MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1080.0871ec
pp. ec1–ec15

e-companion

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Electronic Companion—"Inventory Models for Substitutable Products: Optimal Policies and Heuristics" by Mahesh Nagarajan and S. Rajagopalan, *Management Science*, DOI 10.1287/mnsc.1080.0871.

Appendix

Inventory Models for Substitutable Products: Optimal Policies and Heuristics

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Proof of Theorem 2.1

The demand for the two products are *pD* and *(1-p)D* respectively. We represent the actual demand realization by d_i. For any realization of the random variable *p*, the demands are $d_1 = pD$ and $d_2 = (1-p)D$. Let r_i^j be the number of demand units of product *i* satisfied by product *j* (*i, j* = 1,2). Note that given demand realizations d_i and initial inventory levels Q_i (where $Q_1 = \alpha D$ and $Q_2 = \beta D$), r_i^j is deterministic. Let J_i be the excess demand of product *i* before allowing for substitution. Let o_i and u_i represent the excess inventory and shortage respectively of the two products, after allowing for demand substitutions. Thus, these represent the actual inventory excesses and shortages at the end of the period. Also, let $\mathbf{Q} = (Q_1, Q_2)$. We now write the profit function $\prod(\mathbf{Q})$. To do so, we use $\rho(\mathbf{Q}, p)$ to express the profit conditioned on a given demand realization *p*.

$$\prod(\mathbf{Q}) = -\sum_{i=1}^{2} cQ_i + \int \rho(\mathbf{Q}, p)dF(p)$$

where,

$$\rho(\mathbf{Q}, p) = \underset{(o_i, u_i, r_i^j, J_i)}{\text{Max}} \quad \sum_{i=1}^{2}\sum_{j=1}^{2} sr_i^j - h\sum_{i=1}^{2} o_i - \pi\sum_{i=1}^{2} u_i \ ,$$

subject to:

$$u_i + \sum_{j=1}^{2} r_i^j = d_i \, ; \qquad \forall\ i$$

$$\sum_{j=1}^{2} r_j^i + o_i = Q_i \qquad \forall\ i$$

$$J_i + r_i^i = d_i \qquad \forall\ i$$

$$r_i^j \leq \gamma J_i \qquad \forall\ i,\ j \neq i$$

$$o_i, r_i^j, u_i, J_i \geq 0\,; i, j = 1{,}2.$$

To show that the profit function $\prod(\mathbf{Q})$ is concave, note that $\rho(\boldsymbol{Q}, p)$ can be rewritten in the form $\rho(\boldsymbol{Q}, p) = \underset{\boldsymbol{x}}{Max}\, \boldsymbol{Px}$ such that $\boldsymbol{Ax} \leq \boldsymbol{Q}, \boldsymbol{x} \geq 0$ where $\boldsymbol{P}$ is a vector of the cost components, $\boldsymbol{A}$ is the coefficient matrix from L.H.S. of the constraints and $\boldsymbol{x}$ represents the sale, overage and underage variables. Note that $\boldsymbol{x}$ and $\boldsymbol{Q}$ are vectors. We first show that $\rho(\boldsymbol{Q}, p)$ is jointly concave in $\boldsymbol{Q}$. Clearly, in what follows, other than a slight abuse of notation, there is no loss in ignoring p. Let $\rho(\boldsymbol{Q}^1) = \boldsymbol{Pa}$ and $\rho(\boldsymbol{Q}^2) = \boldsymbol{Pb}$ where $\boldsymbol{Q}^1$ and $\boldsymbol{Q}^2$ are two different $\boldsymbol{Q}$ vectors such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are feasible. Then, $\boldsymbol{A}[\lambda\boldsymbol{a}+(1\text{-}\lambda)\boldsymbol{b}] \leq \lambda\boldsymbol{Q}^1 + (1\text{-}\lambda)\boldsymbol{Q}^2$ where $\lambda \in [0,1]$. This immediately implies that $\rho(\lambda\boldsymbol{Q}^1 + (1\text{-}\lambda)\boldsymbol{Q}^2) \geq \boldsymbol{P}[\lambda\boldsymbol{a}+(1\text{-}\lambda)\boldsymbol{b}] = \lambda\rho(\boldsymbol{Q}^1) + (1\text{-}\lambda)\rho(\boldsymbol{Q}^2)$.This establishes joint concavity of $\rho(\boldsymbol{Q}, p)$. Therefore,

