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Electronic Companion—“Strategic Inventories in Vertical Contracts”
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Management Science, DOI 10.1287/mnsc.1080.0894.

Proofs of Statements

Proof of Theorem 1:

Part 1 of Theorem 1 follows from the results outlined in the column for Dynamic contracts in Table 1. We now prove these results.

In the analysis of this sequential game, we start with the second period, assuming that the buyer carries inventories I and that the supplier has quoted a wholesale price w_2 . The buyer's problem is now the quadratic:

$$\max_{Q_2} (a - b(Q_2 + I))(Q_2 + I) - w_2 Q_2.$$

It is straightforward to see that $Q_2 = \max\{\frac{a-w_2}{2b} - I, 0\}$.

• **Case (i), $Q_2 = 0$:** We first analyze the case when $Q_2 = 0$. Notice that the suppliers second period wholesale price w_2 is irrelevant. The buyer's second period profit function is simply $(a - bI)I$.

Across the two periods, the buyer then solves the following problem:

$$\max_{q_1, I \geq 0} (a - bq_1)q_1 - w_1(q_1 + I) - hI + (a - bI)I$$

The optimal sales and inventory are:

$$q_1 = \frac{a - w_1}{2b}; \quad I = \frac{a - (w_1 + h)}{2b}.$$

Since there is no quantity bought in the second period, the supplier chooses w_1 to maximize $w_1(q_1 + I)$. The optimal wholesale price, $w_1 = \frac{a}{2} - \frac{h}{4}$ and the supplier's profits are $\frac{(2a-h)^2}{16b}$.

• **Case (ii), $Q_2 > 0$:** Next consider the case when $Q_2 > 0$. The supplier's problem in the second period is:

$$\max_{w_2} w_2 \left(\frac{a - w_2}{2b} - I \right).$$

The optimal $w_2 = \frac{a}{2} - bI$. This implies that $Q_2 = \frac{a}{4b} - \frac{I}{2}$. The second-period profit functions of the supplier and the buyer as a function of buyer's inventories are given by

$$\Pi_{B,2}(I) = \frac{1}{4b} [a - w_2]^2 + w_2 I = \frac{1}{4b} \left[\frac{a}{2} + bI \right]^2 + \left[\frac{a}{2} - bI \right] I, \text{ and} \quad (\text{EC.1})$$

$$\Pi_{S,2}(I) = \frac{1}{2b} \left[\frac{a}{2} - bI \right]^2. \quad (\text{EC.2})$$

Now, moving to the first period, when the supplier quotes a wholesale price w_1 , the buyer's problem becomes

$$\max_{q_1, I \geq 0} (a - bq_1)q_1 - w_1(q_1 + I) - hI + \frac{1}{4b} \left[\frac{a}{2} + bI \right]^2 + \left[\frac{a}{2} - bI \right] I, \quad (\text{EC.3})$$

where the buyer buys $Q_1 = q_1 + I$ in the first period, sells q_1 and carries inventory I to sell in the second period. Using the Lagrangian and complementary slackness conditions, it is rather straightforward to show that the solution exhibits two cases:

- (i) $w_1 + h \geq \frac{3a}{4}$: $I = 0$ $Q_1 = q_1 = \frac{a-w_1}{2b}$.
- (ii) $w_1 + h < \frac{3a}{4}$: $I = \frac{2}{3b} \left[\frac{3a}{4} - w_1 - h \right]$; $q_1 = \frac{a-w_1}{2b}$; $Q_1 = \frac{1}{b} \left[a - \frac{7}{6}w_1 - \frac{2}{3}h \right]$.

Substituting into the supplier's profit functions for the first and second periods, we get

$$\Pi_S = \begin{cases} w_1 \left(\frac{a-w_1}{2b} \right) + \frac{a^2}{8b}, & \text{if } w_1 + h > \frac{3a}{4}; \\ \frac{w_1}{b} \left[a - \frac{7}{6}w_1 - \frac{2}{3}h \right] + \frac{2}{9b} (w_1 + h)^2, & \text{otherwise.} \end{cases}$$

The supplier determines the optimal w_1 and the corresponding profits for each of the two cases $Q_2 > 0$ and $Q_2 = 0$. His profits are maximized for the case when $Q_2 > 0$ and $w_1 + h \leq \frac{3a}{4}$. The optimal $w_1 = \frac{9a-2h}{17}$, with profits equal to $\frac{9a^2-4ah+8h^2}{34b}$.

Part 2 of Theorem 1 follows from the results outlined in the Commitment contract column of Table 1. We now prove these results.

We first solve for the buyer's optimal response to a given w_1 and w_2 . Since his decision variables are Q_1, Q_2 and I , the buyer's decision problem is

$$\max_{Q_1 \geq I \geq 0, Q_2 \geq 0} (a - b(Q_1 - I))(Q_1 - I) + (a - b(Q_2 + I))(Q_2 + I) - hI - w_1 Q_1 - w_2 Q_2.$$

Subsequently, we solve for the supplier's optimal pricing decision.

In solving the problem, we initially ignore the constraint that $Q_1 \geq I$; we will check that that it holds in the solution to the relaxed problem. The solution reduces to two possible cases, which are:

- (i) $w_1 + h \geq w_2$: $Q_1 = q_1 = \frac{a-w_1}{2b}$; $Q_2 = q_2 = \frac{a-w_2}{2b}$; $I = 0$.
- (ii) $w_1 + h < w_2$: $Q_1 = \frac{2(a-w_1)-h}{2b}$; $q_1 = \frac{a-w_1}{2b}$; $Q_2 = 0$; $q_2 = I = \frac{a-w_1-h}{2b}$.

Under both these cases, the constraint $Q_1 \geq I$ is satisfied; so the solution is both feasible and optimal.

The supplier profits (over *both* periods) are $\Pi_S = w_1 Q_1 + w_2 Q_2$, which reduce to

$$\Pi_S = \begin{cases} w_1 \left(\frac{a-w_1}{2b} \right) + w_2 \left(\frac{a-w_2}{2b} \right), & \text{if } w_1 + h \geq w_2; \\ w_1 \left(\frac{2(a-w_1)-h}{2b} \right), & \text{otherwise.} \end{cases}$$

Now the supplier optimizes his profit function over w_1 and w_2 . For the first case (assuming $w_1 + h \geq w_2$), we get that the optimal prices are $w_1 = w_2 = \frac{a}{2}$ giving the supplier optimal profits of $\frac{a^2}{4b}$. For the second case (assuming $w_1 + h < w_2$), we derive $w_1 = \frac{a}{2} - \frac{h}{4}$ with optimal profits of $\frac{(2a-h)^2}{16b}$. It is straightforward to see that the supplier makes a higher profit by implementing the prices given by the first case. ■

Proof of Proposition 1:

Follows by straightforward comparison of the appropriate profit expressions given in Theorem 1 (see Table 1). ■

