

e - c o m p a n i o n

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—“Optimal Control and Equilibrium Behavior of
Production-Inventory Systems” by Owen Q. Wu and Hong Chen,
Management Science, DOI 10.1287/mnsc.1100.1186.

Online Supplement

Proof of Proposition 1. For initial inventory levels $\mathbf{x}^a \neq \mathbf{x}^b$, let $\boldsymbol{\pi}^a = \{\pi_t^a = (\lambda_t^a, q_t^a, s_t^a) : t \geq 0\}$ and $\boldsymbol{\pi}^b = \{\pi_t^b = (\lambda_t^b, q_t^b, s_t^b) : t \geq 0\}$ be the corresponding optimal controls, and let $\{\mathbf{x}_t^a : t \geq 0\}$ and $\{\mathbf{x}_t^b : t \geq 0\}$ be the corresponding optimal inventory processes.

Now consider the initial inventory $\mathbf{x}^c = (\mathbf{x}^a + \mathbf{x}^b)/2$, and apply a control $\boldsymbol{\pi}^c = \{\pi_t^c = (\lambda_t^c, q_t^c, s_t^c) : t \geq 0\}$ with $\lambda_t^c = (\lambda_t^a + \lambda_t^b)/2$, $q_t^c = (q_t^a + q_t^b)/2$ and $s_t^c = (s_t^a + s_t^b)/2$. It is clear that $\boldsymbol{\pi}^c$ is an admissible control. From the balance equations in (3), the inventory process starting from \mathbf{x}^c under the control $\boldsymbol{\pi}^c$ is $\mathbf{x}_t^c = (\mathbf{x}_t^a + \mathbf{x}_t^b)/2$. Thus, we have

$$\begin{aligned} V(\mathbf{x}^c, \mathbf{p}, \mathbf{k}) &\geq \mathbb{E} \int_0^\infty e^{-Rt} \left[s_t^c p_{2t} - \lambda_t^c p_{1t} - g(q_t^c, \mathbf{x}_t^c, \mathbf{k}_t) - h(\mathbf{x}_t^c, \mathbf{k}_t) \right] dt \\ &> \frac{1}{2} \mathbb{E} \int_0^\infty e^{-Rt} \left[(s_t^a + s_t^b) p_{2t} - (\lambda_t^a + \lambda_t^b) p_{1t} - g(q_t^a, \mathbf{x}_t^a, \mathbf{k}_t) - g(q_t^b, \mathbf{x}_t^b, \mathbf{k}_t) - h(\mathbf{x}_t^a, \mathbf{k}_t) - h(\mathbf{x}_t^b, \mathbf{k}_t) \right] dt \\ &= \frac{1}{2} \left[V(\mathbf{x}^a, \mathbf{p}, \mathbf{k}) + V(\mathbf{x}^b, \mathbf{p}, \mathbf{k}) \right], \end{aligned}$$

where the first inequality is due to the fact that $\boldsymbol{\pi}^c$ is admissible but not necessarily optimal, and the second inequality is from the definition of $\boldsymbol{\pi}^c$, \mathbf{x}^c and the strict convexity of g and h . This proves the strict concavity of $V(\mathbf{x}, \mathbf{p}, \mathbf{k})$ in \mathbf{x} .

When $\mathbf{x} > 0$, from (9), $\lambda = \bar{\lambda}$ is optimal if $\frac{\partial V(\mathbf{x}, \mathbf{p}, \mathbf{k})}{\partial x_1} > p_1$, and $\lambda = \underline{\lambda}$ is optimal if $\frac{\partial V(\mathbf{x}, \mathbf{p}, \mathbf{k})}{\partial x_1} < p_1$. Due to the strict concavity of V , there exists a unique (possibly infinite) threshold $\hat{x}_1(x_2, \mathbf{p}, \mathbf{k})$ such that the optimal procurement control is of a threshold type in the proposition. Moreover, if it is finite, the threshold $\hat{x}_1(x_2, \mathbf{p}, \mathbf{k})$ is determined by equating marginal profit to price. The proof for the properties of the optimal sales rate are parallel and omitted.

The strict concavity of the value function implies that V has different gradient (with respect to \mathbf{x}) at different inventory levels, which in turn implies that there is at most one intersection of the two thresholds $\hat{x}_1(x_2, \mathbf{p}, \mathbf{k})$ and $\hat{x}_2(x_1, \mathbf{p}, \mathbf{k})$.

The optimality equation for q^* follows from (9) and the strict convexity of $g(q, \mathbf{x}, \mathbf{k})$ in q .

The proof of Proposition 2 will use the following lemma.

Lemma 1 *Let $\mathbf{F} = \mathbf{CA} + \frac{\rho^2}{4}\mathbf{I}$ as defined in Proposition 2. Then,*

$$(i) \quad \text{tr}[\mathbf{F}] > \frac{\rho^2}{2};$$

$$(ii) \quad \det[\mathbf{F}] > \frac{\rho^2}{4} \text{tr}[\mathbf{F}] - \frac{\rho^4}{16} > \frac{\rho^4}{16};$$

(iii) $\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}] \geq 0$, where equality holds if and only if \mathbf{F} is a diagonal matrix.

Proof. By definition,

$$\mathbf{F} = \begin{bmatrix} c_{11}A_1 + \frac{c_{11}-c_{12}}{a_2} + \frac{\rho^2}{4} & c_{12}A_2 - \frac{c_{11}-c_{12}}{a_2} \\ c_{12}A_1 - \frac{c_{22}-c_{12}}{a_2} & c_{22}A_2 + \frac{c_{22}-c_{12}}{a_2} + \frac{\rho^2}{4} \end{bmatrix} \equiv \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

(i) $\text{tr}[\mathbf{F}] = c_{11}A_1 + c_{22}A_2 + \frac{c_{11}+c_{22}-2c_{12}}{a_2} + \frac{\rho^2}{2} > \frac{\rho^2}{2}$, where we used $c_{11}A_1 + c_{22}A_2 > 0$, $c_{12} < \sqrt{c_{11}c_{22}}$ and $c_{11} + c_{22} - 2\sqrt{c_{11}c_{22}} \geq 0$.

(ii) $\det[\mathbf{F}] = \det[\mathbf{CA} + \frac{\rho^2}{4}\mathbf{I}] = \det[\mathbf{CA}] + \frac{\rho^2}{4}(\text{tr}[\mathbf{F}] - \frac{\rho^2}{2}) + \frac{\rho^4}{16} > \frac{\rho^2}{4}\text{tr}[\mathbf{F}] - \frac{\rho^4}{16}$, where the inequality follows from $\det[\mathbf{C}] > 0$, $\det[\mathbf{A}] > 0$ and part (i).