$\prod(\mathbf{Q}) = -\sum_{i=1}^{2} cQ_i + \int \rho(\mathbf{Q}, p)\,dF(p)$ is clearly jointly concave in $\boldsymbol{Q}$. ♦

Proof of Theorem 2.2:

Since the global maxima is uniquely attained and the function $\prod(\mathbf{Q})$ is concave and continuously differentiable, we check the first order conditions for $\Pi_i(\alpha, \beta)$, i = 1,2. Note that we have shown the explicit formulation of $\Pi_i(\alpha,\beta)$. Taking derivatives, canceling common terms and simplifying we have:

$$\frac{d\Pi_1}{d\alpha} = -h[\int_0^{\alpha} dF(p) + \int_{\alpha}^{1} \gamma dF(p)] + \pi\int_{\alpha}^{1}(1-\gamma)dF(p) + s\int_{\alpha}^{1}(1-\gamma)dF(p) - c = 0. \qquad \text{(A2.1)}$$

$$\frac{d\Pi_1}{d\beta} = -h[\int_{1-\beta}^{1} dF(p) + \int_{0}^{1-\beta} \gamma dF(p)] + \pi\int_{0}^{1-\beta}(1-\gamma)dF(p) + s\int_{0}^{1-\beta}(1-\gamma)dF(p) - c = 0. \qquad \text{(A2.2)}$$

$$\frac{d\Pi_2}{d\alpha} = -h\int_0^{\hat{\alpha}} dF(p) + \int_{\hat{\alpha}}^{\hat{\beta}}(s+\pi)dF(p) + \int_{\hat{\beta}}^{1}[(1-\gamma)(s+\pi) - \gamma h]dF(p) - c = 0. \qquad \text{(A2.3)}$$

$$\frac{d\Pi_2}{d\beta} = \int_0^{\hat{\alpha}}((1-\gamma)(s+\pi) - \gamma h)dF(p) + \int_{\hat{\alpha}}^{\hat{\beta}}(s+\pi)dF(p) - h\int_{\hat{\beta}}^{1} dF(p) - c = 0. \qquad \text{(A2.4)}$$

Note the following:

- The maximum is unique and the first order conditions are satisfied uniquely.
- The function $\prod(\mathbf{Q})$ is actually a family of functions, parametrized by γ.

Noting that $F(\alpha^*) = \int_0^{\alpha^*} dF(p)$, it is easy to check from the first order condition (A2.1) that $F(\alpha^*) \geq 0.5$ or $\alpha^* \geq F^{-1}(0.5)$ if $\gamma \leq \gamma^*$. It is easy to deduce from (A2.1) and (A2.2) that $F(\alpha^*) + F(1-\beta^*) = 1$. It then follows that $F(1-\beta^*) \leq 0.5$ and $(1-\beta^*) \leq F^{-1}(0.5)$ or $\beta^* \geq 1- F^{-1}(0.5)$ and so $\alpha^* + \beta^* \geq 1$. So, when $\gamma \leq \gamma^*$, (A2.1) and (A2.2) are indeed the appropriate first order conditions and the result follows.

One can also check directly that $\gamma \leq \gamma^* \Leftrightarrow F(\alpha^*) \geq F(1-\beta^*)$ which implies the result.

Equations (1) and (2) in the paper respectively follow directly from simplifying (A2.1) and (A2.2). Simplifying (A2.3) and (A2.4) and combining terms, we get respectively:

$$(1-\gamma)+\gamma F(\hat{\beta}^*)-F(\hat{\alpha}^*)=\frac{h+c}{(s+\pi+h)}$$

$$F(\hat{\beta}^*)-\gamma F(\hat{\alpha}^*)=\frac{h+c}{(s+\pi+h)}$$

Expressions (3) and (4) follow from the above two equations. One can also observe that $F(\hat{\beta}^*)+F(\hat{\alpha}^*)=1$, which is consistent with the definitions of $\hat{\alpha}$ and $\hat{\beta}$. ♦

Proof of Lemma 2.1: If $(Q_1 + Q_2) \geq H$, then $(Q_1 + Q_2) \geq d_1 + d_2$ which implies that then $(Q_i - d_i) \geq (d_j - Q_j), i,j = 1,2, i \neq j$. Then $(Q_i - d_i)^+ \geq \gamma_j(d_j - Q_j)^+, i,j = 1,2, i \neq j$ because $\gamma_i < 1$. This has two implications.

First, $(\gamma_i(d_i - Q_i)^+ - (Q_j - d_j)^+)^+ = 0$ in (2.5), i.e. substitution demand of product i will be fully satisfied from the excess inventory of product j.

