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Appendix EC.1: Online Appendix

Proof of Proposition 4.1:

(a) Let S_i^{*c} denote the largest maximizing set in $E^1 \cup E^2$ in (4) for μ_i^{*c}. Decompose $S_i^{*c} = S_i^1 \cup S_i^2$, $S_i^1 \subset E^1, S_i^2 \subset E^2$. Let $\alpha = \sum_{m \in S_i^1} \frac{\lambda_i^m}{(\nu^m)^2} / \sum_{m \in S_i^1 \cup S_i^2} \frac{\lambda_i^m}{(\nu^m)^2}$.

$$\mu_i^{*c} = \sum_{l \in S_i^1 \cup S_i^2} \frac{\lambda_i^l}{\nu^l} + \frac{1}{\alpha W_i(S_i^1) + (1-\alpha) W_i(S_i^2)} \leq \sum_{l \in S_i^1} \frac{\lambda_i^l}{\nu^l} + \frac{1}{W_i(S_i^1)} + \sum_{l \in S_i^2} \frac{\lambda_i^l}{\nu^l} + \frac{1}{W_i(S_i^2)} \leq \mu_i^{*1} + \mu_i^{*2}$$

where the equality follows from simple algebra, and the first inequality since $\frac{1}{\alpha W_i(S_i^1)+(1-\alpha)W_i(S_i^2)} \leq \frac{1}{W_i(S_i^1)} + \frac{1}{W_i(S_i^2)}$ as can be verified by multiplying both sides by $(\alpha W_i(S_i^1) + (1-\alpha) W_i(S_i^2))$. The last inequality follows directly from (4).

(b) Follows from part (a) by induction, choosing $E^1 = \{1, \ldots, J-1\}$, and $E^2 = \{J\}$.

(c) Monotonicity and joint convexity are easily verified: for given demand rates, the lower bound for any given set $S \subset E$, is jointly convex, as the composition of a jointly convex function and a linear function. Moreover, the maximum of $2^J - 1$ jointly convex functions is convex.

(d) Monotonicity is straightforward. Let $\widehat{\mu}_i^*$ denote the capacity required when increasing λ_i^l to $\widehat{\lambda}_i^l > \lambda_i^l$, leaving everything else unchanged. The demand rate $\widehat{\lambda}_i^l$ may be viewed as the aggregate arrival rate of *two* classes, class l with rate λ_i^l, and $\widehat{l}$ with $\widehat{\lambda}_i^l - \lambda_i^l$, both with waiting time standard $w_i^l = w_i^{\widehat{l}}$. As in Corollary 4.2(d) , let r_2 be the priority rule associated with the enlarged system and its capacity level $\widehat{\mu}_i^*$. If $\widehat{\mu}_i^* < \mu_i^*$, let $\widehat{r}$ denote the modification of this rule, obtained by giving class $\widehat{l}$ the lowest priority in any of the absolute priority rules over which r_2 randomizes. Clearly, the expected waiting time for all of the original classes $\{1, \ldots, J\}$ do not increase when switching from r_2 to $\widehat{r}$ and therefore are at or below the required standard $w_i^l, l \in E$. This contradicts $\mu_i^* > \widehat{\mu}_i^*$.

If class l is residual at firm i, the marginal capacity cost is clearly 0. Otherwise, the existence of $\frac{\partial \mu_i^*}{\partial \lambda_i^l}$ follows from the fact that $\mu_i^* = \sum_{l \in S_i^*} \frac{\lambda_i^l}{\nu^l} + \frac{1}{W_i(S_i^*)}$ for the *same* set S_i^* in neighborhood of the demand volumes of λ_i^l. The expression for $\frac{\partial \mu_i^*}{\partial \lambda_i^l}$ follows from simple calculus; the conditions about the marginal capacity value being larger or smaller than the expected amount of work, a customer of class l is adding to the system are immediate from the sign of the second term to the right of (9).