(iii) When \mathbf{F} is a diagonal matrix, it has zero off-diagonal elements and, therefore, $A_1 = \frac{c_{22}-c_{12}}{c_{12}a_2}$ and $A_2 = \frac{c_{11}-c_{12}}{c_{12}a_2}$, which imply that $f_{11} = f_{22} = \frac{c_{11}c_{22}-c_{12}^2}{c_{12}a_2} + \frac{\rho^2}{4}$. Then, $\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}] = 0$.

When \mathbf{F} is not a diagonal matrix, we show that $\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}] > 0$, or equivalently,

$$(f_{11} - f_{22})^2 > -4f_{12}f_{21}. \quad (\text{A.1})$$

Consider four cases:

(a) $f_{12}f_{21} > 0$. It is clear that (A.1) holds in this case.

(b) $f_{12}f_{21} = 0$. Since \mathbf{F} is not diagonal, exactly one of the two equalities $A_2 = \frac{c_{11}-c_{12}}{c_{12}a_2}$ and $A_1 = \frac{c_{22}-c_{12}}{c_{12}a_2}$ holds, implying that $c_{11}A_1 - c_{22}A_2 + \frac{c_{11}-c_{22}}{a_2} \neq 0$ or $f_{11} - f_{22} \neq 0$. Hence (A.1) holds.

(c) $f_{21} > 0$ and $f_{12} < 0$, or

$$A_1 > \frac{c_{22}-c_{12}}{c_{12}a_2} \quad \text{and} \quad A_2 < \frac{c_{11}-c_{12}}{c_{12}a_2}. \quad (\text{A.2})$$

The second inequality in (A.2) implies $c_{11} > c_{12}$ because $A_2 > 0$. We can then derive

$$(c_{11} - c_{12})A_1 > (c_{22} - c_{12})A_2. \quad (\text{A.3})$$

To see this, note that (A.3) clearly holds if $c_{22} \leq c_{12}$. When $c_{22} > c_{12}$, (A.2) implies that $(c_{11} - c_{12})A_1 > \frac{(c_{11}-c_{12})(c_{22}-c_{12})}{c_{12}a_2} > (c_{22} - c_{12})A_2$, i.e., (A.3) holds. Thus,

$$c_{11}A_1 - c_{22}A_2 + \frac{c_{11}-c_{22}}{a_2} > c_{12}A_1 - c_{12}A_2 + \frac{c_{11}-c_{22}}{a_2} > \frac{c_{22}-c_{12}+c_{12}-c_{11}+c_{11}-c_{22}}{a_2} = 0,$$

where the first inequality follows from (A.3) and the second follows from (A.2). Hence,

$$(f_{11} - f_{22})^2 = \left(c_{11}A_1 - c_{22}A_2 + \frac{c_{11}-c_{22}}{a_2}\right)^2 > \left(c_{12}A_1 - c_{12}A_2 + \frac{c_{11}-c_{22}}{a_2}\right)^2 = (f_{21} - f_{12})^2 \geq -4f_{12}f_{21}.$$

(d) $f_{21} < 0$ and $f_{12} > 0$. The argument is completely analogous to case (c).

Proof of Proposition 2. We first show that conditions in (23) establish a rational expectations equilibrium. From the last three equations in (22), we have

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{A}\mathbf{p}_t + \mathbf{a} + \mathbf{I}_1\mathbf{k}_t. \quad (\text{A.4})$$

Differentiating the first equation in (22), we have

$$d\mathbf{p}_t = \mathbf{C}_1 d\mathbf{X}_t - \mathbf{D}_1 d\mathbf{k}_t.$$

Substituting (A.4) and the factor process in (19) into the above equation yields

$$d\mathbf{p}_t = [\mathbf{C}_1 \mathbf{A}(\mathbf{p}_t + \mathbf{A}^{-1} \mathbf{a}) + (\mathbf{C}_1 \mathbf{I}_1 - \mathbf{D}_1 \mathbf{K}) \mathbf{k}_t] dt - \mathbf{D}_1 \boldsymbol{\sigma}_0(\mathbf{k}_t) d\mathbf{w}_t. \quad (\text{A.5})$$

To find the rational expectations equilibrium, we match the coefficients in (A.5) with those in (20), which exactly leads to conditions in (23). Next, we prove the existence, uniqueness, and other equilibrium properties.

1. We first show that \mathbf{F} has positive real eigenvalues and linearly independent eigenvectors. Let ξ_1 and ξ_2 denote the eigenvalues of \mathbf{F} , which are the roots to the equation:

$$\det[\mathbf{F} - \xi \mathbf{I}] = \xi^2 - \text{tr}[\mathbf{F}] \xi + \det[\mathbf{F}] = 0.$$

When \mathbf{F} is a diagonal matrix, Lemma 1(iii) implies $\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}] = 0$. Thus, $\xi_1 = \xi_2 = \frac{c_{11}c_{22} - c_{12}^2}{c_{12}a_2} + \frac{\rho^2}{4} > 0$, and \mathbf{F} has two linearly independent eigenvectors, e.g., $[0, 1]$ and $[1, 0]$.

When \mathbf{F} is not a diagonal matrix, from Lemma 1(iii), $\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}] > 0$, meaning that \mathbf{F} has two different real eigenvalues, and therefore, \mathbf{F} has two linearly independent eigenvectors. Furthermore, from Lemma 1, the eigenvalues must be greater than $\frac{\rho^2}{4}$:

$$\min\{\xi_1, \xi_2\} = \frac{\text{tr}[\mathbf{F}] - \sqrt{\text{tr}[\mathbf{F}]^2 - 4\det[\mathbf{F}]}}{2} > \frac{\text{tr}[\mathbf{F}] - \sqrt{\text{tr}[\mathbf{F}]^2 - \text{tr}[\mathbf{F}]\rho^2 + \frac{\rho^4}{4}}}{2} = \frac{\text{tr}[\mathbf{F}] - (\text{tr}[\mathbf{F}] - \frac{\rho^2}{2})}{2} = \frac{\rho^2}{4}. \quad (\text{A.6})$$

2. Next, we show that there exists a solution to $\mathbf{B}^2 - \rho\mathbf{B} = \mathbf{CA}$ that satisfies (21), and that the solution \mathbf{B} has negative real eigenvalues. Let $\boldsymbol{\Xi} = \text{diag}[\xi_1, \xi_2]$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2]$, where \mathbf{v}_i is the eigenvector corresponding to ξ_i , and \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. Then, we have $\mathbf{F} = \mathbf{V}\boldsymbol{\Xi}\mathbf{V}^{-1}$. We define $\sqrt{\boldsymbol{\Xi}} := \text{diag}[\sqrt{\xi_1}, \sqrt{\xi_2}]$, and $\sqrt{\mathbf{F}} := \mathbf{V}\sqrt{\boldsymbol{\Xi}}\mathbf{V}^{-1}$. Note that $\text{tr}[\sqrt{\mathbf{F}}] = \text{tr}[\sqrt{\boldsymbol{\Xi}}]$ and $\det[\sqrt{\mathbf{F}}] = \det[\sqrt{\boldsymbol{\Xi}}]$.