Second, it follows that $U_{ij} = [(Q_i - d_i)^+ - \gamma_j(d_j - Q_j)^+] \geq 0$ and so the terms in U_{ij} in (5) separate into terms corresponding to products i and j. Hence, after rearranging some terms, equation (2.5) can be restated as $\Pi(Q_1,Q_2) = \Pi_1(Q_1) + \Pi_2(Q_2)$, where,

$$\Pi_i(Q_i)=(s_i-c_i)Q_i-(s_i+h_i)E[(Q_i-d_i)^+]-(\pi_i-\gamma_i(s_j+h_j+\pi_i))E[(d_i-Q_i)^+]. \quad \Diamond$$

Proof of Corollary 2.1:

In what follows, for ease of exposition, we drop the (*) from the superscript of Q. We use the fact that the two products are symmetric and thus the order quantities for the two products are equal i.e. $Q_1 = Q_2$. For this specific model i.e. random D and p, we introduce a few notations. We denote the distribution and probability density function of D by G and g respectively. Next we denote the distribution of p by K.

We have then from equation (7) in the paper

$$\int_L^H K(\frac{Q_1}{D})dG(D) = \frac{s+\pi-c-\gamma(s+\pi+h)}{s+\pi+h-\gamma(s+\pi+h)} = 1-\frac{h+c}{(1-\gamma)(s+\pi+h)} = T \text{ (say)}. \qquad \text{(A2.5)}$$

We now prove the two statements.

Proof of (1): To ensure that the integral is well defined, the above expression is essentially equivalent to

$$\int_{\max(L,\frac{Q_1}{b})}^{\min((\frac{Q_1}{a},H)} K(\frac{Q_1}{D})dG(D) = T \ .$$

To simplify this expression, we need to simplify the limits in the above integral. To do so, it is convenient to separate out two cases corresponding to $bL \le aH$ and $aH < bL$ respectively. Noting that $a \le 0.5$ and $0.5 \le b$, we need only consider the cases $H>2bL$ and $H \le 2bL$. In what follows, we discuss the case $H>2bL$ and p and D are uniformly distributed wherein we can re-write the above expression as

$$\int_L^{\frac{Q_1}{b}} dG(D) + \int_{\frac{Q_1}{b}}^H K(\frac{Q_1}{D})dG(D) = T \ .$$

When $H \le 2bL$, the derivation is somewhat easier and we omit the details.

Solving $\int_L^{\frac{Q_1}{b}} dG(D) + \int_{\frac{Q_1}{b}}^H K(\frac{Q_1}{D})dG(D) = T$ we obtain the following:

$Q_1 \ln(\frac{Hb}{Q_1}) + Q_1 = T(H-L)(b-a) + aH + L(b-a)$. We note $l(Q_1) = Q_1 \ln(\frac{Hb}{Q_1}) + Q_1$ is an increasing function in Q_1 and thus if we require that $Q_1 \geq H/2$ this immediately implies

$$T(H-L)(b-a) + aH + L(b-a) \geq l(\frac{H}{2}) = \frac{H}{2}(\ln(2b)+1)$$

$$\Leftrightarrow T \geq \frac{\frac{H}{2}(1+\ln(2b)) - aH - L(b-a)}{(H-L)(b-a)} = X(say)$$

We note that since $b \geq 0.5$, $1 \geq$ X.

Thus, $T \geq X \Leftrightarrow \gamma \leq \gamma^* = 1 - \frac{c+h}{(1-X)(s+\pi+h)}$, which concludes the proof of (1). ♦

Proof of (2): We now have $K(x) = x^\theta, \theta \geq 2$ and D is uniform [L, H]. In the case H< 4L, equation (A2.5) reduces to the following expression: $\frac{1}{H-L}\int_L^H (\frac{Q_1}{D})^\theta dD = T$.

Solving and setting $Q_1 \geq H/2$ implies $T \geq \frac{(\frac{H}{2})^\theta (H^{\theta-1} - L^{\theta-1})}{(H-L)(\theta-1)(HL)^{\theta-1}} = Y(say)$

We note that $0 < Y < 1$ and thus $T \geq Y \Leftrightarrow \Leftrightarrow \gamma \leq \gamma^* = 1 - \frac{c+h}{(1-Y)(s+\pi+h)}$.