(e) Since the same bottleneck set S_i^* prevails for all values $\{\lambda_i^l\}$, if class l is residual for some demand volume $(\lambda_i^l)^0$, it is residual for all possible values, and the capacity μ_i^* is invariant with respect to λ_i^l. Otherwise, the assumption ensures that for the same set S_i^*, $\mu_i^* = \sum_{l \in S_i^*} \frac{\lambda_i^l}{\nu^l} + \frac{1}{W_i(S_i^*)}$ for *all* values of λ_i^l. Differentiating (7) with respect to λ_i^l, we obtain

$$\frac{\partial^2 \mu_i^*}{\partial (\lambda_i^l)^2} = \frac{2w_i^l (b_i^l)^2}{(\nu^l)^2 \left(\widetilde{\lambda_S^w}\right)^3} \left(\sum_{m \in S} \frac{\lambda_i^m w_i^m \nu^m}{(\nu^m)^2 \nu^l} - w_i^l \nu^l \sum_{m \in S} \frac{\lambda_i^m}{(\nu^m)^2 \nu^l} \right) \tag{EC.1}$$

The concavity and convexity properties follow readily.

■

Proof of Corollary 4.2: For $\mu_i^0 = \mu_i^*$, define $\overline{\mathcal{W}}_i$ and $\mathcal{W}_i$ as in the proof of Lemma 4.1, and the set function $b_i(\cdot)$ as in (2). Part (a) is immediate from (4). Part (b): Assume the maximum in (4) is achieved for two sets S,T. Note that $b_i(S \cup T) \le \sum_{l \in S \cup T} \rho_i^l w_i^l = \sum_{l \in S} \rho_i^l w_i^l + \sum_{l \in T} \rho_i^l w_i^l - \sum_{l \in S \cap T} \rho_i^l w_i^l \le b_i(S) + b_i(T) - b_i(S \cap T) \le b_i(S \cup T)$, where the first two inequalities follow from $w_i \in \mathcal{W}_i$ and the last inequality from the supermodularity of the b_i function. Thus $\sum_{l \in S \cup T} \rho_i^l w_i^l = b_i(S \cup T)$, i.e., the maximum in (4) is achieved for $S \cup T$. Part (c): Since $w = \{w_i^l, l \in E\} \in \overline{\mathcal{W}}_i$, the claim follows, as shown in the proof of Lemma 4.1. Part(d): $w \notin \overline{\mathcal{W}}_i$, but the proof of Lemma 4.1 shows that a vector $x \ge 0$ exists such that $w' = w - x \in \overline{\mathcal{W}}_i$. $x^l = 0$ for $l \in S^*$, since $b_i(S_i^*) = \sum_{l \in S_i^*} \rho_i^l w_i^l \ge \sum_{l \in S_i^*} \rho_i^l (w_i^l - x^l) \ge b_i(S_i^*)$, where the first equality follows from the fact that μ_i^* is achieved at S^*, and the last inequality from $w \in \mathcal{W}_i$. The optimality of rules r_1 and r_2 follows again from the proof of Lemma 4.1 and the fact that S and T achieve the maximum. Since, by Carathéodory's Theorem (see e.g. Bazaraa and Shetty (1979)), each point is a J-dimensional polyhedron can be written as a convex combination of no more than $J+1$ extreme points, at most $J+1$ absolute priority rules needed to be randomized.

■

Proof of Proposition 5.1:

(a) Let A^l denote the $N \times N$ matrix with $A_{ii}^l = 2b_i^l$ and $A_{ij}^l = -\beta_{ij}^l, i \ne j$. By (D) it is easily verified that A^l is invertible with $(A^l)^{-1} > 0$ (see e.g. Bernstein and Federgruen (2002)). Let κ^l and κ^{Dl} be the