Proof of Proposition 2:

Formally, the consumer surplus in any given period is given by $\frac{a}{2}q_i^2$ where q_i is the sales quantity in period i . Thus the total consumer surplus (CS) across both periods for the dynamic contract is:

$$CS^d = \frac{185a^2 - 188ah + 104h^2}{2312b}$$

and for the commitment contract is $CS^c = \frac{a^2}{16b}$. The difference between the two is:

$$CS^d - CS^c = \frac{81a^2 - 376ah + 208h^2}{4624b}$$

which is always greater than zero for $h < a/4$. The difference in total welfare between the dynamic and commitment contract is computed as:

$$\frac{191a^2 - 1392ah + 2512h^2}{4624b}$$

which can be shown to be always non-negative for $h < a/4$. ■

Proof of Theorem 2: (dynamic contract)

The equilibrium result for dynamic contracts in Theorem 2 follows from the results outlined in the column for dynamic contracts in Table 2 which we now prove. We proceed in a backward fashion beginning from the last period. Let $\mathbf{1}_{\mathcal{X}}$ be an indicator function which takes the value one if the condition represented by \mathcal{X} is true; otherwise it takes the value zero. Given K_2, w_2 , and I , the buyer solves the following optimization problem for period 2.

$$\max_{Q_2 \geq 0} \Pi_{B,2}(Q_2; I, K_2, w_2) = (a - b(Q_2 + I))(Q_2 + I) - w_2 Q_2 - K_2 \mathbf{1}_{Q_2 > 0}. \quad (\text{EC.4})$$

Given the buyer's response to K_2 and w_2 , the supplier solves

$$\begin{aligned} \max_{w_2, K_2} \quad & K_2 \mathbf{1}_{Q_2 > 0} + w_2 Q_2(w_2, I) \\ \text{s.t.} \quad & \end{aligned} \quad (\text{EC.5})$$

$$\Pi_{B,2}(I, K_2, w_2) \geq 0$$

The following Lemma describes the optimal policy structure for the second period:

LEMMA EC.1. *The optimal second period two-part tariff is $(K_2^*(I), w_2^*)$ where*

$$\begin{aligned} K_2^*(I) &= \frac{a^2}{4b} - (a - bI)I \\ w_2^* &= 0 \end{aligned}$$

The optimal purchase quantity for the buyer is $Q_2^ = \max\{\frac{a}{2b} - I, 0\}$. The corresponding optimal profits of the buyer and supplier are as follows:*

$$\begin{aligned} \Pi_{B,2}(I) &= (a - bI)I \\ \Pi_{S,2}(I) &= \frac{a^2}{4b} - (a - bI)I. \end{aligned}$$

Proof: : Solving for the optimal purchase quantity for the buyer (and ignoring K_2 for the moment), we get:

$$Q_2 = \max \left\{ \frac{a - w_2}{2b} - I, 0 \right\} \quad (\text{EC.6})$$

If $Q_2 > 0$, then the optimal second period profits for the buyer are:

$$\Pi_{B,2}(I, K_2, w_2) = \left(a - b \left(\frac{a - w_2}{2b} \right) \right) \left(\frac{a - w_2}{2b} \right) - w_2 \left(\frac{a - w_2}{2b} - I \right) - K_2.$$

Clearly, $Q_2 > 0$ if and only if, $\Pi_{B,2}$ above is at least as large as the profits that the buyer would earn with $Q_2 = 0$ and selling from inventory I . That is, we require that

$$\Pi_{B,2}(I, K_2, w_2) \geq (a - bI)I.$$

Substituting for $\Pi_{B,2}(I, K_2, w_2)$ and simplifying, we get that

$$K_2 \leq \frac{(a - w_2)^2}{4b} - (a - w_2 - bI)I.$$

Define

$$K_2(I) \equiv \frac{(a - w_2)^2}{4b} - (a - w_2 - bI)I.$$

Then we have that if $K_2 \leq K_2(I)$, the buyer will purchase a non-negative quantity in period 2.

The supplier, in the second period, solves the following problem:

$$\max_{K_2, w_2} \quad K_2(I) \mathbf{1}_{Q_2 > 0} + w_2 Q_2$$

s.t.

$$\Pi_{B,2}(I, K_2, w_2) \geq (a - bI)I$$

where Q_2 is given by (EC.6). Notice that if the supplier sets w_2 such that $\frac{a - w_2}{2b} \leq I$ then $Q_2 = 0$ and the supplier makes zero profits in the second period. So let us assume that $w_2 < a - 2bI$ such that $Q_2 > 0$. Substituting for $K_2(I)$ and Q_2 in the supplier's profit function and optimizing for w_2 , we get that $w_2 = 0$. Of course, we need $I \leq \frac{a}{2b}$ for $Q_2 > 0$. Since $w_2 = 0$, it is straightforward to see that the buyer will obey this constraint; else he loses $(w_1 + h)$ per unit at the margin. Thus,

$$K_2(I) = \frac{a^2}{4b} - (a - bI)I,$$

and

$$Q_2 = \frac{a}{2b} - I.$$

Substituting into the expressions for the profits of the buyer and the supplier, we get the buyer's profits to be:

$$\Pi_{B,2}(I) = (a - bI)I \quad (\text{EC.7})$$

and the supplier's profits as:

$$\Pi_{S,2}(I) = \frac{a^2}{4b} - (a - bI)I. \quad (\text{EC.8})$$

■

We now solve for the first period prices and quantities. In the first period, the supplier announces K_1 and w_1 . The buyer then decides to purchase Q_1 , sells $q_1 \leq Q_1$, and perhaps carries inventory of I into the next period. He solves the following optimization problem:

$$\max_{Q_1 \geq I \geq 0} (a - b(Q_1 - I))(Q_1 - I) - hI - w_1Q_1 - K_1 \mathbf{1}_{Q_1 > 0} + \Pi_{B,2}(I) \quad (\text{EC.9})$$

Then the buyer's first period strategy (assuming for the moment that K_1 is small enough to induce purchases) is given by the following Lemma which is straightforward to derive (we skip the proof).

LEMMA EC.2. *The optimal purchase quantity Q_1 and optimal inventory I are given as follows:*

$$(Q_1, I) = \begin{cases} \left(\frac{2(a-w_1)-h}{2b}, \frac{a-w_1-h}{2b} \right) & \text{if } w_1 < a - h \\ \left(\frac{a-w_1}{2b}, 0 \right) & \text{if } w_1 \geq a - h \end{cases}$$

The supplier needs to determine K_1 and w_1 . he will do so to extract all of the buyer's current and future profits, where the latter is given by $\Pi_{B,2}(I)$. Thus, he solves the following problem:

$$\max_{K_1, w_1} K_1 \mathbf{1}_{Q_1 > 0} + w_1 Q_1(w_1) + \Pi_{S,2}(I(w_1)) \quad (\text{EC.10})$$

s. t.

$$\Pi_B \geq \Pi_{B,2}(I = 0)$$

where $Q_1(w_1)$ and $I(w_1)$ are as given by Lemma EC.2.