This proves (2) of corollary 2.1. ♦

Proof of Theorem 3.1:

To prove concavity, we use an approach similar to the one for the single period problem. The concavity of the single period profit function $\Pi_1(\alpha_1, \beta_1)$ in (α_1, β_1) implies that $G_1(I_1^1, I_1^2)$ is concave in (I_1^1, I_1^2). Assume inductively that $X_{n-1}(\alpha_{n-1}, \beta_{n-1})$ is concave in

$(\alpha_{n-1}, \beta_{n-1})$. Thus $G_{n-1}(I_{n-1}^1, I_{n-1}^2)$ is concave in (I_{n-1}^1, I_{n-1}^2). We want to prove that $X_n(\alpha_n, \beta_n)$ is concave in (α_n, β_n). Let, similar to the single period formulation, r_i^j be the number of demand units of product *i* satisfied by product *j* (*i, j* = 1,2) in period *n*. Also let J_i be the excess demand of product *i* before allowing for substitution, o_i and u_i represent the excess inventory and shortage respectively of the two products, after allowing for demand substitutions in period *n*. For a given realization *p* of the demand in the *n*-th period, $X_n(\alpha_n, \beta_n)$ can be re-written as

$$X_n(\alpha_n, \beta_n) = E_p(\rho(\alpha_n, \beta_n, p)),$$

where $\rho(\alpha_n, \beta_n, p)$ is the profit obtained by stocking up to (α_n, β_n) in period *n* and the demand realization is *p* and an optimal policy is followed from period n-1 onwards and $E_p(\rho(\alpha_n, \beta_n, p)$ is the expectation over all realizations of *p*. Note that $\rho(\alpha_n, \beta_n, p)$ is a redefinition of the profit function used in the proof of the concavity of the single-period problem. Thus $\rho(\alpha_n, \beta_n, p) = \underset{Ax \le Q}{Max}[P.x + G_{n-1}(o_1, o_2)]$, where as before *P* represents the vector whose components are the per unit overage, underage and revenue, *x* represents the vector whose components are r_i^j, J_i, u_i, o_i and *Q* is the vector whose components include $\alpha_n D, \beta_n D$. Thus *P.x* corresponds to the allocation problem in the n-th period and $G_{n-1}(o_1, o_2)$ is the profit obtained by following an optimal policy from the n-1th period onwards, starting with excess inventory (o_1, o_2) carried over from the nth period. Note that G_{n-1} is concave and *Px* is linear. Thus $\rho(\alpha_n, \beta_n)$ is concave in (α_n, β_n) for reasons similar to those in Theorem 2.1 (i.e. in the proof Theorem 2.1 we had a linear objective function;

it is easy to see that the same logic holds when the objective function is concave, as in this case) and the concavity of $X_n(\alpha_n, \beta_n)$ follows♦

Proof of Theorem 3.2:

The proof of part (1) uses two key Lemmas 3.1 and 3.2 that follow.

Lemma 3.1. $\alpha_n^* + \beta_n^* > 1 \ \forall n \ \textit{if} \ \gamma < \gamma^*$.

Proof of Lemma 3.1: We know from section 2 that in a single period problem, if $\gamma < \gamma^*$, it suffices to look at the region $\alpha_1 + \beta_1 > 1$ so that $\alpha_1^* + \beta_1^* > 1$. Let us consider the profit function of a 2 period problem:

$$X_2(\alpha_2, \beta_2) = \Pi_1(\alpha_2, \beta_2) - c(\alpha_2 + \beta_2) + E_p[G_1(I_1^1, I_1^2)].$$

where $E_p(G_1(I_1^1, I_1^2))$ is the expected profit of the one-period problem with starting inventories I_1^1 and I_1^2 computed over all realizations of p. Let us evaluate $\frac{\partial X_2(.,.)}{\partial \alpha_2}$ and $\frac{\partial X_2(.,.)}{\partial \beta_2}$ at (α_1^*, β_1^*), i.e. when $\alpha_2 = \alpha_1^*$ and $\beta_2 = \beta_1^*$. We know the following:

(1) $(\frac{\partial \Pi_1}{\partial \alpha_2}$ - $c)$ and $(\frac{\partial \Pi_1}{\partial \beta_2}$ - $c)$ evaluated at (α_1^*, β_1^*) are zero.

(2) $E_p(I_1^1) < \alpha_1^*$ and $E_p(I_1^2) < \beta_1^*$ if $\alpha_2 = \alpha_1^*$ and $\beta_2 = \beta_1^*$.

