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Electronic Companion—"Is Regime Switching in Stock Returns Important in Portfolio Decisions?" by Jun Tu, *Management Science*, DOI 10.1287/mnsc.1100.1181.

Appendix. A Gibbs Sampling Procedure for Regime Switching Models

We propose a procedure to handle high-dimensional regime switching models based on a Bayesian Gibbs sampling procedure. First, since the state variable s_t is not observable, following Chib (1996), we simulate states $S^* = \{s_t^*\}_{t=1}^T$, and group the full sample data into two sets, $\{R^i\}_{i=1}^2$, according to the associated states, where $R^i = \{Y^i,\, X^i\}$, $\;Y^i = \{y_t | s_t^* = i\}'$, a $T^i \times m$ matrix, $X^i = \{x_t | s_t^* = i\}'$, a $T^i \times k$ matrix, and T^i is the number of observations in the set R^i. In addition, define $Z^i = (\iota_T^i,\, X^i)$, a $T^i \times (k+1)$ matrix, where ι_T^i denotes a T^i-vector of ones. Also define $A^i = (\alpha^i,\, B^i)'$, a $(k+1) \times m$ matrix and $a^i = vec(A^i)$. Then the regression model (3) can be written as:

$$Y^i = Z^i A^i + U^i, \tag{EC.1}$$

where $U^i = \{u_t | s_t^* = i\}'$, a $T^i \times m$ matrix. The likelihood function of R^i can be factored as:

$$p(Y^i,\, X^i \mid E^i,\, V^i) = p(Y^i \mid A^i,\, \Sigma^i,\, X^i)\, p(X^i \mid E_2^i,\, V_{22}^i), \tag{EC.2}$$

where:

$$\begin{aligned} p(Y^i \mid A^i,\, \Sigma^i,\, X^i) &\propto |\Sigma^i|^{-\frac{T}{2}} exp\left\{-\frac{1}{2} tr\left[(Y^i - Z^i A^i)'(Y^i - Z^i A^i)(\Sigma^i)^{-1}\right]\right\} \\ &\propto |\Sigma^i|^{-\frac{T}{2}} exp\left\{-\frac{T}{2} tr\widehat{\Sigma}^i(\Sigma^i)^{-1} - \frac{1}{2} tr\left(\left(A^i - \widehat{A}^i\right)'(Z^i)'Z^i\left(A^i - \widehat{A}^i\right)(\Sigma^i)^{-1}\right)\right\} \\ &\propto |\Sigma^i|^{-\frac{T}{2}} exp\left\{-\frac{T}{2} tr\widehat{\Sigma}^i(\Sigma^i)^{-1} - \frac{1}{2} tr\left[\left(a^i - \hat{a}^i\right)'\left((\Sigma^i)^{-1} \otimes (Z^i)'Z^i\right)\left(a^i - \hat{a}^i\right)\right]\right\}, \end{aligned} \tag{EC.3}$$

and

$$\begin{aligned} p(X^i \mid E_2^i,\, V_{22}^i) &\propto |V_{22}^i|^{-\frac{T}{2}} exp\left\{-\frac{1}{2} tr\left((X^i - \iota_{T^i}(E_2^i)')'(X^i - \iota_{T^i}(E_2^i)')\right)(V_{22}^i)^{-1}\right\} \\ &\propto |V_{22}^i|^{-\frac{T^i}{2}} exp\left\{-\frac{T}{2} tr\widehat{V_{22}}^i V_{22}^{-1} - \frac{T^i}{2} tr\left(\left(E_2^i - \widehat{E_2}^i\right)\left(E_2^i - \widehat{E_2}^i\right)'(V_{22}^i)^{-1}\right)\right\}. \end{aligned} \tag{EC.4}$$

