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## A Appendix

**Proof of Proposition 1:** Assume that the first four steps have been completed. The derivative of  $\pi^B$  with respect to  $K_t^B$  is equal to,

$$\frac{\partial \pi^B}{\partial K_t^B} = rG_t(K_t) - c^B \quad (13)$$

with  $K_t = K_t^A + K_t^B \geq 0$ . Hence  $\pi^B$  is concave in  $K_t^B$  and the optimal capacity of the operator is equal to  $[\phi_t - K^A]^+$  when all other capacity levels are fixed. It remains to compute the value of  $K^A$  which maximizes  $\tilde{\pi}^B(K^A) := \pi^B(K^A, K_1^B(K^A), \dots, K_N^B(K^A)) = \pi^B(K^A, [\phi_1 - K^A]^+, \dots, [\phi_N - K^A]^+)$ .

For a given  $\tau \in [0, \dots, N+1]$ , consider values of  $K^A$  in the interval  $[\phi_\tau, \phi_{\tau+1}]$  (with  $\phi_0 = 0, \phi_{N+1} = +\infty$ ). For all  $t \leq \tau$ ,  $K_t^B = [\phi_t - K^A]^+ = 0$  and  $K_t = K^A$ . For all  $t \geq \tau + 1$ ,  $K_t^B$  and  $K_t$  are respectively equal to  $\phi_t - K^A$  and  $\phi_t$ . Hence the profit function is equal to:

$$\begin{aligned} \tilde{\pi}^B(K^A) &= \sum_{t=1}^{\tau} \left( r \int_0^{K^A} x f_t(x) dx + (rG_t(K^A) - \gamma)K^A \right) \\ &\quad + \sum_{\tau+1}^N \left( r \int_0^{\phi_t} x f_t(x) dx + (rG_t(\phi_t) - c^B)\phi_t \right) \\ &\quad + (N - \tau)(c^B - \gamma)K^A. \end{aligned} \quad (14)$$

This function is twice differentiable for  $\phi_\tau \leq K^A < \phi_{\tau+1}$ , and we have for the first and second order derivatives, (we assume in the rest of the proof that derivatives computed on the lower-bound (resp. upper-bound) of an interval correspond to the right (resp. left) hand derivatives)

$$\tilde{\pi}^{B'}(K^A) = r\tilde{G}_\tau(K^A) - c^B\tau + N(c^B - \gamma) \quad (15)$$

$$\tilde{\pi}^{B''}(K^A) = -r \sum_{t=1}^{\tau} f_t(K^A). \quad (16)$$

Hence, over the interval  $[\phi_\tau, \phi_{\tau+1}]$ ,  $\tilde{\pi}^B$  is strictly concave, increasing if and only if  $K^A \leq \kappa_\tau$ . Furthermore, noting that  $r\tilde{G}_{\tau+1}(\phi_{\tau+1}) = r\tilde{G}_\tau(\phi_{\tau+1}) + c^B$ , one can easily check using (15) that  $\lim_{x \rightarrow \phi_{\tau+1}^+} \tilde{\pi}^{B'}(x) = \lim_{x \rightarrow \phi_{\tau+1}^-} \tilde{\pi}^{B'}(x)$ . Thus,  $\tilde{\pi}^{B'}$  is continuous and  $\tilde{\pi}^B$  is strictly concave. Since furthermore  $\tilde{\pi}^{B'}(0) = (r - c^B) + N(c^B - \gamma) > 0$ , the first order condition  $r\tilde{G}_\tau(K^A) = c^B\tau - N(c^B - \gamma)$  subject to  $K^A \in [\phi_\tau, \phi_{\tau+1}]$  has a unique solution which corresponds to the optimal capacity level. In other words, there is a unique  $\tau^*$  such that  $\kappa_{\tau^*} \in [\phi_{\tau^*}, \phi_{\tau^*+1}]$  and  $K^{A*} = \tau^*$ . Note finally that  $\kappa_t$  is non-increasing in  $t$  and  $\phi_t$  is non-decreasing in  $t$  so that  $\tau^*$  is also the largest  $t$  such that  $\phi_t \leq \kappa_{t-1}$  (or equivalently, the smallest  $t$  such that  $\phi_{t+1} > \kappa_t$ ).  $\square$

