MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1090.1032ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Product Differentiation, Store Differentiation, and Assortment Depth" by Stephen F. Hamilton and Timothy J. Richards, *Management Science*, DOI 10.1287/mnsc.1090.1032.

1 Special Case: Logarithmic Subutility

Consider n retailers equally-spaced around the circumfrance of a circle of unit length. Consumers are distributed uniformly about the circle and incur unit transportation costs in traveling to retailers to purchase goods. The products sold at retailers are symmetric in the sense of Spence-Dixit-Stiglitz and consumers purchase multiple products on each shopping occasion.

1.1 The Consumer's Problem

Consumers compare utility at each retailer with no uncertainty over available products and prices and decide where to shop subject to transaction cost of $\$t$ per unit of distance. The aggregate utility function for each consumer is separable between the retail products, $x \in (0, m]$, and a numeraire good, x_0, and takes the logarithmic form:

$$U(x, v, x_0) = \ln\left(\int_{i\in v} x_i^\theta di\right) + x_0, \tag{1}$$

with $0 < \theta \leq 1$. The logarithmic form of the subutility function in (1) facilitates closed-form solutions and is consistent with the nested logit demand specification of Anderson and de Palma (1992).

For each product, inverse demand is given by

$$p_i = \frac{\theta x_i^{\theta-1}}{\int_{i\in v} x_i^\theta}, \tag{2}$$

where p_i is the retail price of product i. Equations (2) can be inverted to recover the demand functions,

$$x_i(p, v) = \frac{\theta p_i^{\frac{-1}{1-\theta}}}{\int_{i\in v} p_i^{\frac{-\theta}{1-\theta}}}. \tag{3}$$

Letting y denote income of the representative consumer, indirect utility is

$$u(p, v, y) = \ln\left[\theta^\theta \left(\int_{i\in v} p_i^{\frac{-\theta}{1-\theta}}\right)^{1-\theta}\right] - \theta + y. \tag{4}$$

The aggregate demand facing each multi-product firm depends on the decision made by the representative consumer at each point on the line segment regarding where to shop. The location of the consumer who is indifferent between purchasing from either of the two retailers (δ^*) is defined implicitly by

$$u(p, v, y) - \delta t = v(\overline{p}, \overline{v}, y) - t\,(1-\delta)\,,$$

where $\overline{p}$ is the vector of prices at the rival firm and $\overline{v}$ is the number of products at the rival firm. Substituting (4) into this expression and solving for δ^* yields

$$\delta^*(p, v; \overline{p}, \overline{v}) = \frac{1}{2n} + \frac{1}{2t}\left[\ln\left[\theta^\theta\left(\int_{i\in v} p_i^{\frac{-\theta}{1-\theta}}\right)^{1-\theta}\right] - \ln\left[\theta^\theta\left(\int_{i\in \overline{v}} \overline{p}_i^{\frac{-\theta}{1-\theta}}\right)^{1-\theta}\right]\right]. \quad (5)$$

1.2 The Firm's Problem

Profit per customer for the representative firm is

$$\pi(p, v) = \int_{i\in v} (p_i - c) x_i(p, v),$$

where $x_i(p, v)$ is given by (3). Total profit for the representative retailer is

$$\Pi(p, v; \overline{p}, \overline{v}) = 2\delta^*(p, v; \overline{p}, \overline{v})\pi(p, v) - vF. \quad (6)$$

Dropping arguments, the first-order necessary condition with respect to p_i is

$$\frac{-x_i}{t}\pi + 2\delta^*\left(\frac{\partial \pi}{\partial p_i}\right) = 0. \quad (7)$$

where $\partial\delta^*/\partial p_i = -x_i/2t$ follows immediately from equation (5).

The first-order necessary condition with respect to v is

$$\frac{p_v x_v}{t}\left(\frac{1-\theta}{\theta}\right)\pi + 2\delta^*\left(\frac{\partial \pi}{\partial v}\right) - F = 0. \quad (8)$$

where $\partial\delta^*/\partial v = p_v x_v(1-\theta)/2\theta t$ follows from differentiation of equation (5).