The joint prior distribution of the set of all parameters, denoted as θ, is:

$$p_0(\theta) = p_0(\alpha^1|\Sigma^1)p_0(\alpha^2|\Sigma^2)p_0(\Sigma^1)p_0(\Sigma^2)p_0(B^1)p_0(B^2)p_0(E_2^1)p_0(E_2^2)p_0(V_{22}^1)p_0(V_{22}^2)p_0(P,Q), \tag{EC.5}$$

where:

$$p_0(\alpha^i|\Sigma^i) \propto |\Sigma^i|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(\alpha^i)'\left(\frac{\sigma_\alpha^2}{(s^i)^2}\Sigma^i\right)^{-1}(\alpha^i)\right\}, \tag{EC.6}$$

$$p_0(\Sigma^i) \propto |\Sigma^i|^{-\frac{\nu_\Sigma+m+1}{2}} exp\left\{-\frac{1}{2}trH^i(\Sigma^i)^{-1}\right\}, \tag{EC.7}$$

$$p_0(B^i) \propto 1, \tag{EC.8}$$

$$p_0(E_2^i) \propto 1, \tag{EC.9}$$

$$p_0(V_{22}^i) \propto |V_{22}^i|^{-\frac{k+1}{2}}, \tag{EC.10}$$

$H^i = (s^i)^2(\nu_\Sigma - m - 1)I_m$, $\nu_\Sigma = 30$, $(s^i)^2 = tr((Y^i - Z^i\widehat{A}^i)'(Y^i - Z^i\widehat{A}^i)/T^i)/m$, $\widehat{A}^i = ((Z^i)'Z^i)(Z^i)'Y^i$, and we assume that the prior distributions of $(P, 1-P)$, and $(1-Q, Q)$ are two independent Dirichlet distributions on the two-dimensional simplex, i.e.,

$$(P, 1-P) \backsim D(220, 20), \quad (1-Q, Q) \backsim D(40, 200). \tag{EC.11}$$

This is corresponding to a belief centering around $P = 91.67\%$ and $Q = 83.33\%$. Robustness checks show that the results of this paper are qualitatively invariant to different specification of the priors on P and Q. In addition, consider the transformation:

$$(\alpha^i)'\left(\frac{\sigma_\alpha^2}{(s^i)^2}\Sigma^i\right)^{-1}\alpha^i = (a^i)'\left((\Sigma^i)^{-1} \otimes D^i\right)a^i, \tag{EC.12}$$

where $a^i = vec(A^i)$ and D^i is a $(k+1)\times(k+1)$ matrix whose $(1,1)$ element is $(s^i)^2/\sigma_\alpha^2$ and whose other elements are all zero. Then, it follows that the likelihood in (EC.2) – (EC.4) can be combined with the prior in (EC.6) – (EC.11) to obtain the posterior distribution:

$$p(\theta|R) \propto p(R|\theta)p_0(\theta).$$

Since both the likelihood function conditioning on the states and the prior can be factored into two independent parts on (a^i, Σ^i) and (E_2^i, V_{22}^i), respectively, the posteriors on (a^i, Σ^i) and (E_2^i, V_{22}^i) are independent as well. Hence, the joint posterior of the regression parameters is:

$$\begin{aligned} p(a^i, \Sigma^i\,|R^i) \propto |\Sigma^i|^{-\frac{k+1}{2}} exp\left\{-\frac{1}{2}(a^i)'\left((\Sigma^i)^{-1}\otimes D^i\right)a^i - \frac{1}{2}tr\left(\left(a^i - \hat{a}^i\right)'\left((\Sigma^i)^{-1}\otimes (Z^i)'Z^i\right)\left(a^i - \hat{a}^i\right)\right)\right\} \\ \times |\Sigma^i|^{-\frac{T^i+\nu_\Sigma+m-k+1}{2}} exp\left\{-\frac{1}{2}tr\left(H^i + T^i\widehat{\Sigma}^i\right)(\Sigma^i)^{-1}\right\}. \end{aligned} \tag{EC.13}$$

