MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1100.1146ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Contracting for Collaborative Services" by Guillaume Roels, Uday S. Karmarkar, and Scott Carr, *Management Science*, DOI 10.1287/mnsc.1100.1146.

Proofs of Statements

EC.1. Proof of Proposition 1

1. The derivative of $x^{FB}(\alpha)$ with respect to α equals

$$\frac{dx^{FB}(\alpha)}{d\alpha} = \frac{x^{FB}(\alpha)}{1-\alpha-\beta}\left(\ln(x^{FB}(\alpha)) + \frac{1-\beta}{\alpha}\right).$$

Hence, for any α, $dx^{FB}(\alpha)/d\alpha \geq 0$ if and only if $\ln(x^{FB}(\alpha)) + (1-\beta)/\alpha \geq 0$. When $dx^{FB}(\bar{\alpha})/d\alpha = 0$, the derivative of this function with respect to α equals $(dx^{FB}(\bar{\alpha})/d\alpha)/(x^{FB}(\bar{\alpha})) - (1-\beta)/\bar{\alpha}^2 = -(1-\beta)/\bar{\alpha}^2 \leq 0$. Accordingly, the function $\ln(x^{FB}(\alpha)) + (1-\beta)/\alpha$ is decreasing in the neighborhood of $\bar{\alpha}$, showing that $x^{FB}(\bar{\alpha})$ is a local maximum. Furthermore, $dx^{FB}(0)/d\alpha \geq 0$. Hence, $x^{FB}(\alpha)$ is either increasing-decreasing or always increasing.

Whenever $x^{FB}(\alpha) \geq e^{-1}$, we have $\ln(x^{FB}(\alpha)) \geq -1$ and therefore $dx^{FB}(\alpha)/d\alpha \geq (x^{FB}(\alpha))/(1-\alpha-\beta)\,(-1+(1-\beta)/\alpha) > 0$, where the second equality follows from the fact that $\alpha < 1-\beta$. Hence $x^{FB}(\alpha)$ is increasing whenever $x^{FB}(\alpha) \geq e^{-1}$.

2. The derivative of $y^{FB}(\alpha)$ with respect to α equals

$$\frac{dy^{FB}(\alpha)}{d\alpha} = \frac{y^{FB}(\alpha)}{1-\alpha-\beta}\left(\ln(x^{FB}(\alpha)) + 1\right).$$

Hence, for any α, $dy^{FB}(\alpha)/d\alpha \leq 0$ if and only if $\ln(x^{FB}(\alpha)) + 1 \leq 0$. From the first part of the proof, we know that $x^{FB}(\alpha)$ is either always increasing or increasing-decreasing. If $dx^{FB}(\alpha)/\alpha \geq 0$ for all α, the function $\ln(x^{FB}(\alpha)) + 1$ crosses zero at most once, and if it does, say at $\bar{\alpha}$, it crosses it from below. Hence, if $\ln(x^{FB}(\bar{\alpha})) + 1 = 0$, $y^{FB}(\bar{\alpha})$ is a local minimum.

Suppose now that $x^{FB}(\alpha)$ is first increasing and then decreasing with α. For this to be true, we must have $x^{FB}(\alpha) \leq e^{-1}$ for all α. Otherwise, from the first part of the proof, $x^{FB}(\alpha)$ would always be increasing. Whenever $x^{FB}(\alpha) \leq e^{-1}$, we have $\ln(x^{FB}(\alpha)) \leq -1$ and therefore $dy^{FB}(\alpha)/d\alpha \geq (y^{FB}(\alpha))/(1-\alpha-\beta)\,(-1+1) = 0$. Hence $y^{FB}(\alpha)$ is decreasing whenever $x^{FB}(\alpha) \leq e^{-1}$ and therefore when $x^{FB}(\alpha)$ is increasing-decreasing with α.

3. The derivative of $\Pi^{FB}(\alpha, \beta)$ with respect to α equals

$$\frac{\partial \Pi^{FB}(\alpha,\beta)}{\partial \alpha} = \frac{\Pi^{FB}(\alpha,\beta)}{1-\alpha-\beta}\ln(x^{FB}(\alpha,\beta)).$$

Hence, for any α, $\partial\Pi^{FB}(\alpha,\beta)/\partial\alpha \geq 0$ if and only if $\ln(x^{FB}(\alpha,\beta)) \geq 0$. The monotone relationship of $\Pi^{FB}(\alpha,\beta)$ on β is proved in a similar way. Finally, $\Pi^{FB}(\alpha,\beta)$ is jointly convex because the convexity of the output function $\mu x^{\alpha} y^{\beta}$ with respect (α,β) is preserved under maximization (Boyd and Vandenberghe 2004, p. 80). □

