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Online Technical Appendix

First-Best Vertically Integrated Channel

Consider a vertically integrated manufacturer who first produces a quantity and then sets up the price for the product in an uncertain demand environment. Like in the model with decentralized channel, the demand is given by $q = \theta_s - p$, with $s = h$ or l with equal probability. This firm may have access to an information system that gives out signals $\tilde{k} = \tilde{h}, \tilde{l}$; with possibly imperfect reliability, i.e., ρ is free parameter between 0 and 1. The firm's profit function is

$$\Pi_{\tilde{k}} = \Pr(h \mid \tilde{k})[p_{\tilde{k}}(\min(z - p_{\tilde{k}}), Q_{\tilde{k}})] + \Pr(l \mid \tilde{k})[p_{\tilde{k}}(\min(1 - p_{\tilde{k}}), Q_{\tilde{k}})] - cQ_{\tilde{k}},$$

where $Q_{\tilde{k}}$ and $p_{\tilde{k}}$ are signal dependent amount of production and retail price, respectively. Firm can either (1) produce and price for selling out exactly in high state but this amounts to overstocking in low state or (2) produce and price for selling out exactly in low state, in which case it obviously understocks for the high state in which it cannot meet all the demand or (3) produce and price so that it has more than enough if low state is realized but cannot meet all the demand if high state is realized. We start by ruling out (3) first. For the high signal, the profit function in the pricing stage becomes

$$\Pi_{\tilde{h}} = \left(\frac{1+\rho}{2}\right)[p_{\tilde{h}} Q_{\tilde{h}}] + \frac{1-\rho}{2}[p_{\tilde{h}}(1 - p_{\tilde{h}})].$$

For a given $Q_{\tilde{h}}$ it leads to an optimal price of $(z(1+\rho) + Q_{\tilde{h}}(1-\rho))/(2(1+\rho))$. Going back to the quantity stage, however, we see that it leads to a profit function which is quadratic function in $Q_{\tilde{h}}$ with a strictly positive second derivative in $Q_{\tilde{h}}$. This means the function has an interior minimum and the profit maximizing quantity can be found only at one (or both) the corners of this regime which correspond to (1) or (2) above. We call these Overstocking and Understocking respectively. A similar conclusion is reached for low signal also. Thus the optimal production quantity and price for the vertically integrated firm can be found possibly in analyzing the Overstocking and Understocking cases.

The profit function for an understocking equilibrium is simply $(p_u - c)(1 - p_u)$, since the firm always produces and prices for low state which it can always sell. This is irrespective of the received signal. It is easy to see that it leads to an optimal price of $(1+c)/2$ and an optimal profit of $\Pi_u^* = (1/4)(1-c)^2$.

The profit function for Overstocking case is signal-dependent, and denoting the signal by $\tilde{k}$, it can be expressed as

$$\Pi_{o\tilde{k}} = \Pr(h \mid \tilde{k})(z - p_{o\tilde{k}})(p_{o\tilde{k}}) + \Pr(l \mid \tilde{k})(1 - p_{o\tilde{k}})(p_{o\tilde{k}}) - (z - p_{o\tilde{k}})c.$$

Conditional on the signal, the firm fixes the price and it can be easily seen that the profit maximizing price and resulting profit for the high signal is $p_{o\tilde{h}} = \frac{1}{4}((z+1) + \rho(z-1) + 2c)$ leading to

$$\Pi^*_{o\tilde{h}} = \tfrac{1}{16}[(z+1)^2 + 4c(1 + c + \rho(z-1) - 3z) + \rho^2(z-1)^2 + 2\rho(z^2-1)].$$

Similarly, for low signal $p_{o\tilde{l}} = \frac{1}{4}((z+1) - \rho(z-1) + 2c)$ leading to

$$\Pi^*_{o\tilde{l}} = \tfrac{1}{16}[(z+1)^2 + 4c(1 + c - \rho(z-1) - 3z) + \rho^2(z-1)^2 - 2\rho(z^2-1)].$$

What the firm will do for any given signal is then dependent on whether understocking or overstocking is better strategy to follow for any given ρ. After a little bit of algebra, we find three types of strategies emerge in equilibrium for different values of ρ. These are (i) Overstocking if $\tilde{h}$ and Overstocking if $\tilde{l}$, (ii) Overstocking if $\tilde{h}$ and Under-stocking if $\tilde{l}$, and (iii) Under-stocking if $\tilde{h}$ and Under-stocking if $\tilde{l}$.

The expected profits from the three types of strategies can simply be calculated by giving equal-probability weights to the two signals. Finally, we can calculate the first derivatives of the three types of equilibrium profits as below:

$$\frac{\partial\left(\frac{1}{2}\Pi^*_{o\tilde{h}}+\frac{1}{2}\Pi^*_{o\tilde{l}}\right)}{\partial\rho}=\frac{1}{8}\rho(z-1)^2>0$$

$$\frac{\partial(\frac{1}{2}\Pi^*_{o\tilde{h}}+\frac{1}{2}\Pi^*_{u})}{\partial\rho}=\frac{1}{16}[(z^2+1)+\rho(z-1)^2+2c(z-1)]>0 \qquad \text{and} \qquad \frac{\partial(\Pi^*_u)}{\partial\rho}=0.$$

Therefore, for the vertically integrated firm, expected profits are (weakly) increasing in the signal reliability parameter ρ.

Channel with Inventory ($\rho=0$) of §3.2

Denote the inventory bought by the retailer as Q and the retail price as p. There are three possible types of channel equilibria given the inventory and pricing choices of the retailer:

(i) A case of "understocking" in which the retailer chooses inventory that is exactly equal to the demand in the low state but less than the realized demand in the high state. Thus $(1-p)=Q<(z-p)$.

(ii) A case of "overstocking" in which the retailer chooses inventory that is exactly equal to the realized demand if the high state of demand were to be realized, but has excess inventory if the low state were to be realized. Thus $(1-p)<Q=(z-p)$.