Let $F^i = D^i + (Z^i)'Z^i$, and $Q^i = (Z^i)'Z^i - (Z^i)'Z^i(F^i)^{-1}(Z^i)'Z^i$. By completing the square on a^i, we have:

$$p(a^i, \Sigma^i \,|R^i) \propto |\Sigma^i|^{-\frac{k+1}{2}} exp\left\{-\frac{1}{2}\left[\left(a^i - \tilde{a}^i\right)' \left((\Sigma^i)^{-1} \otimes F^i\right)\left(a^i - \tilde{a}^i\right)\right]\right\} \times |\Sigma^i|^{-\frac{T^i+\nu_\Sigma+m-k+1}{2}} exp\left\{-\frac{1}{2} tr\left(H^i + T^i\widehat{\Sigma}^i + (\widehat{A}^i)'Q^i\widehat{A}^i\right)(\Sigma^i)^{-1}\right\}, \tag{EC.14}$$

where $\tilde{a}^i = (I_m \otimes (F^i)^{-1}(Z^i)'Z^i)\,\hat{a}^i$. Hence,

$$(\Sigma^i)^{-1} \,|\, R \sim W\left(T^i + \nu_\Sigma - k,\, \left(H^i + T^i\widehat{\Sigma}^i + (\widehat{A}^i)'Q^i\widehat{A}^i\right)^{-1}\right), \tag{EC.15}$$

and

$$a^i \,|\, (\Sigma^i)^{-1},\, R^i \sim N\left(\tilde{a}^i,\, \Sigma^i \otimes (F^i)^{-1}\right). \tag{EC.16}$$

In addition, the joint posterior distribution of E_2^i and V_{22}^i is:

$$p\left(E_2^i,\, V_{22}^i|R^i\right) \propto |V_{22}^i|^{-\frac{T^i+k+1}{2}} exp\left\{-\frac{T^i}{2} tr\widehat{V_{22}}^i (V_{22}^i)^{-1} - \frac{T^i}{2} tr\left(\left(E_2^i - \widehat{E_2}^i\right)\left(E_2^i - \widehat{E_2}^i\right)'(V_{22}^i)^{-1}\right)\right\}. \tag{EC.17}$$

As a result, we have:

$$(V_{22}^i)^{-1} \,|\, R^i \sim W\left(T^i - 1,\, \left(T^i\widehat{V_{22}}^i\right)^{-1}\right) \tag{EC.18}$$

and

$$E_2^i \,|\, V_{22}^i,\, R^i, \sim N\left(\widehat{E_2}^i,\, \frac{1}{T^i}V_{22}^i\right). \tag{EC.19}$$

Finally, the posterior distributions of P and Q are,

$$(P, 1-P) \backsim D(S_{11} + 220, S_{12} + 20),\;\; (1-Q, Q) \backsim D(S_{21} + 40, S_{22} + 200), \tag{EC.20}$$

where S_{ij}, $i, j = 1, 2$, is the total number of one-step transitions from state i to state j. Then the procedure to draw samples from the joint posterior distribution is as follows:

1) draw states for each of the month t and sort the full sample data R into two sets, R^i, $i = 1,\, 2$.
2) $(\Sigma^i)^{-1} \,|\, R^i \sim W\left(T^i + \nu_\Sigma - k,\, \left(H^i + T^i\widehat{\Sigma}^i + (\widehat{A}^i)'Q^i\widehat{A}^i\right)^{-1}\right)$,
3) $a^i \,|\, (\Sigma^i)^{-1},\, R^i \sim N\,(\tilde{a}^i,\, \Sigma^i \otimes (F^i)^{-1})$,
4) $(V_{22}^i)^{-1} \,|\, R^i \sim W\left(T^i - 1,\, \left(T^i\widehat{V_{22}}^i\right)^{-1}\right)$,

5) $E_2^i \,|\, V_{22}^i,\, R^i, \sim N\left(\widehat{E_2}^i,\, \frac{1}{T^i} V_{22}^i\right)$,

6) $(P, 1-P) \backsim D(S_{11}+220, S_{12}+20)$, and $(1-Q, Q) \backsim D(S_{21}+40, S_{22}+200)$,,

7) Repeat steps $1) - 6)$.