We show that $\mathbf{B} = \frac{\rho}{2}\mathbf{I} - \sqrt{\mathbf{F}}$ is a solution to $\mathbf{B}^2 - \rho\mathbf{B} = \mathbf{CA}$:

$$\mathbf{B}^2 - \rho\mathbf{B} = \frac{\rho^2}{4}\mathbf{I} + \mathbf{F} - \rho\sqrt{\mathbf{F}} - \frac{\rho^2}{2}\mathbf{I} + \rho\sqrt{\mathbf{F}} = -\frac{\rho^2}{4}\mathbf{I} + \mathbf{F} = \mathbf{CA}.$$

Furthermore, \mathbf{B} satisfies (21) because

$$\text{tr}[\mathbf{B}] = \text{tr}[\frac{\rho}{2}\mathbf{I} - \sqrt{\mathbf{F}}] = \rho - \text{tr}[\sqrt{\boldsymbol{\Xi}}] = \rho - \sqrt{\xi_1 + \xi_2 + 2\sqrt{\xi_1\xi_2}} < \rho - \sqrt{\frac{\rho^2}{2} + 2\sqrt{\frac{\rho^4}{16}}} = 0,$$

where the last inequality follows from Lemma 1, and

$$\det[\mathbf{B}] = \det[\sqrt{\mathbf{F}}] - \frac{\rho}{2}\text{tr}[\sqrt{\mathbf{F}}] + \frac{\rho^2}{4} = \det[\sqrt{\boldsymbol{\Xi}}] - \frac{\rho}{2}\text{tr}[\sqrt{\boldsymbol{\Xi}}] + \frac{\rho^2}{4} = (\sqrt{\xi_1} - \frac{\rho}{2})(\sqrt{\xi_2} - \frac{\rho}{2}) > 0,$$

where the last inequality follows from (A.6). The above conditions imply that the eigenvalues of \mathbf{B}

have negative real parts. The eigenvalues are actually real, because

$$\begin{aligned}\mathrm{tr}[\mathbf{B}]^2 - 4\det[\mathbf{B}] &= (\rho - \mathrm{tr}[\sqrt{\boldsymbol{\Xi}}])^2 - 4\det[\sqrt{\boldsymbol{\Xi}}] + 2\rho\mathrm{tr}[\sqrt{\boldsymbol{\Xi}}] - \rho^2 = \mathrm{tr}[\sqrt{\boldsymbol{\Xi}}]^2 - 4\det[\sqrt{\boldsymbol{\Xi}}] \\ &= (\sqrt{\xi_1} + \sqrt{\xi_2})^2 - 4\sqrt{\xi_1\xi_2} \geq 0.\end{aligned}$$

When $\xi_1 \neq \xi_2$, \mathbf{B} has linearly independent eigenvectors; when $\xi_1 = \xi_2$, from Lemma 1(iii), \mathbf{F} and \mathbf{B} are diagonal matrices, and \mathbf{B} also has linearly independent eigenvectors.

We show in passing that $\mathbf{B}_1 = (\mathbf{B} - \rho\mathbf{I})^{-1}$ also has negative real eigenvalues. This will be used later. First, by (21),

$$\det[\mathbf{B}_1] = (\det[\mathbf{B} - \rho\mathbf{I}])^{-1} = (\rho^2 - \mathrm{tr}[\mathbf{B}]\rho + \det[\mathbf{B}])^{-1} > 0, \quad (\text{A.7})$$

Secondly,

$$\mathrm{tr}[\mathbf{B}_1] = \mathrm{tr}[(\mathbf{B} - \rho\mathbf{I})^{-1}] = \frac{\mathrm{tr}[\mathbf{B} - \rho\mathbf{I}]}{\det[\mathbf{B} - \rho\mathbf{I}]} = \frac{\mathrm{tr}[\mathbf{B}] - 2\rho}{\det[\mathbf{B} - \rho\mathbf{I}]} < 0, \quad (\text{A.8})$$

where the second equality is because the matrices here are all two by two matrices. Thirdly,

$$\mathrm{tr}[\mathbf{B}_1]^2 - 4\det[\mathbf{B}_1] = \frac{(\mathrm{tr}[\mathbf{B}] - 2\rho)^2 - 4(\rho^2 - \mathrm{tr}[\mathbf{B}]\rho + \det[\mathbf{B}])}{\det[\mathbf{B} - \rho\mathbf{I}]^2} = \frac{\mathrm{tr}[\mathbf{B}]^2 - 4\det[\mathbf{B}]}{\det[\mathbf{B} - \rho\mathbf{I}]^2} \geq 0.$$

Hence, both \mathbf{B} and \mathbf{B}_1 have negative real eigenvalues.

3. Now, we prove that $\mathbf{B}^2 - \rho\mathbf{B} = \mathbf{CA}$ has a unique solution that satisfies (21). Suppose \mathbf{B}_a and \mathbf{B}_b are two such solutions. Then,

$$\mathbf{CA} = \mathbf{B}_b^2 - \rho\mathbf{B}_b = \mathbf{B}_b(\mathbf{B}_b^2 - \rho\mathbf{B}_b)\mathbf{B}_b^{-1} = \mathbf{B}_b(\mathbf{B}_a^2 - \rho\mathbf{B}_a)\mathbf{B}_b^{-1} = (\mathbf{B}_b\mathbf{B}_a\mathbf{B}_b^{-1})^2 - \rho\mathbf{B}_b\mathbf{B}_a\mathbf{B}_b^{-1}.$$

Thus, $\mathbf{B}_c \equiv \mathbf{B}_b\mathbf{B}_a\mathbf{B}_b^{-1}$ is also a solution to $\mathbf{B}^2 - \rho\mathbf{B} = \mathbf{CA}$ and it also satisfies (21): $\mathrm{tr}(\mathbf{B}_c) = \mathrm{tr}(\mathbf{B}_a) < 0$ and $\det(\mathbf{B}_c) = \det(\mathbf{B}_a) > 0$. Write $\mathbf{B}^2 - \rho\mathbf{B} = \mathbf{CA} \equiv \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ in component form:

$$\begin{aligned}b_{11}^2 - \rho b_{11} + b_{12}b_{21} &= C_{11}, & b_{22}^2 - \rho b_{22} + b_{12}b_{21} &= C_{22}, \\ b_{12}(\mathrm{tr}[\mathbf{B}] - \rho) &= C_{12}, & b_{21}(\mathrm{tr}[\mathbf{B}] - \rho) &= C_{21}.\end{aligned} \quad (\text{A.9})$$

The last equation in (A.9) implies that $b_{c21}(\mathrm{tr}[\mathbf{B}_c] - \rho) = b_{a21}(\mathrm{tr}[\mathbf{B}_a] - \rho)$. Since $\mathrm{tr}(\mathbf{B}_c) = \mathrm{tr}(\mathbf{B}_a) < 0$, we must have $b_{c21} = b_{a21}$. Similarly, we have $b_{c12} = b_{a12}$. Next we show that the diagonal elements of \mathbf{B}_c and \mathbf{B}_a are equal as well.