EC.2. Proof of Proposition 2

For any FF-O contract, (x^{FF-O}, y^{FF-O}) is the unique Nash equilibrium of the subgame; hence the incentive-compatibility constraints are satisfied. Moreover, in equilibrium, $t(x^{FF-O}, V(x^{FF-O}, y^{FF-O})) = c_V y^{FF-O} + U$, and the vendor's participation constraint is satisfied. The contract is optimal when (x^{FF-O}, y^{FF-O}) maximize $V(x,y) - (c_B + \phi_{B1})x - c_V y - \phi_{B0}$ because the total surplus is then equal to the first-best profit, discounted by the buyer's verification costs. □

EC.3. Proof of Proposition 3

For any FF-E contract, (x^{FF-E}, y^{FF-E}) is the unique Nash equilibrium of the subgame; hence the incentive-compatibility constraints are satisfied. Moreover, in equilibrium, the vendor receives $U + c_V y^{FF-E} + \phi_{V0}$ when exerting an effort level y^{FF-E}, and its participation constraint is therefore tight. The contract is optimal because (x^{FF-E}, y^{FF-E}) maximize the first-best total surplus, discounted by the vendor's verification costs. □

EC.4. Proof of Proposition 5

1. Follows from 2.
2. We only prove the first case; the second case is treated in a similar way. First, observe that

$$(\alpha+\beta-2\alpha\beta)^2 - 4\alpha\beta(1-\alpha)(1-\beta) = (\alpha-\beta)^2 \geq 0, \tag{EC.1}$$

which leads to the following inequality

$$2\sqrt{\alpha\beta(1-\alpha)(1-\beta)} \leq \alpha+\beta-2\alpha\beta. \tag{EC.2}$$

The first derivative of $b(\alpha)$ with respect to α is equal to

$$b'(\alpha) = \frac{\beta(1-\beta)}{2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}(\alpha-\beta)^2}\left(2\sqrt{\alpha\beta(1-\alpha)(1-\beta)} - (\alpha+\beta-2\alpha\beta)\right) \leq 0$$

where the inequality follows from (EC.2).

The second-derivative of $b(\alpha)$ is equal to

$$b''(\alpha)=\frac{-\beta(1-\beta)\left(4\alpha(1-\alpha)\left(2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}-(\alpha+\beta-2\alpha\beta)\right)+(\alpha-\beta)^2\right)}{4\alpha(1-\alpha)\sqrt{\alpha\beta(1-\alpha)(1-\beta)}(\alpha-\beta)^3}.$$

Hence, in order to show that $b''(\alpha)\geq 0$ for $\alpha\leq\bar{\alpha}$, it suffices to show that $4\alpha(1-\alpha)\left(2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}-(\alpha+\beta-2\alpha\beta)\right)+(\alpha-\beta)^2\geq 0$ if and only if $\alpha\leq\beta$, as long as $\alpha\leq\bar{\alpha}$. Using (EC.1), this condition can be rewritten as $-4\alpha(1-\alpha)(\alpha-\beta)^2/\left(2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}+(\alpha+\beta-2\alpha\beta)\right)+(\alpha-\beta)^2\geq 0$, or equivalently, as follows:

$$F(\alpha)\doteq(\beta-\alpha)(1-2\alpha)+2\alpha(1-\alpha)\left(\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}-1\right)\geq 0$$

if and only if $\alpha\leq\beta$, as long as $\alpha\leq\bar{\alpha}$.

When $\alpha<\beta$, $\alpha\leq 1/2$ because $\alpha+\beta<1$; therefore, $(\beta-\alpha)(1-2\alpha)\geq 0$; moreover, because $\beta<1-\alpha$ and $\beta>\alpha$, $\beta(1-\beta)>\alpha(1-\alpha)$; as a result $F(\alpha)>0$ when $\alpha<\beta$.

If $\beta\geq 1-\beta$, $F(\alpha)>0$ for all $\alpha\in[0,1-\beta)$, i.e., $b''(\alpha)\geq 0$ for all $\alpha\in[0,1-\beta)$. Suppose now that $\beta\leq 1/2$. It turns out that $F(\alpha)$ has two roots. One of them is β and let us denote by $\bar{\alpha}$ the second root. Because $F(\alpha)>0$ for $\alpha<\beta$ and $F(\alpha)=(1-2\beta)^2\geq 0$ when $\alpha=1-\beta$, the second root $\bar{\alpha}$ lies somewhere in $(\beta,1-\beta]$. In addition, because, for any $\alpha\in(\beta,1/2]$, $(\beta-\alpha)(1-2\alpha)\leq 0$ and $\beta(1-\beta)<\alpha(1-\alpha)$, and therefore $F(\alpha)<0$, the second root $\bar{\alpha}$ is greater than 1/2. As a result, if $\beta\leq 1/2$, there exists some $\bar{\alpha}\in(\max\{1/2,\beta\},1-\beta]$ such that $F(\alpha)\geq 0$ for $\alpha\leq\beta$, $F(\alpha)\leq 0$ for $\alpha\in[\beta,\bar{\alpha}]$, and $F(\alpha)\geq 0$ for $\alpha\geq\bar{\alpha}$, or alternatively, such that $b''(\alpha)\geq 0$ for all $\alpha\leq\bar{\alpha}$ and $b''(\alpha)\leq 0$ for all $\alpha\geq\bar{\alpha}$.

