MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1080.0979ec
pp. ec1–ec4

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Electronic Companion—"When Is Price Discrimination Profitable?" by Eric T. Anderson and James D. Dana, Jr., *Management Science*, DOI 10.1287/mnsc.1080.0979.

Technical Appendix to
"When is Price Discrimination Profitable?"

November 21, 2008

Abstract

This technical appendix to our paper, "When is Price Discrimination Profitable?," provides a proof of Lemma 2. However the proof is quite standard. This appendix is included for the benefit of readers who might not be familiar with it. The proof follows almost identically the proof of Hermalin (2006). The main difference is that Hermalin (2006) does not consider a quality constraint, however Hermalin's proof is unaffected by the addition of this constraint.

Proof of Lemma 2

We will first argue that any pricing scheme $p(\theta), q(\theta)$ that satisfies the incentive compatibility and participation constraints must have the property that $q(\theta)$ is non-decreasing and must imply that

$$v(\theta) = \int_{\theta_L}^{\theta_1} V_\theta(q(t), t)dt. \tag{T-1}$$

Note that it must be true that $v(\theta_L) = 0$ since otherwise the incentive constraint would be violated. That is, some consumers for whom $\theta < \theta_L$ would prefer to announce they were type θ_L.

Consider two arbitrary consumer types, θ_a and θ_b. One must be larger than the other, so there is no loss of generality in assuming it is θ_b. Incentive compatibility requires that

$$v(\theta_b) \geq V(q(\theta_a), \theta_b) - p(\theta_b) \tag{T-2}$$

and

$$v(\theta_a) \geq V(q(\theta_b), \theta_a) - p(\theta_a). \tag{T-3}$$

Using $v(\theta)$ from the participation constraint to substitute out for $p(\theta_a)$ and $p(\theta_b)$, we can these as

$$v(\theta_b) \geq v(\theta_a) + V(q(\theta_a), \theta_b) - V(q(\theta_a), \theta_a) \tag{T-4}$$

and

$$v(\theta_a) \geq v(\theta_b) + V(q(\theta_b), \theta_a) - V(q(\theta_b), \theta_b), \tag{T-5}$$

or

$$v(\theta_b) \geq v(\theta_a) + \int_{\theta_a}^{\theta_b} V_\theta(q(\theta_a), t)dt \tag{T-6}$$

and

$$v(\theta_a) \geq v(\theta_b) + \int_{\theta_b}^{\theta_a} V_\theta(q(\theta_b), t)dt. \tag{T-7}$$

These imply

$$\int_{\theta_a}^{\theta_b} V_\theta(q(\theta_b), t)dt \geq v(\theta_b) - v(\theta_a) \geq \int_{\theta_a}^{\theta_b} V_\theta(q(\theta_a), t)dt \tag{T-8}$$

and ignoring the middle term we this implies

$$\int_{\theta_b}^{\theta_a} V_\theta(q(\theta_b), t) - V_\theta(q(\theta_a), t)dt \geq 0 \tag{T-9}$$

and using the fundamental theorem of calculus

$$\int_{\theta_b}^{\theta_a} \int_{q(\theta_a)}^{q(\theta_b)} V_{q\theta}(z, t)dzdt \geq 0 \tag{T-10}$$

which implies that $q(\theta)$ is non-decreasing (that is, $q(\theta_b) \geq q(\theta_a)$ for all θ_a and θ_b such that $\theta_b \geq \theta_a$).

Equation (T-8) also implies that for all $\epsilon > 0$,

$$\frac{1}{\epsilon} \int_{\theta+\epsilon}^{\theta} V_\theta(q(\theta+\epsilon), t)dt \geq \frac{v(\theta+\epsilon) - v(\theta)}{\epsilon} \geq \frac{1}{\epsilon} \int_{\theta}^{\theta+\epsilon} V_\theta(q(\theta), t)dt. \tag{T-11}$$

Taking the limit as $\epsilon \to 0$ yields

$$V_\theta(q(\theta), \theta) \geq v'(\theta) \geq V_\theta(q(\theta), \theta) \tag{T-12}$$

(see Hermalin, 2006 for additional discussion on taking this limit) or

$$v'(\theta) = V_\theta(q(\theta), \theta) \tag{T-13}$$

almost everywhere. Finally, since $v(\theta_L) = 0$,

$$v(\theta) = \int_{\theta_L}^{\theta} V_\theta(q(t), t)dt. \tag{T-14}$$

So incentive compatibility and participation constraints imply (T-1) and that $q(\theta)$ is non-decreasing.

