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Proofs of Statements

Below we provide the proofs for Propositions 1-4, and derive Equation (24) which gives an upper bound for the number of brands.

EC.1. Proof of Proposition 1.

PROPOSITION 1. Maximizing (12) subject to (4), the optimal closed-loop advertising strategy of brand $i \in I$ is

$$u_i{*} = \frac{\sqrt{M - A_i}}{2}\left((\rho_i + \xi_{ii})\phi_i^i - \sum_{k=1}^{N} \xi_{ki}\phi_k^i\right), \tag{13}$$

and its value function

$$V_i = \sum_{j=1}^{N} \phi_j^i A_j, \tag{14}$$

where the coefficients ϕ_j^i are obtained from the following relations:

$$r\phi_i^i = m_i + \sum_{j\in I,\, j\neq i} \frac{1}{2}\left((\rho_j + \xi_{jj})\phi_j^j - \sum_{k\in I} \xi_{kj}\phi_k^j\right)\left((\rho_j + \xi_{jj})\phi_j^i - \sum_{k\in I} \xi_{kj}\phi_k^i\right). \tag{15}$$

$$\begin{aligned} r\phi_j^i = &-\frac{1}{4}\left((\rho_i + \xi_{ii})\phi_i^i - \sum_{k\in I} \xi_{ki}\phi_k^i\right)^2 \\ &+ \sum_{s\in I, s\neq j} \frac{1}{2}\left((\rho_s + \xi_{ss})\phi_s^s - \sum_{k\in I} \xi_{ks}\phi_k^s\right)\left((\rho_s + \xi_{ss})\phi_s^i - \sum_{k\in I} \xi_{ks}\phi_k^i\right),\ j \neq i. \end{aligned} \tag{16}$$

EC.1.1. Proof of the Main Result

Proof of Proposition 1. The objective for brand $i \in I = \{1, 2, .., N\}$ is

$$\underset{u_i(t)}{Max} \int_0^{\infty} e^{-rt}(m_i A_i(t) - u_i(t)^2)dt, \tag{A1}$$

s.t. $$\frac{dA_i(t)}{dt} = \rho_i u_i(t)\sqrt{M(t) - A_i(t)} - \sum_{\substack{j=1\\ j\neq i}}^{N} \xi_{ij} u_j(t)\sqrt{M(t) - A_j(t)},\ i \in I, \tag{A2}$$

$$\text{where } M(t) \equiv \sum_{i=1}^{N} A_i(t) \,. \tag{A3}$$

We shall drop the time subscripts hereafter. Note that (A2) can be rewritten as

$$\frac{dA_i}{dt} = (\rho_i + \xi_{ii}) u_i \sqrt{M - A_i} - \sum_{j=1}^{N} \xi_{ij} u_j \sqrt{M - A_j} \,,\; i \in I \,. \tag{A2'}$$

The Hamilton-Jacobi-Bellman (HJB) equation for brand i is given by

$$rV_i = \max_{u_i} \; m_i A_i - u_i^2 + \sum_{k=1}^{N} \left\{ \frac{\partial V_i}{\partial A_k} \left((\rho_k + \xi_{kk}) u_k \sqrt{M - A_k} - \sum_{j=1}^{N} \xi_{kj} u_j \sqrt{M - A_j} \right) \right\}. \tag{A4}$$

This can be rewritten to make it simpler. First, note that the term

$\sum_{k=1}^{N} \frac{\partial V_i}{\partial A_k} (\rho_k + \xi_{kk}) u_k \sqrt{M - A_k}$ can be written as $\sum_{j=1}^{N} \frac{\partial V_i}{\partial A_j} (\rho_j + \xi_{jj}) u_j \sqrt{M - A_j}$.

Second, note the following transformation:

$$\begin{aligned}
&\sum_{k=1}^{N} \left\{ \frac{\partial V_i}{\partial A_k} \sum_{j=1}^{N} \xi_{kj} u_j \sqrt{M - A_j} \right\} \\
&= \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{\partial V_i}{\partial A_k} \xi_{kj} u_j \sqrt{M - A_j} \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial V_i}{\partial A_k} \xi_{kj} u_j \sqrt{M - A_j} \\
&= \sum_{j=1}^{N} u_j \sqrt{M - A_j} \sum_{k=1}^{N} \frac{\partial V_i}{\partial A_k} \xi_{kj}
\end{aligned}$$