Suppose $b_{c11} \neq b_{a11}$ are two different solutions to the first equation in (A.9). As $\mathrm{tr}(\mathbf{B}_c) = \mathrm{tr}(\mathbf{B}_a)$, we have $b_{c22} \neq b_{a22}$, which are two different solutions to the second equation in (A.9). Thus, $b_{c11} + b_{a11} = \frac{\rho}{2}$ and $b_{c22} + b_{a22} = \frac{\rho}{2}$. Since $\mathrm{tr}[\mathbf{B}_c] = \mathrm{tr}[\mathbf{B}_a]$, we must have $b_{c11} = b_{a22}$ and $b_{a11} = b_{c22}$. Consequently, the first two equations in (A.9) have the same set of roots, implying $C_{11} = C_{22}$. This leads to $b_{a11}^2 - \rho b_{a11} = b_{a22}^2 - \rho b_{a22}$ or $(\mathrm{tr}[\mathbf{B}_a] - \rho)(b_{a11} - b_{a22}) = 0$. Since $\mathrm{tr}[\mathbf{B}_a] < 0$, we must

have $b_{a11} = b_{a22}$. Similarly, we have $b_{c11} = b_{c22}$, but then $b_{c11} \neq b_{a11}$ implies $\text{tr}[\mathbf{B}_c] \neq \text{tr}[\mathbf{B}_a]$, a contradiction.

Hence, we have $\mathbf{B}_a = \mathbf{B}_c \equiv \mathbf{B}_b \mathbf{B}_a \mathbf{B}_b^{-1}$. This implies that \mathbf{B}_a and \mathbf{B}_b commute. Therefore,

$$e^{\mathbf{B}_a t} e^{\mathbf{B}_b t} = e^{(\mathbf{B}_a + \mathbf{B}_b)t}, \quad \forall t \geq 0.$$

Since both \mathbf{B}_a and \mathbf{B}_b are stable matrices, the above quantity approaches to zero matrix as $t \rightarrow \infty$, implying $\mathbf{B}_a + \mathbf{B}_b$ is also stable. Consequently, $\mathbf{B}_a + \mathbf{B}_b - \rho \mathbf{I}$ is a non-singular matrix.

Now notice that $\mathbf{0} = \mathbf{B}_a^2 - \rho \mathbf{B}_a - \mathbf{B}_b^2 + \rho \mathbf{B}_b = (\mathbf{B}_a - \mathbf{B}_b)(\mathbf{B}_a + \mathbf{B}_b - \rho \mathbf{I})$. As $\mathbf{B}_a + \mathbf{B}_b - \rho \mathbf{I}$ is non-singular, we must have $\mathbf{B}_a - \mathbf{B}_b = \mathbf{0}$. This proves the uniqueness of \mathbf{B} .

4. We prove that $\mathbf{D} = \mathbf{C}_1 \mathbf{I}_1 - \mathbf{B}_1 \mathbf{D} \mathbf{K}$ uniquely determines \mathbf{D} . Horn and Johnson (1991) present the Kronecker products representation for the matrix equation (see Chapter 4 of their book). Thereby we can rewrite the above matrix equation in vector form:

$$\text{vec}[\mathbf{D}] = \text{vec}[\mathbf{C}_1 \mathbf{I}_1] - (\mathbf{K}^\top \otimes \mathbf{B}_1) \text{vec}[\mathbf{D}].$$

Let ζ_1 and $\zeta_2 < 0$ be the eigenvalues of \mathbf{B}_1 , and let λ_1 and λ_2 be the eigenvalues of \mathbf{K} . Then the eigenvalues of $\mathbf{K}^\top \otimes \mathbf{B}_1$ are $\zeta_1 \lambda_1$, $\zeta_1 \lambda_2$, $\zeta_2 \lambda_1$ and $\zeta_2 \lambda_2$ (see Horn and Johnson 1991). We have shown at the end of step 2 that $\zeta_1 < 0$ and $\zeta_2 < 0$. We assumed in the special case that the real parts of λ_1 and λ_2 are negative. Thus, all the eigenvalues of $\mathbf{K}^\top \otimes \mathbf{B}_1$ have positive real parts, and therefore $\det[\mathbf{K}^\top \otimes \mathbf{B}_1 + \mathbf{I}] \neq 0$. Hence, \mathbf{D} is uniquely determined from the above equation.

5. We now derive the equilibrium processes. The equilibrium price process is given in (20) with \mathbf{B} and \mathbf{D} solved above. To derive the inventory process, we have, from the first equation of (22):

$$\mathbf{p}_t = \mathbf{C}_1 (\mathbf{X}_t + \mathbf{C}^{-1}(\mathbf{B}\mathbf{m} - \mathbf{c})) - \mathbf{D}_1 \mathbf{k}_t. \quad (\text{A.10})$$

Substituting (A.10) into (A.4), we have

$$\frac{d\mathbf{X}_t}{dt} = \mathbf{A} \mathbf{C}_1 (\mathbf{X}_t + \mathbf{C}^{-1}(\mathbf{B}\mathbf{m} - \mathbf{c})) + \mathbf{a} + (\mathbf{I}_1 - \mathbf{A} \mathbf{D}_1) \mathbf{k}_t.$$