We can, following Geweke and Zhou (1996), start the above Gibbs sampling procedure from any arbitrary initial value in the support of the posterior density. Let $g = M + Q$ denote the total number of iterations of the above loop. To eliminate the impact of the initial value, we disregard the first M draws of the burning period, and use the other Q draws as the draws from the posterior distribution. Then it is strait forward to compute relevant values, such as the posterior means, the posterior standard errors, the predictive means and the predictive covariances. In addition, one may be worried about the speed of convergence. Starting from different initial values, it turns out that the results converge fast and become virtually the same when the length of the burning period to be as small as $M = 2,000$. In this paper, M is set to be $10,000$ and Q is set to be $10,000$.

References

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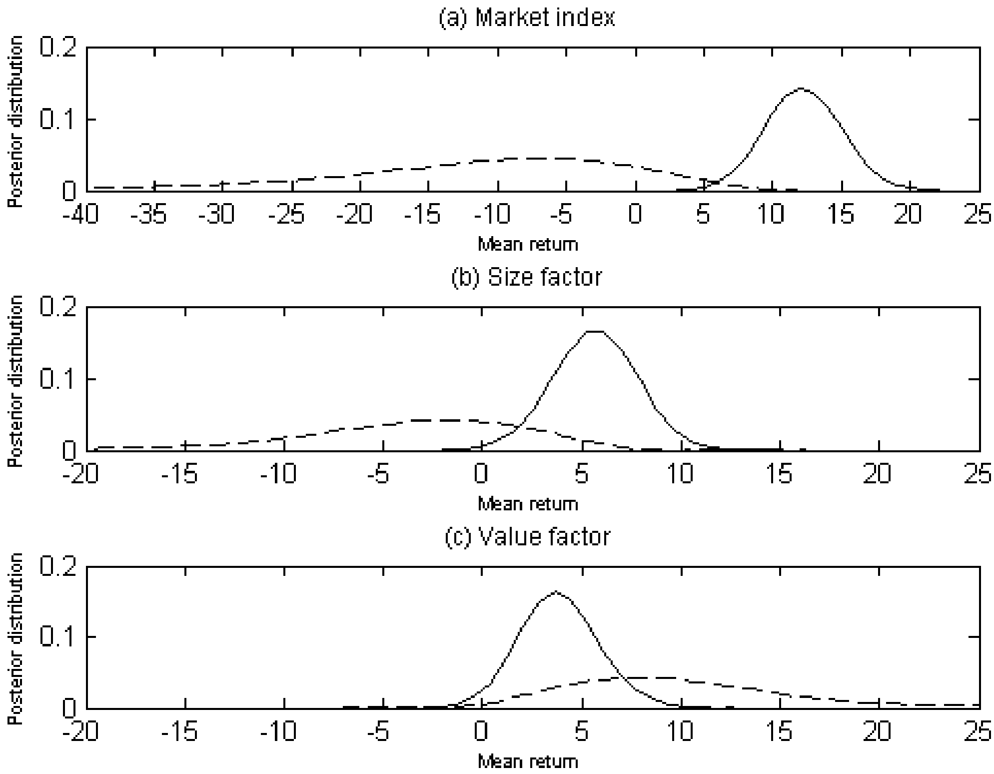

Figure EC.1 Posterior Distribution of Mean Returns.

Notes. This figure displays in percentage points the posterior distributions of annualized mean returns for the Fama-French three factors: (a) the market index, (b) the size factor, and (c) the value factor, based on the monthly returns of the Fama-French three factors from July 1963 to February 2006. The solid curves are posterior distributions for the bull market, while the dashed curves are posterior distributions for the bear market.