(iii) An "in-between" case in which the retailer the retailer understocks if the high state were to be realized, but overstocks if the low state were to be realized. Thus in this case $(1-p)<Q<(z-p)$.

We begin the analysis by showing that the in-between case (iii) described above can never be part of an equilibrium. Given a contract (w, R), the retailer's objective function is $\pi_{rb}=\frac{1}{2}pQ+\frac{1}{2}p(1-p)+\frac{1}{2}R(Q-(1-p))-wQ$. From this we can solve $p=(1+R+Q)/2$ and substituting this into the objective function we get a quadratic in Q with a minimum $(\partial^2\pi_{rb}/\partial Q^2=\frac{1}{4}>0)$. Therefore there is no interior maximum for Q and the inventory choice by the retailer will be at the boundary characterized either by the understocking or the overstocking choice analyzed below. It can also be noted that it will never be the case in equilibrium that the retailer chooses an inventory greater than the maximum possible demand $(Q>z-p)$ because in that case the retailer will always be better off reducing the inventory or the retail price charged. Similarly, the retailer will never choose an inventory less than the minimum possible demand $(Q<1-p)$, because in that case the retailer will always be better off increasing the inventory or the price.

Understocking Equilibrium

Based on the w offered by the manufacturer, the retailer chooses the inventory and the retail price. The retailer's profit function is $\pi_{ru}=(1-p_u)(p_u-w_u)$. From this the optimal retail price is $p_u(w_u)=(1+w_u)/2$ and the retailer profit becomes $\pi_{ru}(w_u)=(1-w_u)^2/4$. While choosing the contract w_u the manufacturer also has to account for the possibility that the retailer might deviate to any choice of inventory and price other than the equilibrium one. Specifically, the retailer might deviate to over-stocking of inventory (because it is the case that the deviation to the case of in-between inventory is not optimal). Denoting the deviation by d, and given w_u, suppose the retailer deviates to choosing $(1-p_d)<Q=(z-p_d)$.

Let us begin with the case where the retailer chooses the deviation price p_d so that there is demand in the off-equilibrium path irrespective of the state. The deviation profit will be $\pi_{rd}=\frac{1}{2}(z-p_d)p_d+\frac{1}{2}(1-p_d)p_d-(z-p_d)w_u$. Thus for given w_u we will have the optimal deviation price to be $p_d(w_u)=(1+z+2w_u)/4$. Thus if the channel equilibrium were to involve understocking, then this understocking equilibrium must provide the retailer with at least as much profit as $\pi_{rd}(p_d(w_u))$.

The manufacturer's problem is to choose w_u to maximize $\pi_{mu}=(1-p_u)(w_u-c)$ subject to $w_u>0$ and the retailer getting atleast the profit available from the best possible deviation. The corresponding Lagrangian will be,

$$L_{mu}=(1-p_u)(w_u-c)+\mu_{wu}w_u+\mu_d(\pi_{ru}(w_u)-\pi_{rd}(w_u))$$

where the μ's are the corresponding Lagrangian multipliers. Note that we are looking for a solution with $w_u > 0$ and therefore $\partial L_{mu}/\partial w_u = 0$ and $\mu_{w_u} = 0$. Let us first consider the case for which $\mu_d > 0$ which means that π_{rd} is binding for the retailer. Then $\partial L_{mu}/\partial \mu_d = 0$ from which we get that $w_u = (z+3)/12$. Next from $\partial L_{mu}/\partial w_u = 0$ we can derive $\mu_d = (z-3-6c)/(z-1)$. And $\mu_d > 0 \to z > (3+6c)$. Substituting the wholesale price the equilibrium manufacturer profit for this case turns out to be $\pi_{mu} = (9-z)(z+3-12c)/288$. The retailer profit is $\pi_{ru} = (9-z)^2/576$. Next note that this equilibrium is supported by the off-equilibrium retailer price p_d and the requirement that there is positive demand in both states in the off-equilibrium path. This requires that $(1-p_d) > 0$ which implies $\frac{1}{8}(5-7z/3) > 0 \to z < \frac{15}{7}$. However, comparing this with the earlier condition $z > 3+6c$ we can see that there is no feasible value of z for which this case exists.

Next consider the case when $\mu_d = 0$. This implies that $\partial L_{mu}/\partial \mu_d > 0$, from which we get that $z < 3+6c$. Also given that $w_u > 0$ which implies $\partial L_{mu}/\partial w_u = 0$ and $w_u = (1+c)/2$. We require $(1-p_d) > 0$ and this implies $z < (2-c)$. Thus this case exists for $z < (2-c)$. The equilibrium manufacturer and retailer profits are respectively $\pi^*_{mu} = (1-c)^2/8$ and $\pi^*_{ru} = (1-c)^2/16$.

Now we consider the case in which the possible deviation by the retailer involves demand only in the high state. The deviation profit will be $\pi_{rd} = \frac{1}{2}(z-p_d)p_d - (z-p_d)w_u$. This means $p_d(w_u) = (z/2) + w_u$ and $\pi_{rd} = \frac{1}{8}(z-2w_u)^2$. Therefore, the Lagrangian for the manufacturer's optimization for this case will be,

$$L_{mu} = (1-p)(w_u - c) + \mu_w w_u + \mu_d\left(\frac{(1-w_u)^2}{4} - \frac{(z-2w_u)^2}{8}\right).$$

First consider when $\mu_d > 0$, then $((1-w)^2/4 - (z-2w)^2/8) = 0$. This is a quadratic in w_u which can be solved for $w_{u1} = (z-1) + (1/\sqrt{2})(z-2)$ and $w_{u2} = (z-1) - (1/\sqrt{2})(z-2)$. We can calculate μ_d from the condition $\partial L_{mu}/\partial w_u = 0$ to be $\mu_d = (1-2w_u+c)/(1+w_u-z)$. Substituting w_{u1} for w_u in the equation for μ_d we get that $\mu_d < 0$ which is impossible. Therefore w_{u1} cannot be a feasible solution. Consider the second root w_{u2}. Substituting it into the retailer price response we get $p(w_{u2}) = ((3z/2) - 1) + \sqrt{2}(1-(z/2))$. Now for the equilibrium understocking demand to be positive $(1-p(w_{u2})) > 0$ implies that $z < 0.7387$ which is impossible by assumption. Therefore, there cannot be a deviation by the retailer which involving positive demand only in the high state and with $\mu_d > 0$.

