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e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

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TECHNICAL APPENDIX

FOR "LEARNING, FORGETTING, AND SALES," BY SOFIA B. VILLAS-BOAS AND J. MIGUEL VILLAS-BOAS

PROOF OF PROPOSITION 1: Using (1) and (3) in (2) one obtains

$$\beta^T + \frac{\delta(1-\beta)}{1-(1-\alpha)\delta^T\beta^T} - \frac{(1-\delta)(\delta\beta-\delta^T\beta^T)}{(1-\delta^T)(1-\delta\beta)} = \frac{1-(1-\alpha)\beta^T}{1-(1-\alpha)\delta^T\beta^T}[1-\delta\beta+\delta^T\frac{1-\delta}{1-\delta^T}]. \qquad \text{(i)}$$

Making $\delta \to 1$, using $\lim_{\delta\to 1}\frac{1-\delta}{1-\delta^T} = \frac{1}{T}$, one then obtains

$$T(1-\beta)\beta^T[1+(1-\alpha)(1-\beta-\beta^T)] - (1-\beta^T)(1-(1-\alpha)\beta^T) = 0. \qquad \text{(ii)}$$

Noting that the left hand side is decreasing in T and α, and increasing in β for β large, one obtains that T is increasing in β and decreasing in α. Finally, the equivalent to (ii) with $\delta \neq 1$, is increasing in δ for δ close to one, obtaining that T is increasing in δ. The comparative statics for p can be similarly obtained using (1) and (ii).

PROOF OF PROPOSITION 2: Because the derivative of $V(x_h)$ with respect to k is increasing in T, and as for $k \to 0$ we have that $T \to 0$ as shown below, we have that T is increasing in k. Differentiating $V(x_h)$ with respect to T, multiplying by $(1-e^{-rT})$, and equalizing to zero, one obtains:

$$-re^{-rT}V(x_h) + \alpha u x_h z e^{-(1+r-\beta)zT} + \alpha u[1-\frac{\lambda(1-\alpha)}{1+\lambda-\beta}](e^{-rT} - ze^{-rzT})$$
$$-\alpha(1-\alpha)u(x_h e^{-(1-\beta)zT} - \frac{\lambda}{1+\lambda-\beta})(e^{-(1+r+\lambda-\beta)T} - ze^{-(1+r+\lambda-\beta)zT} + \frac{rk}{2}(ze^{-rzT} - e^{-rT}) = 0. \qquad \text{(iii)}$$

From this one can obtain that when $k \to 0$ we have $T \to 0$. One can then obtain,

$$\lim_{k\to 0} V(x_h) = \alpha u x_h z + \alpha u(1-z)[1-\frac{\lambda(1-\alpha)}{1+\lambda-\beta}] - (1-\alpha)\alpha u(x_h - \frac{\lambda}{1+\lambda-\beta})(1-z) \qquad \text{(iv)}$$

and $\lim_{k\to 0} x_h = \frac{\lambda(1-z)}{1+\lambda(1-z)-\beta}$. In order to find the optimal z for k close to zero, one can maximize

(iv) to obtain

$$z = \frac{1}{\lambda}(2-\beta-\sqrt{(1-\beta)^2+\frac{(1-\beta)(1+\lambda-\beta)}{1-\alpha}}). \tag{v}$$

Differentiating (v) with respect to β, α, and λ one obtains that $\frac{dz}{d\beta} > 0$, $\frac{dz}{d\alpha} < 0$, and $\frac{dz}{d\lambda} < 0$.

In order to consider the comparative statics with respect to T for k small, multiply (iii) by $(1-e^{-rT})$ and divide by T^2 to obtain

$$\begin{aligned}\lim_{k\to 0}\frac{2k}{z(1-z)\alpha u T^2} &= \lambda\frac{z(1+r-\beta)-r}{1+\lambda(1-z)-\beta}+r[1-\frac{\lambda(1-\alpha)}{1+\lambda-\beta}] \\ &+[(1+r+\lambda-\beta)(1+z)-r]\frac{\lambda(1-\alpha)(1-\beta)}{(1+\lambda-\beta)[1+\lambda(1-z)-\beta]}.\end{aligned} \tag{vi}$$

Differentiating (vi) for z in (v) close to one we can obtain $\frac{dT}{d\beta} > 0$, $\frac{dT}{d\alpha} < 0$, $\frac{dT}{d\lambda} < 0$, $\frac{dT}{du} < 0$, and $\frac{dT}{dr} < 0$.

PROOF OF PROPOSITION 3: Differentiating the duration of the sale $(1-z)T$ with respect to a variable y one obtains $\frac{d(1-z)T}{dy} = -T\frac{dz}{dy} + (1-z)\frac{dT}{dy}$. For k close to zero, we can obtain from (vi) $\frac{dT}{dy} = \frac{T}{2(1-z)}\frac{dz}{dy} - \frac{T^3(1-z)}{4k}\frac{d}{dy}(\frac{2k}{T^2(1-z)})$. This then yields that for k close to zero, and the optimal z close to one, the sign of $\frac{d(1-z)T}{dy}$ is the same as the sign of $-\frac{dz}{dy}$ for $y \in \{\beta, \lambda, \alpha\}$, and the same as the sign of $\frac{dT}{dy}$ for $y \in \{u, r\}$ (given the assumption of k converging to zero faster than z converging to one).