Putting these into (A4) yields

$$rV_i = \max_{u_i} \; m_i A_i - u_i^2 + \sum_{j=1}^{N} \frac{\partial V_i}{\partial A_j} (\rho_j + \xi_{jj}) u_j \sqrt{M - A_j} - \sum_{j=1}^{N} u_j \sqrt{M - A_j} \sum_{k=1}^{N} \frac{\partial V_i}{\partial A_k} \xi_{kj} \tag{A5}$$

which can be rewritten as

$$rV_i = \max_{u_i} \; m_i A_i - u_i^2 + \sum_{j \in I} u_j \sqrt{M - A_j} \left((\rho_j + \xi_{jj}) \frac{\partial V_i}{\partial A_j} - \sum_{k \in I} \xi_{kj} \frac{\partial V_i}{\partial A_k} \right). \tag{A6}$$

We equate the derivative of the RHS to zero and obtain the optimal feedback control

$$u_i^* = \frac{\sqrt{M - A_i}}{2}\left((\rho_i + \xi_{ii})\frac{\partial V_i}{\partial A_i} - \sum_{k \in I} \xi_{ki}\frac{\partial V_i}{\partial A_k} \right), \ \forall i \in I . \tag{A7}$$

Assuming that the controls are positive, we insert these controls into the HJB equations to obtain

$$\begin{aligned} rV_i = m_i A_i - \frac{(M - A_i)}{4}\left((\rho_i + \xi_{ii})\frac{\partial V_i}{\partial A_i} - \sum_{k \in \mathbf{I}} \xi_{ki}\frac{\partial V_i}{\partial A_k} \right)^2 \\ + \sum_{j \in \mathbf{I}} \frac{(M - A_j)}{2}\left((\rho_j + \xi_{jj})\frac{\partial V_j}{\partial A_j} - \sum_{k \in I} \xi_{kj}\frac{\partial V_j}{\partial A_k} \right)\left((\rho_j + \xi_{jj})\frac{\partial V_i}{\partial A_j} - \sum_{k \in \mathbf{I}} \xi_{kj}\frac{\partial V_i}{\partial A_k} \right). \end{aligned} \tag{A8}$$

To solve these N simultaneous partial differential equations, we use the following linear value functions, observing that they satisfy the Hamilton-Jacobi equations,

$$V_i = \sum_{j=1}^{N} \phi_j^i A_j , \ \forall i \in I . \tag{A9}$$

Thus, there are a total of N unknown parameters (the ϕ_j^i s) for each of the N firms. To determine these parameters, we insert (A9) into the Hamilton-Jacobi equation and obtain, $\forall i \in I$,

$$\begin{aligned} r\sum_{j=1}^{N} \phi_j^i A_j = m_i A_i - \frac{(M - A_i)}{4}\left((\rho_i + \xi_{ii})\phi_i^i - \sum_{k \in I} \xi_{ki}\phi_k^i \right)^2 \\ + \sum_{j \in I} \frac{(M - A_j)}{2}\left((\rho_j + \xi_{jj})\phi_j^j - \sum_{k \in I} \xi_{kj}\phi_k^j \right)\left((\rho_j + \xi_{jj})\phi_j^i - \sum_{k \in I} \xi_{kj}\phi_k^i \right). \end{aligned} \tag{A10}$$

Recalling that $M(t) \equiv \sum_{i \in I} A_i(t)$, we equate the A_i terms on both sides of this equation.

$$r\phi_i^i = m_i + \sum_{j \in \mathbf{I}, j \neq i} \frac{1}{2}\left((\rho_j + \xi_{jj})\phi_j^j - \sum_{k \in I} \xi_{kj}\phi_k^j \right)\left((\rho_j + \xi_{jj})\phi_j^i - \sum_{k \in I} \xi_{kj}\phi_k^i \right).$$

$$\begin{aligned} r\phi_j^i = -\frac{1}{4}\left((\rho_i + \xi_{ii})\phi_i^i - \sum_{k \in I} \xi_{ki}\phi_k^i \right)^2 \\ + \sum_{s \in I, s \neq j} \frac{1}{2}\left((\rho_s + \xi_{ss})\phi_s^s - \sum_{k \in I} \xi_{ks}\phi_k^s \right)\left((\rho_s + \xi_{ss})\phi_s^i - \sum_{k \in I} \xi_{ks}\phi_k^i \right), j \neq i. \end{aligned}$$

We get such relationships for each of the N brands in I , which are to be solved simultaneously for the unknown coefficients.

EC.2. Proof of Proposition 2.