Finally, consider the case in which $\mu_d = 0$ and $\partial L_{mu}/\partial \mu_d > 0$. As before $w_u > 0$ and therefore $\partial L_{mu}/\partial w_u = 0$ and $\mu_{w_u} = 0$. From this we get $w_u = (1+c)/2$. From $\partial L_{mu}/\partial \mu_d > 0$ we get the condition $z < (1+c) + ((1-c)/\sqrt{2})$. The equilibrium profits are once again $\pi^*_{mu} = (1-c)^2/8$ and $\pi^*_{ru} = (1-c)^2/16$. To summarize, in the understocking equilibrium $w^*_u = (1+c)/2$, $p^*_u = (3+c)s/4$, $\pi^*_{mu} = (1-c)^2/8$, and $\pi^*_{ru} = (1-c)^2/16$.

Overstocking Equilibrium

The retailer's profit function in the overstocking case is

$$\pi_{ro} = p_o\left(\frac{(z-p_o)}{2} + \frac{(1-p_o)}{2}\right) + \frac{R_o}{2}(z-1) - w_o(z-p_o).$$

From this the optimal retail price $p_o(w_o) = (z+1+2w_o)/4$. The manufacturer problem is to maximize the profit function $\pi_{mo} = (w_o - c)(z-p_o) - (R_o/2)(z-1)$ subject to $w_o > 0$, $R_o \geq 0$ and the retailer getting atleast the profit available from the best possible deviation. Note again that the only feasible deviation for the retailer will be to an understocking case. The retailer's off-equilibrium deviation profit will be $\pi_{rd} = (1-p_d)(p_d - w_o)$. Finding the optimal $p_d(w_o)$ and substituting back we get $\pi_{rd}(w_o) = (1-w_o)^2/4$. The Lagrangian for the manufacturer's optimization will be,

$$L_{mo} = (w_o - c)(z-p_o) - \frac{R_o}{2}(z-1) + \mu_{wo} w_o + \mu_R R_o + \mu_d(\pi_{ro}(w_o, R_o) - \pi_{rd}(w_o)).$$

Consider first the case in which $R_o > 0$ and $\mu_d > 0$ which implies that the retailer profit condition is binding. Because $R_o > 0$, $\mu_R = 0$ and $\partial L_{mo}/\partial R_o = 0$ which implies that $\mu_d = 1$. Because $w_o > 0$, $\partial L_{mu}/\partial w_o = 0$ and $\mu_{w_o} = 0$ and from this we can calculate $w_o = (1+c)/2$. The corresponding retail price is $p^*_o = (z+c+2)/4$. Because $\mu_d > 0$ we will have $\partial L_{mu}/\partial \mu_d = 0$ from which we can calculate $R_o = (3+6c-z)/8$. Since we need $R_o > 0$, this results in the condition $z < (3+6c)$. The equilibrium manufacturer profit can be calculated to be $\pi^*_{mo} = (z^2 + 2z - 12zc + 2c^2 + 8c - 1)/16$ and the equilibrium retailer profit is $\pi^*_{ro} = (1-c)^2/16$. Finally, for the demand in the low state to be positive we need $(1-p^*_o) > 0 \Rightarrow z < (2-c)$. Thus the binding condition for existence of this overstocking case is $z < (2-c)$.

Finally, suppose that $\mu_d = 0$. Note as usual because $w_o > 0$ then $\mu_{w_o} = 0$. Now suppose $R_o > 0$, then $\partial L_{mo}/\partial R_o = 0$ and $\mu_{Ro} = 0$. But that will imply that $\mu_d = 1$ which contradicts our assumption. Therefore, $R_o = 0$ if $\mu_d = 0$. Of course, if $\mu_d = 0$, then we must have that $\pi_{ro}(w_o, R_o) - \pi_{rd}(w_o) > 0$. But $\pi_{ro}(w_o, R_o) - \pi_{rd}(w_o) = -((z-1)(4z-3(1-c)))/8 < 0$ always. Therefore this case never exists.

The General Case $(0 < \rho < 1)$ of §3.4

In this analysis, we proceed like in the channel with inventory case above, however we need to analyze both types of equilibrium (overstocking and understocking), for both types of possible signal ($\tilde{h}$ or $\tilde{l}$). We start with the $\tilde{h}$ signal followed by the $\tilde{l}$ signal.

Understocking given $\tilde{h}$. Based on $w_{\tilde{h}}$ offered by the manufacturer, the retailer chooses the inventory and the retail price. The retailer's profit function is $\pi_{r\tilde{h}u} = (1-p_{\tilde{h}})(p_{\tilde{h}} - w_{\tilde{h}})$. From this the optimal retail price is $p_{\tilde{h}}(w_{\tilde{h}}) = (1+w_{\tilde{h}})/2$ and the retailer profit becomes $\pi_{r\tilde{h}u}(w_{\tilde{h}}) = (1-w_{\tilde{h}})^2/4$. In choosing the contract $w_{\tilde{h}}$ the manufacturer must guard against two possible deviations from the equilibrium. Either the retailer can deviate to overstocking with positive demand in both states or the retailer can deviate to overstocking with positive demand only in the high state. We derive the equilibrium for the case in which deviation that is relevant for the manufacturer to guard against in supporting the equilibrium wholesale price is the one in which the retailer may deviate to overstocking with positive demand in both states.

