MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1100.1245ec

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Selling with Binding Reservations in the Presence of Strategic Consumers" by Nikolay Osadchiy and Gustavo Vulcano, *Management Science*, DOI 10.1287/mnsc.1100.1245.

Selling with Binding Reservations in the Presence of Strategic Consumers
Online Appendix

Nikolay Osadchiy Gustavo Vulcano

Leonard N. Stern School of Business, New York University, 44 West 4th St., New York, NY 10012

{nosadchi, gvulcano}@stern.nyu.edu

A1. Technical results and proofs

A1.1. Preliminary results

We compile a few results that will be used in the main proofs.

Lemma A1 *The following results hold:*

(i) Let $N(x)$ be a Poisson random variable with mean $x>0$. For a nonnegative integer n,

$$\frac{d}{dx}\mathbb{P}(N(x)\leq n) = -\mathbb{P}(N(x)=n).$$

(ii) Let $N(x)$ be a Poisson random variable with mean $x>0$. Let a and b be two nonnegative constants, and let $N\geq 1$ be an integer. Then, there exists $0\leq\beta(N)\leq 1$ such that

$$\left|\mathbb{P}(N(b)\leq n)-\mathbb{P}(N(a)\leq n)\right|\leq\beta(n)\,|b-a|.$$

In particular, $\beta(n)\triangleq\mathbb{P}(N(n)=n)$.

(iii) For any constants $a,b,$and c, $|\max\{a,c\}-\max\{b,c\}|\leq|a-b|$.

Proof: Results (i) and (ii) are proved in Caldentey and Vulcano (2007) (see Lemmas A1 and A3 in the corresponding Online Appendix). The third result is easy to verify. ∎

Next lemma introduces some useful bounds for future reference:

Lemma A2 *The following bounds hold for $\tau\in[0,T]$:*

(i) $1\leq\exp(w(T-\tau))\leq\exp(wT)$.

(ii) A (random) lower bound for the leftover inventory at time T is

$$\underline{Q}_T \triangleq (Q_0 - N(\bar{\Lambda}_{H_B}(T)))^+ \leq_{st} (Q_0 - N(\Lambda_{H_B}(T)))^+ = Q_T,$$

where

$$\bar{\Lambda}_{H_B}(\tau) \triangleq \bar{\lambda}\tau \max_{t\in[0,\tau]} \bar{F}(p_h, t), \tag{A1}$$

and where "$\leq_{st}$" stands for the stochastic order relational operator.[1]

(iii) For a strict ordering rationing rule, $\Pi_H(\tau) \geq \mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1) > 0$.

(iv) For a strict ordering rationing rule, $\Pi_H(\tau) \leq \mathbb{P}(Q_\tau > 0|H)$.

(v) $\mathbb{P}(Q_\tau > 0|H) \geq \mathbb{P}(N(\bar{\Lambda}_{H_B}(\tau)) \leq Q_0 - 1) \geq \mathbb{P}(N(\bar{\Lambda}_{H_B}(T)) \leq Q_0 - 1)$, *where* $\bar{\Lambda}_{H_B}(\tau)$ *and* $\bar{\Lambda}_{H_B}(T)$ *are given by (A1).*

Proof: Bounds in (i) follow from the monotonicity of the exponential function.

Bound (ii) comes from the fact that an upper bound for the total number of buy-nows is given by $\bar{\Lambda}_{H_B}(T)$.

For bounds (iii) and (iv) observe that

$$\begin{aligned}
\Pi_H(\tau) &= \mathbb{P}(N(\Lambda_{H_B}(T) + \Lambda_{H_R}(\tau)) \leq Q_0 - 1) \\
&\geq \mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1) \; > \; 0, \text{ and} \\
\Pi_H(\tau) &\leq \mathbb{P}(N(\Lambda_{H_B}(T)) \leq Q_0 - 1) \\
&= \mathbb{P}(Q_0 - N(\Lambda_{H_B}(T)) > 0) \\
&\leq \mathbb{P}(Q_0 - N(\Lambda_{H_B}(\tau)) > 0) = P(Q_\tau > 0|H).
\end{aligned}$$

To prove bound (v) recall that $\mathbb{P}(Q_\tau > 0|H) = \mathbb{P}(N(\Lambda_{H_B}(\tau)) \leq Q_0 - 1)$. The first bound now follows from the fact that an upper bound for the mean number of buy-nows up to time τ is $\bar{\Lambda}_{H_B}(\tau)$ as defined in (A1). The last bound holds since $T \geq \tau$, and since the cumulative distribution is decreasing in the mean. ∎

[1] Two random variables X and Y are such that $X \leq_{st} Y$ if $\mathbb{P}(X > x) \leq \mathbb{P}(Y > x), \forall x$.

A1.2. Consumer purchasing behavior in the stochastic game

A1.2.1. Strict priority rationing rules

Summary of the results

For technical purposes we assumed that the distribution of customers' valuations $F(v,t)$ is continuously differentiable function for all t. This implies that $F(v,t)$ is K_F-Lipschitz continuous in the first argument, for some constant K_F, for all t.[2]

Lemma A3 argues that $\Pi_H(\tau)$ is Lipschitz continuous, which is critical for the equilibrium proof. Since a PE is characterized by the fixed-point condition $\mathcal{R}(H^*) = H^*$, we conclude that a symmetric purchasing equilibrium of this game $H^* \in \mathcal{H}$ is in fact continuous. Proposition A2 shows another required property: the best-response strategies are K-Lipschitz continuous functions in $[0,T]$, for an appropriate constant $K > 0$.