Equations (7) and (8) can be simplified as follows. Substituting the demand functions (3) into variable profit gives

$$\pi = \frac{\theta \int_{i\in v}(p_i - c) p_i^{\frac{-1}{1-\theta}}}{\int_{i\in v} p_i^{\frac{-\theta}{1-\theta}}}.$$

Differentiating this expression with respect to p_i and v respectively and factoring terms gives

$$\frac{\partial \pi}{\partial p_i} = \frac{c x_i}{(1-\theta)} \left(\frac{1}{p_i} - \frac{\theta \int_{i \in v} p_i^{\frac{-1}{1-\theta}}}{\int_{i \in v} p_i^{\frac{-\theta}{1-\theta}}} \right)$$

and

$$\frac{\partial \pi}{\partial v} = p_v x_v c \left(\frac{1}{p_v} - \frac{\int_{i \in v} p_i^{\frac{-1}{1-\theta}}}{\int_{i \in v} p_i^{\frac{-\theta}{1-\theta}}} \right).$$

Now impose symmetry. In the symmetric equilibrium, $p_i = p = \overline{p}$ and $v = \overline{v}$, market share for the representative firm is $\delta^* = 1/2$, profit per customer is

$$\pi = (p - c) v x,$$

and the price and variety effects on profit per customer are

$$\left. \frac{\partial \pi}{\partial p_i} \right|_{p_i = p} = \frac{c x}{p}$$

and

$$\left. \frac{\partial \pi}{\partial v} \right|_{p_i = p} = 0,$$

respectively, where $x_i = x$ is the symmetric output per brand.

The effect of brand proliferation on profit per customer in the logarithmic case is zero, because sales of new variants cannibalize completely on sales of existing products. This outcome, which is peculiar to the logarithmic case, facilitates closed-form solutions in the symmetric case. On substitution of terms,

$$\frac{1}{n} \left(1 - \left(\frac{p-c}{p} \right) \right) = \frac{\theta}{t} \left(\frac{p-c}{p} \right), \tag{9}$$

$$\frac{\theta}{vpt} (1-\theta)(p-c) = F. \tag{10}$$

Notice that the left-hand side of equation (10) embodies only the market-share effect of an increase in assortment depth (recall that $vpx = \theta$). A deeper assortment has no effect on category demand and hence no value on the intra-retailer margin.

The short-run oligopoly equilibrium is given by the simultaneous solution to (9) and (10). Solving these equations yields the equilibrium prices and assortment depth,

$$p^e = \frac{c(\theta n + t)}{\theta n}, \qquad v^e = \frac{\theta(1-\theta)}{(\theta n + t)F}. \tag{11}$$

In (11), equilibrium prices in the category rise following either an increase in t or a decrease in n; however, retailers respond to an increase in t by reducing assortment depth and to an decrease in n by increasing assortment depth.

An increase in product differentiation has an ambiguous effect on assortment depth in (11). In response to an increase in product differentiation, retailers provide a deeper assortment for relatively non-differentiated variants, $\psi(n,t) \equiv -t/n + \sqrt{t(n+t)/n^2} \leq \theta$, but provide a shallower assortment for sufficiently differentiated variants, $\theta < \psi(n,t)$.

Total category profit in the short-run is given by

$$\Pi^e = \frac{\theta(t-(1-\theta)n)}{n(\theta n+t)}, \tag{12}$$

which decreases with the degree of product differentiation. Greater product differentiation reduces category profitability by decreasing category sales, $vpx = \theta$. Category profit increases with the degree of store differentiation, irrespective of whether this occurs through a rise in t or a decrease in n.

The long-run equilibrium is derived by equating profits to zero in (12). This yields the equilibrium number of retailers,

$$n^* = t/(1-\theta).$$

The number of retailers is increasing in the degree of store differentiation, but decreasing in the degree of product differentiation in the category. Substituting the equilibrium number of firms into (11) yields the equilibrium prices, $p^* = c/\theta$, and the equilibrium assortment depth, $v^* = \theta(1-\theta)^2/Ft$.

The long-run effects of changes in the various parameters on equilibrium prices and assortment depth are qualitatively similar to the short-run outcomes. Greater product differentiation decreases retail sales, but has ambiguous implications for assortment depth. A rise in the degree of product differentiation in the category leads to a deeper assortment when $\theta > 1/3$, but to a shallower assortment in categories with more differentiated variants ($\theta < 1/3$).