In deviating to overstocking with positive demand in both states (denoted by subscript d), and given $w_{\tilde{h}}$, if the deviation price is $p_{\tilde{h}d}$, then the deviation profit will be

$$\pi_{r\tilde{h}d} = \Pr(h \mid \tilde{h})(z - p_{\tilde{h}d})p_{\tilde{h}d} + \Pr(l \mid \tilde{h})(1 - p_{\tilde{h}d})p_{\tilde{h}d} - (z - p_{\tilde{h}d})w_{\tilde{h}}.$$

This leads to an optimal deviation price to be $p_{\tilde{h}d}(w_{\tilde{h}}) = (1 + z + 2w_{\tilde{h}} + (z-1)\rho)/4$. Thus if the channel equilibrium were to involve understocking, then this understocking equilibrium must provide the retailer with atleast as much profit as $\pi_{r\tilde{h}d}(p_{\tilde{h}d}(w_{\tilde{h}}))$.

The manufacturer's problem is to choose $w_{\tilde{h}}$ to maximize $\pi_{m\tilde{h}u} = (1-p_{\tilde{h}})(w_{\tilde{h}} - c)$ subject to $w_{\tilde{h}} > 0$ and the retailer getting atleast the profit available from the best possible deviation. The corresponding Lagrangian is

$$L_{m\tilde{h}u} = (1-p_{\tilde{h}})(w_{\tilde{h}} - c) + \mu_{w_{\tilde{h}}} w_{\tilde{h}} + \mu_{\tilde{h}d}(\pi_{r\tilde{h}u}(w_{\tilde{h}}) - \pi_{r\tilde{h}d}(p_{\tilde{h}d}(w_{\tilde{h}}))).$$

We consider only $w_{\tilde{h}} > 0$ and therefore $\partial L_{m\tilde{h}u}/\partial w_{\tilde{h}} = 0$ and $\mu_{w_{\tilde{h}}} = 0$. Let us first consider the case for which $\mu_{\tilde{h}d} = 0$, which means that the deviation constraint is not binding for the retailer. Solving $\partial L_{m\tilde{h}u}/\partial w_{\tilde{h}} = 0$ gives us $w_{\tilde{h}u1} = (1+c)/2$. In order for $\pi_{r\tilde{h}u}(w_{\tilde{h}}) > \pi_{r\tilde{h}d}(p_{\tilde{h}d}(w_{\tilde{h}}))$, we must have $(1 < z < (3 + 6c - \rho(4 + 2c - \rho))/(1+\rho)^2)$. This is one possible solution that leads to a manufacturer profit of $\pi^*_{m\tilde{h}u1} = (1-c)^2/8$. Next we consider the case when the deviation constraint is binding, i.e., $(\pi_{r\tilde{h}u}(w_{\tilde{h}})) = \pi_{r\tilde{h}d}(p_{\tilde{h}d}(w_{\tilde{h}}))$, and that implies $\mu_{\tilde{h}d} \geq 0$. Solving this along with $\partial L_{m\tilde{h}u}/\partial w_{\tilde{h}} = 0$ gives us

$$w_{\tilde{h}u2} = \frac{3 + z(1+\rho)^2 + \rho(2-\rho)}{4(3-\rho)} \qquad \text{and} \qquad \mu_{\tilde{h}d} = \frac{z(\rho+1)^2 - 3(1+2c) + \rho(4+2c-\rho)}{(3(3-2\rho)+\rho^2)(z-1)}.$$

Note that the Lagrange multiplier must be positive, which is satisfied for $z > (3 + 6c - \rho(4+2c-\rho))/(1+\rho)^2$. This solution gives the manufacturer profit as

$$\pi^*_{m\tilde{h}u2} = \frac{(\rho^2 - 2z\rho - z\rho - z - 3 - 2\rho + 12c - 4c\rho)(6\rho - 9 + 2z\rho + z\rho^2 + z - \rho^2)}{32(3-\rho)^2}.$$

Thus, subject to the regularity condition, an understocking equilibrium given the signal $\tilde{h}$ may involves $w^*_{\tilde{h}u} = w_{\tilde{h}u1}$, $\pi^*_{m\tilde{h}u} = \pi^*_{m\tilde{h}u1}$, or, $w^*_{\tilde{h}u} = w_{\tilde{h}u2}$ and $\pi^*_{m\tilde{h}u} = \pi^*_{m\tilde{h}u2}$ subject to their respective regularity conditions for existence.

Next, we need to find conditions for which the equilibrium outcomes identified above is immune to the retailer deviating to a case where he sells only in the h state. Notice that the deviation profit for the retailer in this case is $\pi_{rd} = \Pr(h \mid \tilde{h})(z - p_d)p_d - (z - p_d)w_{\tilde{h}}$. This means

$$p_d(w_{\tilde{h}}) = \frac{z(\rho+1) + 2w_{\tilde{h}}}{2(1+\rho)} \qquad \text{and} \qquad \pi_{rd}(w_{\tilde{h}}) = \frac{[z(1+\rho) - 2w_{\tilde{h}}]^2}{8(1+\rho)}.$$

As long as this deviation profit remains lower than $\pi^*_{r\tilde{h}u1} = (1-c)^2/16$ for $w_{\tilde{h}} = w^*_{\tilde{h}u1} = (1+c)/2$, the first equilibrium identified above remains robust to a retailer deviation of this kind. The condition simplifies to

$$z < \frac{2(1+c)+\sqrt{2(1+\rho)(1-c)^2}}{2(1+\rho)}.$$

Following similar approach, the second equilibrium identified above remains robust to a retailer deviation for

$$z < \frac{(3-\rho)(2\rho+2+4\sqrt{2\rho+2})}{2(7+6\rho-\rho^2)}.$$

Overstocking given $\tilde{h}$. The retailer's profit function in the overstocking case is

$$\pi_{r\tilde{h}o} = \Pr(h \mid \tilde{h})(p_{\tilde{h}})(z-p_{\tilde{h}}) + \Pr(l \mid \tilde{h})[(p_{\tilde{h}})(1-p_{\tilde{h}}) + R_{\tilde{h}}(z-1)] - w_{\tilde{h}}(z-p_{\tilde{h}}).$$