3. One can show that, when the bonus rate is set to b, the total surplus is equal to

$$\Pi(b)=\mu^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha(1-b)}{c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\beta b}{c_V}\right)^{\frac{\beta}{1-\alpha-\beta}}(1-\alpha(1-b)-\beta b).$$

The elasticity of $\Pi(b)$ is equal to

$$\frac{b\Pi'(b)}{\Pi(b)}=\frac{(1-b)^2\beta(1-\alpha)-b^2\alpha(1-\beta)}{(1-\alpha-\beta)(1-\alpha(1-b)-\beta b)(1-b)},$$

which is equal to $(\beta-\alpha)/((1-\alpha-\beta)(2-\alpha-\beta))$ when $b=1/2$. One can easily show that this elasticity is larger than or equal to -1 whenever $\alpha \leq 2-\beta-\sqrt{2(1-\beta)}$, which is always true when $\alpha \leq 1/2$. Similarly, one can show that the elasticity is smaller than or equal to 1 whenever $\beta \leq 1/2$. □

EC.5. Proof of Proposition 6

When $V(x,y)=\mu x^\alpha y^\beta$, one can show that the total surpluses are equal to:

$$\Pi^{FF-O} = \mu^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{c_B+\phi_{B1}}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{c_V}\right)^{\frac{\beta}{1-\alpha-\beta}} (1-\alpha-\beta)-\phi_{B0} \tag{EC.3}$$

$$\Pi^{FF-E/TM} = \mu^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{c_V+\phi_{V1}}\right)^{\frac{\beta}{1-\alpha-\beta}} (1-\alpha-\beta)-\phi_{V0} \tag{EC.4}$$

$$\Pi^{PB}(1/2) = \mu^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{2c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{2c_V}\right)^{\frac{\beta}{1-\alpha-\beta}} (1-\alpha/2-\beta/2). \tag{EC.5}$$

Let $G^{FF-O}(\alpha,\beta) \doteq (c_B/(c_B+\phi_{B1}))^\alpha$, $G^{TM}(\alpha,\beta) \doteq (c_V/(c_V+\phi_{V1}))^\beta$, and $G^{PB}(\alpha,\beta) \doteq (1/2)^{\alpha+\beta}\left((1-\alpha/2-\beta/2)/(1-\alpha-\beta)\right)^{1-\alpha-\beta}$. One can check that the functions $G^{FF-O}(\alpha,\beta)$ and $G^{TM}(\alpha,\beta)$ are both jointly convex. The function $G^{PB}(\alpha,\beta)$ is in contrast jointly concave because the function $F(x)=(1/2)^x\left((1-x/2)/(1-x)\right)^{1-x}$ is concave in $x\in[0,1]$ and because concavity is preserved under affine compositions (Boyd and Vandenberghe 2004, p. 79).

1. When $\phi_{B0}=\phi_{V0}$, $\Pi^{FF-O} \geq \Pi^{FF-E/TM}$ if and only if $G^{FF-O}(\alpha,\beta) \geq G^{TM}(\alpha,\beta)$, that is, if and only if $\alpha \leq \beta \ln\left(c_V/(c_V+\phi_{V1})\right)/\ln\left(c_B/(c_B+\phi_{B1})\right)$. Hence, in the (α,β) space, the region where FF-O (resp. FF-E/TM) contracts generate more surplus than FF-E/TM (resp. FF-O) contracts is a triangle determined by the lines $\alpha=0$ (resp. $\beta=0$), $\alpha+\beta=1$, and $\alpha = \beta \ln\left(c_V/(c_V+\phi_{V1})\right)/\ln\left(c_B/(c_B+\phi_{B1})\right)$.

