves part (a) of Theorem 3 for $n=1$.

Substituting $y^\tau_1(\mathbf{t})$ and $\bar{y}^\tau_1(\mathbf{t},\mathbf{j})$ into (EC.F2), we get

$$W^\tau_1(\mathbf{t},\mathbf{j},x)=\begin{cases}G^\tau(\mathbf{t},\mathbf{j},0,y^\tau_1(\mathbf{t})), & \text{if } x\le y^\tau_1(\mathbf{t}),\\[1ex] G^\tau(\mathbf{t},\mathbf{j},x,x), & \text{if } y^\tau_1(\mathbf{t})<x<\bar{y}^\tau_1(\mathbf{t},\mathbf{j}),\\[1ex] G^\tau(\mathbf{t},\mathbf{j},\bar{y}^\tau_1(\mathbf{t},\mathbf{j}),\bar{y}^\tau_1(\mathbf{t},\mathbf{j}))-s(\mathbf{t},\mathbf{j})(x-\bar{y}^\tau_1(\mathbf{t},\mathbf{j})), & \text{if } x\ge\bar{y}^\tau_1(\mathbf{t},\mathbf{j}).\end{cases}$$

It is seen that $W^\tau_1(\mathbf{t},\mathbf{j},x)$ is concave and nonincreasing in x. This proves part (b) of Theorem 3 for $n=1$.

Next we show that both parts of Theorem 3 hold for n, based on the hypothesis that they hold for $n-1$. We first show that $EW^\tau_{n-1}(\mathbf{t}',\mathbf{k}(\mathbf{t})+\mathbf{M},y-Z^\tau(\mathbf{t},y))$, given in (EC.F1), is nonincreasing and concave in y. Recall that $y-Z^\tau(\mathbf{t},y)=(y-D^\tau(\mathbf{t}))^+$, where $D^\tau(\mathbf{t})=D(\mathbf{t},\mathbf{k}(\mathbf{t}))$, which is a nondecreasing and convex function of y for each realized $D^\tau(\mathbf{t})$. Using the fact that if $f(y)$ is nonincreasing and concave in y and $g(y)$ is nondecreasing and convex in y, then $f(g(y))$ is nonincreasing and concave in y, we conclude via our hypothesis that for every realized $D^\tau(\mathbf{t})=d$, $W^\tau_{n-1}(\mathbf{t}',\mathbf{k}(\mathbf{t})+\mathbf{M},(y-d)^+)$ is nonincreasing and concave in y. Since the expectation operator preserves the concavity and nonincreasing property, $EW^\tau_{n-1}(\mathbf{t}',\mathbf{k}(\mathbf{t})+\mathbf{M},(y-D^\tau(\mathbf{t}))^+)$ is nonincreasing and concave in y. This further implies that the expression within the maximum operator of (EC.F1) is a concave function of y for given $\mathbf{j}$ and x. Repeating the same approach for $n=1$, we can show that the optimal balanced ordering policy is characterized two control limits $y^\tau_n(\mathbf{t})\le\bar{y}^\tau_n(\mathbf{t},\mathbf{j})$ as follows:

$$y^\tau_n(\mathbf{t},\mathbf{j},x)=\begin{cases}y^\tau_n(\mathbf{t}), & \text{if } x\le y^\tau_n(\mathbf{t}),\\[1ex] x, & \text{if } y^\tau_n(\mathbf{t})<x<\bar{y}^\tau_n(\mathbf{t},\mathbf{j}),\\[1ex] \bar{y}^\tau_n(\mathbf{t},\mathbf{j}), & \text{if } x\ge\bar{y}^\tau_n(\mathbf{t},\mathbf{j}).\end{cases}$$

Table EC.3 Cost Parameters in Example 2

	Generation-0		Generation-1	
Component	Profit margin	Overage cost	Profit margin	Overage cost
1	11	9	9.8	8
2	11	6	9.5	5

This completes the induction step for part (a) of Theorem 3 Substituting $y^{\tau}_{n+1}(\mathbf{t}, \mathbf{j}, x)$ into (EC.F1), and following the similar argument as that for $n = 1$, one sees that $W^{\tau}_{n}(\mathbf{t}, \mathbf{j}, x)$ is nonincreasing and concave function of x. This completes the induction step for part (b) of Theorem 3. Finally, using the standard convergence property of the discounted DP, we let $n \to \infty$ and conclude that both parts of Theorem 3 hold true. □