A standard way to prove the existence of a symmetric equilibrium $H(\tau)$ is to verify that the set of purchasing strategies $\mathcal{H}$ has the fixed-point property (i.e., every continuous mapping $\mathcal{R}: \mathcal{H} \to \mathcal{H}$ has a fixed-point; see Cheney (2001), Section 7.1 for details) and that the best-response mapping $\mathcal{R}$ is continuous in $\mathcal{H}$. These properties are formally verified for our case in Theorem A1.

Statement of results and proofs

Lemma A3 *For any rationing rule defined by $\xi(\cdot) \in \Xi$, and for any strategy profile $H \in \mathcal{H}$, $\Pi_H(\tau)$ is differentiable and $|\Pi'_H(\tau)| \le K_\Pi < \infty$, for all τ.*

Proof: The proof amounts to computing the derivative of $\Pi_H(\cdot)$. If the rationing rule is based on a strict ordering, then $\Pi_H(\tau) = \mathbb{P}(N(\Lambda_{H_R}(\tau) + \Lambda_{H_B}(T)) \le Q_0 - 1)$ and by Lemma A1(i),

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi_H(\tau)\right| &= \left|\mathbb{P}(N(\Lambda_{H_B}(T) + \Lambda_{H_R}(\tau)) = Q_0 - 1)\frac{\mathrm{d}\Lambda_{H_R}(\tau)}{\mathrm{d}\tau}\right| \\ &\le \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^T \mathbb{1}\{\xi(t) > \xi(\tau)\}\lambda(t)F(H(t),t)\mathrm{d}t\right)\right|.\end{aligned}$$

[2] A scalar function F is K_F-Lipschitz continuous in x if for all x_1, x_2, $|F(x_1,t) - F(x_2,t)| \le K_F|x_1 - x_2|, \forall t$. A sufficient condition to verify Lipschitz continuity of F is continuous differentiability, i.e., the fact that $\frac{\partial}{\partial v}F(v,t) \equiv f(v,t)$ is continuous. Standard distributions like the uniform, beta, exponential, normal, and Cauchy verify this property.

Note that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^T 1\!\!1\{\xi(t)>\xi(\tau)\}\lambda(t)F(H(t),t)\mathrm{d}t\right)\right| = \lim_{\mathrm{d}\tau\longrightarrow 0}\frac{\left|\int_0^T \left(1\!\!1\{\xi(t)>\xi(\tau+\mathrm{d}\tau)\}-1\!\!1\{\xi(t)>\xi(\tau)\}\right)\lambda(t)F(H(t),t)\mathrm{d}t\right|}{\mathrm{d}\tau}$$
$$\leq \lim_{\mathrm{d}\tau\longrightarrow 0}\frac{\left|\sum_{i=1}^N \lambda(\tau_i)F(H(\tau_i),\tau_i)\mathrm{d}\tau+o(\mathrm{d}\tau)\right|}{\mathrm{d}\tau}\leq\bar{\lambda}N,$$

where $\tau_1,\ldots,\tau_N$ are such that $\xi(\tau)=\xi(\tau_i),\ i=1,\ldots,N$. Since $\xi\in\Xi$, it has a finite number of local extrema and hence N is finite. Thus, $\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi_H(\tau)\right|\leq\bar{\lambda}N$. ∎

Next lemma characterizes an important feature of the elements of $\mathcal{H}$, namely, there is always a range of consumers with valuations above p_h that prefer to reserve an item, irrespective of their arrival times:

Lemma A4 *For all $H\in\mathcal{H}$ there is a valuation $v_H>p_h$ such that $\mathcal{R}(H)(\tau)\geq v_H$ for all $\tau\in[0,T]$. The infimum of v_H over H satisfies $\tilde{v}=\inf_{H\in\mathcal{H}}\{v_H\}\geq\frac{p_h\bar{g}}{\bar{g}-1}$, where $\bar{g}=\exp(wT)/\mathbb{P}(N(\bar{\lambda}T)\leq Q_0-1)$.*

Proof: First, we need to show that $g_H(\tau)$ in (7) is a well defined function in $[0,T]$. By (i), (iii), and (iv) of Lemma A2, the following upper and lower bounds can be established:

$$1\leq g_H(\tau)\leq\frac{\exp(wT)}{\mathbb{P}(N(\bar{\lambda}T)\leq Q_0-1)}.$$

Because the LHS in (7) is decreasing in v_τ and has a vertical asymptote at $v_\tau=p_h$, and since $g_H(\tau)$ is bounded above, we can assert that there exists such $\tilde{v}>p_h$. ∎

The following proposition characterizes $\mathcal{R}(H)$. It is a simple consequence of the continuity of $g_H(\tau)$, and the shape of the LHS in (7): It is monotonically decreasing, with minimum $1/(1-p_h)$ achieved when $v_\tau=1$.

Proposition A1 *A consumer places a reservation (i.e., condition (7) is satisfied) when*

$$v_\tau\leq\frac{p_h g_H(\tau)}{g_H(\tau)-1}\quad \text{for } g_H(\tau)>\frac{1}{1-p_h}. \tag{A2}$$

Thus, a consumer arriving at time τ with valuation v_τ places a reservation if and only if $v_\tau\leq\mathcal{R}(H)(\tau)$, where

$$\mathcal{R}(H)(\tau)\triangleq\begin{cases}1 & \text{if } g_H(\tau)\leq\frac{1}{1-p_h}\\ \frac{p_h g_H(\tau)}{g_H(\tau)-1} & \text{if } g_H(\tau)>\frac{1}{1-p_h}.\end{cases}$$

This best-response mapping $\mathcal{R}(H)(\tau)$ is continuous in τ.