N-vectors with $\kappa_i^{Dl} = a_i^l(w_i^l) - \sum_{j\neq i} \alpha_{ij}^l(w_j^l) + b_i^l\left(c_i^l + \frac{\gamma_i}{\nu^l}\right)$ and $\kappa_i^l = \kappa_i^{Dl} + \frac{\gamma_i b_i^l}{\nu^l} \frac{\sum_{m\in S_i^*} \frac{\lambda_i^m w_i^m}{\nu^m \nu^l} - w_i^l \widetilde{\lambda_s}}{\left(\sum_{m\in S_i^*} \frac{\lambda_i^m w_i^m}{\nu^m}\right)^2}$. p^* satisfies (17), which in matrix form, by (6) can be written as $A^l(p_1^l, \ldots, p_N^l)^T = \kappa^l$. Applying Theorem 5.1 to the setting in which dedicated service is provided at all firms it follows that the equilibrium p^D satisfies (17) with the second term to the left replaced by 0. Thus, $(A^l)(p_1^{Dl}, \ldots, p_N^{Dl})^T = \kappa^{Dl}$ and $(p_1^{*l}, \ldots, p_N^{*l}) = (A^l)^{-1}\kappa^l \geq (A^l)^{-1}\kappa^{Dl} = (p_1^{Dl}, \ldots, p_N^{Dl})$, where the inequality follows $(A^l)^{-1} \geq 0$ and $\kappa^l \geq \kappa^{Dl}$ since $w_i^l \nu^l \geq W_i(S_i^*), \forall i = 1, \ldots, N$.

(b) Analogous to the proof of part(a) except that $\kappa^l < \kappa^{Dl}$.

(c) Analogous to the proof of part (a) replacing κ^l by $\widehat{\kappa}^l$ where $\widehat{\kappa}_1^l = \kappa_1^l$, and $\widehat{\kappa}_i^l = \kappa_i^{Dl}, \forall i = 2, \ldots, N$

Proof of Theorem 5.2: With a fixed price vector p, the profit function π_i can be written as $\pi_i(p, w) = \min_{S\subset E} \pi_i^S(p, w)$ where

$$\pi_i^S(p, w) = \sum_{m\in E} (p_i^m - c_i^m)\lambda_i^m(p^m, w^m) - \gamma_i \left(\sum_{m\in S} \frac{\lambda_i^m(p^m, w^m)}{\nu^m} + \frac{\sum_{m\in S} \frac{\lambda_i^m(p^m, w^m)}{(\nu^m)^2}}{\sum_{m\in S} \frac{\lambda_i^m(p^m, w^m)}{\nu^m} w_i^m} \right)$$

As in the proof of Theorem 5.1, it suffices to show that each of the functions π_i^S is jointly concave in $(w_i^1, \ldots, w_i^J)$, so that π_i, as the minimum of these functions, is jointly concave. We, again, show concavity of π_i^S by verifying that its Hessian has negative diagonal elements and is diagonally dominant. To that end, let $\widetilde{\lambda_S^W} = \sum_{m\in S} \frac{\lambda_i^m w_i^m}{\nu^m}$. If $l \notin S$, $\frac{\partial \pi_i^S}{\partial w_i^l} = a_i^{l\prime}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right)$ and thus for all $k \in E$, $\frac{\partial^2 \pi_i^S}{\partial w_i^l \partial w_i^k} = 0$, and $\frac{\partial^2 \pi_i^S}{\partial (w_i^l)^2} = a_i^{l\prime\prime}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) \leq 0$. If $l \in S$, we obtain after some algebra,

$$\frac{\partial \pi_i^S}{\partial w_i^l} = a_i^{l\prime}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) - \gamma_i a_i^{l\prime} \frac{\frac{1}{(\nu^l)^2}\sum_{m\in S} \frac{\lambda_i^m w_i^m}{\nu^m} - \frac{w_i^l}{\nu^l}\widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^2} + \gamma_i \frac{\frac{\lambda_i^l}{\nu^l}\widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^2} \tag{EC.2}$$

$$= a_i^{l\prime}\left(p_i^l - c_i^l - \gamma_i \frac{\partial \mu_i^*}{\partial \lambda_i^l}\right) + \gamma_i \nu^l \frac{\lambda_i^l\,(\nu^l)^2}{\sum_{m\in S} \lambda_i^m/(\nu^m)^2} \frac{1}{W_i^2(S)}$$

$$\frac{\partial^2 \pi_i^S}{\partial (w_i^l)^2} = a_i^{l\prime\prime}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) - \frac{\gamma_i a_i^{l\prime\prime}}{\nu^l} \frac{\sum_{m\in S} \frac{\lambda_i^m}{\nu^m}\left(\frac{w_i^m}{\nu^l} - \frac{w_i^l}{\nu^m}\right)}{\left(\widetilde{\lambda_S^w}\right)^2} \tag{EC.3}$$