Appendix G. An Example of Extended IPA

Example 2. We suppress (t_1, t_2) in this example. Demand D follows the uniform distribution between 0 and 10. The initial configuration is $\mathbf{j} = (1, 1)$ and $x = 3$. Cost parameters are summarized in Table EC.3. We assume $c_i - s_i = 1$ for $i = 1, 2$. For component 1, $\Delta G_1(1, 3, y) = 0$ has three roots: $y = 1.82$, 3.88, and 7.03 (these roots actually depend on x). For component 2, $\Delta G_2(1, 3, y) = 0$ has a single root, $y = 9.46$. The next several steps of IPA are summarized in Table EC.4. As seen, Config-$(0, 0)$ dominates if y is in interval $[0, 1.82)$ and $[3.88, 7.03)$. According to (5.5), the first limit for Config-$(0, 0)$ is 5.95. Since $x = 3 < 5.95$, the optimal inventory level for Config-$(0, 0)$ is 5.95, which happens to belong to $[3.88, 7.03)$. Because the pair $((0, 0), 5.95)$ is the only local optimum, it is also the global optimum. Here, both components are upgraded, which means that the initial inventory $x = 3$ is salvaged for both components and the new inventory for Config-$(0, 0)$ is brought to the first control limit.

Appendix H. Proof of Theorem 4

Using the standard induction approach, we truncate $W(\mathbf{t}, \mathbf{j}, x)$ to $W_n(\mathbf{t}, \mathbf{j}, x)$, where $W_n(\mathbf{t}, \mathbf{j}, x)$ is the n-period counterpart of $W(\mathbf{t}, \mathbf{j}, x)$. When $n = 1$, $W_1(\mathbf{t}, \mathbf{j}, x) = \max_{\mathbf{k}\in\{0,1\}^m,\ y\ge 0}\{G(\mathbf{t}, \mathbf{j}, x, \mathbf{k}, y)\}$ is decreasing on x, by Theorem 3. We assume that $W_n(\mathbf{t}, \mathbf{j}, x)$ is nonincreasing on x for some n. For $n + 1$, we write the truncated optimality equation as

$$W_{n+1}(\mathbf{t}, \mathbf{j}, x) = \max_{\mathbf{k}\in\{0,1\}^m,\, y\ge 0}\{J_{n+1}(\mathbf{t}, \mathbf{j}, x, \mathbf{k}, y)\}, \tag{EC.H1}$$

where

$$\begin{aligned} J_{n+1}(\mathbf{t}, \mathbf{j}, x, \mathbf{k}, y) = {} & r(\mathbf{t}, \mathbf{k})E[Z(\mathbf{t}, \mathbf{k}, y)] - H(\mathbf{t}, \mathbf{k})(y - E[Z(\mathbf{t}, \mathbf{k}, y)]) - c(\mathbf{t}, \mathbf{k})y - \sum_{j_i = k_i} \Delta_i(t_i, j_i)(x - y)^+ \\ & - \sum_{j_i \ne k_i} \Delta_i(t_i, j_i)x + \lambda \sum_{\mathbf{t}'} \sum_{\mathbf{M}} p^{\mathbf{M}}(\mathbf{t}' \mid \mathbf{t}) E W_n(\mathbf{t}', \mathbf{k} + \mathbf{M}, y - Z(\mathbf{t}, \mathbf{k}, y)). \end{aligned} \tag{EC.H2}$$

We need to prove that, if the density function $f(\xi)$ of demand $D(\mathbf{t}, \mathbf{k})$ is log concave, then $J_{n+1}(\mathbf{t}, \mathbf{j}, x, \mathbf{k}, y)$ is quasi-concave in y, with other parameters fixed. We resort to Proposition 3.1 in

Table EC.4 The Result of Generalized IPA for Example 2

	Dominate configurations in sub-intervals				
Component	$[0, 1.82)$	$[1.82, 3.88)$	$[3.88, 7.03)$	$[7.03, 9.46)$	$[9.46, \infty)$
1	0	1	0	1	1
2	0	0	0	0	1
Optimal inventory level	5.95	6.0	5.95	6.0	6.0
Local optimum?	No	No	Yes	No	No