The supplier's optimal response is derived as follows. First consider the constraint in (EC.10). Recall that, if the buyer chooses $Q_1 > 0$, then

$$\Pi_B = (a - b(Q_1 - I))(Q_1 - I) - hI - w_1Q_1 - K_1 + \Pi_{B,2}(I).$$

Also observe that $\Pi_{B,2}(I = 0) = 0$. Substituting for $\Pi_{B,2}(I)$ from Lemma EC.1, the constraint in (EC.10) can be written as:

$$K_1 \leq (a - b(Q_1 - I))(Q_1 - I) - hI - w_1Q_1 + (a - bI)I.$$

Since the supplier is maximizing profits, he will set

$$K_1 = (a - b(Q_1 - I))(Q_1 - I) - hI - w_1Q_1 + (a - bI)I.$$

Substituting for K_1 (we will write it as $K_1(w_1)$ since Q_1 and I are functions of w_1) in the optimization problem of the supplier, we get that the supplier solves the following problem:

$$\max_{w_1} K_1(w_1) + w_1Q_1(w_1) + \Pi_{S,2}(I(w_1)).$$

From Lemma EC.2 we need to consider two cases. First suppose $w_1 < a - h$. Then substituting for $Q_1(w_1)$ and $I(w_1)$ and simplifying, we write the supplier's objective function as:

$$\max_{w_1} \frac{2a^2 - w_1^2}{4b} - h \left(\frac{a - w_1 - h}{2b} \right).$$

Equating the first order condition with respect to w_1 to zero, we get $w_1 = h$. Since we require that $w_1 < a - h$, this price is feasible whenever $h < a/2$ which holds. Substituting this price into expressions for $Q_1(w_1)$ and $I(w_1)$, we get

$$I = \frac{a - 2h}{2b} \quad \text{and} \quad Q_1 = \frac{2a - 3h}{2b}.$$

So $Q_1 - I = \frac{a-h}{2b}$. Substituting into $K_1(w_1)$, we get

$$\begin{aligned} K_1 &= (a - b(Q_1 - I))(Q_1 - I) - hI - w_1Q_1 + (a - bI)I \\ &= \frac{a^2}{2b} - \frac{3ah}{2b} + \frac{5h^2}{4b}. \end{aligned}$$

Substituting for the optimal I into $K_2(I)$ and $Q_2(I)$ in Lemma EC.1, we get $K_2 = h^2/b$ and $Q_2 = h/b$. Therefore, the supplier's overall profits given by

$$K_1 + K_2 + w_1Q_1 + w_2Q_2 = \frac{a^2}{2b} - \frac{ah}{2b} + \frac{3h^2}{4b},$$

which is less than the first-best profits of $a^2/2b$. The buyer makes zero profits.

From Lemma EC.2, the other case is $w_1 \geq a - h$ with $Q_1 = \frac{a-w_1}{2b}$ and $I = 0$. Working out this case, we see that the unconstrained optimal $w_1 = 0$ is infeasible as we require that $w_1 \geq a - h$. So the supplier sets $w_1 = a - h$. Then, $Q_1 = h/2b$ and $I = 0$. Substituting to get the fixed fees, we get that $K_1 = h^2/4b$ and $K_2 = a^2/4b$. The supplier's total profits are

$$K_1 + K_2 + w_1Q_1 = \frac{a^2 - h^2 + 2ah}{4b}.$$

It is easy to see this case is dominated by the earlier case discussed (when $Q_1 > 0$ and $I > 0$) for $h < a/4$.

So in equilibrium the supplier sets $K_1 = \frac{a^2}{2b} - \frac{3ah}{2b} + \frac{5h^2}{4b}$, $K_2 = h^2/b$, $w_1 = h$ and $w_2 = 0$. ■

Proof of Theorem 2: (commitment contract)

The equilibrium results for the commitment contract in Theorem 2 follows from the results outlined in the column for commitment contracts in Table 2. We now prove these results.

We derive the optimal policy structure working backwards starting with the buyer's response. The buyer solves the following problem:

$$\begin{aligned} \max_{Q_1 \geq I \geq 0, Q_2 \geq 0} & (a - b(Q_1 - I))(Q_1 - I) - w_1Q_1 - K_1 \mathbf{1}_{Q_1 > 0} \\ & + (a - b(Q_2 + I))(Q_2 + I) - hI - w_2Q_2 - K_2 \mathbf{1}_{Q_2 > 0} \end{aligned} \quad (\text{EC.11})$$

The buyer's optimal response is given by the following Lemma.

LEMMA EC.3. *Given (K_1, w_1) and (K_2, w_2) , the buyer chooses one of the following actions:*

No.	Actions			Buyer's Profit	Necessary Conditions on w_1, w_2, h
	Q_1	I	Q_2		
1	0	0	$\frac{a-w_2}{2b}$	$\Pi_B^{(1)} = -K_2 + \frac{(a-w_2)^2}{4b}$	$w_2 < a$
2	$\frac{a-w_1}{2b}$	0	0	$\Pi_B^{(2)} = -K_1 + \frac{(a-w_1)^2}{4b}$	$a - h \leq w_1 < a$
3	$\frac{2(a-w_1)-h}{2b}$	$\frac{a-w_1-h}{2b}$	0	$\Pi_B^{(3)} = -K_1 + \frac{(a-w_1)^2}{4b} + \frac{(a-w_1-h)^2}{4b}$	$w_1 < a - h$
4	$\frac{a-w_1}{2b}$	0	$\frac{a-w_2}{2b}$	$\Pi_B^{(4)} = -K_1 - K_2 + \frac{(a-w_1)^2}{4b} + \frac{(a-w_2)^2}{4b}$	$\{w_1, w_2\} < a$, and $w_1 + h \geq w_2$

Proof: : The above table identifies the necessary conditions on w_1 , w_2 , and h for each of the actions specified to be implemented. Obviously, for the supplier to implement a particular action, he also needs to choose the right values of K_1 and K_2 , which we will address when we solve the supplier's problem.

There are three possible strategies for the buyer assuming that the participation constraint of non-negative profit is met. Either (i) $Q_1 = 0$, $Q_2 > 0$; or (ii) $Q_1 > 0$, $Q_2 = 0$; or (iii) $Q_1 > 0$, $Q_2 > 0$.

We now consider each of these cases.