2. When $\phi_{V0}=0$ and $b=1/2$, $\Pi^{PB}(1/2) \geq \Pi^{FF-E/TM}$ if and only if $G^{PB}(\alpha,\beta) \geq G^{TM}(\alpha,\beta)$. Suppose that (α_1,β_1) and (α_2,β_2) are such that $G^{PB}(\alpha_i,\beta_i) \geq G^{TM}(\alpha_i,\beta_i)$, for $i=1,2$. For any $\lambda\in[0,1]$, we have

$$\begin{aligned} G^{PB}(\lambda\alpha_1+(1-\lambda)\alpha_2, \lambda\beta_1+(1-\lambda)\beta_2) &\geq \lambda G^{PB}(\alpha_1,\beta_1)+(1-\lambda)G^{PB}(\alpha_2,\beta_2) \\ &\geq \lambda G^{TM}(\alpha_1,\beta_1)+(1-\lambda)G^{TM}(\alpha_2,\beta_2) \end{aligned}$$

$$\geq G^{TM}(\lambda\alpha_1+(1-\lambda)\alpha_2,\lambda\beta_1+(1-\lambda)\beta_2),$$

where the first inequality follows by concavity of $G^{PB}(\alpha,\beta)$, the second by assumption on (α_1,β_1) and (α_2,β_2), and the third by convexity of $G^{TM}(\alpha,\beta)$. Hence, the region $\{(\alpha,\beta):\Pi^{PB}(1/2)\geq \Pi^{FF-E/TM}\}$ is convex when $\phi_{V0}=0$.

Similar, when $\phi_{B0}=0$, $\Pi^{PB}\geq\Pi^{FF-O}$ if and only if $G^{PB}(\alpha,\beta)\geq G^{FF-O}(\alpha,\beta)$ and one can show that the region $\{(\alpha,\beta):\Pi^{PB}(\alpha,\beta)\geq\Pi^{FF-O}(\alpha,\beta)\}$ is convex when $\phi_{B0}=0$. Because the intersection of two convex regions is convex (Boyd and Vandenberghe 2004, p. 36), $\mathcal{R}^{PB}$ is convex. One can finally check that, when $\alpha=\beta=0$, $G^{PB}(\alpha,\beta)=G^{FF-O}(\alpha,\beta)=G^{TM}(\alpha,\beta)=1$; therefore, $\mathcal{R}^{PB}$ is anchored at the point $(\alpha,\beta)=(0,0)$ when $\phi_{B0}=\phi_{V0}=0$, and therefore when $\phi_{B0}\geq 0$ and $\phi_{V0}\geq 0$ given that Π^{FF-O} and $\Pi^{FF-E/TM}$ are decreasing with ϕ_{B0} and ϕ_{V0}.

3. Similar to Proposition 1, one can show that Π^{FF-O}, Π^{TM} and $\Pi^{PB}(b)$ are all convex with respect to α and β, and so is therefore the function $\max\{\Pi^{FF-O},\Pi^{TM},\max_b\Pi^{PB}(b)\}$ given that convexity is preserved under pointwise maximization (Boyd and Vandenberghe 2004, p. 80). Hence, the function $\max\{\Pi^{FF-O},\Pi^{TM},\max_b\Pi^{PB}(b)\}$ has convex sublevel sets. □

EC.6. Proof of Proposition 7

Similar to (2), the expected first-best surplus under output uncertainty is maximized when the effort levels are chosen ex-post and is equal to

$$\mathbb{E}[\Pi^{FB}(\epsilon_0,\epsilon_1)]=\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right]\left(\frac{\alpha}{c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\beta}{c_V}\right)^{\frac{\beta}{1-\alpha-\beta}}(1-\alpha-\beta)$$

and therefore $\mathbb{E}[\Pi^{FB}(\epsilon_0,\epsilon_1)]/\Pi^{FB}=\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right]/\mu^{\frac{1}{1-\alpha-\beta}}$.

Under a PB contract, the buyer solves the following program:

$$\begin{aligned}
\max_{b,s}\quad & \mathbb{E}\left[(1-b)\epsilon_0+(1-b)\epsilon_1\left(x(\epsilon_0,\epsilon_1)\right)^{\alpha}\left(y(\epsilon_0,\epsilon_1)\right)^{\beta}-c_Bx(\epsilon_0,\epsilon_1)-s\right]\\
& \mathbb{E}\left[b\epsilon_0+b\epsilon_1\left(x(\epsilon_0,\epsilon_1)\right)^{\alpha}\left(y(\epsilon_0,\epsilon_1)\right)^{\beta}-c_Vy(\epsilon_0,\epsilon_1)+s\right]\geq U \qquad (IR_V)\\
x(\epsilon_0,\epsilon_1)&=\arg\max_{\bar{x}}\left\{(1-b)\epsilon_0+(1-b)\epsilon_1\bar{x}^{\alpha}\left(y(\epsilon_0,\epsilon_1)\right)^{\beta}-c_B\bar{x}-s\right\} \quad (IC_B) \qquad \text{(EC.6)}\\
y(\epsilon_0,\epsilon_1)&=\arg\max_{\bar{y}}\left\{b\epsilon_0+b\epsilon_1\left(x(\epsilon_0,\epsilon_1)\right)^{\alpha}\bar{y}^{\beta}-c_V\bar{y}+s\right\} \qquad (IC_V).
\end{aligned}$$

After solving this program, we obtain from (EC.5) that $\mathbb{E}[\Pi^{PB}(\epsilon_0,\epsilon_1)] = \Pi^{PB}\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right]/\mu^{\frac{1}{1-\alpha-\beta}}$.