It follows from (2) above that $\frac{\partial E_p[G_1(I_1^1, I_1^2)]}{\partial \alpha_2}$ and $\frac{\partial E_p[G_1(I_1^1, I_1^2)]}{\partial \beta_2}$ are both greater than zero when evaluated at (α_1^*, β_1^*). Hence, $\frac{\partial X_2}{\partial \alpha_2}$ and $\frac{\partial X_2}{\partial \beta_2}$ are greater than zero at

(α_1^*, β_1^*) and since $X_2(\alpha_2, \beta_2)$ is concave, we need only consider $(\alpha_2, \beta_2) \geq (\alpha_1^*, \beta_1^*)$ and so $(\alpha_2^* + \beta_2^*) \geq 1$.

The proof for any *n* follows exactly as above by induction ♦

Lemma 3.2. When $\gamma < \gamma^*$ we have $\dfrac{\partial^2 X_n}{\partial \beta_n \partial \alpha_n} = 0 \ \forall n$.

Proof of Lemma 3.2: From (A2.1) and (A2.2), we know that the result is true for *n* = 1. Consider the two-period problem, with n = 2 representing that there are two periods to go. Then,

$$G_2(I_2^1, I_2^2) = \underset{(\alpha_2,\beta_2)\geq(I_2^1,I_2^2)}{Max} [\Pi_1(\alpha_2, \beta_2) - c(\alpha_2 + \beta_2 - I_2^1 - I_2^2) + \delta \int_0^{1-\beta_2} G_1(\alpha_2 - p - \gamma(1-p-\beta_2), 0) dF$$
$$+ \delta \int_{1-\beta_2}^{\alpha_2} G_1(\alpha_2 - p, \beta_2 - (1-p)) dF + \delta \int_{\alpha_2}^{1} G_1(0, \beta_2 - (1-p) - \gamma(p - \alpha_2)) dF] \qquad (A3.1)$$

where $G_1(x, y)$ is the optimal profit in the 1-period problem with starting inventories (*x, y*) of products 1 and 2 respectively prior to ordering and $\Pi_1(\alpha_2, \beta_2)$ is the one-period profit with an inventory of α_2 and β_2 respectively for products 1 and 2. That is,

$$G_1(x, y) = \underset{(\alpha_1,\beta_1)\geq(x,y)}{Max} [\Pi_1(\alpha_1, \beta_1) - c(\alpha_1 + \beta_1 - x - y)]$$

From the analysis of the one-period problem, $\dfrac{\partial^2 \Pi_1(\alpha_n, \beta_n)}{\partial \alpha_n \partial \beta_n} = 0$ for any inventory levels α_n and β_n prior to demand realization. Now,

$$G_1(x, y) = \underset{(\alpha_1,\beta_1)}{Max.} \{\Pi_1(\max(\alpha_1, x), \max(\beta_1, y)) - c[(\alpha_1 - x)^+ + (\beta_1 - y)^+]\}$$

$$\frac{\partial G_1(x, y)}{\partial x} = \left\{ \frac{\partial \Pi_1(\max(\alpha_1, x), \max(\beta_1, y))}{\partial x} - c \frac{\partial((\alpha_1 - x)^+)}{\partial x} \right\} \qquad (A3.2)$$

Note that x may take different values such as $(\alpha_2 - p - \gamma(1- p - \beta_2))$ or $(\alpha_2 - p)$ or 0 depending on the three different scenarios above. G_1 is a function of only the states given by the initial inventory levels x and y and so from (A3.2) $\partial G_1/\partial x$ is not a function of α_2 or β_2. Hence,

$$\frac{\partial}{\partial \alpha_2}\left(\frac{\partial G_1}{\partial x}\right) = \frac{\partial}{\partial \beta_2}\left(\frac{\partial G_1}{\partial x}\right) = 0 \qquad \text{(A3.3)}$$

for any starting inventory level x and y. A similar comment holds for $\partial G_1/\partial y$. Let

$$Z_1 = \int_0^{1-\beta_2} G_1(\alpha_2 - p - \gamma(1-p-\beta_2),0)dF + \int_{1-\beta_2}^{\alpha_2} G_1(\alpha_2 - p, \beta_2 - (1-p))dF$$
$$+ \int_{\alpha_2}^{1} \mathrm{G}_1(0, \beta_2 \text{ - } (1\text{ - } p) - \gamma(p-\alpha_2))dF \;]$$