PROPOSITION 2. In mature markets, a brand's optimal closed-loop advertising strategy is

$$u_i^* = \frac{\rho_i \sqrt{1-A_i}}{2(N-1)} \left(N\phi_i^i - \sum_{j=1}^{N} \phi_j^i \right), \tag{17}$$

and its value function

$$V_i = \phi_0^i + \sum_{j=1}^{N} \phi_j^i A_j , \qquad \forall i \in I , \tag{18}$$

where the coefficients (ϕ_0^i , ϕ_j^i) are obtained from the following ($N+1$) relations:

$$r(\sum_{j=0}^{N} \phi_j^i) = m_i , \tag{19}$$

$$r\phi_i^i = m_i - \frac{\rho_i^2}{4(N-1)^2} \left(N\phi_i^i - \sum_{k\in I} \phi_k^i \right)^2 , \tag{20}$$

$$r\phi_j^i = -\frac{\rho_j^2}{2(N-1)^2} \left(N\phi_j^j - \sum_{k\in I} \phi_k^j \right) \left(N\phi_j^i - \sum_{k\in I} \phi_k^i \right), \ \forall j \in I, j \neq i . \tag{21}$$

EC.2.1. Proof of the Main Result

Proof of Proposition 2. The Hamilton-Jacobi-Bellman (HJB) equation for brand i is given by

$$rV_i = \max_{u_i} \ \ m_i A_i - u_i^2 + \sum_{j\in I} \frac{\partial V_i}{\partial A_j} \left(\frac{N}{N-1} \rho_j u_j \sqrt{1-A_j} - \frac{1}{N-1} \sum_{k\in I} \ \rho_k u_k \sqrt{1-A_k} \right), \tag{B1}$$

which can be rewritten as

$$rV_i = \max_{u_i} \ \ m_i A_i - u_i^2 + \sum_{j\in I} \ \frac{\rho_j u_j \sqrt{1-A_j}}{N-1} \left(N \frac{\partial V_i}{\partial A_j} - \sum_{k\in I} \ \frac{\partial V_i}{\partial A_k} \right). \tag{B2}$$

We obtain the optimal feedback controls

$$u_i{*} = \frac{\rho_i \sqrt{1-A_i}}{2(N-1)} \left(N \frac{\partial V_i}{\partial A_i} - \sum_{k\in I} \ \frac{\partial V_i}{\partial A_k} \right), \ \ \forall i \in I . \tag{B3}$$

Assuming that the controls are positive, we insert these controls into the HJB equations to obtain

$$rV_i = m_i A_i - \frac{\rho_i^{\,2}(1-A_i)}{4(N-1)^2}\left(N\frac{\partial V_i}{\partial A_i} - \sum_{k\in I}\frac{\partial V_i}{\partial A_k}\right)^2 + \sum_{j\in I}\frac{\rho_j^{\,2}(1-A_j)}{2(N-1)^2}\left(N\frac{\partial V_j}{\partial A_j} - \sum_{k\in I}\frac{\partial V_j}{\partial A_k}\right)\left(N\frac{\partial V_i}{\partial A_j} - \sum_{k\in I}\frac{\partial V_i}{\partial A_k}\right). \quad \text{(B4)}$$

To solve these N simultaneous partial differential equations, we use the following linear value functions, observing that they satisfy the Hamilton-Jacobi equations,

$$V_i = \phi_0^i + \sum_{j=1}^{N}\phi_j^i A_j, \qquad \forall i\in I. \quad \text{(B5)}$$

To determine the unknown parameters, we insert (B5) into (B4) and obtain, $\forall i\in I$,

$$r(\phi_0^i + \sum_{j=1}^{N}\phi_j^i A_j) = m_i A_i - \frac{\rho_i^{\,2}(1-A_i)}{4(N-1)^2}\left(N\phi_i^i - \sum_{k\in I}\phi_k^i\right)^2 + \sum_{j\in I}\frac{\rho_j^{\,2}(1-A_j)}{2(N-1)^2}\left(N\phi_j^j - \sum_{k\in I}\phi_k^j\right)\left(N\phi_j^i - \sum_{k\in I}\phi_k^i\right) \quad \text{(B6)}$$

$$\Rightarrow r(\sum_{j=0}^{N}\phi_j^i - \sum_{j=1}^{N}\phi_j^i(1-A_j)) = m_i - m_i(1-A_i) - \frac{\rho_i^{\,2}(1-A_i)}{4(N-1)^2}\left(N\phi_i^i - \sum_{k\in I}\phi_k^i\right)^2 + \sum_{j\in I}\frac{\rho_j^{\,2}(1-A_j)}{2(N-1)^2}\left(N\phi_j^j - \sum_{k\in I}\phi_k^j\right)\left(N\phi_j^i - \sum_{k\in I}\phi_k^i\right). \quad \text{(B7)}$$