Cheng and Sethi (1999), which states that if the density function $f(\xi)$ is log concave and function $K(y)$ is quasi-concave in y, then

$$J(y)=\int_0^\infty K(y-\xi)f(\xi)\,d\xi$$

quasi-concave in y. To apply the above result, we condition on demand $D(\mathbf{t},\mathbf{k})=\xi$, and rewrite (EC.H2) as

$$J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k},y)=\int_0^\infty K_{n+1}(y-\xi)f(\xi)\,d\xi, \tag{EC.H3}$$

where, by (EC.H2), $K_{n+1}(y)$ satisfies

$$K_{n+1}(y)=[r(\mathbf{t},\mathbf{k})-c(\mathbf{t},\mathbf{k})](y+\xi)-[r(\mathbf{t},\mathbf{k})+H(\mathbf{t},\mathbf{k})]y^+ +\sum_{j_i=k_i}\Delta_i(t_i,j_i)\min(y+\xi-x,0)$$
$$-\sum_{j_i\neq k_i}\Delta_i(t_i,j_i)x+\lambda\sum_{\mathbf{t}'}\sum_{\mathbf{M}}p^{\mathbf{M}}(\mathbf{t}'\mid\mathbf{t})W_n(\mathbf{t}',\mathbf{k}+\mathbf{M},y^+). \tag{EC.H4}$$

Next, we show $K_{n+1}(y)$ is increasing on y when $y\le 0$ and decreasing on y when $y\ge 0$, which implies that $K_{n+1}(y)$ is unimodel (i.e., quasi-concave) on y, for any given ξ. We consider three cases.

Case 1. $y\le 0$: In this case, $y^+=0$ and (EC.H4) becomes

$$K_{n+1}(y)=[r(\mathbf{t},\mathbf{k})-c(\mathbf{t},\mathbf{k})](y+\xi)+\sum_{j_i=k_i}\Delta_i(t_i,j_i)\min(y+\xi-x,0)$$
$$-\sum_{j_i\neq k_i}\Delta_i(t_i,j_i)x+\lambda\sum_{\mathbf{t}'}\sum_{\mathbf{M}}p^{\mathbf{M}}(\mathbf{t}'\mid\mathbf{t})W_n(\mathbf{t}',\mathbf{k}+\mathbf{M},0).$$

It is easily seen that the first and second terms of the above expression are increasing functions of y, and the third and fourth terms are independent of y. Therefore $K_{n+1}(y)$ is an increasing function of y.

Case 2. $0\le y\le x-\xi$: We have $y^+=y$ and $\min(y+\xi-x,0)=y+\xi-x$. Thus,

$$K_{n+1}(y)=-\Big[r(\mathbf{t},\mathbf{k})+H(\mathbf{t},\mathbf{k})-\sum_{j_i=k_i}\Delta_i(t_i,j_i)\Big]y+\Big[r(\mathbf{t},\mathbf{k})-c(\mathbf{t},\mathbf{k})+\sum_{j_i=k_i}\Delta_i(t_i,j_i)\Big]\xi$$
$$-\sum_i\Delta_i(t_i,k_i)x+\lambda\sum_{\mathbf{t}'}\sum_{\mathbf{M}}p^{\mathbf{M}}(\mathbf{t}'\mid\mathbf{t})W_n(\mathbf{t}',\mathbf{k}+\mathbf{M},y). \tag{EC.H5}$$

Note that

$$\sum_{j_i=k_i}\Delta_i(t_i,j_i)\le\sum_i\Delta_i(t_i,k_i)=c(\mathbf{t},\mathbf{k})-s(\mathbf{t},\mathbf{k})\le c(\mathbf{t},\mathbf{k})+H(\mathbf{t},\mathbf{k}).$$

Hence the first term of (EC.H5) is decreasing on y. The second and third terms are independent of y. The last term is a decreasing function of y by our hypothesis of W_n. Therefore, $K_{n+1}(y)$ is decreasing on y in Case 2.