The equation $\mathbf{B}^2 - \rho \mathbf{B} = \mathbf{C} \mathbf{A}$ is equivalent to $\mathbf{B} = \mathbf{C}_1 \mathbf{A}$. From Lemma 2(i), \mathbf{C}_1 is symmetric, and therefore, $\mathbf{B}^\top = \mathbf{A} \mathbf{C}_1$. Using this relation, we have $\mathbf{I} = \mathbf{A} \mathbf{C}_1 \mathbf{C}_1^{-1} \mathbf{A}^{-1} = \mathbf{B}^\top \mathbf{C}^{-1} (\mathbf{B} - \rho \mathbf{I}) \mathbf{A}^{-1}$. Then, the equilibrium inventory process becomes:

$$\begin{aligned} \frac{d\mathbf{X}_t}{dt} &= \mathbf{B}^\top (\mathbf{X}_t - \mathbf{C}^{-1}(\mathbf{c} - \mathbf{B}\mathbf{m})) - \mathbf{B}^\top \mathbf{C}^{-1} (\mathbf{B} - \rho \mathbf{I}) \mathbf{m} + (\mathbf{I}_1 - \mathbf{A} \mathbf{D}_1) \mathbf{k}_t. \\ &= \mathbf{B}^\top (\mathbf{X}_t - \mathbf{m}_x) + (\mathbf{I}_1 - \mathbf{A} \mathbf{D}_1) \mathbf{k}_t, \end{aligned}$$

where $\mathbf{m}_x = \mathbf{C}^{-1}(\mathbf{c} - \rho\mathbf{m})$. This, together with (A.10), implies

$$\begin{aligned}\mathbf{p}_t &= \mathbf{C}_1(\mathbf{X}_t - \mathbf{C}^{-1}(\mathbf{c} - \rho\mathbf{m}) + \mathbf{C}^{-1}(\mathbf{B} - \rho\mathbf{I})\mathbf{m}) - \mathbf{D}_1\mathbf{k}_t \\ &= \mathbf{C}_1(\mathbf{X}_t - \mathbf{m}_x) + \mathbf{m} - \mathbf{D}_1\mathbf{k}_t.\end{aligned}$$

6. Finally, we show that the equilibrium processes are uniformly bounded for $t \geq 0$, and in particular, inventory and price processes are bounded above zero under certain parameter settings.

For simplicity, assume $\mathbf{X}_0 = \mathbf{m}_x$. Then, the explicit solution for \mathbf{X}_t satisfying (25) is:

$$\begin{aligned}\mathbf{X}_t &= e^{\mathbf{B}^\top t}\mathbf{X}_0 + \int_0^t e^{\mathbf{B}^\top(t-u)}(-\mathbf{B}^\top\mathbf{m}_x + (\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1)\mathbf{k}_u)du, \\ &= \mathbf{m}_x + \int_0^t e^{\mathbf{B}^\top(t-u)}(\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1)\mathbf{k}_u du,\end{aligned}\tag{A.11}$$

To ensure that the inventory is always positive, we need to show that the second term in (A.11) is uniformly bounded by $\|\mathbf{m}_x\|$. Let $\|\mathbf{v}\|$ denote the super norm of $\mathbf{v} \in \mathcal{Z}^2$, the complex space, and let $\|\mathbf{Q}\| = \sup\{\|\mathbf{Q}\mathbf{v}\| : \|\mathbf{v}\| \leq 1, \mathbf{v} \in \mathcal{Z}^2\}$ be the norm of linear transformation \mathbf{Q} on \mathcal{Z}^2 . Then, we have

$$\begin{aligned}\left\|\int_0^t e^{\mathbf{B}^\top(t-u)}(\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1)\mathbf{k}_u du\right\| &\leq \int_0^t \|e^{\mathbf{B}^\top(t-u)}\| \|(\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1)\| \|\mathbf{k}_u\| du \\ &\leq K \|(\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1)\| \int_0^t \|e^{\mathbf{B}^\top u}\| du,\end{aligned}$$

where $\|\mathbf{k}_t\| \leq K$ for $t \geq 0$. We need to show that $\int_0^t \|e^{\mathbf{B}^\top u}\| du$ is also uniformly bounded. To see this, recall in Step 2 we have shown that \mathbf{B}^\top has two linearly independent eigenvectors $\tilde{\mathbf{V}} = [\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2]$, with negative real eigenvalues $\tilde{\mathbf{\Xi}} = \text{diag}[\tilde{\xi}_1, \tilde{\xi}_2]$. Then, $e^{\mathbf{B}^\top t} = \tilde{\mathbf{V}}e^{\tilde{\mathbf{\Xi}}t}\tilde{\mathbf{V}}^{-1}$, and we have

$$\int_0^t \|e^{\mathbf{B}^\top u}\| du \leq \int_0^t \|\tilde{\mathbf{V}}\| \|e^{\tilde{\mathbf{\Xi}}u}\| \|\tilde{\mathbf{V}}^{-1}\| du \leq \|\tilde{\mathbf{V}}\| \|\tilde{\mathbf{V}}^{-1}\| \int_0^t (|e^{\tilde{\xi}_1 u}| + |e^{\tilde{\xi}_2 u}|) du \equiv K_1, \quad \forall t \geq 0.$$

Hence, when the factor process \mathbf{k}_t is such that

$$\mathbf{k}_t \leq K \leq \frac{\|\mathbf{m}_x\|}{K_1 \|\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1\|}, \quad \forall t \geq 0,$$

the second term in (A.11) is uniformly bounded by $\|\mathbf{m}_x\|$, ensuring that $\mathbf{X}_t \geq 0$ for $t \geq 0$.

Using a similar approach, we can show that the price process is also uniformly bounded. Consider the process $\mathbf{z}_t \equiv \mathbf{p}_t + \mathbf{D}_1\mathbf{k}_t$, and assume $\mathbf{z}_0 = \mathbf{p}_0 = \mathbf{m}$. Using (A.5) and the factor process in (19), we derive the following equation for \mathbf{z}_t :

$$d\mathbf{z}_t = [\mathbf{B}(\mathbf{z}_t - \mathbf{m}) + (\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1)\mathbf{k}_t] dt.$$

The explicit solution to the above equation is:

$$\mathbf{z}_t = \mathbf{m} + \int_0^t e^{\mathbf{B}(t-u)}(\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1)\mathbf{k}_u du.$$

Thus,

$$\mathbf{p}_t = \mathbf{m} + \int_0^t e^{\mathbf{B}(t-u)}(\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1)\mathbf{k}_u du - \mathbf{D}_1\mathbf{k}_t.$$

To ensure positive price process, we need to show that the last two terms above are uniformly bounded by $\|\mathbf{m}\|$. We have

$$\begin{aligned} \left\| \int_0^t e^{\mathbf{B}(t-u)}(\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1)\mathbf{k}_u du - \mathbf{D}_1\mathbf{k}_t \right\| &\leq \int_0^t \|e^{\mathbf{B}(t-u)}\| \|\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1\| \|\mathbf{k}_u\| du + \|\mathbf{D}_1\| \|\mathbf{k}_t\| \\ &\leq K \left(\|\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1\| \int_0^t \|e^{\mathbf{B}u}\| du + \|\mathbf{D}_1\| \right) \equiv K \cdot K_2 \end{aligned}$$

Hence, when the factor process \mathbf{k}_t is such that

$$\mathbf{k}_t \leq K \leq \|\mathbf{m}\|/K_2,$$

we can ensure that $\mathbf{p}_t \geq 0$ for $t \geq 0$.

The proof of Proposition 2 used part (i) of the following lemma. The rest of the lemma will be used in the proof of Proposition 3.