Equating the powers of $(1-A_i)$, we get

$$r(\sum_{j=0}^{N}\phi_j^i) = m_i, \quad \text{(B8)}$$

$$r\phi_i^i = m_i - \frac{\rho_i^{\,2}}{4(N-1)^2}\left(N\phi_i^i - \sum_{k\in I}\phi_k^i\right)^2, \quad \text{(B9)}$$

$$r\phi_j^i = -\frac{\rho_j^{\,2}}{2(N-1)^2}\left(N\phi_j^j - \sum_{k\in I}\phi_k^j\right)\left(N\phi_j^i - \sum_{k\in I}\phi_k^i\right), \forall j\in I,\, j\neq i. \quad \text{(B10)}$$

These $N+1$ relationships are obtained for each of the N brands in I, thereby resulting in $N(N+1)$ equations to be solved for the $N(N+1)$ unknown coefficients.

EC.3. Proof of Proposition 3.

PROPOSITION 3. For each brand in a mature market, the equilibrium share is given by

$$\bar{A}_i = 1 - \frac{N-1}{B_i \times \sum_{j=1}^{N} B_j^{-1}}, \qquad \forall i \in I, \tag{22}$$

where $B_i = \frac{\rho_i^2}{2(N-1)}\left(N\phi_i^i - \sum_{j=1}^{N}\phi_j^i\right)$.

EC.3.1. Proof of the Main Result

Proof of Proposition 3. We insert the optimal controls into the dynamic equations to get

$$\frac{dA_i}{dt} = N(1-A_i)B_i - \sum_{j\in\mathbf{I}} (1-A_j)B_j, \quad \forall i \in I, \tag{C1}$$

where $B_i \equiv \frac{\rho_i^2}{2(N-1)^2}(N\phi_i^i - \sum_{k\in\mathbf{I}} \phi_k^i)$. To find the steady state market shares, we solve $\frac{dA_i}{dt} = 0$. Let $\bar{A}_i$ denote the awareness share in steady state for brand i. It follows from (C1) that

$$1 - \bar{A}_i = \frac{1}{NB_i}\sum_{j\in I} (1-\bar{A}_j)B_j \tag{C2}$$

$$\Rightarrow \sum_{i\in I} (1-\bar{A}_i) = \sum_{i\in I} \frac{1}{NB_i}\sum_{j\in I} (1-\bar{A}_j)B_j$$
$$\Rightarrow \frac{N-1}{\sum_{i\in I}\frac{1}{B_i}} = \frac{1}{N}\sum_{j\in I} (1-\bar{A}_j)B_j$$
$$\Rightarrow \frac{N-1}{B_i\sum_{i\in I}\frac{1}{B_i}} = \frac{1}{NB_i}\sum_{j\in I} (1-\bar{A}_j)B_j$$

Inserting the last expression back into (C2), we obtain the desired result:

$$\bar{A}_i = 1 - \frac{N-1}{B_i\sum_{j\in I}\frac{1}{B_j}}, \qquad \forall i \in I. \tag{C3}$$

EC.4. Proof of Proposition 4.

PROPOSITION 4. In mature markets, the category ad spending increases as the number of brands increases.

EC.4.1. Proof of the Main Result

Proof of Proposition 4. From Proposition 2, the value function is $V_i = \phi_0 + \phi_1 A_i + \sum_{j\neq i} \phi_2 A_j$ for symmetric brands, where $\forall i \in I$, $\phi_0^i = \phi_0$, $\phi_i^i = \phi_1$, $\phi_j^i(\forall j \neq i) = \phi_2$. To determine ($\phi_0$, ϕ_1, ϕ_2) in terms of the model parameters, we apply Proposition 2 to get the relations:

$$r(\phi_0 + \phi_1 + (N-1)\phi_2) = m\,, \tag{D1}$$

$$r\phi_1 = m - \frac{\rho^2(\phi_1 - \phi_2)^2}{4}\,, \tag{D2}$$

$$r\phi_2 = \frac{\rho^2(\phi_1 - \phi_2)^2}{2(N-1)}\,. \tag{D3}$$

Subtracting (D3) from (D2), we get a quadratic equation in ($\phi_1 - \phi_2$), whose roots are given by

$$\phi_1 - \phi_2 = \left(\pm\sqrt{r^2 + \frac{m\rho^2(N+1)}{N-1}} - r\right) \Big/ \left(\frac{\rho^2(N+1)}{2(N-1)}\right). \tag{D4}$$