Case (i) $Q_1 = 0$, $Q_2 > 0$: In this case the optimization problem for the buyer is

$$\max_{Q_2} -K_2 - w_2 Q_2 + (a - bQ_2)Q_2.$$

Solving for Q_2 gives $Q_2 = \frac{a-w_2}{2b}$ which is non-zero only when $w_2 < a$. Substituting back into the profit function gives us that

$$\Pi_B = -K_2 + \frac{(a - w_2)^2}{4b}.$$

Case (ii) $Q_1 > 0$, $Q_2 = 0$: In this case the buyer does not purchase any quantity in the second period. This implies that he may choose to carry over inventory from the first period and sell it in the second. Thus the buyer's objective function is:

$$\max_{Q_1, I} -K_1 - w_1 Q_1 + (a - b(Q_1 - I))(Q_1 - I) - hI + (a - bI)I.$$

The optimal inventory is given as $I = \frac{Q_1}{2} - \frac{h}{4b}$. Thus if $Q_1 \leq \frac{h}{2b}$ then $I = 0$ and no inventory is carried; otherwise $I > 0$.

Now suppose $I = 0$; substituting back into the profit function and solving for optimal Q_1 , we see that $Q_1 = \frac{a-w_1}{2b}$. First we require that $w_1 < a$ for $Q_1 > 0$. Furthermore, we need that $I = 0$ which implies that $Q_1 \leq \frac{h}{2b}$. This implies that $w_1 \geq a - h$.

Next consider the situation when $I > 0$. Substituting the expression for I in the buyer's objective function and solving for the optimal Q_1 by equating the first order condition w.r.t. Q_1 to zero, we get that,

$$Q_1 = \frac{2(a - w_1) - h}{2b}; \quad I = \frac{a - w - h}{2b}.$$

Notice that for $Q_1 > \frac{h}{2b}$ we would need that $w_1 < a - h$.

Thus case (ii) has two solutions: if $w_1 \geq a - h$ then $Q_1 = \frac{a-w_1}{2b}$ and $I = 0$; else if $w_1 < a - h$ then $Q_1 = \frac{2(a-w_1)-h}{2b}$ and $I = \frac{a-w-h}{2b}$. Substituting back into the profit function, we derive that the buyer's profits in the former case is $\Pi_B = -K_1 + \frac{(a-w_1)^2}{4b}$ and in the latter it is $\Pi_B = -K_1 + \frac{(a-w_1)^2}{4b} + \frac{(a-w_1-h)^2}{4b}$.

Case (iii) $Q_1 > 0, Q_2 > 0$: In this case the buyer's optimization problem is:

$$\begin{aligned} \max_{Q_1, Q_2, I} \quad & -K_1 - K_2 - w_1 Q_1 - w_2 Q_2 \\ & + [a - b(Q_1 - I)](Q_1 - I) + [(a - b(Q_2 + I)](Q_2 + I) - hI \end{aligned}$$

Given that $Q_1, Q_2 > 0$, the analysis of this case is similar to the linear contract under commitments.

Recall that for that case we derived that if $w_1 + h \geq w_2$, then $Q_1 = \frac{a-w_1}{2b}$, $Q_2 = \frac{a-w_2}{2b}$, and $I = 0$ giving

$$\Pi_B = -K_1 - K_2 + \frac{(a-w_1)^2}{4b} + \frac{(a-w_2)^2}{4b}.$$

If $w_1 + h < w_2$ then the buyer's optimal action would be $Q_1 > 0$ and $Q_2 = 0$ and we are back to case (ii).

To summarize, the buyer acts to choose one of four outcomes— one outcome derived in case (i), two in case (ii), and one in case (iii)— subject to the participation constraint for non-negative profits. These correspond to the four cases in Lemma EC.3. ■

The supplier then optimizes the following objective function

$$\max_{K_1, K_2, w_1, w_2} K_1 \mathbf{1}_{Q_1 > 0} + w_1 Q_1 + K_2 \mathbf{1}_{Q_2 > 0} + w_2 Q_2$$

subject to the buyer maximizing his own profit function as per the choices listed in Lemma EC.3.

Now observe that the buyer's action (1) is a special case of his action (4), which Pareto-dominates it. That is, the supplier will always choose to force the buyer to pick (4) instead of (1). Similarly, buyer's action (2) is also Pareto-dominated by his action (4). Thus the supplier only needs to consider the buyer's actions (3) and (4) in deciding his pricing scheme.

Let $\Pi_B^{(k)}$ denote buyer's profits under action (k) outlined in Lemma EC.3. Now consider the situation that the supplier wishes to enforce buyer's action (3). Then he solves the following problem:

$$\max_{K_1, w_1} \quad K_1 + w_1 \frac{2(a - w_1) - h}{2b} \quad (\text{EC.12a})$$

$$\text{s.t.} \quad \Pi_B^{(3)} \geq 0 \quad (\text{EC.12b})$$

$$\Pi_B^{(3)} \geq \Pi_B^{(1)} \quad (\text{EC.12c})$$

$$\Pi_B^{(3)} \geq \Pi_B^{(2)} \quad (\text{EC.12d})$$

$$\Pi_B^{(3)} > \Pi_B^{(4)} \quad (\text{EC.12e})$$

$$w_1 < a - h \quad (\text{EC.12f})$$

Notice that,

$$\text{Constraint (EC.12b)} \Rightarrow K_1 \leq \frac{(a - w_1)^2}{4b} + \frac{(a - w_1 - h)^2}{4b}.$$

Since the supplier is maximizing over K_1 , he will choose

$$K_1 = \frac{(a - w_1)^2}{4b} + \frac{(a - w_1 - h)^2}{4b}.$$

Substituting this into Constraint (EC.12c), we have

$$\text{Constraint (EC.12c)} \Rightarrow K_2 \geq \frac{(a - w_2)^2}{4}.$$

Constraint (EC.12d) is trivially satisfied. Finally,

$$\text{Constraint (EC.12e)} \Rightarrow K_2 \geq \frac{(a - w_2)^2}{4b} - \frac{(a - w_1 - h)^2}{4b}$$

Thus the constraints (EC.12b)- (EC.12e) reduce to

$$K_1 = \frac{(a - w_1)^2}{4b} + \frac{(a - w_1 - h)^2}{4b}$$

$$K_2 \geq \frac{(a - w_2)^2}{4b}.$$

Substituting for K_1 into the objective function and setting a large enough value for K_2 , the supplier's optimization problem to implement action (3) reduces to:

$$\max_{w_1 < a-h} \frac{(a-w_1)^2}{4b} + \frac{(a-w_1-h)^2}{4b} + w_1 \frac{2(a-w_1)-h}{2b}.$$

The optimal solution is given by:

$$w_1 = 0; \quad K_1 = \frac{a^2}{4b} + \frac{(a-h)^2}{4b},$$

and the buyer chooses

$$Q_1 = \frac{2a-h}{2b}; \quad I = \frac{a-h}{2b}.$$

Since inventory is carried, first-best is not achieved. The supplier extracts all of the channel profits with $\Pi_S = K_1 = \frac{a^2}{4b} + \frac{(a-h)^2}{4b}$.