Lemma 2 *Under a rational expectations equilibrium,*

- (i) $\mathbf{C}_1 = \mathbf{C}_1^\top$, $\det[\mathbf{C}_1] > 0$, $\mathbf{C}_1 < 0$;
- (ii) $(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 > 0$ for any $r > 0$;
- (iii) $\mathbf{B}_1(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 < 0$ for any $r > 0$;
- (iv) $\mathbf{B}\mathbf{C}_1$ and $\mathbf{B}\mathbf{C}$ are symmetric matrices;
- (v) If \mathbf{B} has negative diagonal elements, then $\mathbf{C}^{-1}(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1$ has positive diagonal elements for any $r > 0$.

Proof. (i) The first condition in (23) is equivalent to $\mathbf{B} = \mathbf{C}_1\mathbf{A}$. By the definitions of \mathbf{B}_1 and \mathbf{C}_1 , we have $\mathbf{B} = \mathbf{C}\mathbf{C}_1^{-1} + \rho\mathbf{I}$. Thus, we have $\mathbf{C}_1\mathbf{A} = \mathbf{C}\mathbf{C}_1^{-1} + \rho\mathbf{I}$. Let $\mathbf{C}_1 = [\tilde{c}_{ij}]$, then this equation can be written as

$$\begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{bmatrix} \mathbf{A} = \frac{1}{\det[\mathbf{C}_1]} \mathbf{C} \begin{bmatrix} \tilde{c}_{22} & -\tilde{c}_{12} \\ -\tilde{c}_{21} & \tilde{c}_{11} \end{bmatrix} + \rho\mathbf{I}.$$

Regarding $\det[\mathbf{C}_1]$ as a parameter, we can solve the above linear system for \tilde{c}_{ij} :

$$\begin{aligned} \tilde{c}_{11} &= (c_{11} + (A_2 + \frac{1}{a_2})\det[\mathbf{C}_1])\rho\Delta, \\ \tilde{c}_{22} &= (c_{22} + (A_1 + \frac{1}{a_2})\det[\mathbf{C}_1])\rho\Delta, \\ \tilde{c}_{12} = \tilde{c}_{21} &= (c_{12} + \frac{1}{a_2}\det[\mathbf{C}_1])\rho\Delta, \end{aligned}$$

where $\Delta = (\det[\mathbf{C}_1]\det[\mathbf{A}] - \det[\mathbf{C}]\det[\mathbf{C}_1]^{-1})^{-1}$. Thus, \mathbf{C}_1 is symmetric. Notice that $\Delta^{-1} = \det[\mathbf{B}] - \det[\mathbf{B}^{-1}] = \det[\mathbf{B}] - \det[\mathbf{B} - \rho\mathbf{I}] = -\rho^2 + \text{tr}[\mathbf{B}]\rho < 0$, as $\text{tr}[\mathbf{B}] < 0$ due to (21). We also have $\det[\mathbf{C}_1] = \det[\mathbf{B}_1]\det[\mathbf{C}] > 0$, which follows from (A.7). As all the other parameters in the above expressions for \tilde{c}_{ij} are all positive, we conclude that $\mathbf{C}_1 < 0$.

(ii) For two by two matrix, we have $(\mathbf{B}_1 - r\mathbf{I})^{-1} = \frac{\mathbf{B}_1^{-1}\det[\mathbf{B}_1] - r\mathbf{I}}{\det[\mathbf{B}_1 - r\mathbf{I}]}$. Then,

$$(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 = \frac{\det[\mathbf{B}_1]\mathbf{C} - r\mathbf{C}_1}{r^2 - \text{tr}[\mathbf{B}_1]r + \det[\mathbf{B}_1]} > 0,$$

where the last inequality follows from (A.7), (A.8), $\mathbf{C} > 0$ and $\mathbf{C}_1 < 0$.

(iii) By the definition of \mathbf{B}_1 and using equation $\mathbf{B} = \mathbf{C}_1\mathbf{A}$, we have

$$\mathbf{B}_1(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 = (\mathbf{C}_1^{-1} - r\mathbf{C}_1^{-1}\mathbf{B}_1^{-1})^{-1} = ((1 + r\rho)\mathbf{C}_1^{-1} - r\mathbf{A})^{-1}.$$

Part (i) implies that the elements of \mathbf{C}_1^{-1} have signs $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$. Then the elements of $(1 + r\rho)\mathbf{C}_1^{-1} - r\mathbf{A}$ also have signs $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$. Furthermore, $\det[(1 + r\rho)\mathbf{C}_1^{-1} - r\mathbf{A}] = \det[\mathbf{C}_1^{-1}]\det[(1 + r\rho)\mathbf{I} - r\mathbf{B}] > 0$ as $\det[\mathbf{C}_1] > 0$ and $\det[\mathbf{B} - r\mathbf{I}] > 0$ for any $r > 0$. Hence, $\mathbf{B}_1(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 < 0$.

(iv) $\mathbf{B}\mathbf{C}_1 = (\mathbf{B}_1^{-1} + \rho\mathbf{I})\mathbf{C}_1 = \mathbf{C} + \rho\mathbf{C}_1 = (\mathbf{B}_1^{-1}\mathbf{C}_1)^\top + \rho\mathbf{C}_1 = \mathbf{C}_1(\mathbf{B}^\top - \rho\mathbf{I}) + \rho\mathbf{C}_1 = \mathbf{C}_1\mathbf{B}^\top$.
 $\mathbf{B}\mathbf{C} = \mathbf{B}\mathbf{B}_1^{-1}\mathbf{C}_1 = \mathbf{B}_1^{-1}\mathbf{B}\mathbf{C}_1 = \mathbf{B}_1^{-1}\mathbf{C}_1\mathbf{B}^\top = \mathbf{C}\mathbf{B}^\top$.

(v) Using the definition of \mathbf{C}_1 and (iv), we have $\mathbf{C}^{-1}(\mathbf{B}_1 - r\mathbf{I})^{-1}\mathbf{C}_1 = (\mathbf{C}_1^{-1}(\mathbf{B}_1 - r\mathbf{I})\mathbf{C})^{-1} = (\mathbf{I} - r\mathbf{C}^{-1}\mathbf{B}_1^{-1}\mathbf{C})^{-1} = ((1 + r\rho)\mathbf{I} - r\mathbf{C}^{-1}\mathbf{B}\mathbf{C})^{-1} = ((1 + r\rho)\mathbf{I} - r\mathbf{B}^\top)^{-1}$. Now if \mathbf{B} has negative diagonal elements, then $(1 + r\rho)\mathbf{I} - r\mathbf{B}^\top$ has positive diagonal elements. Since it has positive determinant, $((1 + r\rho)\mathbf{I} - r\mathbf{B}^\top)^{-1}$ has positive diagonal elements.