Our next result shows that the best-response strategies are K-Lipschitz continuous functions in $[0,T]$.

Proposition A2 *For all $H\in\mathcal{H}$, there is a positive constant K (independent of H) such that the best-response strategy $\mathcal{R}(H)(\tau)$ is a K-Lipschitz continuous function.*

Proof: Lemmas A3 and A4 imply continuity of $\mathcal{R}(H)(\tau)$ in τ. Recall from the definition of v_H that $\mathcal{R}(H)(\tau)>p_h$ for all $\tau\in[0,T]$. Also, recall that $\mathcal{R}(H)(\tau)=1$ when $g_H(\tau)\le 1/(1-p_h)$. So, we can concentrate on proving the K-Lipschitz property on the intervals where $g_H(\tau)$ is differentiable and $g_H(\tau)>1/(1-p_h)>1$. Take $\tau_1,\tau_2\in(\underline{\tau},\bar{\tau})$ such that $g_H(\tau)>1/(1-p_h)$ for all $\tau\in(\underline{\tau},\bar{\tau})$. We have that

$$
\begin{aligned}
|\mathcal{R}(H)(\tau_1)-\mathcal{R}(H)(\tau_2)| &= p_h\left|\frac{g_H(\tau_1)}{g_H(\tau_1)-1}-\frac{g_H(\tau_2)}{g_H(\tau_2)-1}\right| \\
&= p_h\left|\int_{\tau_2}^{\tau_1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g_H(x)}{g_H(x)-1}\right)\mathrm{d}x\right| = p_h\left|\int_{\tau_2}^{\tau_1}\frac{g'_H(x)}{(g_H(x)-1)^2}\mathrm{d}x\right|. \qquad \text{(A3)}
\end{aligned}
$$

For the numerator in the integrand, by expanding the derivatives we get:

$$
g'_H(\tau)=\frac{(\exp(w(T-\tau))\mathbb{P}(Q_\tau>0|H))'\,\Pi_H(\tau)-\exp(w(T-\tau))\mathbb{P}(Q_\tau>0|H)\Pi'_H(\tau)}{\Pi_H^2(\tau)},
$$

where from the fact that

$$
\mathbb{P}(Q_\tau>0)=\mathbb{P}(N(\Lambda_{H_B}(\tau))\le Q_0-1),
$$

and property (i) in Lemma A1,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}(\exp(w(T-\tau))\mathbb{P}(Q_\tau>0|H)) &= -\exp(w(T-\tau))\left[w\mathbb{P}(Q_\tau>0|H)+\mathbb{P}(N(\Lambda_{H_B}(\tau))=Q_0-1)\frac{\mathrm{d}}{\mathrm{d}\tau}\Lambda_{H_B}(\tau)\right] \\
&= -\exp(w(T-\tau))\left[w\mathbb{P}(Q_\tau>0|H)+\mathbb{P}(N(\Lambda_{H_B}(\tau))=Q_0-1)\lambda(\tau)\bar{F}(H(\tau),\tau)\right].
\end{aligned}
$$

Therefore,

$$
g'_H(\tau) = -\frac{\exp(w(T-\tau))}{\Pi_H^2(\tau)}\left(\Pi_H(\tau)\lambda(\tau)\bar{F}(H(\tau),\tau)\mathbb{P}(N(\Lambda_{H_B}(\tau))=Q_0-1)+\mathbb{P}(Q_\tau>0|H)(w\Pi_H(\tau)+\Pi'_H(\tau))\right)
$$

Upper bounding the probabilities in the numerator by 1, and using bounds (i) and (iii) of Lemma A2, and Lemma A3, we get

$$\begin{aligned}|g'_H(\tau)| &\leq \frac{\exp(w(T-\tau))}{\Pi_H^2(\tau)}\left(\Pi_H(\tau)(\lambda(\tau)+w)+\Pi'_H(\tau)\right)\\ &\leq \exp(wT)\left(\frac{\bar\lambda+w}{\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)}+\frac{K_\Pi}{\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)^2}\right).\end{aligned} \tag{A4}$$

For the denominator in the integrand of (A3), since $g_H(\tau)>1/(1-p_h)$, we have

$$(g_H(\tau)-1)^2>\left(\frac{p_h}{1-p_h}\right)^2. \tag{A5}$$

Plugging bounds (A4) and (A5) back into (A3), we get

$$|\mathcal{R}(H)(\tau_1)-\mathcal{R}(H)(\tau_2)| = K|\tau_1-\tau_2|,$$

where the constant $K=\exp(wT)\frac{(1-p_h)^2}{p_h}\left(\frac{\bar\lambda+w}{\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)}+\frac{K_\Pi}{\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)^2}\right)$ is independent of τ_1 and τ_2, and is guaranteed to be finite because $\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)>0$. ∎

Our main theorem is included next. It is a technical version of Theorem 1 in the main body of the paper:

Theorem A1 *For any rationing rule defined by $\xi(\cdot)\in\Xi$, and for any strategy profile $H\in\mathcal{H}$, the set of strategies $\mathcal{H}$ equipped with the uniform norm $\|X\|=\sup_{0\leq\tau\leq T}\{|X(\tau)|\}$ in $[0,T]$ exhibits the fixed-point property. For all $H,\tilde H\in\mathcal{H}$, the mapping $\mathcal{R}$ satisfies:*

$$\|\mathcal{R}(H)-\mathcal{R}(\tilde H)\|\leq\frac{3(1-p_h)^2\exp(wT)\,\beta(Q_0-1)\,\bar\lambda\,K_F\,T}{p_h\,\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)^2}\|H-\tilde H\|,$$

where $\beta(Q_0-1)=\mathbb{P}(N(Q_0-1)=Q_0-1)$. In addition, if

$$\frac{3(1-p_h)^2\exp(wT)\,\beta(Q_0-1)\,\bar\lambda\,K_F\,T}{p_h\,\mathbb{P}(N(\bar\lambda T)\leq Q_0-1)^2}<1, \tag{A6}$$

then $\mathcal{R}$ is a contraction. In this case, the fixed-point $\mathcal{R}(H^*)=H^*$ is guaranteed to be unique in $\mathcal{H}$ and can be found through the iteration $H^{(n+1)}=\mathcal{R}(H^{(n)})$ starting from an arbitrary $H^{(0)}\in\mathcal{H}$.