Now consider the situation when the supplier wishes to enforce buyer's action (4). Then he solves the following problem:

$$\max_{K_1, K_2, w_1, w_2} \quad K_1 + K_2 + w_1 \frac{a-w_1}{2b} + w_2 \frac{a-w_2}{2b} \quad (\text{EC.13a})$$

$$\text{s.t.} \quad \Pi_B^{(4)} \geq 0 \quad (\text{EC.13b})$$

$$\Pi_B^{(4)} \geq \Pi_B^{(1)} \quad (\text{EC.13c})$$

$$\Pi_B^{(4)} \geq \Pi_B^{(2)} \quad (\text{EC.13d})$$

$$\Pi_B^{(4)} \geq \Pi_B^{(3)} \quad \text{if } w_1 + h \leq a \quad (\text{EC.13e})$$

$$w_1 + h \geq w_2 \quad (\text{EC.13f})$$

$$w_1 \leq a, \quad w_2 \leq a \quad (\text{EC.13g})$$

Notice that,

$$\text{Constraint (EC.13c)} \Rightarrow K_1 \leq \frac{(a-w_1)^2}{4b}$$

$$\text{Constraint (EC.13d)} \Rightarrow K_2 \leq \frac{(a-w_2)^2}{4b}$$

Constraints (EC.13c) and (EC.13d) \Rightarrow Constraint (EC.13b)

$$\text{Constraint (EC.13e)} \Rightarrow K_2 \leq \frac{(a-w_2)^2}{4b} - \frac{(a-w_1-h)^2}{4b} \quad \text{if } w_1 + h \leq a$$

This implies that the supplier will choose the following fixed fees:

$$K_1 = \frac{(a-w_1)^2}{4b} \quad (\text{EC.14})$$

$$K_2 = \begin{cases} \frac{(a-w_2)^2}{4b} & \text{if } w_1 + h > a \\ \frac{(a-w_2)^2}{4b} - \frac{(a-w_1-h)^2}{4b} & \text{otherwise} \end{cases} \quad (\text{EC.15})$$

The supplier's optimization problem to implement action (4) is then given by the objective function (EC.13a) with constraints (EC.14)- (EC.15) and (EC.13f)- (EC.13g). Depending on the value of w_1 , the supplier solves two optimization problems by substituting the appropriate value of K_1 and K_2 into the objective function and then optimizing over w_1 and w_2 . The optimal solution for the two cases that arise is as follows:

- $w_1 + h > a$: The supplier chooses the following:

$$w_1 = a - h; \quad w_2 = 0; \quad K_1 = \frac{h^2}{4b}; \quad K_2 = \frac{a^2}{4b}.$$

The buyer's actions are:

$$Q_1 = \frac{h}{2b}; \quad Q_2 = \frac{a}{2b}; \quad I = 0.$$

The supplier's profits are: $\Pi_S = \frac{a^2}{4b} + \frac{ah}{2b} - \frac{h^2}{4b}$.

- $w_1 + h \leq a$: The supplier chooses the following:

$$w_1 = \frac{a-h}{2b}; \quad w_2 = 0; \quad K_1 = \frac{(a+h)^2}{16b}; \quad K_2 = \frac{a^2}{4b} - \frac{(a-h)^2}{16b}.$$

The buyer's actions are:

$$Q_1 = \frac{a+h}{4b}; \quad Q_2 = \frac{a}{2b}; \quad I = 0.$$

The supplier's profits are: $\Pi_S = \frac{3a^2+2ah-h^2}{8b}$.

Comparing the supplier's strategies under actions (3) and (4), we get that the supplier will prefer to implement action (3) when $h < (1 - 2/\sqrt{6})a$ and action (4) otherwise. The equilibrium solution is summarized in the following table.

	Supplier					Buyer			
	w_1	w_2	K_1	K_2	Π_S	Q_1	I	Q_2	Π_B
$h < (1 - 2/\sqrt{6})a$	0	N.A.	$\frac{a^2}{4b} + \frac{(a-h)^2}{4b}$	large	K_1	$\frac{2a-h}{2b}$	$\frac{a-h}{2b}$	0	0
$h \geq (1 - 2/\sqrt{6})a$	$\frac{a-h}{2}$	0	$\frac{(a+h)^2}{16b}$	$\frac{a^2}{4b} - \frac{(a-h)^2}{16b}$	$\frac{3a^2+3ah-h^2}{8b}$	$\frac{a+h}{4b}$	0	$\frac{a}{2b}$	$\frac{(a-h)^2}{16b}$

Proof of Proposition 3:

The buyer clearly is indifferent between the dynamic and commitment contracts for $h < (1 - 2/\sqrt{6})a$ and prefers the commitment contract for $h \geq (1 - 2/\sqrt{6})a$.

Similarly it is easy to see that the supplier prefers the dynamic contract for $h < (1 - 2/\sqrt{6})a$. For $h \geq (1 - 2/\sqrt{6})a$, it is straightforward to show that the supplier prefers the dynamic contract for $h < \frac{3-\sqrt{2}}{7}a$ and the commitment contract otherwise. Since we require that $h < a/4$, we have the following situation for the supplier: he prefers the dynamic 2-part tariff contract for $h < \frac{3-\sqrt{2}}{7}a$ and the commitment 2-part tariff contract for $h \in [\frac{3-\sqrt{2}}{7}a, \frac{a}{4}]$.

The channel profits under the dynamic contract are given by:

$$\Pi_C = \frac{a^2}{2b} - \frac{ah}{2b} + \frac{3h^2}{4b} \quad (\text{EC.16})$$

Similarly, the channel profits under the commitment contract are

$$\Pi_C = \Pi_B + \Pi_S = \begin{cases} \frac{a^2}{2b} - \frac{ah}{2b} + \frac{h^2}{4b} & h < (1 - 2/\sqrt{6})a \\ \frac{7a^2}{16b} + \frac{ah}{8b} - \frac{h^2}{16b} & h \geq (1 - 2/\sqrt{6})a \end{cases} \quad (\text{EC.17})$$

It is then straightforward to show that the channel profit is higher under dynamic 2-part tariff whenever $h < (1 - 2/\sqrt{6})a$ and under the commitment 2-part tariff whenever $h \geq (1 - 2/\sqrt{6})a$. ■

Proof of Proposition 4: Recall that the channel preferred dynamic contracts whenever $h < 55a/288$. In contrast, with two-part tariffs, the channel preferred the commitment contract for $h < (1 - 2/\sqrt{6})a$ and commitment contract otherwise (Proposition 3). However, comparing the linear and two-part tariff contracts for the channel, we conclude that the channel is always better off with the commitment 2-part tariff contract than with either linear contract. Therefore, the channel profits are maximized as per the results of the 2-part tariff contract - dynamic for $h < (1 - 2/\sqrt{6})a$ and commitment otherwise.