Proof of Proposition 3. Let $\mathbf{C}_1 = [\tilde{c}_1, \tilde{c}_2]$. Then, the solution for \mathbf{D} in Proposition 2(ii) becomes:

$$\text{vec}[\mathbf{D}] = \begin{bmatrix} -\kappa_1\mathbf{B}_1 + \mathbf{I} & 0 \\ 0 & -\kappa_2\mathbf{B}_1 + \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{c}_1 \\ -\tilde{c}_2 \end{bmatrix} = \begin{bmatrix} -(\kappa_1\mathbf{B}_1 - \mathbf{I})^{-1}\tilde{c}_1 \\ (\kappa_2\mathbf{B}_1 - \mathbf{I})^{-1}\tilde{c}_2 \end{bmatrix}.$$

Using Lemma 2(ii), the elements of \mathbf{D} have signs $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$. Using Lemma 2(iii) and $\mathbf{D}_1 = \mathbf{B}_1\mathbf{D}$, the elements of \mathbf{D}_1 have signs $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$. Using Lemma 2(v), the elements of $\mathbf{C}^{-1}\mathbf{D}$ have signs $\begin{bmatrix} - & \\ & + \end{bmatrix}$.

(i) Let $\Delta_1 > 0$ and $\Delta_2 > 0$ denote positive supply and demand shocks, respectively. Let $\mathbf{D}_1 = [\tilde{d}_{ij}]$.

Then, the impact on raw material price is $-\tilde{d}_{11}\Delta_1 - \tilde{d}_{12}\Delta_2$ and the impact on finished goods price is $-\tilde{d}_{21}\Delta_1 - \tilde{d}_{22}\Delta_2$. Part (i) follow immediately from the signs of \tilde{d}_{ij} .

(ii) The result follows immediately from the signs of \mathbf{D} .

(iii) Using the equilibrium condition (23), we have $\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1 = \mathbf{C}_1^{-1}(\mathbf{C}_1\mathbf{I}_1 - \mathbf{B}\mathbf{D}_1) = \mathbf{C}_1^{-1}((\mathbf{B}_1^{-1} - \mathbf{B})\mathbf{D}_1 + \mathbf{D}_1\mathbf{K}) = \mathbf{C}_1^{-1}\mathbf{D}_1(\mathbf{K} - \rho\mathbf{I}) = \mathbf{C}^{-1}\mathbf{D}(\mathbf{K} - \rho\mathbf{I})$. $\mathbf{K} - \rho\mathbf{I}$ is a diagonal matrix with negative diagonal elements. Thus, the signs of $\mathbf{C}^{-1}\mathbf{D}$ implies that the elements of $\mathbf{I}_1 - \mathbf{A}\mathbf{D}_1$ have signs $\begin{bmatrix} + & \\ & - \end{bmatrix}$.

Proof of Proposition 4. In a finite-horizon model, under the belief of price in (28) in the paper, the key equilibrium condition becomes

$$\mathbf{B}_t(\mathbf{p}_t - \mathbf{m}_t) + \mathbf{D}_t\mathbf{k}_t - \rho\mathbf{p}_t = -\mathbf{c} + \mathbf{C}\mathbf{X}_t.$$

Solving for \mathbf{p}_t , we have

$$\begin{aligned} \mathbf{p}_t &= \mathbf{C}_{1t}\mathbf{X}_t - \mathbf{D}_{1t}\mathbf{k}_t + \mathbf{B}_{1t}(\mathbf{B}_t\mathbf{m}_t - \mathbf{c}) \\ &= \mathbf{C}_{1t}\mathbf{X}_t - \mathbf{D}_{1t}\mathbf{k}_t + \mathbf{f}_t, \end{aligned} \tag{A.12}$$

where we used $\mathbf{f}_t = \mathbf{B}_{1t}\mathbf{B}_{1t}^{-1}\mathbf{m}_t - \mathbf{B}_{1t}(\mathbf{c} - \rho\mathbf{m}_t) = \mathbf{B}_{1t}[(\mathbf{B}_t - \rho\mathbf{I})\mathbf{m}_t - \mathbf{c} + \rho\mathbf{m}_t] = \mathbf{B}_{1t}(\mathbf{B}_t\mathbf{m}_t - \mathbf{c})$. The relation in (A.12) also can be expressed as

$$\mathbf{X}_t = \mathbf{C}_{1t}^{-1}(\mathbf{p}_t + \mathbf{D}_{1t}\mathbf{k}_t - \mathbf{f}_t). \tag{A.13}$$

To derive the price dynamics under the belief in (28), we differentiate (A.12) and obtain

$$d\mathbf{p}_t = \mathbf{C}_{1t}d\mathbf{X}_t + (d\mathbf{C}_{1t})\mathbf{X}_t - \mathbf{D}_{1t}d\mathbf{k}_t - (d\mathbf{D}_{1t})\mathbf{k}_t + d\mathbf{f}_t.$$

Using (A.4), (A.13), and $d\mathbf{k}_t = \mathbf{K}\mathbf{k}_tdt + \boldsymbol{\sigma}_0(\mathbf{k}_t)d\mathbf{w}_t$ defined in (19) in the paper, we have

$$\begin{aligned} d\mathbf{p}_t &= \mathbf{C}_{1t}(\mathbf{A}\mathbf{p}_t + \mathbf{a} + \mathbf{I}_1\mathbf{k}_t)dt + (d\mathbf{C}_{1t})\mathbf{C}_{1t}^{-1}(\mathbf{p}_t + \mathbf{D}_{1t}\mathbf{k}_t - \mathbf{f}_t) - \mathbf{D}_{1t}(\mathbf{K}\mathbf{k}_tdt + \boldsymbol{\sigma}_0(\mathbf{k}_t)d\mathbf{w}_t) \\ &\quad - (d\mathbf{D}_{1t})\mathbf{k}_t + d\mathbf{f}_t \\ &= [\mathbf{C}_{1t}\mathbf{A}dt + (d\mathbf{C}_{1t})\mathbf{C}_{1t}^{-1}]\mathbf{p}_t + [\mathbf{C}_{1t}\mathbf{I}_1dt + (d\mathbf{C}_{1t})\mathbf{C}_{1t}^{-1}\mathbf{D}_{1t} - \mathbf{D}_{1t}\mathbf{K}dt - d\mathbf{D}_{1t}]\mathbf{k}_t \\ &\quad + \mathbf{C}_{1t}\mathbf{a}dt - (d\mathbf{C}_{1t})\mathbf{C}_{1t}^{-1}\mathbf{f}_t + d\mathbf{f}_t - \mathbf{D}_{1t}\boldsymbol{\sigma}_0(\mathbf{k}_t)d\mathbf{w}_t \\ &= (\mathbf{B}_t(\mathbf{p}_t - \mathbf{m}_t) + \mathbf{D}_t\mathbf{k}_t)dt + \boldsymbol{\sigma}(t, \mathbf{p}_t, \mathbf{k}_t)d\mathbf{w}_t, \end{aligned} \tag{A.14}$$