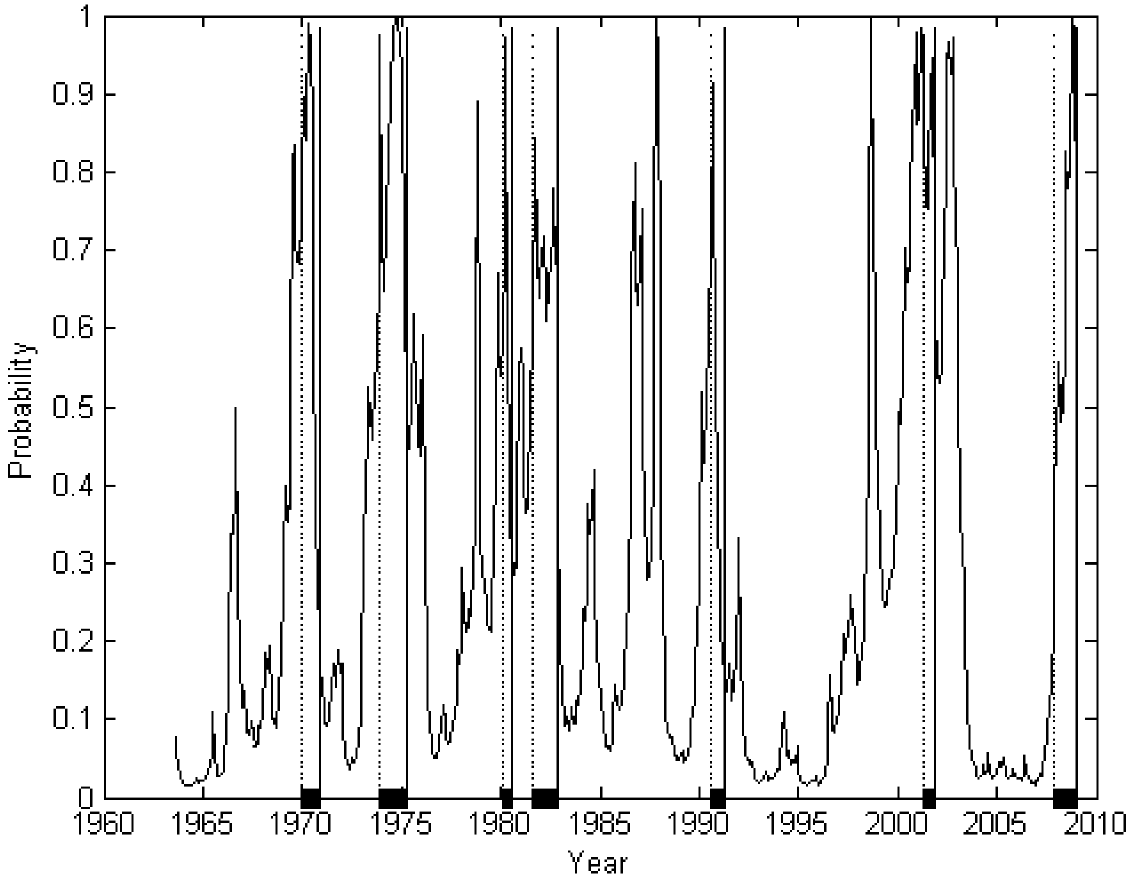

Figure EC.2 Probability in the Bear Regime.

Notes. This figure plots the empirical probability of being in the bear regime from July 1963 through December 2008. The vertical dotted and solid lines represent the National Bureau of Economic Research (NBER) peaks and troughs (except the last solid line, which does not represent the trough but represents the last month of our sample period), respectively.