Under a TM contract, the vendor cedes its decision rights to the buyer. Specifically, the buyer decides how much effort the vendor must exert and pays the vendor a price p per unit of effort, in addition to a fixed fee s. That is, the buyer optimizes the following problem:

$$\max_{s,p} \mathbb{E}\left[\epsilon_0 + \epsilon_1 \left(x(\epsilon_0,\epsilon_1)\right)^{\alpha}\left(y(\epsilon_0,\epsilon_1)\right)^{\beta} - c_B x(\epsilon_0,\epsilon_1) - p y(\epsilon_0,\epsilon_1) - s\right]$$

$$s.t. \qquad \mathbb{E}\left[s + (p - c_V - \phi_{V1}) y(\epsilon_0,\epsilon_1) - \phi_{V0}\right] \geq U \qquad (IR_V)$$

$$(x(\epsilon_0,\epsilon_1), y(\epsilon_0,\epsilon_1)) = \arg\max_{\bar{x},\bar{y}}\left\{\epsilon_0 + \epsilon_1 \bar{x}^{\alpha}\bar{y}^{\beta} - c_B\bar{x} - p\bar{y} - s\right\} \ (IC_B),$$

which generates the following total surplus:

$$\mathbb{E}[\Pi^{TM}(\epsilon_0,\epsilon_1)] = \mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right]\left(\frac{\alpha}{c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\beta}{c_V+\phi_{V1}}\right)^{\frac{\beta}{1-\alpha-\beta}}(1-\alpha-\beta) - \phi_{V0}.$$

As a result, when $\phi_{V0} = 0$, using (EC.4), we find that $\mathbb{E}[\Pi^{TM}(\epsilon_0,\epsilon_1)]/\Pi^{TM} = \mathbb{E}\left[\Pi^{FB}(\epsilon_0,\epsilon_1)\right]/\Pi^{FB}$.

Under a FF-E contract, the buyer specifies ex-ante the vendor's effort level y, in exchange for a fixed fee s, so as to maximize its expected profit, subject to the vendor's participation constraint (IR_V) and its ex-post optimal choice of effort (IC_B), that is,

$$\max_{s,y} \quad \mathbb{E}\left[\epsilon_0 + \epsilon_1\left(x(\epsilon_0,\epsilon_1)\right)^{\alpha} y^{\beta} - c_B x(\epsilon_0,\epsilon_1) - s\right]$$

$$s.t. \qquad s - (c_V + \phi_{V1})y - \phi_{V0} \geq U \qquad (IR_V)$$

$$x(\epsilon_0,\epsilon_1) = \arg\max_{\bar{x}}\left\{\epsilon_0 + \epsilon_1\bar{x}^{\alpha}y^{\beta} - c_B\bar{x} - s\right\} \ (IC_B),$$

which generates an expected total profit equal to

$$\mathbb{E}[\Pi^{FF-E}(\epsilon_0,\epsilon_1)] = \left(\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\left(\frac{\beta}{c_V+\phi_{V1}}\right)^{\frac{\beta}{1-\alpha-\beta}}\left(\frac{\alpha}{c_B}\right)^{\frac{\alpha}{1-\alpha-\beta}}(1-\alpha-\beta) - \phi_{V0}.$$

As a result, when $\phi_{V0} = 0$, using (EC.4), $\mathbb{E}[\Pi^{FF-E}(\epsilon_0,\epsilon_1)]/\Pi^{FF-E} = \left(\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\right)^{\frac{1-\alpha}{1-\alpha-\beta}}/\mu^{\frac{1}{1-\alpha-\beta}}$.

Finally, under a FF-O contract, the buyer specifies ex-ante its effort level x, a fixed fee s, as well as an output level T below which the fixed fee is not transferred. The buyer chooses the contract parameters so as to maximize its expected profit, subject to the vendor's participation constraint (IR_V) and the vendor's ex-post optimal choice of effort (IC_V), that is,

$$\begin{aligned} \max_{s,x,y,T} \; & \mathbb{E}\left[\epsilon_0+\epsilon_1 x^\alpha \left(y(\epsilon_0,\epsilon_1)\right)^\beta-(c_B+\phi_{B1})x-\phi_{B0}-s\mathbb{1}\left\{\epsilon_0+\epsilon_1 x^\alpha \left(y(\epsilon_0,\epsilon_1)\right)^\beta \geq T\right\}\right] \\ s.t. \; & \mathbb{E}\left[s\mathbb{1}\left\{\epsilon_0+\epsilon_1 x^\alpha \left(y(\epsilon_0,\epsilon_1)\right)^\beta \geq T\right\}-c_V y(\epsilon_0,\epsilon_1)\right]\geq U \qquad (IR_V) \\ & y(\epsilon_0,\epsilon_1)=\arg\max_{\bar{y}} s\mathbb{1}\left\{\epsilon_0+\epsilon_1 x^\alpha \bar{y}^\beta \geq T\right\}-c_V\bar{y} \qquad (IC_V). \end{aligned} \tag{EC.7}$$