**Proof of Proposition 2:** Note first that  $D_1 \leq_{st} D_2$  implies that  $\phi_2 \geq \phi_1$ . In order to apply Proposition 1, we need to compare  $\phi_2$  with  $\kappa_1$ . We have

$$\begin{aligned}\phi_2 > \kappa_1 &\Leftrightarrow F_2^{-1}(1 - c^B/r) > G_1^{-1}[(2\gamma - c^B)/r] \\ &\Leftrightarrow \gamma > \tilde{\gamma}\end{aligned}$$

It follows then that  $t^* = 1$  if and only if  $\gamma > \tilde{\gamma}$  and the result is obtained from the direct application of Proposition 1.  $\square$

**Proof of Proposition 3:** Note first that  $F_\theta^{-1}(\epsilon)$  and  $1 - F_{-\theta}(\epsilon)$  are respectively non-decreasing and non-increasing in  $\theta \geq 0$  from the first order stochastic dominance  $A_\theta \geq_{st} A_{\theta'}$  for  $\theta \geq \theta'$ . It follows that  $1 - F_{-\theta}(F_\theta(\epsilon))$  and hence  $\tilde{\gamma}$  are non-increasing in  $\theta > 0$ . The second part of the proposition follows from Proposition 2 by defining  $\tilde{\theta} := \min_{\theta \geq 0} \{\gamma > \tilde{\gamma}\}$ . A similar approach shows the first part.  $\square$

**Proof of Proposition 4:** In each period, the game between the operator and the contractor is equivalent to the capacity game between a manufacturer and a subcontractor described by Van Mieghem (1999), where the market demand for the subcontractor is zero. More precisely, consider the production-subcontracting subgame in Section 3.1 of Van Mieghem (1999) and the notations within. When the market demand for the subcontractor is zero ( $D_S = 0$ ), the supplier does not produce goods for this market ( $x_S = 0$ ) but produces at capacity for the manufacturer ( $x_t^S = K_S$ ). It follows that the manufacturer outsources the surplus of his market demands ( $x_t = \min([D_M - K_M]^+, K_S)$ ) and the capacity investment game described in Section 3.2 of Van Mieghem (1999) (the choice of the capacities  $(K_M, K_S)$ ) is equivalent to our operator-contractor game (determination of  $(K_t^B, K_t^A)$ ). Van Mieghem (1999) shows that a unique Nash equilibrium exists for the capacity investment game. We can hence deduce a similar result for the operator-contractor capacity game. Furthermore, noting that

$$\pi_t^A = p \int_{K_t^B}^{K_t} (x - K_t^B) f(x) dx + p K_t^A [1 - F(K_t)] - c^A K_t^A$$

the best-response curve of the contractor is given by the positive solution of the following first order condition

$$\frac{\partial \pi_t^A}{\partial K_t^A} = p(1 - F(K_t)) - c^A = 0 \tag{17}$$

for a given  $K_t^B$  (where  $K_t = K_t^A + K_t^B$ ), and with  $K_t^A = 0$  when  $p < c^A$  that is when (17) does not admit a solution.

Similarly, the best-response curve of the operator is given by the following first order condition, after some simplifications,

$$\frac{\partial \pi_t^B}{\partial K_t^B} = r[1 - F(K_t)] - c^B + p[F(K_t) - F(K_t^B)] = 0 \quad (18)$$

subject to  $K_t^B \leq K_t$ . When  $p > rc^A/c^B > c^A$ , plugging (17) in (18), we obtain

$$F(K_t^B) = 1 - \frac{c^A + c^B}{p} + \frac{rc^A}{p^2}. \quad (19)$$

Note that the right hand-side of (19) is negative or null if and only if  $rc^A \leq c_0^2$ , with  $c_0 := (c^A + c^B)/2$ , and  $p \in [p_1, p_2]$  where  $p_1 = c_0 \left(1 - \sqrt{1 - rc^A/c_0^2}\right)$  and  $p_2 = c_0 \left(1 + \sqrt{1 - rc^A/c_0^2}\right)$ . Noting that  $p_2 < rc^A/c^B$ , we can then deduce  $K_t^B$  and  $K_t^A$  of Proposition 4 from (17) and (19) when  $p > rc^A/c^B$ , which also implies that the constraint  $K_t^B \leq K_t$  is satisfied. On the other hand when  $p < rc^A/c^B$ , the constraint  $K_t^B \leq K_t$  is binding and  $K_t^A = 0$  so that  $K_t^{B*} = F_t^{-1}(1 - c^B/r)$ .