Proof: To prove that $\mathcal{H}$ has the fixed-point property, we apply the Schauder-Tychonoff Fixed-Point Theorem (see Cheney (2001), Chapter 7 for details). For this, we need to show that $\mathcal{H}$ is a compact convex set. Convexity is immediate from the definition of $\mathcal{H}$. To check compactness, we apply the Arzelà-Ascoli Theorem II (Cheney (2001), Chapter 7), that is, we need to show that $\mathcal{H}$ is closed, bounded, and equicontinuous. Take a sequence $\{H^{(n)}\}_{n\geq 1}$ of strategies in $\mathcal{H}$ that converges point-wise to H. In order to verify the K-Lipschitz property of H note that from the Proposition, for $n \geq 1$ and $\tau_1, \tau_2 \in [0,T]$,

$$|H^{(n)}(\tau_1) - H^{(n)}(\tau_2)| \leq K\,|\tau_1 - \tau_2|.$$

By the continuity of the absolute value and the point-wise convergence of $H^{(n)}$ to H we conclude

$$|H(\tau_1) - H(\tau_2)| \leq K\,|\tau_1 - \tau_2|,$$

which proves the closedness of $\mathcal{H}$. The boundedness of $\mathcal{H}$ follows from the fact $H(\tau) \in [0,1]$ for all $H \in \mathcal{H}$. Equicontinuity, on the other hand, follows directly from the fact that the elements of $\mathcal{H}$ are K-Lipschitz continuous. In fact, to prove equicontinuity of $\mathcal{H}$ we need to show that for $\epsilon > 0$ there is $\delta > 0$ such that:

$$\text{For all } H \in \mathcal{H} \text{ and } \tau_1, \tau_2 \in [0,T], \text{ such that } |\tau_1 - \tau_2| < \delta \text{ then } |H(\tau_1) - H(\tau_2)| < \epsilon.$$

For this, take $\delta = \frac{\epsilon}{K}$ and use the K-Lipschitz continuity of H as follows:

$$\text{For all } H \in \mathcal{H} \text{ and } \tau_1, \tau_2 \in [0,T] \text{ such that } |\tau_1 - \tau_2| < \delta,\ |H(\tau_1) - H(\tau_2)| \leq K|\tau_1 - \tau_2| < K\,\delta = \epsilon.$$

This proves that $\mathcal{H}$ has the fixed-point property.

We now prove that the best-response $\mathcal{R}$ mapping is continuous in $\mathcal{H}$. We start by observing that we can rewrite the best response mapping in (8) as

$$\mathcal{R}(H)(\tau) = \frac{p_h \max\{g_H(\tau), 1/(1-p_h)\}}{\max\{g_H(\tau), 1/(1-p_h)\} - 1}.$$

Therefore,

$$|\mathcal{R}(H)_\tau - \mathcal{R}(\tilde{H})_\tau| = \left| \frac{p_h \max\{g_H(\tau), 1/(1-p_h)\}}{\max\{g_H(\tau), 1/(1-p_h)\} - 1} - \frac{p_h \max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\}}{\max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\} - 1} \right|$$

$$
\begin{aligned}
&= p_h \left|\max\{g_H(\tau), 1/(1-p_h)\} \left(\max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\} - 1\right) \right.\\
&\qquad \left. - \max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\} \left(\max\{g_H(\tau), 1/(1-p_h)\} - 1\right)\right| \\
&\qquad \times \frac{1}{\left(\max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\} - 1\right) \left(\max\{g_H(\tau), 1/(1-p_h)\} - 1\right)} \\
&\leq \frac{(1-p_h)^2}{p_h} \left|\max\{g_{\tilde{H}}(\tau), 1/(1-p_h)\} - \max\{g_H(\tau), 1/(1-p_h)\}\right| \\
&\leq \frac{(1-p_h)^2}{p_h} \left|g_H(\tau) - g_{\tilde{H}}(\tau)\right|, \qquad \text{(A7)}
\end{aligned}
$$

where the first inequality is verified from the fact that $\max\{g(\tau), 1/(1-p_h)\} \geq 1/(1-p_h) > 1$, and the second inequality holds from Lemma A1, part (iii).