From this the optimal retail price is $p_{\tilde{h}}(w_{\tilde{h}}) = (z+1+2w_{\tilde{h}}+(z-1)\rho)/4$. The manufacturer problem is to maximize the profit function $\pi_{mo\tilde{h}} = (w_{\tilde{h}}-c)(z-p_{\tilde{h}}) - \Pr(l \mid h)R_{\tilde{h}}(z-1)$ subject to $w_{\tilde{h}} > 0$, $R_{\tilde{h}} \geq 0$ and the retailer getting at least the profit available from the best possible deviation to an understocking case. The retailer's deviation profit will be $\pi_{r\tilde{h}d} = (1-p_{\tilde{h}d})(p_{\tilde{h}d}-w_{\tilde{h}})$. Finding the optimal $p_{\tilde{h}d}(w_{\tilde{h}})$ and substituting back we get $\pi_{r\tilde{h}d}(w_{\tilde{h}}) = (1-w_{\tilde{h}})^2/4$. The Lagrangian for the manufacturer's optimization will be,

$$L_{m\tilde{h}o} = (w_{\tilde{h}}-c)(z-p_{\tilde{h}}) - \Pr(l \mid \tilde{h})R_{\tilde{h}}(z-1) + \mu_{w_{\tilde{h}}} w_{\tilde{h}} + \mu_{R\tilde{h}} R_{\tilde{h}} + \mu_{\tilde{h}d}(\pi_{r\tilde{h}o}(w_{\tilde{h}}, R_{\tilde{h}}) - \pi_{r\tilde{h}d}(w_{\tilde{h}})).$$

Starting with the case of $\mu_{\tilde{h}d} = 0$ and the deviation constraint not binding: since $w_{\tilde{h}} > 0$ then we must have $\mu_{w_{\tilde{h}}} = 0$. Now suppose $\partial L_{m\tilde{h}o}/\partial R_{\tilde{h}} = 0$, which gives us $\mu_{R\tilde{h}} = ((1-\rho)(z-1))/2 > 0$. This implies that $R_{\tilde{h}} = 0$. Now substituting these into $\mu_{w_{\tilde{h}}} = 0$, we get $w_{\tilde{h}1} = \frac{1}{4}(3z-1+2c-(z-1)\rho)$. Substituting $R_{\tilde{h}} = 0$ and $w_{\tilde{h}} = w_{\tilde{h}1}$, we get the deviation constraint $(\pi_{r\tilde{h}o}(w_{\tilde{h}}, R_{\tilde{h}}) - \pi_{r\tilde{h}d}(w_{\tilde{h}}))$. For this solution to be valid, this must be non-negative. This and the condition for demand to be positive in both states gives us a regularity conditions in z as

$$z \leq \min\left(\frac{(3-\rho)(1-c)}{4(1-\rho)}, \; \frac{7-2c+\rho}{5+\rho}\right).$$

Under this solution, the manufacturer's expected profit is $\pi^*_{m\tilde{h}o1} = (1-3z+z\rho-\rho+2c)^2/32$.

Consider next the case in which $R_{\tilde{h}} > 0$ and $\mu_{\tilde{h}d} > 0$, which implies that the retailer profit condition is binding. Solving $\partial L_{m\tilde{h}o}/\partial R_{\tilde{h}} = 0$, $\partial L_{m\tilde{h}o}/\partial w_{\tilde{h}} = 0$ and the binding profit condition together, we get $w_{\tilde{h}} = (1+c)/2$, $\mu_{\tilde{h}d} = 1$ and

$$R_{\tilde{h}} = \frac{3+6c-z-\rho^2(z-1)-2\rho(z+c+2)}{8(1-\rho)}.$$

The corresponding retail price is $p^*_{\tilde{h}} = (z+c+2+\rho(z-1))/4$. From the positive demand condition in off-equilibrium path, we get a condition: $z < (2-c+\rho)/(1+\rho)$. In addition, the requirement for positive $R_{\tilde{h}}$ gives us another condition,

$$z \leq \frac{3+6c-\rho(4+2c-\rho)}{1+\rho^2}.$$

Under these regularity conditions, the equilibrium manufacturer profit can be calculated to be

$$\pi^*_{m\tilde{h}o2} = \frac{(2z-1+8c-12cz+2c^2+z^2)+(z-1)^2\rho^2+2\rho(z^2+2c(z-1)-1)}{16}$$

and the equilibrium retailer profit is $\pi^*_{ro\tilde{h}} = (1-c)^2/16$. Depending upon the parameter values, one or the other solution obtained above may not exist. Whenever both solutions exist, we can compare $\pi^*_{m\tilde{h}o1}$ and $\pi^*_{m\tilde{h}o2}$ and show that for

$$\rho < \rho^{\tilde{h}} = \frac{2[8z(c+z)+(1-c)^2]^{1/2}-(5z+1+2c)}{(z-1)},$$

$\pi^*_{m\tilde{h}o1}$ is higher for the manufacturer and hence for that, $w_{\tilde{h}} = \frac{1}{4}(3z-1+2c-(z-1)\rho)$ and $R_{\tilde{h}} = 0$. Otherwise,

$$w_{\tilde{h}} = \frac{1+c}{2} \qquad \text{and} \qquad R_{\tilde{h}} = \frac{3+6c-z-\rho^2(z-1)-2\rho(z+c+2)}{8(1-\rho)}.$$