The optimal vendor's ex-post level is equal to $y^{FF-O}(\epsilon_0,\epsilon_1)=\epsilon_1^{-1/\beta}\left((T-\epsilon_0)^+\right)^{1/\beta}x^{\frac{-\alpha}{\beta}}$, where $(z)^+=\max\{0,z\}$. Because the buyer's profit function is linear decreasing with s, the optimal fee s makes the (IR_V) constraint tight, giving the buyer an expected profit equal to:

$$\mathbb{E}[T-\epsilon_0]^+-(c_B+\phi_{B1})x-\phi_{B0}-c_V x^{\frac{-\alpha}{\beta}}\left(\mathbb{E}[T-\epsilon_0]^+\right)^{\frac{1}{\beta}}\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]-U.$$

The buyer's profit function is concave in $\mathbb{E}[T-\epsilon_0]^+$ and is maximized at $\mathbb{E}[T-\epsilon_0]^+=(\beta/c_V)^{\frac{\beta}{1-\beta}}x^{\frac{\alpha}{1-\beta}}\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]^{\frac{-\beta}{1-\beta}}$, yielding the buyer $\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]^{\frac{-\beta}{1-\beta}}x^{\frac{\alpha}{1-\beta}}(\beta/c_V)^{\frac{\beta}{1-\beta}}(1-\beta)-(c_B+\phi_{B1})x-\phi_{B0}-U$. The buyer's maximum expected profit is attained when $x=\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]^{\frac{-\beta}{1-\beta}}(\alpha/(c_B+\phi_{B1}))^{\frac{1-\beta}{1-\alpha-\beta}}(\beta/c_V)^{\frac{\beta}{1-\alpha-\beta}}$, and is equal to

$$\mathbb{E}[\Pi^{FF-O}(\epsilon_0,\epsilon_1)]=\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]^{\frac{-\beta}{1-\beta}}\left(\frac{\alpha}{c_B+\phi_{B1}}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{\beta}{c_V}\right)^{\frac{\beta}{1-\alpha-\beta}}(1-\alpha-\beta). \tag{EC.8}$$

As a result, when $\phi_{B0}=0$, using (EC.4), $\mathbb{E}[\Pi^{FF-O}(\epsilon_0,\epsilon_1)]/\Pi^{FF-O}=\left(\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]\right)^{\frac{-\beta}{1-\beta}}/\mu^{\frac{1}{1-\alpha-\beta}}$.

The end of the proof consists in showing that:

$$\left(\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]\right)^{\frac{-\beta}{1-\beta}}\leq\left(\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\leq\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right].$$

By Jensen's inequality, $\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]\geq(\mathbb{E}[\epsilon_1])^{\frac{-1}{\beta}}$ and $\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\geq(\mathbb{E}[\epsilon_1])^{\frac{1}{1-\alpha}}$, showing that

$$\left(\mathbb{E}\left[\epsilon_1^{\frac{-1}{\beta}}\right]\right)^{\frac{-\beta}{1-\beta}}\leq(\mathbb{E}[\epsilon_1])^{\frac{1}{1-\alpha-\beta}}\leq\left(\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\right)^{\frac{1-\alpha}{1-\alpha-\beta}}. \tag{EC.9}$$

The second inequality is showed by using Hölder's inequality: $\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha}}\right]\leq\left(\mathbb{E}\left[\epsilon_1^{\frac{1}{1-\alpha-\beta}}\right]\right)^{\frac{1-\alpha-\beta}{1-\alpha}}$. □

EC.7. Proof of Proposition 8

1. If the vendor's unit cost of effort is equal to $g(z)c_V$, the first-best total surplus is equal to $\Pi^{FB}(x,y,z) = V(x,y) - c_B x - c_V y g(z) - c_{PI} z$, which is strictly concave because $g''(z) > 0$. The optimal process improvement effort z^{FB}, if positive, must solve the first-order optimality condition: $-c_V y g'(z) = c_{PI}$. For each contract, the vendor's profit can be written as follows:

$$\Pi_V^{FF-O}(y,z) = s\mathbb{1}\left\{V(x,y) \geq V^{FF-O} \text{ and } x \geq x^{FF-O}\right\} - c_V y g(z) - c_{PI} z$$

$$\Pi_V^{FF-E}(y,z) = s\mathbb{1}\left\{y \geq y^{FF-E}\right\} - c_V y g(z) - \phi_{V1} y - c_{PI} z - \phi_{V0}$$

$$\Pi_V^{TM}(y,z) = s + (p - c_V g(z) - \phi_{V1}) y - c_{PI} z - \phi_{V0}$$

$$\Pi_V^{PB}(y,z) = s + bV(x,y) - c_V y g(z) - c_{PI} z.$$

For all contracts, the first-order optimality conditions with respect to z yields that $-c_V y g'(z) = c_{PI}$, similar to the first-best first-order optimality condition.