Because the value function $V_i = \phi_0 + \phi_1 A_i + \sum_{j\neq i} \phi_2 A_j$ can be written as $V_i = (\phi_0 + \phi_2) + (\phi_1 - \phi_2)A_i$ after using the identity $\sum_{j\neq i} A_j = 1 - A_i$, we expect $\phi_1 - \phi_2 > 0$, and so only the positive root should be used. Further simplification thus yields

$$\phi_1 - \phi_2 = \frac{2m}{r + \sqrt{r^2 + \frac{m\rho^2(N+1)}{N-1}}} > 0\,. \tag{D5}$$

Next, we express (ϕ_0, ϕ_1, ϕ_2) in terms of $(\phi_1 - \phi_2)$ because the latter term depends only on model parameters via (D5). To this end, we multiply (D2) by 2 and (D3) by $N-1$ and add them to obtain

$$2\phi_1 + (N-1)\phi_2 = 2m/r$$
$$\Rightarrow \phi_2 = \frac{2}{N+1}\left(\frac{m}{r} - (\phi_1 - \phi_2)\right) \qquad \text{(D6)}$$
$$\Rightarrow \phi_1 = \frac{1}{N+1}\left(\frac{2m}{r} + (N-1)(\phi_1 - \phi_2)\right)$$

and

$$\phi_0 = \frac{m}{r} - \frac{1}{N+1}\left(\frac{2mN}{r} - (N-1)(\phi_1 - \phi_2)\right), \qquad \text{(D7)}$$

after substituting ϕ_1 and ϕ_2 from (D6) into (D1).

Finally, because $u^* = \frac{\rho\sqrt{1-A}}{2}(\phi_1 - \phi_2)$ from Proposition 2 and $\frac{\partial(\phi_1 - \phi_2)}{\partial N} > 0$ from (D5), we find that $\frac{\partial \mathrm{u}^*}{\partial \mathrm{N}} > 0$. Hence, the total category advertising Nu^* increases with the number of brands.

EC.5. Derivation of an Upper Bound for N.

REMARK 5.1. We obtain an upper bound

$$N^* = 3 + \frac{2r}{\sqrt{r^2 + m\rho^2} - r}, \qquad (24)$$

which reveals that product categories will sustain three (or more) brands.

EC.5.1. Derivation of the Main Result

Derivation of Upper Bound. The total category value is

$$\sum_{i\in\mathbf{I}} V_i = \sum_{i\in\mathbf{I}} \left((\phi_0 + \phi_2) + (\phi_1 - \phi_2)A_i\right) = N(\phi_0 + \phi_2) + (\phi_1 - \phi_2). \qquad \text{(E1)}$$

Although $\phi_1 - \phi_2 > 0$ from (D5), the term $(\phi_0 + \phi_2)$ can become negative as the number of brands increases. Hence, the product category can sustain only a limited number of brands before a shakeout occurs when some brands do not advertise, letting awareness to decline to zero, and exiting the market. To determine the sustainable number of brands, we first observe that

$$\sum_{i\in I} V_i = \int_0^\infty e^{-rt}(m - \sum_{i\in I} u_i*(t)^2)dt = \frac{m}{r} - \frac{\rho^2(\phi_1-\phi_2)^2(N-1)}{4r} \geq 0\,. \qquad \text{(E2)}$$

Next, inserting $(\phi_1 - \phi_2)$ from (D5) into (E2) and simplifying the algebra, we obtain the condition:

$$\frac{m\rho^2 N(3-N)}{N-1} + 2r(r + \sqrt{r^2 + \frac{m\rho^2(N+1)}{N-1}}) \geq 0$$
$$\Rightarrow N(3-N) + \frac{2(N+1)}{\sqrt{1+\frac{m\rho^2(N+1)}{r^2(N-1)}} - 1} \geq 0. \qquad \text{(E3)}$$

It is easy to see that for $N \leq 3$ condition (E3) is satisfied. It also reveals that an upper bound on N that satisfies (E3) is given by

$$\mathrm{N}^* = 3 + \frac{2}{\sqrt{1+\frac{\mathrm{m}\rho^2}{\mathrm{r}^2}} - 1}\,. \qquad \text{(E4)}$$