Similarly, we conclude that the buyer's profit is always maximized as per the linear contract with the buyer preferring the dynamic contract for $h < 21a/152$ and the commitment contract otherwise.

Finally, the supplier profits are maximized as per the 2-part tariff contracts. ■

Proof of Proposition 5:

Before proceeding with the proof of the Proposition, we show the following lemma which states that the optimal solution for at least one i-curve is an interior solution.

LEMMA EC.4. *There always exists at least one $q_i^* \in (k_{i-1}, k_i)$, for $i = 1, \dots, N$.*

Proof: Consider some $i \in [2, N - 1]$. If $q_i^* \in (k_{i-1}, k_i)$ we are done. Suppose $q_i^* \geq k_i$. Recall that $q_{i+1}^* > q_i^*$ which implies that $q_{i+1}^* > k_i$. Then either there exists some $j > i$ such that $q_j^* \in (k_{j-1}, k_j)$, in which case we have shown the lemma, or $q_j^* \geq k_j$ for all $j < N$. Now consider $j = N$. Since $q_{N-1}^* \geq k_{N-1}$, we must have that $q_N^* > k_{N-1}$. However, by definition, we have that $q_N^* < k_N$, which implies that $q_N^* \in (k_{N-1}, k_N)$ which proves the lemma.

Alternately, suppose that $q_i^* \leq k_{i-1}$. Observe then that $q_{i-1}^* < k_{i-1}$. Then either there exists some $j < i$ such that $q_j^* \in (k_{j-1}, k_j)$, in which case we have shown the lemma, or $q_j^* \leq k_{j-1}$ for all $j > 1$. Now consider $j = 2$. Since $q_2^* \leq k_1$, we have that $q_1^* < k_1$. However, by definition, we have that $q_1^* > k_0 = 0$, which implies that $q_1^* \in (k_0, k_1)$ which proves the lemma. ■

Proof of Proposition: For a given wholesale price w , the buyer's optimization problem can be determined by solving N (constrained) optimizations on the i-curve, for $i = 1, \dots, N$ and selecting the one that gives maximal profit. The buyer's (constrained) optimization problem for any i-curve can be stated as $\max_{k_{i-1} \leq q \leq k_i} (a_i - b_i q)q - wq$.

Let $\pi_B^i(q)$ be the buyer's profit when he chooses a quantity q on the i-curve. Let $\pi_S^i(w)$ be the supplier's profits when the wholesale price is w and the buyer buys a quantity by optimizing on the i-curve. Let w_i^* and q_i^* be the unconstrained equilibrium wholesale price and quantity for the i-curve. From linearity of the i-curve, we have that $w_i^* = \frac{a_i}{2}$ and $q_i^* = \frac{a_i}{4b_i}$.

We will prove the proposition by contradiction. Let the optimal price and quantity be w^* and

$q^*(w^*)$. Without loss of generality assume $q^*(w^*) \in (k_{i-1}, k_i]$. Now suppose that $w^* \neq \frac{a_i}{2}$. Since w_i^* is the equilibrium solution when buyer optimizes on the i -curve, we have that $\pi_S^i(w^*) < \pi_S^i(w_i^*)$. Now if $q_i^* \in (k_{i-1}, k_i)$, then $w^* \neq \frac{a_i}{2}$ cannot be the optimal price. Otherwise, either $q_i^* \geq k_i$ or $q_i^* \leq k_{i-1}$. We consider each of these separately.

- $q_i^* \geq k_i$: Since $k_N = \frac{a_N}{b_N}$ and $q_N^* < k_N$. We have a contradiction for $i = N$. Therefore, $i \leq N - 1$.

Then we must have that $q^*(w^*) = k_i$. Now we need to consider two cases.

Case (a): The buyer optimizes on the i -curve to choose k_i . Notice that $w = a_i - 2b_i k_i$ induces $q = k_i$. Since the supplier's profit function is concave on the i -curve, we must have that

$$\pi_S^i(w^*) \leq \pi_S^i(a_i - 2b_i k_i).$$

However, it is not clear that at this price ($w = a_i - 2b_i k_i$) the buyer will indeed choose $q = k_i$ if he were to optimize on the $(i+1)$ -curve given by $P^{i+1}(q) = a_{i+1} - b_{i+1}q$. To evaluate this recall that on the $(i+1)$ -curve, the buyer will choose a quantity $q = \frac{a_{i+1} - w}{2b_{i+1}}$ for a wholesale price w . For $w = a_i - 2b_i k_i$ and using the fact that $k_i = \frac{a_{i+1} - a_i}{b_{i+1} - b_i}$ and $b_i > b_{i+1}$, we see that

$$q = \frac{a_{i+1} - w}{2b_{i+1}} = \frac{(b_{i+1} + b_i)k_i}{2b_{i+1}} > k_i.$$

Thus,

$$\pi_B^i(k_i) < \pi_B^{i+1}\left(\frac{(b_{i+1} + b_i)k_i}{2b_{i+1}}\right)$$

and the buyer will rather optimize on the $(i+1)$ curve when $w = a_i - 2b_i k_i$. But what about the supplier? Since the buyer buys a larger quantity on the $(i+1)$ -curve, we must have that

$$\pi_S^i(a_i - 2b_i k_i) \leq \pi_S^{i+1}(a_i - 2b_i k_i).$$

But, appealing to concavity of the profit function, we have already established that

$$\pi_S^i(w^*) \leq \pi_S^i(a_i - 2b_i k_i).$$

Combining this with the optimality of $w_{i+1}^* = \frac{a_{i+1}}{2}$ for the $(i+1)$ -curve, we have that

$$\pi_S^i(w^*) \leq \pi_S^{i+1}(a_i - 2b_i k_i) \leq \pi_S^{i+1}(w_{i+1}^*).$$

So when $w = a_i - 2b_i k_i$, it is neither in the interest of the buyer nor the supplier to optimize on the i -curve; it is Pareto optimal to be on the $(i+1)$ -curve which, as shown, will lead to a quantity different from k_i . Then $q^*(w^*) \notin (k_{i-1}, k_i]$ which is a contradiction to the assumption. Notice that, in the process, we have eliminated the solution being at the kink k_i when optimizing on the i -curve.