We proceed as follows:

$$
\begin{aligned}
|g_H(\tau) - g_{\tilde{H}}(\tau)| &= \exp(w(T-\tau)) \left| \frac{\mathbb{P}(Q_\tau > 0|H)}{\Pi_H(\tau)} - \frac{\mathbb{P}(Q_\tau > 0|\tilde{H})}{\Pi_{\tilde{H}}(\tau)} \right| \\
&\leq \exp(wT) \left| \frac{\mathbb{P}(Q_\tau > 0|H)\Pi_{\tilde{H}}(\tau) - \mathbb{P}(Q_\tau > 0|\tilde{H})\Pi_H(\tau)}{\Pi_H(\tau)\Pi_{\tilde{H}}(\tau)} \right| \\
&\leq \frac{\exp(wT)}{\mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1)^2} \left| \mathbb{P}(Q_\tau > 0|H)\Pi_{\tilde{H}}(\tau) - \mathbb{P}(Q_\tau > 0|\tilde{H})\Pi_H(\tau) \right| \\
&\leq \frac{\exp(wT)}{\mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1)^2} \left[\left| \mathbb{P}(Q_\tau \geq 1|H) - \mathbb{P}(Q_\tau > 0|\tilde{H}) \right| + |\Pi_H(\tau) - \Pi_{\tilde{H}}(\tau)| \right], \qquad \text{(A8)}
\end{aligned}
$$

where the first inequality holds from Lemma A2 part (i), the second one holds from Lemma A2 part (iii), and the last one from the fact that for any pair of reals a, b, and for any reals c, d, such that $|c|, |d| \leq 1$,

$$
|ad - cb| = |(a-c)d + (d-b)c| \leq |a-c| + |d-b|.
$$

Taking the first module in equation (A8), we have

$$
\begin{aligned}
\left| \mathbb{P}(Q_\tau > 0|H) - \mathbb{P}(Q_\tau > 0|\tilde{H}) \right| &= \left| \mathbb{P}(N(\Lambda_{H_B}(\tau)) \leq Q_0 - 1) - \mathbb{P}(N(\Lambda_{\tilde{H}_B}(\tau)) \leq Q_0 - 1) \right| \\
&\leq \beta(Q_0 - 1) \left| \Lambda_{H_B}(\tau) - \Lambda_{\tilde{H}_B}(\tau) \right| \\
&= \beta(Q_0 - 1) \left| \int_0^\tau \lambda(t)(F(\tilde{H}(t), t) - F(H(t), t)) \mathrm{d}t \right| \\
&\leq \beta(Q_0 - 1)\, \bar{\lambda}\, K_F \int_0^\tau |\tilde{H}(t) - H(t)| \mathrm{d}t \\
&\leq \beta(Q_0 - 1)\, \bar{\lambda}\, K_F\, T ||\tilde{H} - H||, \qquad \text{(A9)}
\end{aligned}
$$

where the first inequality holds from Lemma A1, part(ii); the second inequality holds from the K_F-Lipschitz continuity of the c.d.f. F, the third one from the definition of uniform norm for the function space $\mathcal{H}$, and the last one from the observation: $\tau \le T$.

For the second module in equation (A8), note that the expressions for $\Pi_H(\cdot)$ and $\Pi_{\tilde{H}}(\cdot)$ depend on the parameter τ. Using (6) and Lemma A1(ii), and following arguments similar to the ones in (A9), we get:

$$\begin{aligned}
&\left|\mathbb{P}(N(\Lambda_{H_R}(\tau)+\Lambda_{H_B}(T)) \le Q_0-1) - \mathbb{P}(N(\Lambda_{\tilde{H}_R}(\tau)+\Lambda_{\tilde{H}_B}(T)) \le Q_0-1)\right| \\
&\qquad \le \beta(Q_0-1)\left|\Lambda_{H_R}(\tau)+\Lambda_{H_B}(T) - (\Lambda_{\tilde{H}_R}(\tau)+\Lambda_{\tilde{H}_B}(T))\right| \\
&\qquad \le \beta(Q_0-1)\left(\left|\Lambda_{H_R}(\tau)-\Lambda_{\tilde{H}_R}(\tau)\right| + \left|\Lambda_{H_B}(T)-\Lambda_{\tilde{H}_B}(T)\right|\right) \\
&\qquad \le 2\,\beta(Q_0-1)\,\bar{\lambda}\,K_F\,T||\tilde{H}-H||. \qquad \text{(A10)}
\end{aligned}$$

Plugging bounds (A9) and (A10) into (A8), and then (A8) into (A7), we get:

$$|\mathcal{R}(H)_\tau - \mathcal{R}(\tilde{H})_\tau| \le \frac{3(1-p_h)^2\exp(wT)\,\beta(Q_0-1)\,\bar{\lambda}\,K_F\,T}{p_h\,\mathbb{P}(N(\bar{\lambda}T)\le Q_0-1)^2}||\tilde{H}-H||,$$

where $\beta(Q_0-1) = \mathbb{P}(N(Q_0-1) = Q_0-1)$.

From this result, we conclude that $\mathcal{R}$ is continuous which together with the fixed-point property of the set $\mathcal{H}$ guarantee the existence of a PE, if the rationing rule is based on a strict priority ordering (as defined in Section 4). ∎

The condition (A6) of Theorem A1 is easily satisfied by most practical cases of the problem. Indeed, the LHS decreases if Q_0 and λT grow proportionally large.

A1.2.2. RA rationing rule

Summary of the results

Under the RA-s rationing rule the probability $c(H)$ of getting a reserved item is the same for any arrival $\tau \in [0, T_S]$. Then, for the exponential utility function (1), and according to condition (3), a consumer places a reservation when

$$\frac{v_\tau}{v_\tau - p_h} \ge \frac{\exp(w(T_S-\tau))\mathbb{P}(Q_\tau > 0|H)}{c(H)} \triangleq g_H^{RA}(\tau), \quad \tau \in [0, T_S], \qquad \text{(A11)}$$

where

$$c(H) = \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \left[\sum_{n=0}^{k} \mathbb{P}(N(\Lambda_{H_R}^{RA}(T_S)) = n) + \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda_{H_R}^{RA}(T_S)) = n) \right]. \qquad \text{(A12)}$$

Observe that $c(H)$ is guaranteed to be strictly positive.[3]