2. If the vendor needs to work $g(z)y$ hours to provide y units of effort, $\Pi^{FB}(x,y,z)$, $\Pi_V^{FF-O}(y,z)$, and $\Pi_V^{PB}(y,z)$ are the same as in Case 1. Under a FF-E and TM contracts, the vendor's profit is equal to:

$$\Pi_V^{FF-E}(y,z) = s\mathbb{1}\left\{y g(z) \geq y^{FF-E}\right\} - (c_V + \phi_{V1}) y g(z) - c_{PI} z - \phi_{V0}$$

$$\Pi_V^{TM}(y,z) = s + (p - c_V - \phi_{V1}) g(z) y - c_{PI} z - \phi_{V0}.$$

In both cases, the vendor's profit is decreasing with z. The equilibrium vendor's effort level in process improvement is thus equal to zero.

3. If the service output is equal to $V(x,y)g(z)$, the first-best total surplus is equal to $\Pi^{FB}(x,y,z) = V(x,y)g(z) - c_B x - c_V y - c_{PI} z$, which is concave when $g''(z) < 0$ and $V(x,y)$ is supermodular. The optimal process improvement effort z^{FB}, if positive, must satisfy $V(x,y)g'(z) = c_{PI}$, whereas the optimal effort y^{FB} must satisfy $dV(x,y)/dy g(z) = c_V$. Combining these two conditions yields: $c_{PI} g(z^{FB}) \frac{\partial V(x^{FB}, y^{FB})}{\partial y} = c_V g'(z^{FB}) V(x^{FB}, y^{FB})$.

With a FF-O contract, the vendor's profit can be written as follows:

$$\Pi_V^{FF-O}(y,z) = s\mathbb{1}\left\{V(x,y)g(z) \geq V^{FF-O} \text{ and } x \geq x^{FF-O}\right\} - c_V y - c_{PI} z,$$

and is maximized when $V(x,y)g(z) \geq V^{FF-O}$. Minimizing the vendor's costs $c_V y + c_{PI} z$ subject to the constraint that $V(x,y)g(z) \geq V^{FF-O}$ yields the same necessary condition as in the first-best solution.

Under a PB contract, the vendor's profit is equal to

$$\Pi_V^{PB}(y,z) = s + bV(x,y)g(z) - c_V y - c_{PI} z.$$

The optimal process improvement effort z^{PB}, if positive, must satisfy $bV(x,y)g'(z) = c_{PI}$, whereas the optimal effort y^{PB} must satisfy $b\frac{\partial V(x,y)}{\partial y} g(z) = c_V$, which yields: $c_{PI} g(z^{PB}) \frac{\partial V(x^{PB},y^{PB})}{\partial y} = c_V g'(z^{PB}) V(x^{PB}, y^{PB})$.

With a FF-E and TM contracts, the vendor's profits can be written as

$$\Pi_V^{FF-E}(y,z) = s\mathbb{1}\left\{y \geq y^{FF-E}\right\} - (c_V + \phi_{V1})y - c_{PI} z - \phi_{V0}$$
$$\Pi_V^{TM}(y,z) = s + (p - c_V - \phi_{V1})y - c_{PI} z - \phi_{V0}$$

and they are decreasing with z. Hence, the optimal process improvement effort level is in this case equal to zero. $\square$

EC.8. Proof of Proposition 9

If (x^{FB}, y^{FB}) maximizes the first-best total surplus with one vendor $V(x,y) - c_B x - c_V y$, the triplet (x^{FB}, y^{FB}, y^{FB}) will maximize the first-best total surplus with two vendors $\overline{V}(x, y_1, y_2) - c_B x - (c_V/2)(y_1 + y_2)$, because the first-order optimality conditions are identical:

$$\frac{\partial \overline{V}(x^{FB}, y^{FB}, y^{FB})}{\partial x} - c_B = 0 \Leftrightarrow \frac{\partial V(x^{FB}, y^{FB})}{\partial x} - c_B = 0$$
$$\frac{\partial \overline{V}(x^{FB}, y^{FB}, y^{FB})}{\partial y_i} - \frac{c_V}{2} = 0 \Leftrightarrow \frac{\partial V(x^{FB}, g(y^{FB}, y^{FB}))}{\partial y} \frac{\partial g(y^{FB}, y^{FB})}{\partial y_i} - \frac{c_V}{2} = 0, \quad \forall i = 1,2$$
$$\Leftrightarrow \frac{1}{2} \frac{\partial V(x^{FB}, y^{FB})}{\partial y} - \frac{c_V}{2} = 0.$$