We now show that any pricing scheme $p(\theta), q(\theta)$ that satisfies (T-1), and has the property that $q(\theta)$ is everywhere non-decreasing, must satisfy the incentive compatibility and participation constraints.

Clearly (T-1) implies that the participation constraint is satisfied. Incentive compatibility (see T-4 and T-5) requires that

$$V(q(\theta_b), \theta_b) - V(q(\theta_b), \theta_a) \geq v(\theta_b) - v(\theta_a) \tag{T-15}$$

for all θ_a and θ_b. First suppose $\theta_b > \theta_a$, so that $q(\theta_b) \geq q(t)$ for all $t \in [\theta_a, \theta_b)$. Using (T-1) and the fundamental theorem of calculus, (T-15) can be written

$$\int_{\theta_a}^{\theta_b} V_\theta(q(\theta_b), t)dt \geq \int_{\theta_L}^{\theta_b} V_\theta(q(t), t)dt - \int_{\theta_L}^{\theta_a} V_\theta(q(t), t)dt \tag{T-16}$$

where the right hand side equals

$$\int_{\theta_a}^{\theta_b} V_\theta(q(t), t)dt \tag{T-17}$$

so (T-15) becomes

$$\int_{\theta_a}^{\theta_b} V_\theta(q(\theta_b), t) - V_\theta(q(t), t)dt \geq 0 \tag{T-18}$$

which must hold because $q(\theta_b) \geq q(t)$ for all $t \in [\theta_a, \theta_b)$ and $V_{q\theta} > 0$ implies $V_\theta(q(\theta_b), t) - V_\theta(q(t), t) \geq 0$.

Now suppose $\theta_b \leq \theta_a$. Then (T-15) can be rewritten as

$$\int_{\theta_b}^{\theta_a} V_\theta(q(\theta_b), t) - V_\theta(q(t), t)dt \leq 0 \tag{T-19}$$

which holds because now $q(\theta_a) \geq q(t)$ for all $t \in [\theta_b, \theta_a)$ and $V_{q\theta} > 0$.

Therefore any pricing scheme $p(\theta), q(\theta)$ that satisfies (T-1) and has the property that $q(\theta)$ is everywhere non-decreasing must also satisfy the incentive compatibility and participation constraints.

Taken together, we now have that any pricing scheme $p(\theta), q(\theta)$ satisfies (T-1) and has the property that $q(\theta)$ is everywhere non-decreasing if and only if it satisfies the incentive compatibility and participation constraints.

So substituting $v(\theta)$ from the participation constrain into the objective function, (4), we can write the firm's problem as

$$\max_{\theta_L, q(\theta), v(\theta)} \int_{\theta_L}^{\theta_1} \left[V(q(\theta), \theta) - c(q(\theta)) - v(\theta)\right] dF(\theta) \tag{T-20}$$

subject to (T-1), $q(\theta)$ being non-decreasing, and $q(\theta) < 1$. Finally, substituting for $v(\theta)$ using (T-1) and applying intergration by parts, we can write the firm's problem as

$$\max_{\theta_L, q(\theta)} \int_{\theta_L}^{\theta_1} [V(q(\theta), \theta) - c(q(\theta)) - J(\theta) V_\theta(q(\theta), \theta)] \, dF(\theta) \tag{T-21}$$

subject to $q(\theta)$ being non-decreasing and $q(\theta) < 1$.

References

[1] Hermalin, Benjamin. 2006. "Second Degree Price Discrimination with a Continuum of Types," lecture notes, http://www.faculty.haas.berkeley.edu/hermalin/.