$$+ 2\gamma_i \left(a_i^{l'}\right)^2 \frac{w_i^l}{(\nu^l)^2} \frac{\sum_{m\in S} \frac{\lambda_i^m}{\nu^m}\left(\frac{w_i^m}{\nu^l} - \frac{w_i^l}{\nu^m}\right)}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$- \gamma_i a_i^{l'} \frac{\left(\sum_{m\in S} \frac{\lambda_i^m w_i^m}{\nu^m}\right)\left(\frac{1}{(\nu^l)^2}\frac{\lambda_i^l}{\nu^l} - \frac{1}{\nu^l}\widetilde{\lambda}_s\right) - 2\frac{\lambda_i^l}{(\nu^l)^2}\sum_{m\in S}\frac{\lambda_i^m}{\nu^m}\left(\frac{w_i^m}{\nu^l} - \frac{w_i^l}{\nu^m}\right)}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$+ \gamma_i a_i^{l'} \frac{\left(\sum_{m\in S} \frac{\lambda_i^m w_i^m}{\nu^m}\right)\left(\frac{1}{(\nu^l)^2}\frac{\lambda_i^l}{\nu^l} + \frac{1}{\nu^l}\widetilde{\lambda}_s\right) - 2\frac{\lambda_i^l w_i^l}{(\nu^l)^2}\widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$- 2\gamma_i \frac{\left(\frac{\lambda_i^l}{\nu^l}\right)^2 \widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^3}$$

Note that the first and last elements in (EC.3) are negative, while all other terms vanish as the demand rates increase. Thus, $\frac{\partial^2 \Pi_i^S}{(\partial w_i^l)^2} < 0$ when the demand volumes $\{\lambda_i^m\}$ are sufficiently large.

$$\frac{\partial^2 \pi_i^S}{\partial w_i^l \partial w_i^k} = -\gamma_i \frac{\left(a_i^{l'}\right)\left(a_i^{k'}\right)}{\nu^l \nu^k} \frac{\left(\frac{w_i^k}{\nu^l} - \frac{w_i^l}{\nu^k}\right)\left(\widetilde{\lambda_S^w}\right) - 2w_i^k \sum_{m\in S}\frac{\lambda_i^m}{\nu^m}\left(\frac{w_i^m}{\nu^l} - \frac{w_i^l}{\nu^m}\right)}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$- \gamma_i \frac{\left(a_i^{l'}\right)}{\nu^l \nu^k} \frac{\frac{\lambda_i^k}{\nu^l}\left(\widetilde{\lambda_S^w}\right) - 2\lambda_i^k \sum_{m\in S}\frac{\lambda_i^m}{\nu^m}\left(\frac{w_i^m}{\nu^l} - \frac{w_i^l}{\nu^m}\right)}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$+ \gamma_i \left(a_i^{l'}\right) \frac{\frac{\lambda_i^l}{\nu^l(\nu^k)^2}\left(\widetilde{\lambda_S^w}\right) - 2\frac{w_i^k \lambda_i^l}{\nu^k \nu^l}\sum_{m\in S}\frac{\lambda_i}{(\nu^m)^2}}{\left(\widetilde{\lambda_S^w}\right)^3}$$

$$- 2\gamma_i \frac{\left(\frac{\lambda_i^l}{\nu^l}\right)\left(\frac{\lambda_i^k}{\nu^k}\right)\widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^3}.$$

Thus,

$$-\left|\frac{\partial^2 \pi_i^S}{\partial (w_i^l)^2}\right| + \sum_{k\neq l}\left|\frac{\partial^2 \pi_i^S}{\partial w_i^l \partial w_i^k}\right| < a_i^{l''}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) + 2\gamma_i \frac{\frac{\lambda_i^l}{\nu^l}\left(\sum_{m\in S}\frac{\lambda_i^m}{\nu^m} - 2\frac{\lambda_i^l}{\nu^l}\right)\widetilde{\lambda}_s}{\left(\widetilde{\lambda_S^w}\right)^3} + o(\lambda). \qquad \text{(EC.4)}$$