Understocking given $\tilde{l}$. Given $w_{\tilde{l}}$ from the manufacturer, the retailer chooses the inventory and the retail price. The retailer's profit function is $\pi_{ru} = (1-p_{\tilde{l}})(p_{\tilde{l}} - w_{\tilde{l}})$. From this the optimal retail price is $p_{\tilde{l}}(w_{\tilde{l}}) = (1+w_{\tilde{l}})/2$ and the retailer profit becomes $\pi_{r\tilde{l}u}(w_{\tilde{l}}) = (1-w_{\tilde{l}})^2/4$. Once again, all deviations from the understocking equilibrium need to be considered. Just as we did for the $\tilde{h}$ signal, we again derive the equilibrium considering the case where the relevant off-equilibrium deviation is that of the retailer moving to overstocking when there is positive demand in both states of demand. Given $w_{\tilde{l}}$, let the retailer choose the deviation price $p_{\tilde{l}d}$ so that there is demand in the off-equilibrium path irrespective of the state. The deviation profit will be $\pi_{r\tilde{l}d} = \Pr(h \mid \tilde{l})(z - p_{\tilde{l}d})p_{\tilde{l}d} + \Pr(l \mid \tilde{l})(1-p_{\tilde{l}d})p_{\tilde{l}d} - (z-p_{\tilde{l}d})w_{\tilde{l}}$. This leads to an optimal deviation price of $p_{\tilde{l}d}(w_{\tilde{l}}) = (1+z+2w_{\tilde{l}} - (z-1)\rho)/4$. Thus if the channel equilibrium were to involve understocking, then this understocking equilibrium must provide the retailer with at least as much profit as $\pi_{r\tilde{l}d}(p_{\tilde{l}d}(w_{\tilde{l}}))$.

The manufacturer's problem is to choose $w_{\tilde{l}}$ to maximize $\pi_{m\tilde{l}u} = (1-p_{\tilde{l}})(w_{\tilde{l}} - c)$ subject to $w_{\tilde{l}} > 0$ and the retailer getting at least the profit available from the best possible deviation. The corresponding Lagrangian will be:

$$L_{m\tilde{l}u} = (1-p_{\tilde{l}})(w_{\tilde{l}} - c) + \mu_{w_{\tilde{l}}} w_{\tilde{l}} + \mu_{\tilde{l}d}\big(\pi_{ru}(w_{\tilde{l}}) - \pi_{rd}(p_{\tilde{l}d}(w_{\tilde{l}}))\big).$$

Let us first consider the case for which $\mu_{\tilde{l}d} = 0$ which means that the deviation constraint is not binding for the retailer. Solving $\partial L_{m\tilde{l}u}/\partial w_{\tilde{l}} = 0$ gives us $w_{l\tilde{u}1} = (1+c)/2$. In order for $\pi_{rd} \geq 0$, we must have

$$1 < z < \frac{3+6c+\rho(4+2c+\rho)}{1-\rho)^2}.$$

This is one possible solution that leads to a manufacturer profit of $\pi^*_{m\tilde{l}u1} = (1-c)^2/8$ and $\pi^*_{r\tilde{l}u1} = (1-c)^2/16$. Next we consider the case when the deviation profit constraint is binding. This implies $\mu_{\tilde{l}d} \geq 0$. Solving this along with $\partial L_{m\tilde{l}u}/\partial w_{\tilde{l}} = 0$ gives us

$$w_{l\tilde{u}2} = \frac{3+z(1-\rho)^2 - \rho(2+\rho)}{4(3+\rho)} \qquad \text{and} \qquad \mu_{\tilde{l}d} = \frac{z(1-\rho)^2 - 3(1+2c) - \rho(4+2c+\rho)}{(3(3+2\rho)+\rho^2)(z-1)}.$$

Note that for $\mu_{\tilde{l}d} \geq 0$, we must have

$$z > \frac{3+6c+\rho(4+2c+\rho)}{1-\rho^2},$$

which is the regularity condition for existence of this solution. Thus given the signal $\tilde{l}$, an understocking equilibrium may involve $w^*_{\tilde{l}u1} = (1+c)/2$ and no returns price, or

$$w_{l\tilde{u}2} = \frac{3+z(1-\rho)^2 - \rho(2+\rho)}{4(3+\rho)}$$

and no returns price.

Finally, to identify condition when the above outcome is immune to a retailer deviation involving demand only in high state, we start by identifying that the deviation profit will be $\pi_{rd} = \Pr(h \mid \tilde{l})(z - p_d)p_d - (z-p_d)w_{\tilde{l}}$. This means

$$p^*_d(w_{\tilde{l}}) = \frac{z(1-\rho)+2w_{\tilde{l}}}{2(1-\rho)} \qquad \text{and} \qquad \pi_{rd}(w_{\tilde{l}}) = \frac{[z(1-\rho)-2w_{\tilde{l}}]^2}{8(1-\rho)}.$$

Notice that this is valid only if $p^*_d(w_{\tilde{l}}) < z$ as otherwise demand in High state is also negative which is meaningless. Subject to this regularity condition, comparing the equilibrium profit with the deviation profit, we get that whenever

$$\frac{2(1+c) - [2(1-c)^2(1-\rho)]^{1/2}}{2(1-\rho)} \leq z \leq \frac{2(1+c) + [2(1-c)^2(1-\rho)]^{1/2}}{2(1-\rho)},$$

the identified equilibrium is immune to this deviation.

Overstocking given $\tilde{l}$. The retailer's profit function in the overstocking case is

$$\pi_{r\tilde{l}o} = \Pr(h \mid \tilde{l})(p_{\tilde{l}})(z-p_{\tilde{l}}) + \Pr(l \mid \tilde{l})[(p_{\tilde{l}})(1-p_{\tilde{l}}) + R_{\tilde{l}}(z-1)] - w_{\tilde{l}}(z-p_{\tilde{l}}).$$