Table EC.1 Portfolio Weights in November 2006

Portfolio	$c=\infty$						$c=5$					
	$\sigma_\alpha=0$		$\sigma_\alpha=1\%$		$\sigma_\alpha=\infty$		$\sigma_\alpha=0$		$\sigma_\alpha=1\%$		$\sigma_\alpha=\infty$	
	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS
MKT	42.0	37.0	-13.8	87.1	-59.9	136.5	42.0	37.0	1.2	33.5	0.0	0.0
SMB	29.0	13.0	24.3	73.8	22.0	162.6	29.0	13.0	0.1	0.0	-0.0	0.0
HML	88.0	24.0	64.2	19.6	49.4	42.6	88.0	24.0	1.3	13.9	-0.0	-0.0
S1B1	0	0	-76.0	-2.4	-142.8	-25.3	0	0	-80.1	5.8	-117.4	-2.0
S1B2	0	0	32.9	-1.2	61.8	6.2	0	0	28.4	0.6	25.9	-3.3
S1B3	0	0	-8.7	-12.3	-16.3	-36.2	0	0	-0.4	0.0	-0.0	-0.0
S1B4	0	0	67.8	-27.6	127.4	-34.3	0	0	71.8	-29.6	114.4	-36.6
S1B5	0	0	24.9	15.8	46.9	38.0	0	0	31.4	21.2	43.0	35.7
S2B1	0	0	-7.6	-8.0	-14.3	-22.2	0	0	-13.4	0.3	-7.6	-5.4
S2B2	0	0	-19.0	-10.5	-35.7	-26.5	0	0	-20.3	-1.8	-15.7	-1.2
S2B3	0	0	42.1	15.1	79.0	43.1	0	0	28.6	19.1	15.1	41.7
S2B4	0	0	9.3	-13.3	17.6	-27.1	0	0	11.9	-1.9	0.0	0.0
S2B5	0	0	-15.2	-0.1	-28.5	-20.9	0	0	-0.5	1.9	-0.0	-0.0
S3B1	0	0	-20.2	-0.7	-38.0	-7.1	0	0	-18.4	4.5	-9.0	9.0
S3B2	0	0	4.2	0.1	7.8	9.8	0	0	-0.3	0.8	-0.0	0.0
S3B3	0	0	-25.4	9.1	-47.7	8.1	0	0	-17.6	13.0	-3.6	3.6
S3B4	0	0	-26.8	4.7	-50.3	-5.5	0	0	-6.9	5.9	0.0	-0.0
S3B5	0	0	13.3	-21.9	25.0	-36.8	0	0	15.7	-15.7	0.0	-0.0
S4B1	0	0	68.7	-21.1	129.0	-18.0	0	0	55.7	-22.0	65.0	-25.8
S4B2	0	0	-51.6	7.6	-96.9	6.3	0	0	-50.1	5.8	-61.4	11.2
S4B3	0	0	3.6	-4.1	6.7	-1.9	0	0	-0.4	-0.1	-0.0	0.0
S4B4	0	0	22.3	4.3	41.8	6.6	0	0	20.8	0.6	8.9	-8.5
S4B5	0	0	-12.9	1.2	-24.3	1.2	0	0	-6.9	-0.2	-0.2	0.2
S5B1	0	0	35.1	5.0	66.0	31.2	0	0	0.9	2.1	11.9	13.0
S5B2	0	0	2.6	-2.0	5.0	1.5	0	0	-1.0	-4.2	-0.0	0.0
S5B3	0	0	26.6	-4.2	50.0	3.1	0	0	12.2	-8.3	0.0	0.0
S5B4	0	0	-7.9	5.7	-14.8	9.7	0	0	-1.8	0.6	0.0	-0.0
S5B5	0	0	-5.4	8.3	-10.2	2.9	0	0	0.6	0.9	-0.8	0.7

Notes. This table presents the optimal portfolio weights per $100 for a mean-variance utility investor with risk aversion coefficient equal to 10, under no margin requirements ($c=\infty$) and 20% margin requirements ($c=5$). For easier assessment of the resulting changes from incorporating regime switching, we report the differences (under the subtitle "RMS," RSM - SSM) between the portfolio weights under RSM and those under SSM (under the subtitle "SSM"). The mispricing priors imposed on the Fama-French three-factor model are $\sigma_\alpha=0$, 1%, and ∞, respectively. The results are for November 2006.