Then, taking derivatives and canceling common terms using (A3.2)

$$\frac{\partial Z_1}{\partial \alpha_2} = \int_0^{1-\beta_2} \frac{\partial G_1(\alpha_2 - p - \gamma(1-p-\beta_2),0)}{\partial \alpha_2}dF + \int_{1-\beta_2}^{\alpha_2} \frac{\partial G_1(\alpha_2 - p, \beta_2 - (1-p))}{\partial \alpha_2}dF$$
$$+ \int_{\alpha_2}^{1} \frac{\partial \mathrm{G}_1(0, \beta_2 \text{ - } (1\text{ - } p) - \gamma(p-\alpha_2))}{\partial \alpha_2}dF \;]$$

Let $Y_1 = (\alpha_2 - p - \gamma(1- p - \beta_2))$, $Y_2 = (\alpha_2 - p)$, $Y_3 = (\beta_2 - (1 - p))$, $Y_4 = (\beta_2 - (1 - p) - \gamma(p - \alpha_2))$. Then,

$$\frac{\partial Y_1}{\partial \alpha_2} = 1, \frac{\partial Y_2}{\partial \alpha_2} = 1, \frac{\partial Y_3}{\partial \alpha_2} = 0, \frac{\partial Y_4}{\partial \alpha_2} = \gamma.$$

So we have:

$$\frac{\partial Z_1}{\partial \alpha_2} = \int_0^{1-\beta_2} \frac{\partial G_1(Y_1,0)}{\partial Y_1}dF + \int_{1-\beta_2}^{\alpha_2} \frac{\partial G_1(Y_2,Y_3)}{\partial Y_2}dF + \int_{\alpha_2}^{1} \gamma \frac{\partial \mathrm{G}_1(0,Y_4)}{\partial Y_4}dF \;]$$

From (A3.3), the partials within the integrals are not functions of α_2 or β_2 and so

$\frac{\partial^2 Z_1}{\partial \alpha_2 \partial \beta_2} = 0$. Now consider the function $G_2(I_2^1, I_2^2)$. Since $\frac{\partial^2 \Pi_1}{\partial \alpha_2 \partial \beta_2} = 0$ as noted earlier

and the remaining part of the objective function (A3.1) for $G_2(.)$ is separable in α_2 and β_2,

$\frac{\partial^2 X_2}{\partial \beta_2 \partial \alpha_2} = 0$. By induction, we have the proof of the lemma.♦

We now prove the theorem. Part (1) of the theorem i.e. that when there *n* periods to go, it is optimal follow a simple decoupled base stock policy now follows from Lemma 3.2 together with the fact that $X_n(\alpha, \beta)$ is concave

To prove part (2) we demonstrate that the base stock levels are monotonically increasing with *n,* the number of periods till the end of the horizon. We demonstrate this for the case when n = 2. The proof of the general case is by induction and follows exactly the same logic. In what follows, we show that $\alpha_2^* \geq \alpha_1^*$. Due to separability and symmetry, this implies that $\beta_2^* \geq \beta_1^*$.

We know that $G_2(I_2^1, I_2^2) = \underset{(\alpha_2,\beta_2)\geq(I_2^1,I_2^2)}{Max} \{K(\alpha_2, \beta_2)\}$ where

$K(\alpha_2, \beta_2) = X_2(\alpha_2, \beta_2) + c(I_2^1 + I_2^2)$.

Thus we have that,

$$K(\alpha_2, \beta_2) = \Pi_1(\alpha_2, \beta_2) - c(\alpha_2 + \beta_2 - I_2^1 - I_2^2) + \int_0^{1-\beta_2} G_1(\alpha_2 - p - \gamma(1 - p - \beta_2), 0)dF$$

$$+ \int_{1-\beta_2}^{\alpha_2} G_1(\alpha_2 - p, \beta_2 - 1 - p)dF + \int_{\alpha_2}^{1} G_1(0, \beta_2 - 1 - p - \gamma(p - \alpha_2))dF$$

We know from Theorem 3.1 that $K(\alpha_2,\beta_2)$ is jointly concave and the unique root of the equation $\frac{\partial K(\alpha_2,\beta\ )}{\partial \alpha_2}=0$ is α_2^*. In fact,

$$\frac{\partial K(\alpha_2,\beta_2)}{\partial \alpha_2}=-c+\frac{\partial \Pi_1}{\partial \alpha_2}+\int_0^{1-\beta_2}\frac{\partial G_1(\alpha_2-p-\gamma(1-p-\beta_2),0)}{\partial \alpha_2}dF+$$

$$\int_{1-\beta_2}^{\alpha_2}\frac{\partial G_1(\alpha_2-p,\beta_2-(1-p))}{\partial \alpha_2}dF\ +\int_{\alpha_2}^{1}\frac{\partial G_1(0,\beta_2-(1-p)-\gamma(p-\alpha_2))}{\partial \alpha_2}dF$$