Case (b): The buyer optimizes on the $(i+1)$ -curve to choose k_i ; observe that this will only happen if $q_{i+1}^* \leq k_i$. Now when the buyer optimizes on the $(i+1)$ curve, he chooses a quantity $q = \frac{a_{i+1}-w}{2b_{i+1}}$ which by assumption equals k_i . We will now show that, if the supplier sets w such that $q = k_i$ while the buyer optimizes on the $(i+1)$ -curve, at this w the buyer can do better by optimizing on the i -curve instead. That is,

$$\pi_B^i \left(\frac{a_i - w}{2b_i} \right) > \pi_B^{i+1}(k_i).$$

Observe that for a given w the buyer's optimal profit when optimizing on the i -curve

$$\pi_B^i \left(\frac{a_i - w}{2b_i} \right) = \frac{(a_i - w)^2}{4b_i}.$$

The buyer's profits from choosing k_i while optimizing on the $(i+1)$ -curve is given by

$$\begin{aligned} \pi_B^{i+1}(k_i) &= (a_{i+1} - b_{i+1}k_i)k_i - wk_i \\ &= \frac{a_i - a_{i+1}}{b_i - b_{i+1}} \left(\frac{a_{i+1}b_i - a_i b_{i+1}}{b_i - b_{i+1}} - w \right). \end{aligned}$$

But w is such that $\frac{a_{i+1}-w}{2b_{i+1}} = k_i$ which implies that $w = a_{i+1} - \left(\frac{a_i - a_{i+1}}{b_i - b_{i+1}} \right) 2b_{i+1}$. Substituting for w in the simplified expressions for the inequality $\pi_B^i \left(\frac{a_i - w}{2b_i} \right) > \pi_B^{i+1}(k_i)$, we see that the inequality is true as long as $(b_i - b_{i+1})^2 > 0$ which always holds. Thus, if the supplier chooses a wholesale price such that $q^*(w^*) = k_i$, the buyer is better off optimizing on the i -curve rather than on the $(i+1)$ -curve; notice that this implies the solution cannot be at the kink k_i . Either the unconstrained optimal quantity $q_i^* = \frac{a_i}{4b_i}$ is feasible which will prove the proposition or $q_i^* \leq k_{i-1}$ (for $i > 1$)¹⁹ which implies that $q^*(w^*) \notin (k_{i-1}, k_i]$ as assumed and hence a contradiction.

¹⁹ For $i = 1$, then $q_1^* > k_0$.

• $q_i^* \leq k_{i-1}$: For $i = 1$, we know that $q_1^* > k_0 = 0$, so we have a contradiction. So $i > 1$. From the concavity of the profit function, if the buyer optimizes on the i -curve, $q^*(w^*) = k_{i-1}$. But this means that $q^*(w^*) \notin (k_{i-1}, k_i]$ and we again have a contradiction.

From the lemma we know that there always exists at least one $q_i^* \in (k_{i-1}, k_i)$, for $i = 1, \dots, N$; suppose that the maximum $\pi_B^i(q_i^*)$ over all such feasible q_i^* is obtained on the k -curve. Then $q^*(w^*) = q_k^*$ and $w^* = \frac{a_k}{2}$. Notice that throughout we have ruled out the solution being at a kink. Each time we suppose that the solution is at a kink, k_i , the buyer and supplier find it better to move to either the $(i-1)$ or the $(i+1)$ -curve. ■

Proof of Proposition 6:

Suppose that the supplier offers $w_1 = \frac{a_i}{2}$ in the first period. We know from our previous analysis of the dynamic problem under linear demand that the buyer will purchase a quantity $q_1 = \frac{a_i}{4b_i} + I$, where $I > 0$, sell the quantity $\frac{a_i}{4b_i}$ this period and carry inventory I . Now suppose that the supplier offers $w_2 = \frac{a_i}{2}$ in the second period (This is feasible, but not optimal.). Then the buyer will purchase the quantity $q_2 = \frac{a_i}{4b_i} - I$, and sell $\frac{a_i}{4b_i}$ this period. The supplier's profits from this strategy are $\pi^S = \frac{a_i}{2} \cdot \left(\frac{a_i}{4b_i} + I\right) + \frac{a_i}{2} \cdot \left(\frac{a_i}{4b_i} - I\right) = 2 \cdot \frac{a_i^2}{8b_i}$. Thus the supplier can ensure a minimum profit of $2 \cdot \frac{a_i^2}{8b_i}$. ■

Proof of Theorem 3:

The proof is by contradiction. Suppose that there exists an equilibrium in which the buyer does not carry inventories across the first and second periods. Let the equilibrium wholesale prices be w_1^* and w_2^* , and the purchase quantities be q_1^* and q_2^* (Since inventories are not carried, purchase and sales quantities in each period are identical.). Since the second period begins without any inventory, the optimal wholesale price is $w_2^* = \frac{a_i}{2}$ for some $i \in \{1, 2, \dots, N\}$, with the corresponding induced purchase quantity being $q_2^* = \frac{a_i}{4b_i}$, by Proposition 5. The supplier makes a second-period profit of $\pi_2^S = \frac{a_i^2}{8b_i}$. Suppose that the supplier offers w_1 in the first period. In order to ensure that the buyer does not carry inventory into the second period, it must be the case that $w_1 \neq \frac{a_i}{2}$ (by Proposition 6). But by Proposition 5, $w_1 \neq \frac{a_i}{2}$ leads to supplier profits $\pi_1^S < \frac{a_i^2}{8b_i}$. Thus in the proposed equilibrium

without inventories, $\pi^S = \pi_1^S + \pi_2^S < 2 \cdot \frac{a_i^2}{8b_i}$. This cannot be an equilibrium, since the supplier can make profits of at least $2 \cdot \frac{a_i^2}{8b_i}$ by Proposition 6. Since we picked an arbitrary equilibrium under which the buyer does not carry inventories, no such equilibrium exists: the supplier can always do strictly better by inducing an equilibrium in which the buyer carries inventories. ■

Proof of Theorem 4:

We focus on the finite (n -period) horizon problem, with the per-period multiplicative discount factor of δ , where $0 < \delta \leq 1$. The proof extends in a straightforward way to the *discounted* infinite horizon ($0 < \delta < 1$).

We prove this result by contradiction, using an outcomes-based argument. Suppose that such a contract exists, and the supplier can make first-best profits which are $\frac{1-\delta^n}{1-\delta} \cdot R(q_{fb})$ for the discounted, finite horizon ($\delta < 1$), $n \cdot R(q_{fb})$ for the undiscounted, finite horizon, and $\frac{1}{1-\delta} \cdot R(q_{fb})$ for the discounted, infinite horizon. Then the *outcome* of the contract will need to satisfy the following conditions:

1. *Sales and Purchase Quantities, and Inventories:* The quantities sold by the buyer in the market in each period *must* be $q_1 = q_2 = \dots = q_n = q_{fb}$. These are the *unique* set of sales quantities that implement the first-best solution in each period, and generate per-period, channel-revenue maximizing sales of $R(q_{fb})$. Further, since inventories will lead to channel losses via holding costs, we must have $I_1 = I_2 = \dots = I_{n-1} = 0$, where I_j is the inventory carried by the buyer from period j to $j+1$. Thus, the buyer's *purchase* quantities in each period (induced by the dynamic contract) must equal the sales quantities for that period; i.e., $Q_1 = Q_2 = \dots = Q_n = q_{fb}$, ensuring that inventories are not carried.