Table EC.2 Portfolio Weights in October 2008

Portfolio	$c=\infty$						$c=5$					
	$\sigma_\alpha=0$		$\sigma_\alpha=1\%$		$\sigma_\alpha=\infty$		$\sigma_\alpha=0$		$\sigma_\alpha=1\%$		$\sigma_\alpha=\infty$	
	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS	SSM	RMS
MKT	33.0	-46.0	-14.1	-40.4	-51.6	-136.5	33.0	-46.0	0.0	-54.6	-0.0	-1.9
SMB	28.0	-21.0	25.1	-57.8	23.8	-154.2	28.0	-21.0	-0.1	-32.6	0.0	0.0
HML	80.0	-31.0	50.1	-23.9	30.2	-0.8	80.0	-31.0	7.8	18.4	-0.0	0.0
S1B1	0	0	-75.2	40.4	-137.7	22.7	0	0	-76.4	41.5	-113.6	12.7
S1B2	0	0	31.9	-16.2	58.4	-7.3	0	0	26.6	-10.9	21.4	-14.6
S1B3	0	0	-14.2	21.7	-26.0	35.3	0	0	-2.5	10.0	-0.0	0.0
S1B4	0	0	69.9	-25.5	128.0	42.4	0	0	68.8	-24.5	109.6	6.4
S1B5	0	0	23.1	-24.9	42.2	-61.5	0	0	25.1	-27.0	30.1	-30.1
S2B1	0	0	-14.5	13.6	-26.5	34.1	0	0	-15.3	14.5	-11.5	11.5
S2B2	0	0	-16.1	12.5	-29.4	20.0	0	0	-14.2	10.6	-1.8	0.3
S2B3	0	0	36.3	-32.0	66.5	-61.4	0	0	28.5	-24.1	14.8	-14.8
S2B4	0	0	-1.1	7.3	-2.1	34.4	0	0	0.9	5.2	0.0	-0.0
S2B5	0	0	-16.8	16.6	-30.7	17.9	0	0	-1.2	1.0	0.0	-0.0
S3B1	0	0	-19.2	15.6	-35.2	26.5	0	0	-18.1	14.6	-5.6	-13.5
S3B2	0	0	9.3	-16.6	17.0	-33.0	0	0	2.8	-10.1	0.0	-4.6
S3B3	0	0	1.2	-3.6	2.3	-25.7	0	0	1.8	-4.2	-0.0	-18.8
S3B4	0	0	-17.2	10.5	-31.5	-8.2	0	0	-2.5	-4.2	-0.0	-13.5
S3B5	0	0	22.9	-2.1	41.9	54.7	0	0	23.2	-2.4	15.5	34.6
S4B1	0	0	69.5	-29.0	127.2	30.2	0	0	57.3	-16.8	65.2	12.9
S4B2	0	0	-49.5	23.3	-90.5	8.4	0	0	-44.4	18.2	-60.8	17.1
S4B3	0	0	-24.8	18.6	-45.3	36.6	0	0	-20.0	13.8	-13.1	2.2
S4B4	0	0	31.8	-16.1	58.2	-20.3	0	0	25.2	-9.5	15.5	-9.1
S4B5	0	0	-21.2	10.6	-38.8	-3.7	0	0	-16.4	5.9	-12.1	-5.7
S5B1	0	0	29.5	-27.5	54.1	-30.4	0	0	4.9	-2.9	9.3	-9.3
S5B2	0	0	10.0	-8.4	18.3	3.8	0	0	0.1	1.5	-0.0	-0.0
S5B3	0	0	15.5	-9.7	28.3	-2.9	0	0	4.2	1.7	-0.0	0.0
S5B4	0	0	-8.0	2.3	-14.6	-8.1	0	0	-3.7	-2.0	-0.1	0.1
S5B5	0	0	-1.9	1.9	-3.4	-22.6	0	0	-0.0	0.1	-0.0	-9.9

Notes. This table presents the optimal portfolio weights per $100 for a mean-variance utility investor with risk aversion coefficient equal to 10, under no margin requirements ($c=\infty$) and 20% margin requirements ($c=5$). For easier assessment of the resulting changes from incorporating regime switching, we report the differences (under the subtitle "RMS," RSM - SSM) between the portfolio weights under RSM and those under SSM (under the subtitle "SSM"). The mispricing priors imposed on the Fama-French three-factor model are $\sigma_\alpha=0$, 1%, and ∞, respectively. The results are for October 2008.