A consumer arriving at time τ with valuation v_τ places a reservation if and only if $v_\tau \leq \mathcal{R}(H)(\tau)$, where

$$\mathcal{R}(H)(\tau) \triangleq \begin{cases} 1 & \text{if } \; g_H^{RA}(\tau) \leq \frac{1}{1-p_h} \\ \frac{p_h g_H^{RA}(\tau)}{g_H^{RA}(\tau)-1} & \text{if } \; g_H^{RA}(\tau) > \frac{1}{1-p_h}. \end{cases}$$

We can now establish results analogous to the ones for the strict ordering rationing rules:

- There is an infimum $\tilde{v} > p_h$ such that $\mathcal{R}(H)(\tau) \geq \tilde{v}$, for all $H \in \mathcal{H}$ (Lemma A5).
- For all $H \in \mathcal{H}$, $\mathcal{R}(H)(\tau)$ is K-Lipschitz continuous (Proposition A3).
- There exists a symmetric equilibrium $H \in \mathcal{H}$ (Theorem A2).

Like in the case of rationing rules based on strict priority orderings, in this benchmark scenario the probability $c(H)$ (formerly $\Pi_H(\tau)$) is also differentiable, which is critical for the existence of an equilibrium.

Unfortunately, the exact analysis for all rationing rules under consideration is not exhaustive in the sense that there are instances of the problems for which we do not have a guaranteed method for computing the equilibrium, although in our experiments we have always been able to find a PE using the basic iteration $H^{(n+1)} = \mathcal{R}(H^{(n)})$.[4]

[3] Note that our formulation is slightly different from the one in Aviv and Pazgal (2008) (see equation (2) in page 344 and Theorem 2 in page 348 therein). They use the unconditional probability of getting a unit via the random allocation rule (captured by the indicator $\mathbb{1}\{\mathcal{A}|Q_T\}$), but do not account for the fact that at the moment t of making the decision there are units available; i.e., $Q_t > 0$. However, if Q_0 is relatively large, then $\mathbb{P}(Q_t > 0) \approx 1$, and hence both formulations are indeed equivalent.

[4] An alternative implementation is to use the small step-size version of the iteration in Theorem 1 (see Bertsekas and Tsitsiklis (1996), Chapter 4). The so-called Robbins-Monro stochastic approximation algorithm is of the form:

$$H^{(n+1)} = \theta H^{(n)} + (1-\theta)\mathcal{R}(H^{(n)}),$$

where $\theta \in [0,1)$ is empirically selected.

Statement of results and proofs

Lemma A5 *For all $H \in \mathcal{H}$ and $\tau \in [0, T_S]$: $g_H^{RA}(\tau) > 1$, and there is a valuation $v_H > p_h$, such that $R(H)(\tau) \geq v_H$. The infimum $\tilde{v} = \inf_{H \in H}\{v_H\}$ satisfies $\tilde{v} = \inf_{H \in \mathcal{H}}\{v_H\} \geq \frac{p_h \bar{g}^{RA}}{\bar{g}^{RA}-1}$, where $\bar{g}^{RA} = \exp(wT_S)/\mathbb{P}(N(\bar{\lambda}T_S) \leq \underline{Q}_{T_S} - 1)$ and $\underline{Q}_{T_S} \triangleq (Q_0 - N(\bar{\Lambda}_{H_B}(T_S)))^+$, for $\bar{\Lambda}_{H_B}(T_S)$ given by (A1).*

Proof: Note that $c(H) < \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) = \mathbb{P}(Q_{T_S} > 0|H)$, hence

$$g_H^{RA}(\tau) > \frac{\mathbb{P}(Q_\tau \geq 1|H)}{\mathbb{P}(Q_{T_S} > 0|H)} > 0.$$

To prove the bound

$$g_H^{RA}(\tau) \leq \frac{\exp(wT_S)}{\mathbb{P}(N(\bar{\lambda}T_S) \leq \underline{Q}_{T_S} - 1)}, \tag{A13}$$

observe that

$$\begin{aligned} c(H) &\geq \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \sum_{n=0}^{k} \mathbb{P}(N(\Lambda_{H_R}(T_S) = n) \\ &= \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H)\mathbb{P}(N(\Lambda_{H_R}(T_S)) \leq k) \\ &= \mathbb{P}(N(\Lambda_{H_R}(T_S)) \leq Q_{T_S}) \\ &\geq \mathbb{P}(N(\Lambda_{H_R}(T_S)) \leq Q_{T_S} - 1) \\ &\geq \mathbb{P}(N(\bar{\lambda}T_S) \leq \underline{Q}_{T_S} - 1). \end{aligned}$$

The bound (A13) follows from (i) in Lemma A2, and the argument concludes as in the proof of Lemma A4. ∎

Proposition A3 *For the exponential utility function (1) and for all $H \in \mathcal{H}$, there is a positive constant K (independent of H) such that the best-response strategy $\mathcal{R}(H)(\tau)$ is a K-Lipschitz continuous function.*

Proof: The proof is analogous to Proposition A2. Representations (A3) and (A4) hold, $\Pi_H(\tau) = c(H)$ and $\Pi'_H(\tau) = 0$. From this and (A5):

$$|\mathcal{R}(H)(\tau_1) - \mathcal{R}(H)(\tau_2)| \leq \frac{(1-p_h)^2}{p_h} \frac{(w+\bar{\lambda})\exp(wT_S)}{\mathbb{P}(N(\bar{\lambda}T_S) \leq \underline{Q}_{T_S} - 1)} |\tau_1 - \tau_2| \triangleq K|\tau_1 - \tau_2|,$$

where the bound on $c(H)$ follows from Lemma A5. The constant K is independent of τ_1 and τ_2, and is guaranteed to be finite because $\mathbb{P}(N(\bar{\lambda}T_S) \leq Q_{T_S} - 1) > 0$. ∎