2. *Transfer Payments from Buyer to Supplier:* The payment from buyer to supplier can be via fixed fees, unit prices, a combination of the two, or any other non-linear device. In making our argument here, we are only concerned with the *total* transfer payments. The buyer can and will choose not to participate in any period if he expects to make a loss by accepting the supplier's terms. This is because, under dynamic contracting, future periods are not contractible in the current

period, and past-period contracts have expired. To ensure the buyer's participation, his total per-period transfer payment to the supplier, say H_i in period i , must be bounded from above by his sales revenues of $R(q_{fb})$ per period. Thus the supplier's total profits are bounded from above by $\frac{1-\delta^n}{1-\delta} \cdot R(q_{fb})$, the maximum total payment he can get from the buyer. By assumption, the supplier's profits must attain this bound under the contract. Thus, the total payment to the supplier by the buyer, H_i , in each period must be exactly $R(q_{fb})$ – the single-period, first-best profit.

The *outcome* of the optimal dynamic contract that *both* implements the first-best solution *and* extracts away all of the buyer's residual profits over the n periods must be as follows: Total emoluments transferred from buyer to supplier are $H_1 = H_2 = \dots = H_n = R(q_{fb})$, purchase quantities are $Q_1 = Q_2 = \dots = Q_n = q_{fb}$, and Inventories are $I_1 = I_2 = \dots = I_{n-1} = 0$, where I_j is the inventory carried by the buyer from period j to $j + 1$.

Now we demonstrate that such an outcome cannot arise from any (sub-game perfect) equilibrium, since the buyer can do better by unilateral deviation. Suppose in the first period, the buyer has purchased the quantity $Q_1 = q_{fb}$, and paid a sum of $H_1 = R(q_{fb})$ to the supplier. If the buyer sells this entire quantity in the first period, he makes zero residual profits in the first period, and then, from the second period onwards, the supplier can implement the rest of the optimal contract. Knowing that this will be the outcome *if* he sells all his purchased quantities in the first period, the buyer can try to do better by selling some of his purchased quantity in the first period, and carrying the rest as inventory. Observe that, once the buyer has purchased the quantity q_{fb} in the first period, he is free to sell that quantity or carry it forward to future periods: The supplier has no credible enforcing mechanism to ensure sales of the entire purchased quantity within the buyer's period of purchase. Punishments via future contracts are neither credible (history-dependency in the finite horizon fails to meet the subgame-perfection criterion) nor feasible (since the buyer's residual profits in each period are already driven to his participation constraint). As an example, the buyer could sell whatever he carries forward from the first period in the second period. We analyze the result of such an unilateral deviation.²⁰ Under this strategy, the buyer's optimization

²⁰ Other deviations are possible, such as selling the purchased quantity gradually, over multiple periods. Some of these

is given by:

$$\max_{q_1} \Pi(q_1) = R(q_1) + \delta \cdot R(q_{fb} - q_1) - h(q_{fb} - q_1) - H_1,$$

subject to the constraint $0 \leq q_1 \leq q_{fb}$, where q_1 is the quantity sold in the first period, $(q_{fb} - q_1)$ is the inventory carried and sold in the second period, and $H_1 = R(q_{fb})$ is the first-period payment to the supplier.²¹ Observe that H_1 was paid by the buyer in the first period to procure the quantity q_{fb} , and is a sunk cost when the buyer decides on how much to sell. Setting aside the constraint on the range of q_1 for now, the First Order Conditions with respect to q_1 are:

$$\Gamma(q_1) = R'(q_1) - \delta \cdot R'(q_{fb} - q_1) + h = 0 \quad (\text{EC.18})$$

Since $R'(q_{fb}) = 0$ (which is the first-best solution in the static case) and $\delta \cdot R'(0) > h$ (by assumption), $\Gamma(0) > 0 > \Gamma(q_{fb})$. Thus there exists at least one $q_1 \in (0, q_{fb})$ such that $\Gamma(q_1) = 0$. Let q_1^* be the largest such q_1 .

Since $\Gamma(q_1^*) = 0$ and $\Gamma(q_{fb}) < 0$, it is clear that the buyer's profits at this maximizing solution are $\Pi(q_1^*) > \Pi(q_{fb}) = 0$ (the latter equality holds by construction of H_1). Hence the buyer will carry inventories (given by $I = q_{fb} - q_1^*$), and sell only a part of his first period purchase q_{fb} in the first period. Faced with the threat of residual profits of $\Pi(q_1^*)$ (with the buyer carrying inventories), the supplier can only implement those contracts in the second period that guarantee the buyer at least $\Pi(q_1^*) = R(q_1^*) + \delta \cdot R(q_{fb} - q_1^*) - h(q_{fb} - q_1^*) - H_1$ in profits.

Thus, *the contractual outcomes specified by conditions (1) and (2) above, are inconsistent with the requirement of subgame perfection. But we showed that conditions (1) and (2) must be satisfied by any contract that generates first-best profits to the supplier. Hence, by contradiction, no such contract that generates first-best profits to the supplier is feasible.* ■

Proof of Theorem 5:

may yield even higher residual profits to the buyer than the deviation we analyze. However, positive residual buyer profits under the simple deviation we consider will be sufficient to demonstrate that the posited first-best dynamic contract is an infeasible equilibrium under subgame-perfection.

²¹ We fix the strategies and outcomes from periods 3 to n as in the posited contract, so the buyer's profits are zero from period 3 onwards.

This is similar to the proof of Theorem 4 except that the per period transfer payment is of the form $H_i = K_i + w_i q_{fb}$. ■

Proof of Theorem 6: The simplest proof of this result is to construct a commitment contract that (i) ensures buyer-participation, (ii) implements the first-best solution and (iii) extracts away all of the buyer's residual profits. Variants of "selling the firm" to the buyer, at a fee equal to the total discounted first-best profits, will satisfy all three conditions.

We know that the first-best profits in each period are $R(q_{fb})$. Thus, the total discounted first-best profits over the n period horizon are $\sum_{i=1}^n \delta^{i-1} \cdot R(q_{fb})$, which simplifies to $\frac{1-\delta^n}{1-\delta} \cdot R(q_{fb})$ for $0 < \delta < 1$, and $n \cdot R(q_{fb})$ when $\delta = 1$. Over the infinite horizon, the total discounted first-best profits are $\sum_{i=1}^{\infty} \delta^{i-1} \cdot R(q_{fb})$, which simplifies to $\frac{R(q_{fb})}{1-\delta}$.

A commitment contract with upfront fees equal to the total discounted first-best profits over the horizon (as derived above) and marginal unit-cost pricing (i.e., providing any quantity the buyer desires at zero incremental cost), will accomplish the supplier's objectives. The buyer will optimize and buy the quantity q_{fb} every period, to make profits of $R(q_{fb})$ every period. The buyer's total optimal discounted profits over the horizon will be equal to the upfront fee paid to the supplier, and so his residual profits will be driven to zero. The supplier makes first-best channel profits. ■