**Proof of Proposition 5:** For any given  $\gamma$ , Proposition 1 implies that the corresponding optimal capacity levels  $K^A$  and  $K^B$  are equal to  $\kappa_1$  and 0 respectively, since  $\kappa_1 > \phi^1$  from  $c^B > \gamma$ . But when  $K^B$  is zero, the contract is equivalent to the supply chain wholesale contract for a supplier-vendor in Lariviere and Porteus (2001), where the operator is a retailer, the contractor a manufacturer, and  $K^A$  a quantity of products. Applying Theorem 1 of Lariviere and Porteus (2001) leads then to the result.  $\square$

**Proof of Proposition 6:** We show in the following the first part of the result. The proof for the two last parts is similar. Assume then that  $J_1(\phi_2) < (2c^A - c^B)/r$ .

For  $\gamma > \tilde{\gamma}$ , the optimal capacity verifies  $\gamma = (rG_1(K^A) + c^B)/2$  from Proposition 2. The profit of the contractor is then equal to  $\pi^A(K^A) = (\gamma - c^A)K^A = (rG_1(K^A) + c^B - 2c^A)K^A$ . The first order condition yields

$$J_1(K^A) = \frac{2c^A - c^B}{r}. \quad (20)$$

Since  $J_1$  is non-increasing when it is positive and since  $J_1(0) = 1 \geq (2c^A - c^B)/r$ , Equation (20) admits a unique solution such that  $J_1(K^{A*}) = 2c^A - c^B/r > J_1(\phi_2)$ . It follows that  $K^{A*} < \phi_2$  which is equivalent to  $\gamma^* > \tilde{\gamma}$ . In other words there exists a Stackelberg equilibrium such that  $\gamma^* > \tilde{\gamma}$ .

Consider now prices such that  $\gamma \leq \tilde{\gamma}$ . Following a similar approach using the second part of Proposition 2, we have  $\gamma = r/2\tilde{G}_2(K^A)$  and we can deduce the following first order condition in  $K^A$ ,

$$J_1(K^A) + J_2(K^A) = \frac{2c^A}{r}. \quad (21)$$

But if  $J_1(\phi_2) < (2c^A - c^B)/r$ , then  $J_1(\phi_2) + J_2(\phi_2) < 2c^A/r$  since  $J_2(\phi_2) = G_2(\phi_2) - \phi_2 f_2(\phi_2) < c^B/r$  from the definition of  $\phi_2$ . Thus, the optimal capacity is in this case the smallest  $K^A$  such that  $\gamma = r/2\tilde{G}_2(K^A) \leq \tilde{\gamma}$ , which is equal to  $\phi_2$  from the definition of  $\tilde{\gamma}$ .

It follows that the Stackelberg game has a unique equilibrium such that  $K^{A*}$  satisfies Equation (20) with

$$\gamma^* = (rG_1(K^{A*}) + c^B)/2 > \tilde{\gamma}.$$

$K_1^{B*}$  and  $K_2^{B*}$  are then given by the first part of Proposition 2.  $\square$