Theorem A2 *For the RA rationing rule, the set of strategies $\mathcal{H}$ equipped with the uniform norm $\|X\| = \sup_{0\leq\tau\leq T_S}\{|X(\tau)|\}$ in $[0, T_S]$ exhibits the fixed-point property. In addition, for all $H, \tilde{H} \in \mathcal{H}$, the mapping $\mathcal{R}$ satisfies:*

$$\|\mathcal{R}(H) - \mathcal{R}(\tilde{H})\| \leq K_H \|H - \tilde{H}\|,$$

where K_H is finite. Therefore, $\mathcal{R}$ is a continuous mapping and there always exists a PE. In addition, if $K_H < 1$ then $\mathcal{R}$ is a contraction. In this case, the fixed-point $\mathcal{R}(H^*) = H^*$ is guaranteed to be unique in $\mathcal{H}$ and can be found through the iteration $H^{(n+1)} = \mathcal{R}(H^{(n)})$ starting from an arbitrary $H^{(0)} \in \mathcal{H}$.

Proof: From the proof of Theorem 1, $\mathcal{H}$ has the fixed-point property. To prove the continuity of the best-response mapping $\mathcal{R}(\mathcal{H})$, representation (A7) holds, and therefore

$$|\mathcal{R}(\mathcal{H})_\tau - \mathcal{R}(\tilde{\mathcal{H}})_\tau| \leq \frac{(1-p_h)^2}{p_h} |g_H^{RA}(\tau) - g_{\tilde{H}}^{RA}(\tau)|.$$

Now

$$\begin{aligned}
|g_H^{RA}(\tau) - g_{\tilde{H}}^{RA}(\tau)| &= \exp(w(T_S-\tau)) \left| \frac{\mathbb{P}(Q_\tau > 0|H)}{c(H)} - \frac{\mathbb{P}(Q_\tau \geq 1|\tilde{H})}{c(\tilde{H})} \right| \\
&\leq \exp(wT_S) \left| \frac{\mathbb{P}(Q_\tau \geq 1|H)c(\tilde{H}) - \mathbb{P}(Q_\tau > 0|\tilde{H})c(H)}{c(H)c(\tilde{H})} \right| \\
&\leq \frac{\exp(wT_S)}{\mathbb{P}(N(\bar{\lambda}T_S) \leq Q_{T_S} - 1)^2} \left| \mathbb{P}(Q_\tau > 0|H)c(\tilde{H}) - \mathbb{P}(Q_\tau > 0|\tilde{H})c(H) \right| \\
&\leq \frac{\exp(wT_S)}{\mathbb{P}(N(\bar{\lambda}T_S) \leq Q_{T_S} - 1)^2} \left[|\mathbb{P}(Q_\tau > 0|H) - \mathbb{P}(Q_\tau > 0|\tilde{H})| + |c(\tilde{H}) - c(H)| \right].
\end{aligned}$$

Let

$$\Lambda_{H_B}^{RA}(T_S) \triangleq \int_0^{T_S} \lambda(t)\bar{F}(H(t), t)\mathrm{d}t.$$

The first module in brackets can be bounded by (A9). Consider the second module and recall $c(H) = \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \left[\sum_{n=0}^{k} \mathbb{P}(N(\Lambda_{H_R}^{RA}(T_S)) = n) + \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda_{H_R}^{RA}(T_S)) = n) \right]$. Therefore:

$|c(\tilde{H}) - c(H)|$

$$= \left| \sum_{k=1}^{Q_0} \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \le k) + \sum_{k=1}^{Q_0} \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) \right.$$
$$\left. - \sum_{k=1}^{Q_0} \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \le k) - \sum_{k=1}^{Q_0} \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n) \right|$$
$$\le |A| + |B|,$$

where

$$|A| = \left| \sum_{k=1}^{Q_0} [\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \le k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \le k)] \right|$$
$$\le \sum_{k=1}^{Q_0} \left| \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \le k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)\mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \le k) \right|$$
$$\le \sum_{k=1}^{Q_0} \Big[|\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)|$$
$$+ |\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \le k) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \le k)| \Big]$$
$$\le \sum_{k=1}^{Q_0} \Big[|\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) \le Q_0 - k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) \le Q_0 - k)|$$
$$+ |\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) \le Q_0 - k - 1) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) \le Q_0 - k - 1)| + |\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \le k) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \le k)| \Big]$$
$$\le |\Lambda^{RA}_{\tilde{H}_B}(T_S) - \Lambda^{RA}_{H_B}(T_S)| \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1)] + |\Lambda^{RA}_{\tilde{H}_R}(T_S) - \Lambda^{RA}_{H_R}(T_S)| \sum_{k=1}^{Q_0} \beta(k)$$
$$\le \bar{\lambda} K_F T_S K_A \|\tilde{H} - H\|,$$

with $K_A = \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1) + \beta(k)]$ being finite. We are assuming in the previous derivation $\beta(-1) = 0$.

Consider term $|B|$:

$$|B| = \left| \sum_{k=1}^{Q_0} [\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) \right.$$
$$\left. - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n)] \right|$$
$$\le \sum_{k=1}^{Q_0} \left| \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n) \right|$$
$$\le \sum_{k=1}^{Q_0} \left[|\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)| + \left| \sum_{n=k+1}^{\infty} \frac{k}{n} (\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n)) \right| \right].$$

By the same argument as for term $|A|$, $\sum_{k=1}^{Q_0} |\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_B}(T_S)) = Q_0 - k) - \mathbb{P}(N(\Lambda^{RA}_{H_B}(T_S)) = Q_0 - k)| \le \bar{\lambda} K_F T_S \|\tilde{H} - H\| \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1)]$.