where the last equality means that the price process is consistent with the firms' beliefs under the rational expectations equilibrium. Matching the coefficients of \mathbf{p}_t on both sides of (A.14), we have

$$\begin{aligned} \mathbf{B}_tdt &= \mathbf{C}_{1t}\mathbf{A}dt + (d\mathbf{C}_{1t})\mathbf{C}_{1t}^{-1} \\ \frac{d\mathbf{C}_{1t}}{dt} &= -\mathbf{C}_{1t}\mathbf{A}\mathbf{C}_{1t} + \mathbf{B}_t\mathbf{C}_{1t} \end{aligned} \tag{A.15}$$

$$\frac{d\mathbf{C}_{1t}}{dt} = -\mathbf{C}_{1t}\mathbf{A}\mathbf{C}_{1t} + \rho\mathbf{C}_{1t} + \mathbf{C}, \tag{A.16}$$

where we used $\mathbf{B}_t\mathbf{C}_{1t} = (\rho\mathbf{I} + \mathbf{B}_{1t}^{-1})\mathbf{C}_{1t} = \rho\mathbf{C}_{1t} + \mathbf{C}$. This proves the first differential equation in the

proposition.

Matching the coefficients of \mathbf{k}_t on both sides of (A.14) and using (A.16), we have:

$$\begin{aligned}\mathbf{D}_t dt &= \mathbf{C}_{1t} \mathbf{I}_1 dt + (d\mathbf{C}_{1t}) \mathbf{C}_{1t}^{-1} \mathbf{D}_{1t} - \mathbf{D}_{1t} \mathbf{K} dt - d\mathbf{D}_{1t} \\ \mathbf{D}_t &= \mathbf{C}_{1t} \mathbf{I}_1 - \mathbf{C}_{1t} \mathbf{A} \mathbf{D}_{1t} + \rho \mathbf{D}_{1t} + \mathbf{C} \mathbf{C}_{1t}^{-1} \mathbf{D}_{1t} - \mathbf{D}_{1t} \mathbf{K} - \frac{d\mathbf{D}_{1t}}{dt} \\ \frac{d\mathbf{D}_{1t}}{dt} &= \mathbf{C}_{1t} \mathbf{I}_1 - \mathbf{C}_{1t} \mathbf{A} \mathbf{D}_{1t} - \mathbf{D}_{1t} (\mathbf{K} - \rho \mathbf{I}),\end{aligned}$$

where we used $\mathbf{C} \mathbf{C}_{1t}^{-1} \mathbf{D}_{1t} = \mathbf{B}_{1t}^{-1} \mathbf{D}_{1t} = \mathbf{D}_t$. This proves the second differential equation in the proposition.

Matching the other terms in (A.14) and using (A.15), we have

$$\begin{aligned}-\mathbf{B}_t \mathbf{m}_t dt &= \mathbf{C}_{1t} \mathbf{a} dt - (d\mathbf{C}_{1t}) \mathbf{C}_{1t}^{-1} \mathbf{f}_t + d\mathbf{f}_t \\ -\mathbf{B}_t \mathbf{m}_t &= \mathbf{C}_{1t} \mathbf{a} + \mathbf{C}_{1t} \mathbf{A} \mathbf{f}_t - \mathbf{B}_t \mathbf{f}_t + \frac{d\mathbf{f}_t}{dt} \\ -\mathbf{B}_t \mathbf{m}_t &= \mathbf{C}_{1t} \mathbf{a} + (\mathbf{C}_{1t} \mathbf{A} - \rho \mathbf{I}) \mathbf{f}_t - (\mathbf{B}_t - \rho \mathbf{I}) \mathbf{f}_t + \frac{d\mathbf{f}_t}{dt} \\ \frac{d\mathbf{f}_t}{dt} &= -(\mathbf{C}_{1t} \mathbf{A} - \rho \mathbf{I}) \mathbf{f}_t - \mathbf{C}_{1t} \mathbf{a} - \mathbf{c}\end{aligned}$$

where we used $\mathbf{B}_t \mathbf{m}_t - (\mathbf{B}_t - \rho \mathbf{I}) \mathbf{f}_t = \mathbf{B}_t \mathbf{m}_t - (\mathbf{B}_t - \rho \mathbf{I}) \mathbf{B}_{1t} (\mathbf{B}_t \mathbf{m}_t - \mathbf{c}) = \mathbf{B}_t \mathbf{m}_t - (\mathbf{B}_t \mathbf{m}_t - \mathbf{c}) = \mathbf{c}$.

The equilibrium price process is parameterized by \mathbf{B}_t , \mathbf{m}_t , \mathbf{D}_t , and $\boldsymbol{\sigma}(t, \mathbf{p}_t, \mathbf{k}_t)$, where \mathbf{B}_t and \mathbf{D}_t are readily available once \mathbf{C}_{1t} and \mathbf{D}_{1t} are determined by the above differential equations. Matching the diffusion coefficient in (A.14) gives $\boldsymbol{\sigma}(t, \mathbf{p}_t, \mathbf{k}_t) = -\mathbf{D}_{1t} \boldsymbol{\sigma}_0(\mathbf{k}_t)$. Since $\mathbf{f}_t = \mathbf{m}_t - \mathbf{B}_{1t}(\mathbf{c} - \rho \mathbf{m}_t)$, we can solve for $\mathbf{m}_t = (\mathbf{I} + \rho \mathbf{B}_{1t})^{-1}(\mathbf{f}_t + \mathbf{B}_{1t} \mathbf{c})$.

To find the equilibrium inventory process, we substitute (A.12) into (A.4) and obtain:

$$\begin{aligned}\frac{d\mathbf{X}_t}{dt} &= \mathbf{A} \mathbf{C}_{1t} \mathbf{X}_t - \mathbf{A} \mathbf{D}_{1t} \mathbf{k}_t + \mathbf{A} \mathbf{f}_t + \mathbf{a} + \mathbf{I}_1 \mathbf{k}_t \\ &= \mathbf{A} \mathbf{C}_{1t} (\mathbf{X}_t + \mathbf{C}^{-1}(\rho \mathbf{m}_t - \mathbf{c})) + \mathbf{A} \mathbf{m}_t + \mathbf{a} + (\mathbf{I}_1 - \mathbf{A} \mathbf{D}_{1t}) \mathbf{k}_t.\end{aligned}$$

where we used $\mathbf{f}_t = \mathbf{C}_{1t} \mathbf{C}^{-1}(\rho \mathbf{m}_t - \mathbf{c}) + \mathbf{m}_t$.

References

Horn, R. A., C. R. Johnson. 1991. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, New York.