**Proof of Proposition 7:** Consider  $\phi_{2,\theta} = F_\theta^{-1}(1 - c^B/r)$  as defined in Proposition 2. (In the following, we sometimes add the subscript  $\theta$  to the quantities that depend on this parameter, for the sake of clarity.). Since the distribution of  $D_1 = A_0$  has an IGFR,  $J_1(\epsilon)$  is non-increasing in  $\epsilon$  when it is positive. Furthermore, from the first order stochastic dominance  $A_\theta >_{st} A_{\theta'}$  for  $\theta > \theta'$ ,  $\phi_{2,\theta}$  is non-decreasing in  $\theta$  and  $J_1(\phi_{2,\theta})$  is non-increasing in  $\theta$  as long as  $J_1(\phi_{2,\theta}) > 0$ . Hence there exist a unique threshold  $\hat{\theta}$  such that  $\hat{\theta}$  is the minimum value of  $\theta$  satisfying  $J_1(\phi_{2,\theta}) \leq (2c^A - c^B)/r$ . For  $\theta > \hat{\theta}$ ,  $J_1(\phi_{2,\theta}) < (2c^A - c^B)/r$  so that at the equilibrium,  $K^{A*}$  and  $\gamma^*$  do not depend on  $\theta$  according to the first part of Proposition 6. Furthermore,  $K_1^{B*} = 0$  and  $K_2^{B*}$  is non-decreasing in  $\theta$  since  $F_\theta(\epsilon)$  decreases in  $\theta$  from the first order stochastic dominance  $A_\theta >_{st} A_{\theta'}$ ,  $\theta > \theta'$ .

It remains to show the first part of the proposition for  $\theta < \hat{\theta}$  which corresponds to the two last cases of Proposition 6. Since  $K_\theta^{A*}$ ,  $K_{2,\theta}^{B*}$  and  $\gamma_\theta^*$  are continuous in  $\theta$ , it suffices to show the result for each case. The second case is similar to the first one. For the last case we have

$$K_1^{B*} = K_2^{B*} = 0 \tag{22}$$

$$J_1(K_\theta^{A*}) + J_{2,\theta}(K_\theta^{A*}) = 2c^A/r \tag{23}$$

$$\gamma^* = \frac{r}{2}\tilde{G}_{2,\theta}(K^{A*}\theta). \tag{24}$$

From the stochastic dominance  $A_\theta >_{hr} A_{\theta'}$  (and hence  $A_\theta >_{st} A_{\theta'}$ ) with  $\theta > \theta'$ , for any  $\epsilon$  such that  $J_1(\epsilon) \geq 0$ , we have  $J_{2,\theta}(\epsilon) > J_{2,\theta'}(\epsilon) \geq J_{2,0}(\epsilon) = J_1(\epsilon) \geq 0$  and  $J_{2,\theta}(\epsilon)$  is non-increasing in  $\epsilon$ . It follows that  $J_1(\epsilon) + J_{2,\theta}(\epsilon)$  is non-increasing in  $\epsilon$  and non-decreasing in  $\theta$  so that  $K_\theta^{A*}$  is non-decreasing in  $\theta$  as long as  $J_1(K_\theta^{A*}) \geq 0$ . Note however that  $J_1(\phi_{2,\hat{\theta}}) + J_{2,\hat{\theta}}(\phi_{2,\hat{\theta}}) = 2c^A/r$  from the definitions of  $\phi_{2,\hat{\theta}}$  and  $\hat{\theta}$ . Hence, for  $\theta \leq \hat{\theta}$ ,  $K_\theta^{A*} \leq \phi_{2,\hat{\theta}}$  and  $J_1(K_\theta^{A*}) \geq J_1(\phi_{2,\hat{\theta}}) = (2c^A - c^B)/r > 0$ .  $\square$

**Proof of Proposition 8:**  $p^*$  is the maxima of  $\pi^A = p \min([D - K^B(p)]^+, K^A(p)) - c^A K^A(p)$ , subject to  $\pi^B(p)$  being larger than the profit of the call center operator when it does not outsource

any calls. We can then restrict  $p$  such that  $p \geq rc^A/c^B$ , since  $K^A$  and then  $\pi^A$  are zero otherwise from the expression of  $K^A$  of Proposition 4.