When the buyer makes its efforts verifiable at a cost $\phi_{B0} + \phi_{B1} x$ and uses two symmetric FF-O contracts, its profit is equal to

$$\Pi_B^{FF-O}(x, V^{FF-O}, s) = \overline{V}(x, y_1, y_2) - (c_B + \phi_{B1})x - 2s\mathbb{1}\left\{\overline{V}(x, y_1, y_2) \geq V^{FF-O}\right\} - \phi_{B0}$$

where the vendors' effort levels individually maximize $\Pi_{Vi}^{FF-O}(y_i) = s\mathbb{1}\left\{\overline{V}(x,y_1,y_2) \geq V^{FF-O}\right\} - (c_V/2)y_i$. The buyer sets the fixed fees s such that the vendors' participation constraints are tight and therefore collects the total surplus, minus the vendors' reservation utilities. Optimizing the contract parameters, one obtains a total surplus identical to the case where there is only one vendor.

When the vendors make their efforts verifiable at a cost $\phi_{V0}/2 + (\phi_{V1}/2)y_i$, $i = 1,2$, and the buyer uses two symmetric FF-E contracts (or TM contracts), the buyer's profit equals

$$\Pi_B^{FF-E}(x, y^{FF-E}, s) = \overline{V}(x,y_1,y_2) - c_B x - s\mathbb{1}\left\{y_1 \geq y^{FF-E}\right\} - s\mathbb{1}\left\{y_2 \geq y^{FF-E}\right\}$$

where the vendors' effort levels maximize $\Pi_{Vi}^{FF-E}(y_i) = s\mathbb{1}\{y_i \geq y^{FF-E}\} - (c_V + \phi_{V1})/2y_i - \phi_{V0}/2$. The buyer sets the fixed fees s such that the vendors' participation constraints are tight and therefore collects the total surplus, minus the vendors' reservation utilities. Optimizing the contract parameters, one obtains a total surplus identical to the case where there is only one vendor.

Finally, with double moral hazard, when the buyer uses two symmetric PB contracts giving a share $b/2$ of the output to each vendor, the buyer's profit is equal to

$$\Pi_B^{PB}(x,b,s) = (1-b)\overline{V}(x,y_1,y_2) - c_B x - 2s.$$

On the other hand, the vendors' profits are equal to $\Pi_{Vi}^{PB}(y_i) = (b/2)\overline{V}(x,y_1,y_2) - (c_V/2)y_i + s$ and are maximized when y_i solves

$$\begin{aligned}
\frac{b}{2}\frac{\partial \overline{V}(x,y_1,y_1)}{\partial y_i} - \frac{c_V}{2} = 0 \;\; \forall i = 1,2 \Leftrightarrow\; & \frac{b}{2}\frac{\partial \overline{V}(x,y_1,y_2)}{\partial y_i} - \frac{c_V}{2} = 0 \;\; \forall i = 1,2 \\
\Leftrightarrow\; & \frac{b}{2}\frac{\partial V(x,g(y_1,y_1))}{\partial y}\frac{\partial g(y_1,y_1)}{\partial y_i} - \frac{c_V}{2} = 0 \;\; \forall i = 1,2 \\
\Leftrightarrow\; & \frac{b}{2}\frac{\partial V(x,y)}{\partial y} - c_V = 0
\end{aligned}$$

where the first identity follows from the fact that both vendors are symmetric, the second identity follows by definition of $\overline{V}$ and the third identity follows from the fact that $\frac{\partial g(y_1,y_2)}{\partial y_i} = 1/2$ when $y_1 = y_2$.

By contrast, with only one vendor, the vendor's profit is equal to $bV(x,y) - c_V y - s$ and is maximized when y solves $b\partial V(x,y)/\partial y - c_V = 0$. Hence, the equilibrium effort level of each vendor is smaller than if there was only one vendor; in particular, the average effort level $g(y_1, y_2)$ with two vendors is smaller than if there was only one vendor. By supermodularity of $\overline{V}(x, y_1, y_2)$, the buyer will therefore provide less effort with two vendors than with one vendor. Because $V(x,y) - c_B x - c_V y$ is strictly concave and $x^{PB} \leq x^{FB}$ and $y^{PB} \leq y^{FB}$, the total surplus achieved with two vendors is then strictly smaller than that achieved with one vendor only. $\square$