Case 3. $y\ge 0$ and $y\ge x-\xi$: We have $y^+=y$ and $\min(y+\xi-x,0)=0$. Then,

$$K_{n+1}(y)=-[r(\mathbf{t},\mathbf{k})+H(\mathbf{t},\mathbf{k})]y+[r(\mathbf{t},\mathbf{k})-c(\mathbf{t},\mathbf{k})]\xi-\sum_{j_i=k_i}\Delta_i(t_i,k_i)x+\lambda\sum_{\mathbf{t}'}\sum_{\mathbf{M}}p^{\mathbf{M}}(\mathbf{t}'\mid\mathbf{t})W_n(\mathbf{t}',\mathbf{k}+\mathbf{M},y)$$

which is decreasing on y, by our hypothesis.

Note that $K_{n+1}(y)$ is continuous at $y=\xi$ and $y=x-\xi$. Therefore, $K_{n+1}(y)$, and henceforth $J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k},y)$, are strictly quasi-concave in y for any given $(\mathbf{t},\mathbf{j},x,\mathbf{k})$.

Now, we characterize the optimal ordering policy for any given $(\mathbf{t},\mathbf{j},x,\mathbf{k})$. From (EC.H2), we find that

$$\frac{\partial J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k},y)}{\partial y}=\begin{cases}\dfrac{\partial J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)}{\partial y} & y\ge x,\\[2ex] \dfrac{\partial J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)}{\partial y}+\displaystyle\sum_{j_i=k_i}\Delta_i(t_i,j_i) & y<x.\end{cases}$$

It can be seen from (EC.H2) that $\partial J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)/\partial y$ is independent of $\mathbf{j}$ if $y\geq x$ but dependent on $\mathbf{j}$ otherwise. Since $J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)$ is strictly quasi-concave in y for any given $(\mathbf{t},\mathbf{j},\mathbf{k})$, let

$$\begin{cases} y^*_{n+1}(\mathbf{t},\mathbf{k}) = \arg_y\left\{\dfrac{\partial J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)}{\partial y}=0\right\},\\ \bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k}) = \arg_y\left\{\dfrac{\partial J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y)}{\partial y}=-\sum_{j_i=k_i}\Delta_i(t_i,j_i)\right\}. \end{cases} \tag{EC.H6}$$

Repeat the similar argument as in Theorem 3 that establishes the optimality of the two-limit policy for $n=1$, one can show that the optimal inventory policy in state $(\mathbf{t},\mathbf{j},x,\mathbf{k})$ is determined by the two thresholds $y^*_{n+1}(\mathbf{t},\mathbf{k})$ and $\bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k})$. This completes the induction step for part (a). To complete the induction step for part (b), we note that under the two-limit policy,

$$\max_{y\geq 0}\{J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k})\} = \begin{cases} J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},y^*_{n+1}(\mathbf{t},\mathbf{k})) - \sum_{j_i\neq k_i}\Delta_i(t_i,j_i)x, & x\leq y^*_{n+1}(\mathbf{t},\mathbf{k}),\\ J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},x) - \sum_{j_i\neq k_i}\Delta_i(t_i,j_i)x, & y^*_{n+1}(\mathbf{t},\mathbf{k}) < x < \bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k}),\\ J_{n+1}(\mathbf{t},\mathbf{j},0,\mathbf{k},\bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k})) - \sum_{j_i\neq k_i}\Delta_i(t_i,j_i)x & \\ \quad - \sum_{j_i=k_i}\Delta_i(t_i,j_i)(x-\bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k})), & \bar{y}^*_{n+1}(\mathbf{t},\mathbf{j},\mathbf{k})\leq x. \end{cases}$$

It is seen that $\max_{y\geq 0}\{J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k})\}$ is nonincreasing in x, and thus,

$$W_{n+1}(\mathbf{t},\mathbf{j},x) = \max_{\mathbf{k}\in\{0,1\}^m,\, y\geq 0}\{J_{n+1}(\mathbf{t},\mathbf{j},x,\mathbf{k})\}$$

is nonincreasing in x. This completes the induction step for part (b). Finally, using the standard convergence property of the discounted DP, we let $n\to\infty$ and conclude that both parts of Theorem 4 holds true. □

Reference

Cheng, F., S. Sethi. 1999. A periodic review inventory model with demand influenced by promotion decisions. *Management Sci.* **45** 1510–1523.