If $\frac{\lambda_i^l}{\nu^l} \geq \frac{1}{2}\sum_{m\in S}\frac{\lambda_i^m}{\nu^m}$, this expression is strictly negative for sufficiently large λ. If $\frac{\lambda_i^l}{\nu^l} \leq \frac{1}{2}\sum_{m\in S}\frac{\lambda_i^m}{\nu^m}$, the right hand side of (EC.4) can be bounded by $a_i^{l''}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) + \frac{2\gamma_i}{(\underline{w}_i)^2(\underline{w\nu})_i}\frac{\frac{\lambda_i^l}{\nu^l}\left(\Lambda - 2\frac{\lambda_i^l}{\nu^l}\right)}{\Lambda^2} + o(\lambda)$, where $\Lambda = \sum_{m\in S}\frac{\lambda_i^m}{\nu^m}$. For fixed Λ this expression is bounded from above by $a_i^{l''}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) + \frac{\gamma_i}{4(\underline{w}_i)^3\underline{\nu}_i} + o(\lambda)$ which is strictly negative in view of the lower bound for $\underline{w}_i$ ■

Proof of Proposition 5.2: Assume to the contrary that for some $i = 1, \ldots, N$, $S_i^* \subsetneq E$ is the bottleneck set of customer classes. Let $l \notin S_i^*$. By (3) and the fact that w^* is an interior point of the feasible region, it is possible to reduce w_i^* without incurring any additional capacity costs, while increasing the firm's variable profits as given by the first term in (7). This contradicts the fact that w^* is a Nash equilibrium. The conclusion regarding the firms' priority rules is immediate from Corollary 4.2(c). ■

Proof of Proposition 5.3:

(a) Since w_i^* is an interior point of the feasible waiting time region, it follows from Proposition 5.2 that $\pi_i(w^*) = \pi_i^E(w^*)$, and $0 = \frac{\partial \pi_i^*}{\partial w_i^l} = \frac{\partial \pi_i^E}{\partial w_i^l}$. Using (EC.2) and adding $\frac{\gamma_i}{(w_i^l)^2}$ to both sides of the equation, we obtain after some algebra that

$$a_i^{l'}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) + \frac{\gamma_i}{(w_i^l)^2} = \gamma_i a_i^{l'} \frac{\frac{1}{(\nu^l)^2}\widetilde{\lambda_E^w} - \frac{w_i^l}{\nu^l}\widetilde{\lambda}}{\left(\widetilde{\lambda_E^w}\right)^2} + \left[\frac{\gamma_i}{(w_i^l)^2} - \gamma_i \frac{\frac{\lambda_i^l}{\nu^l}\widetilde{\lambda}}{\left(\widetilde{\lambda_E^w}\right)^2}\right] \tag{EC.5}$$

where $\widetilde{\lambda_S^w} = \sum_{m \in S} \frac{\lambda_i^m w_i^m}{\nu^m}$. Since $a_i^l(\cdot)$ is decreasing, and $\frac{w_i^{*l}\nu^l}{W_i(E)} \leq 1$, the first term to the right of (EC.5) is negative. The lower bound for $\nu^l w_i^{*l}$ is equivalent to $\frac{1}{(w_i^l)^2} = \frac{(\nu^l)^2}{(w_i^l \nu^l)^2} \leq \frac{\frac{\lambda_i^l}{\nu^l}}{\widetilde{\lambda}} \frac{1}{(W_i(E))^2} = \frac{\frac{\lambda_i^l}{\nu^l}}{\widetilde{\lambda}} \left(\frac{\widetilde{\lambda}}{\widetilde{\lambda_E^w}}\right)^2 = \frac{\frac{\lambda_i^l}{\nu^l}\widetilde{\lambda_s}}{(\widetilde{\lambda_S^w})^2}$, which in turn is equivalent to the second term to the right of (EC.5) being negative as well. We conclude that w_i^{*l} satisfies the equation

$$a_i^{l'}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) = R - \frac{\gamma_i}{(w_i^l)^2} \tag{EC.6}$$

where $R \leq 0$, while class l's equilibrium waiting time under dedicated service is easily verified to satisfy $a_i^{l'}\left(p_i^l - c_i^l - \frac{\gamma_i}{\nu^l}\right) = -\frac{\gamma_i}{(w_i^l)^2}$. The solution of (EC.6) is the (at most unique) intersection of a decreasing and an increasing function, and it is decreasing in R; thus, $w_i^{*l} \leq w_i^{Dl}$.