Note then that

$$\begin{aligned} \frac{\partial \pi^A}{\partial p} &= \int_{K^B}^K G(x)dx + p \left( G(K) \frac{\partial K}{\partial p} - G(K^B) \frac{\partial K^B}{\partial p} \right) \\ &\quad - c^A \frac{\partial(K - K^B)}{\partial p} \\ &= \int_{K^B}^K G(x)dx + p \left( F(K^B) - F(K) \right) \frac{\partial K^B}{\partial p} \end{aligned} \quad (25)$$

where the last equality is true since  $pG(K) = c^A$  from Proposition 4. Since  $p \geq rc^A/c^B$ , we have  $F(K^B) - F(K) \leq 0$ . As a result, if  $\partial K^B/\partial p$  is negative for a given  $p$  then  $\partial \pi^A/\partial p$  is positive, and  $p$  cannot be an equilibrium. A direct computation leads then to

$$\frac{\partial K^B}{\partial p} = 2 \left( \frac{c_0}{p^2} - \frac{rc^A}{p^3} \right) \frac{1}{f(K^B)} \quad (26)$$

which is negative or zero if and only if  $p \leq rc^A/c_0$ . It follows that  $p^* \geq rc^A/c_0$ .

Finally, let us show that  $p^* \leq r$ . From the definition of  $\pi^B$  and Proposition 4, we have

$$\frac{\partial \pi^B}{\partial p} = (r - p)G(K) \frac{\partial K}{\partial p} + \frac{p - r}{p} c^A \frac{\partial K^B}{\partial p} - \int_{K^B}^K G(x)dx. \quad (27)$$

which is negative for  $p \geq r$  ( $\partial K^B/\partial p$  is negative since  $p \geq r \geq rc^A/c_0$ ). In this case,  $\pi^B(p)$  is then less or equal to  $\pi^B(r)$  which is also the profit of the call center operator when it does not outsource calls. Hence  $p^* \leq r$ .  $\square$

**Proof of Proposition 9:** If  $F$  has DFR then for all  $K^B \geq x$ ,  $G(K^B)/f(K^B) \leq G(x)/f(x)$ . As a result,

$$\int_{K^B}^K G(x)dx = \int_{K^B}^K \frac{G(x)}{f(x)} f(x)dx \geq \frac{G(K^B)}{f(K^B)} \left( F(K) - F(K^B) \right) \quad (28)$$

From Equation (25) and Proposition 4 we have

$$\frac{\partial \pi^A}{\partial p} \geq \frac{F(K) - F(K^B)}{f(K^B)} \frac{rc^A}{p^2} > 0 \quad (29)$$

Hence  $\pi^A$  is strictly increasing and  $p^* = r$ , which is also the operator's profit when it does not outsource calls and the operator never prefers outsourcing the fluctuation (Contract 2).  $\square$

**Proof of Proposition 10:** Assume that demand has an uniform distribution over  $[a, b]$ . It follows that when  $x \in [a, b]$ ,  $G(x) = (b - x)/d$  with  $d = b - a$ ,  $\int_0^x G(u)du = (1 - x/2)x/d$  and  $f(G^{-1}(x)) = dx + a$ . Using (25), (26) and Proposition 4, we compute

$$\frac{\partial \pi^A}{\partial p} = \frac{F(K) - F(K^B)}{2ap^2} (3rc^A - c^B p)$$

which is positive if and only if  $p < 3rc^A/c^B$ . From Proposition 8,  $p^* \in [2rc^A/(c^A + c^B), r]$  where  $3rc^A/c^B > 2rc^A/(c^A + c^B)$  and the result directly follows.  $\square$

**Proof of Proposition 11:** Denote by  $p^*$  the equilibrium price of the unconstrained problem. When  $c^A/c^B > p$ , B does not outsource any call from Proposition 4. Combined with the constraint (10) this implies then that if  $c^B/r < \alpha$ ,  $F(K^B) = F(K) = 1 - c^B/r$  (with  $F(K) > 1 - \alpha$ ). Similarly, when  $c^A/r > \alpha$ , the price of the contract with service level agreement should be equal to  $c^A/\alpha$  which is larger than  $r$ . But from Proposition 8,  $p^* \leq r$  and the equilibrium price does not achieve the required capacity level. It follows that for the constrained problem, the profit of B is always higher when going solo and we have  $F(K) = F(K^B) = 1 - \alpha$  (since  $1 - \alpha > 1 - c^B/r$ ). Finally, when  $c^A/r \leq \alpha \leq c^B/r$ , the constraint is binding if and only if the equilibrium price cannot achieve the service constraint, that is if and only if  $p^* \leq c^A/\alpha$ , from which we obtain the result directly.  $\square$