Note first that $\frac{\partial \Pi_1}{\partial \alpha_2}-c$ evaluated at α_1^* equals zero. Further, note that the profit to go function $G_1(x,y)$ for a fixed y reaches its optimal value when $x=\alpha_1^*$ and thus for a fixed y the concavity of G implies $\frac{\partial G}{\partial x}\geq 0$ when $x\leq \alpha_1^*$. We note that in the above expressions in the integral, when we set $\alpha_2=\alpha_1^*$, the argument x is $\leq \alpha_1^*$ This immediately implies $\frac{\partial K(\alpha_1^*,\beta)}{\partial \alpha_2}\geq 0$. Concavity of $K(\alpha_2,\beta_2)$ implies $\alpha_2^*\geq \alpha_1^*$. This proves (2) and thus concludes the proof of Theorem 3.2. ♦

Proof of Theorem 3.3:

Define,

$$T(\alpha,\beta,I_1,I_2,G)=\Pi_1(\alpha,\beta)-c(\alpha+\beta-I_1-I_2)+\delta\int_0^{1-\beta}G(\alpha-p-\gamma(1-p-\beta),0)dF$$

$$+\delta\int_{1-\beta}^{\alpha}G(\alpha-p,\beta-(1-p))dF+\delta\int_{\alpha}^{1}G(0,\beta-(1-p)-\gamma(p-\alpha))dF$$

Then equation (3.1) in the paper can be re-written as $G(I_1,I_2)=\underset{(\alpha,\beta)\geq(I_1,I_2)}{Max}\ T(\alpha,\beta,I_1,I_2,G)$.

Now let $G_0(I_1, I_2)$ be a bounded and continuous function. Define inductively, for n = 1,2… ∞

$$G_{n+1}(I_{n+1}^1, I_{n+1}^2) = \underset{(\alpha_{n+1},\beta_{n+1}) \geq (I_{n+1}^1, I_{n+1}^2)}{Max} \{\Pi_1(\alpha_{n+1}, \beta_{n+1}) - c(\alpha_{n+1} + \beta_{n+1} - I_{n+1}^1 - I_{n+1}^2) + E_p[G_n(I_n^1, I_n^2)]\}.$$

Thus $T(\alpha_{n+1}, \beta_{n+1}, G_n, I_{n+1}^1, I_{n+1}^2) = \Pi_1(\alpha_{n+1}, \beta_{n+1}) - c(\alpha_{n+1} + \beta_{n+1} - I_{n+1}^1 - I_{n+1}^2) + E_p[G_n(I_n^1, I_n^2)]$

and we can write $G_{n+1}(I_{n+1}^1, I_{n+1}^2) = \underset{(\alpha_{n+1},\beta_{n+1}) \geq (I_{n+1}^1, I_{n+1}^2)}{Max} T(\alpha_{n+1}, \beta_{n+1}, G_n, I_{n+1}^1, I_{n+1}^2)$.

In what follows, we demonstrate that $\underset{n\to\infty}{Lim}\, G_n(x, y) = G(x, y), 0 \leq x, y < \infty$. The analysis is similar to Theorem 1 in Bellman, Glicksberg and Gross (1955) and in addition uses the decoupled structure of the policy. It involves constructing a contraction mapping, showing that the convergence is uniform and then demonstrating the existence and uniqueness of a continuous bounded solution in equation (3.1) in the paper.

Let $x_n(I) = \begin{matrix} \alpha_n^*,\ I \leq \alpha_n^* \\ I,\ I > \alpha_n^* \end{matrix}$ and $y_n(I) = \begin{matrix} \beta_n^*,\ I \leq \beta_n^* \\ I,\ I > \beta_n^* \end{matrix}$ thus

$$G_{n+1}(x, y) = T(x_{n+1}(x), y_{n+1}(y), G_n, x, y).$$

Notice that for every non negative *x,y,* $G_{n+1}(x, y) \geq G_n(x, y)$ and

$G_{n+1}(x, y) = T(x_{n+1}(x), y_{n+1}(y), G_n, x, y) \geq T(x_n(x), y_{n,}(y), G_n, x, y)$ and

$G_n(x, y) = T(x_n(x), y_n(y), G_{n-1}, x, y) \geq T(x_{n+1}(x), y_{n+1}(y), G_{n-1}, x, y)$. This immediately implies that

$$\left|G_{n+1}(x, y) - G_n(x, y)\right| \leq \left|T(x_{n+1}(x), y_{n+1}(y), G_n, x, y) - T(x_{n+1}(x), y_{n+1}(y), G_{n-1}, x, y)\right|$$

$$= \delta\left|E_p[G_n(I_n^1, I_n^2) - E_p[G_{n-1}(I_{n-1}^1, I_{n-1}^2)\right|$$

In the above expression, the distribution of the initial inventories (I_n^1, I_n^2) and (I_{n-1}^1, I_{n-1}^2) are exactly the same since the decisions are identical in the two periods, namely $x_{n+1}(x), y_{n+1}(y)$.