From this the optimal retail price $p_{\tilde{l}}(w_{\tilde{l}}) = (z+1+2w_{\tilde{l}} - (z-1)\rho)/4$. The manufacturer problem is to maximize the profit function $\pi_{m\tilde{l}o} = (w_{\tilde{l}} - c)(z-p_{\tilde{l}}) - \Pr(l \mid \tilde{l})R_{\tilde{l}}(z-1)$ subject to $w_{\tilde{l}} > 0$, $R_{\tilde{l}} \geq 0$ and the retailer getting at least the profit available from the best possible deviation to an understocking case. The retailer's deviation profit will be $\pi_{r\tilde{l}d} = (1-p_{\tilde{l}d})(p_{\tilde{l}d} - w_{\tilde{l}})$. Finding the optimal $p_{\tilde{l}d}(w_{\tilde{l}})$ and substituting back we get $\pi_{r\tilde{l}d}(w_{\tilde{l}}) = (1-w_{\tilde{l}})^2/4$. The Lagrangian for the manufacturer's optimization will be,

$$L_{m\tilde{l}o} = \Pr(h \mid \tilde{l})(w_{\tilde{l}} - c)(z-p_{\tilde{l}}) - \Pr(l \mid \tilde{l})R_{\tilde{l}}(z-1) + \mu_{w_{\tilde{l}}} w_{\tilde{l}} + \mu_{R_{\tilde{l}}} R_{\tilde{l}} + \mu_{\tilde{l}d}\big(\pi_{r\tilde{l}o}(w_{\tilde{l}}, R_{\tilde{l}}) - \pi_{r\tilde{l}d}(w_{\tilde{l}})\big).$$

Starting with the case of deviation constraint not binding, since $w_{\tilde{l}} > 0$ then we must have $\mu_{w_{\tilde{l}}} = 0$. Now suppose $\partial L_{m\tilde{l}o}/\partial R_{\tilde{l}} = 0$, which gives us $\mu_{R_{\tilde{l}}} = (1-\rho)(z-1)/2 > 0$. This implies that $R_{\tilde{l}} = 0$. Now substituting these into $\mu_{w_{\tilde{l}}} = 0$, we get $w^*_{\tilde{l}1} = \frac{1}{4}(3z-1+2c+(z-1)\rho)$. Substituting $R^*_{\tilde{l}} = 0$ and $w_{\tilde{l}} = w^*_{\tilde{l}1}$ we can calculate the deviation constraint, which must be non-negative to be valid. However, that requires $(1 \geq z \geq ((3+\rho)(1-c))/(4(1-\rho)))$, which is not possible so this solution is ruled out.

Consider next the case in which $R_{\tilde{l}} > 0$ and $\mu_{\tilde{l}d} > 0$, which implies that the retailer profit condition is binding. Solving $\partial L_{m\tilde{l}o}/\partial R_{\tilde{l}} = 0$, and $\partial L_{m\tilde{l}o}/\partial w_{\tilde{l}} = 0$ and the binding profit condition together, we get $w_{\tilde{l}} = (1+c)/2$, $\mu_{\tilde{l}d} = 1$ and

$$R_{\tilde{l}} = \frac{3+6c-z+\rho^2(z-1)+2\rho(z+c+2)}{8(1+\rho)}.$$

Since $R_{\tilde{l}}$ must be positive, we must have $z \leq (3+6c+\rho(4+2c+\rho))/(1-\rho)^2$. The corresponding retail price is $p^*_{\tilde{l}} = (z+c+2-\rho(z-1))/4$. From the positive demand condition in both states, we get another regularity condition: $z < (2-c-\rho)/(1-\rho)$. Under these conditions, the equilibrium manufacturer profit can be calculated to be

$$\pi^*_{m\tilde{l}o2} = \frac{(2z-1+8c-12cz+2c^2+z^2)+(z-1)^2\rho^2-2\rho(z^2+2c(z-1)-1)}{16}$$

and the equilibrium retailer profit is $\pi^*_{r\tilde{l}o} = (1-c)^2/16$. For $\pi^*_{m\tilde{l}o2} \geq 0$, we must have low enough reliability i.e.,

$$\rho < \rho^l = \frac{z+1+2c-[2(1-c)^2+8cz]^{1/2}}{z-1}.$$

This is the only surviving solution subject to the regularity conditions already mentioned above.

Retail Competition, §4

Channel with Inventory ($\rho = 0$)
In the competitive case also, we need to consider two types of equilibrium as before, as the in-between case can be ruled out for equilibrium (and deviations) for reasons similar to the single retailer channel. We solve for the symmetric equilibrium.

Understocking Equilibrium
Based on the w offered by the manufacturer, the retailers choose the inventory and the retail price. Retailer i's profit function is $\pi_{riu} = [1-p_i+\gamma(p_j-p_i)](p_i-w)$. From this the optimal retail price is $p_1(w) = p_2(w) = (1+w(1+\gamma))/(2+\gamma)$ and retailer i's profit can be expressed as $\pi_{riu}(w)$. Given the contract no retailer's should have the incentive to unilaterally deviate to any choice of inventory and price other than understocking. Like in the single retailer case, once again the deviation possibilities involve overstocking, either with positive demand in both states or only in the high state. We will consider the equilibrium where the relevant off-equilibrium deviation supporting the equilibrium involves the case where there is demand in both states. Later on we will identify conditions when this equilibrium is immune to the deviation to the other type of overstocking. Without loss of generality, let the deviating retailer be retailer 1.

If retailer 1 deviates to overstocking, it chooses $Q=(z-p_1+\gamma(p_2-p_1))$. Let retailer 1's choice of the deviation price be called p_{1d}. The deviation profit function will be

$$\pi_{r1d}=\tfrac{1}{2}(z-p_{1d}+\gamma(p_2(w)-p_{1d}))p_{1d}+\tfrac{1}{2}(1-p_{1d}+\gamma(p_2(w)-p_{1d}))p_{1d}-(z-p_{1d}+\gamma(p_2(w)-p_{1d}))w.$$

Thus for a given w, we will have the optimal deviation price to be

$$p_{1d}(w)=\frac{4w(\gamma+1)^2+z(\gamma+2)+3\gamma+2}{4(\gamma^2+3\gamma+2)}.$$

The equilibrium contract must provide the retailer with atleast as much profit as $\pi_{r1d}(p_{1d}(w))$.