(b) The proof of part (b) is analogous.

(c) Immediate from the fact that the system of equations (EC.6) decomposes on a firm by firm basis.

■

Proof of Theorem 5.3:

(a) As in the proof of Theorem 5.2, one verifies that, for all $S \subseteq E$ π_i^S is jointly concave in $(p_i^1, \ldots, p_i^J)$

and $(w_i^1,\ldots,w_i^J)$, by verifying that the Hessian is dominant diagonal. The analysis is analogous to that of Theorem 5.2, noting that $\frac{\partial^2\pi_i^S}{\partial p_i^l\partial w_i^k}=o(\lambda)$ for $k\neq l$, while $\frac{\partial^2\pi_i^S}{\partial p_i^l\partial w_i^l}=\frac{da_i^l(w_i^l)}{dw_i^l}+o(\lambda)$.

(b) Analogous to the proof of Proposition 5.2.

■

Proof of Theorem 6.1:

The proof proceeds in close similarity to that of Theorem 5.1. Once again, it suffices to show that each of the functions $\pi_i^S(p)$, given by (12) is jointly concave in the vector $\{p_i^1,\ldots,p_i^J\}$, as long as all $\{\lambda_i^m\}$ are in excess of certain minimal threshold values $\{\underline{\lambda}_i^m\}$.

$$\frac{\partial\pi_i^S}{\partial p_i^l}=\lambda_i^l-b_i^l(p_i^l-c_i^l)+\sum_{m\neq l}\varphi_{ii}^{ml}-\gamma_i\sum_{m\in S}\frac{\lambda_i^m}{\partial p_i^l}\frac{1}{\nu^m}\left\{\frac{\sum_{n\in S}\frac{\lambda_i^n w_i^n}{\nu^n\nu^m}-w_i^m\sum_{n\in S}\frac{\lambda_i^m}{\nu^n}}{\left(\widetilde{\lambda_s^w}\right)^2}+1\right\}$$

where $\widetilde{\lambda_s^w}=\sum_{n\in S}\frac{\lambda_i^n w_i^n}{\nu^n}$. Thus,

$$\frac{\partial\pi_i^S}{(\partial p_i^l)^2}=2bi^l+\delta_{ii}^{ll}\qquad\frac{\partial\pi_i^S}{\partial p_i^l\partial p_i^k}=\varphi_{ii}^{lk}+\delta_{ii}^{lk}$$

, where

$$\delta_{ii}^{ll}=\gamma_i\frac{\partial\left\{\sum_{m\in S}\frac{\lambda_i^m}{\partial p_i^l}\frac{1}{\nu^m}\left[\frac{\sum_{n\in S}\frac{\lambda_i^n w_i^n}{\nu^n\nu^m}-w_i^m\sum_{n\in S}\frac{\lambda_i^m}{\nu^n}}{\left(\widetilde{\lambda_S^w}\right)^2}\right]\right\}}{\partial p_i^l}\quad\delta_{ii}^{lk}=\gamma_i\frac{\partial\left\{\sum_{m\in S}\frac{\lambda_i^m}{\partial p_i^l}\frac{1}{\nu^m}\left[\frac{\sum_{n\in S}\frac{\lambda_i^n w_i^n}{\nu^n\nu^m}-w_i^m\sum_{n\in S}\frac{\lambda_i^m}{\nu^n}}{\left(\widetilde{\lambda_S^w}\right)^2}\right]\right\}}{\partial p_i^k}$$

As in the proof of Theorem 5.1, it is possible to show that for any $\epsilon\leq 0$, $|\delta_{ii}^{ll}|\leq\epsilon$ and $|\delta_{ii}^{lk}|\leq\epsilon$ as long as the minimal demand volumes $\{\underline{\lambda}_i^m\}$ are sufficiently large. Invoking condition (D^q) this implies that the Hessian of the function π_i^S has a negative diagonal and it is diagonally dominant, ensuring that it is semi-negative definite. This completes the proof that the function π_i^S is jointly concave. ■