Thus $\underset{0\le x,y\le D}{Max}\left|G_{n+1}(x,y)-G_n(x,y)\right| \le \delta \underset{0\le x,y\le D}{Max}\left|G_n(x,y)-G_{n-1}(x,y)\right| \le \delta^n\left|G_0(x,y)\right|$. The implications are as follows: (i) $\sum(G_{n+1}(x,y)-G_n(x,y))$ converges absolutely and thus $G_n(x,y)$ converges uniformly for every $0\le x,y\le D.$ (ii) $G_n: R^2 \to R^+$ is a contraction mapping and thus it has a unique limit which is continuous. These implications together with the fact that the limit and integration can be interchanged when the integrand is a bounded function imply that $\underset{n\to\infty}{Lim}\, G_n(x,y)=G(x,y)$.

We now characterize the solution of the infinite horizon problem. To do so, we rewrite equation (3.1) in the paper as $G(I_1,I_2)=\underset{(\alpha,\beta)\ge(I_1,I_2)}{Max}[V(\alpha,\beta)+c(I_1+I_2)]$ and solve the equations $\partial V/\partial\alpha=0$ and $\partial V/\partial\beta=0$ simultaneously. Note that

$$\partial V/\partial\alpha=\frac{\partial\Pi_1(\alpha,\beta)}{\partial\alpha}-c+\delta\frac{\int_0^{1-\beta}\partial G(\alpha-p-\gamma(1-p-\beta),0)}{\partial\alpha}dF$$
$$-\delta\int_{1-\beta}^{\alpha}\frac{\partial G(\alpha-p,\beta-(1-p))}{\partial\alpha}dF+\gamma\delta\int_{\alpha}^{1}\frac{\partial G(0,\beta-(1-p)-\gamma(p-\alpha))}{\partial\alpha}dF$$

The expression for $\partial V/\partial\beta$ is similar. Further, note that when $\gamma\le\gamma^*$ and the order quantities are such that $\alpha+\beta\ge1$, we have that $\partial G/\partial\alpha=\partial G/\partial\beta=c$. This immediately implies that

$$\partial V/\partial\alpha=0\Rightarrow F(\alpha_\infty^*)=\frac{(s+\pi-c)-\gamma(s+\pi+h-\delta c)}{(s+\pi+h-\delta c)(1-\gamma)}$$

$$\partial V/\partial\beta=0\Rightarrow F(1-\beta_\infty^*)=\frac{h-c(1+\delta)}{(s+\pi+h-\delta c)(1-\gamma)}$$

Note that $\gamma \le 1 - \frac{2(h+(1-\delta)c)}{s+\pi+h-\delta c} \Rightarrow F(\alpha_\infty^*) = \frac{(s+\pi-c)-\gamma(s+\pi+h-\delta c)}{(s+\pi+h-\delta c)(1-\gamma)} \ge 0.5$. Further, $F(\alpha_\infty^*) + F(1-\beta_\infty^*) = 1$. It then follows that $F(1-\beta_\infty^*) \le 0.5$ and $(1-\beta_\infty^*) \le F^{-1}(0.5)$ or $\beta_\infty^* \ge 1 - F^{-1}(0.5)$ and so $\alpha_\infty^* + \beta_\infty^* \ge 1$. Further, $\partial^2 V / \partial\alpha^2 = [c-(s+\pi+h)](1-\gamma) < 0$ and a similar expression can be derived for β. Thus V is concave with respect to each variable. This together with the fact that the convergence preserves the decoupled structure (i.e. separability) implies that when $\gamma < \gamma^{\otimes} = 1 - \frac{2(h+(1-\delta)c)}{s+\pi+h-\delta c}$, the optimal policy for the infinite horizon problem is to follow a stationary policy of ordering up to α_∞^* and β_∞^* in every period in which the initial inventory is below this order-up-to level. ♦