The manufacturer's problem is to choose w to

$$\begin{aligned}&\text{maximize}\quad \pi_{mu}=\big[(1-p_1+\gamma(p_2-p_1))+(1-p_2+\gamma(p_1-p_2))\big](w-c)\\&\text{subject to}\quad w>0\end{aligned}$$

and the retailer getting at least the profit available from the deviation identified above. The corresponding Lagrangian will be

$$L_{mu}=[(1-p_1+\gamma(p_2-p_1))+(1-p_2+\gamma(p_1-p_2))](w-c)+\mu_w w+\mu_d(\pi_{riu}(w)-\pi_{rid}(p_{id}(w))),$$

where the μ's are the corresponding Lagrangian multipliers. Parallel to the analysis in the single retailer case, we look at the solution candidates one by one based on the relevant first order conditions. One solution candidate is $w_1^*=(1+c)/2$, which exists only if

$$z\in\left(1,\ \frac{4c(\gamma^2+4\gamma+3)+(4\gamma^2+9\gamma+6)}{2+\gamma}\right)$$

and the deviation constraint is not binding. Whenever it exists, it gives

$$\pi_{mu}=\pi^*_{mu1}=\frac{(1+\gamma)(1-c)^2}{2(2+\gamma)}$$

as the manufacturer profit. Another solution candidate is

$$w_2^*=\frac{z(\gamma+2)+7\gamma+6}{8(\gamma^2+4\gamma+3)},$$

which exists only if

$$\left[\frac{4c\gamma^2+4\gamma^2-9\gamma-16c\gamma-12c-6}{2+\gamma}<z<(9+8\gamma)\right]$$

and the deviation constraint is binding. Whenever this exists, it gives

$$\pi_{mu}=\pi^*_{mu2}=\frac{(8\gamma+9-z)(z\gamma+2z+7\gamma+6-8c\gamma^2-32c\gamma-24c)}{32(3+\gamma)(\gamma^2+4\gamma+3)}.$$

It can be shown that $\pi^*_{mu2}\le\pi^*_{mu1}$. Thus the maximum equilibrium manufacturer profits are $\pi^*_{mu}=((1+\gamma)(1-c)^2)/(2(2+\gamma))$ and $w^*=(1+c)/2$.

To see if the equilibrium with w_1^* is immune against one retailer unilaterally deviating to the overstocking case of the type involving only high state demand, we start with calculating that the deviating retailer's profit function will be

$$\pi_{rid2}(w)=\frac{[z(\gamma+2)+\gamma-w(5\gamma+\gamma^2+4)]^2}{(2+\gamma)(3\gamma+\gamma^2+2)}.$$

For the retailer deviating from the equilibrium path to this deviation, however, it is sufficient that $(\pi_{riu}(w)-\pi_{rid2}(w))$ be negative for $w^*=(1+c)/2$. The condition for that turns out to be that

$$z\in\left(\frac{c(8+14\gamma+\gamma^3+7\gamma^2)+(5\gamma^2+10\gamma+\gamma^3+8)\pm2[2((1-c)^2(4+6\gamma^3+13\gamma^2+12\gamma+\gamma^4)-12(1-\gamma))]^{1/2}}{2(\gamma^2+4\gamma+4)}\right).$$

Thus as long as z is between the two roots, the equilibrium identified above is valid. Finally for the manufacturer w_1^* is always the best strategy and it can be shown that no other strategy (that makes the retailer deviation involving the demand in the high state only relevant) is profit improving.

Overstocking Equilibrium

The retailer's profit function in the overstocking case is

$$\pi_{rio} = \tfrac{1}{2}[(z-p_i+\gamma(p_j-p_i))(p_i)+(1-p_i+\gamma(p_j-p_i))(p_i)+R(z-1)]-w(z-p_i+\gamma(p_j-p_i)).$$

From this the optimal retail price $p_i(w)=(2w(1+\gamma)+z+1)/(2(\gamma+2))$. Given this, the manufacturer problem is to maximize the profits given by

$$\pi_{mo}=(w-c)[(z-p_1+\gamma(p_2-p_1))+(z-p_2+\gamma(p_1-p_2))]-R(z-1),$$

subject to $w>0$, $R\geq 0$ and the retailer getting at least the profit available from the best possible deviation. We need to consider deviation to the understocking case. Again, without loss of generality, let retailer 1 deviate and charge a price of p_{1d}. We can show that

$$p_{1d}(w)=\frac{4w(1+\gamma)^2+(3+z)\gamma+4}{4(2+3\gamma+\gamma^2)} \qquad \text{and} \qquad \pi_{r1d}(p_{1d}(w))=\frac{[3\gamma+4+z\gamma-4w(1+\gamma)]^2}{16(\gamma+2)(3\gamma+2+\gamma^2)}.$$

To solve the manufacturer's problem, we can set up the following Lagrangian:

$$\begin{aligned}L_{mo} &= (w-c)[(z-p_1+\gamma(p_2-p_1))+(z-p_2+\gamma(p_1-p_2))]-R(z-1)\\ &\quad +\mu_w w+\mu_R R+\mu_d(\pi_{rio}(w,R)-\pi_{rid}(w)).\end{aligned}$$

Proceeding in the usual manner to consider all solution candidates, we can show that the only solution candidate that can survive involves retailer's deviation constraint to be binding and it involves a positive returns price R. Whenever $z<(3\gamma+4-2c(1+\gamma))/(2+\gamma)$ the manufacturer may offer a wholesale price

$$w^*=\frac{2c(1+\gamma)+2+(\gamma(z+1))}{4(1+\gamma)}$$

and the returns price

$$R^*=\frac{2\gamma^2z+3z\gamma-2z+2(2c+1)\gamma^2+16c\gamma+5\gamma+12c+6}{3\gamma+2+\gamma^2}.$$

This results in a manufacturer profit of

$$\pi^*_{mo}=\frac{(\gamma+2)(z+1)^2+4c(c+c\gamma+\gamma-3z\gamma+4+6z)-4}{8(\gamma+2)}.$$