Consider $\sum_{k=1}^{Q_0} \left| \sum_{n=k+1}^{\infty} \frac{k}{n} (\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n)) \right|$. We have:

$$\left| \sum_{n=k+1}^{\infty} \frac{k}{n} (\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n)) \right|$$

$$\leq \sum_{n=k+1}^{\infty} \left| \frac{k}{n} (\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) = n) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) = n)) \right|$$

$$\leq \sum_{n=k+1}^{\infty} \frac{k}{n} \left[|\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \leq n) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \leq n)| + |\mathbb{P}(N(\Lambda^{RA}_{\tilde{H}_R}(T_S)) \leq n-1) - \mathbb{P}(N(\Lambda^{RA}_{H_R}(T_S)) \leq n-1)| \right]$$

$$\leq k|\Lambda^{RA}_{\tilde{H}_R}(T_S) - \Lambda^{RA}_{H_R}(T_S)| \sum_{n=k+1}^{\infty} \frac{1}{n}(\beta(n) + \beta(n-1)) \;\leq\; k\bar{\lambda} K_F T_S \|\tilde{H} - H\| \sum_{n=k+1}^{\infty} \frac{1}{n}(\beta(n) + \beta(n-1)).$$

From Stirling's approximation, we know that $n! > \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} > \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$ (see Feller (1968), Section 2.9). Hence,

$$\begin{aligned}
\sum_{n=k+1}^{\infty} \frac{1}{n}(\beta(n) + \beta(n-1)) &= \sum_{n=k+1}^{\infty} \frac{1}{n} \left(\frac{e^{-n} n^n}{n!} + \frac{e^{-(n-1)}(n-1)^{(n-1)}}{(n-1)!} \right) \\
&\leq \frac{1}{\sqrt{2\pi}} \sum_{n=k+1}^{\infty} \left(\frac{1}{n^{3/2}} + \frac{1}{n\sqrt{n-1}} \right) \\
&\leq \frac{1}{\sqrt{2\pi}} \sum_{n=k+1}^{\infty} \left(\frac{1}{n^{3/2}} + \frac{1}{(n-1)^{3/2}} \right) \triangleq K_k < \infty. \qquad \text{(A14)}
\end{aligned}$$

Therefore,

$$|B| \leq \bar{\lambda} K_F T_S \|\tilde{H} - H\| \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1) + kK_k].$$

Finally,

$$\begin{aligned}
|\mathcal{R}(\mathcal{H})_\tau - \mathcal{R}(\tilde{\mathcal{H}})_\tau| \leq \frac{(1-p_h)^2}{p_h} \frac{\exp(wT_S)}{\mathbb{P}(N(\bar{\lambda} T_S) \leq \underline{Q}_{T_S} - 1)^2} \bar{\lambda} K_F T_S \\
\times \left(\beta(Q_0 - 1) + \textstyle\sum_{k=1}^{Q_0} [2\beta(Q_0 - k) + 2\beta(Q_0 - k - 1) + \beta(k) + kK_k] \right) \|\tilde{H} - H\|,
\end{aligned}$$

where $\beta(n) = \mathbb{P}(N(n) = n)$ and K_k is given by (A14).

From this result, we conclude that $\mathcal{R}$ is continuous, which together with the fixed-point property of the set $\mathcal{H}$ guarantees existence of a PE. ∎

A1.3. Asymptotic analysis of the game

A1.3.1. FIFO rationing rule

Proof of Theorem 2 Recalling that $\lambda^{(n)}(t) = n\lambda(t)$, we start by defining

$$\Lambda^{(n)}_{H_B}(\tau) \triangleq \int_0^\tau \lambda^{(n)}(s)\, \bar{F}(H(s), s) \mathrm{d}s \quad \text{and} \quad \Lambda^{(n)}_{H_R}(\tau) \triangleq \int_0^\tau \lambda^{(n)}(s)\, F(H(s), s) \mathrm{d}s.$$

For part (i), we follow the argument in Maglaras and Meissner (2006), Section 4.2.2. The functional strong-law-of-large-numbers (FSLLN) for Poisson processes states that if $N \triangleq N_1$ is a unit rate Poisson process, then as $n \longrightarrow \infty$,

$$\frac{N(n\tau)}{n} \longrightarrow \tau \quad \text{a.s., uniformly in } \tau \in [0,T].$$

Noting that

$$\Lambda_{H_B}^{(n)}(\tau) = \int_0^\tau n\lambda(s)\,\bar{F}(H(s),s)\mathrm{d}s = n\Lambda_{H_B}(\tau),$$

and that the counting process of a nonhomogeneous Poisson process with rate $\Lambda_{H_B}^{(n)}$, $N_{\Lambda_{H_B}^{(n)}}(\tau)$, is equal in distribution to $N(n\Lambda_{H_B}(\tau))$, we get

$$\frac{N_{\Lambda_{H_B}^{(n)}}(\tau)}{n} \overset{\mathcal{D}}{=} \frac{N(\Lambda_{H_B}^{(n)}(\tau))}{n} \longrightarrow \Lambda_{H_B}(\tau) \quad \text{a.s., uniformly in } \tau \in [0,T].$$

The result $N(\Lambda_{H_R}^{(n)}(\tau))/n \longrightarrow \Lambda_{H_R}(\tau)$ is verified similarly.

From the continuity of the max function, it follows that as $n \longrightarrow \infty$, and for all $\tau \in [0,T]$,

$$\frac{Q_\tau^{(n)}}{n} = \frac{(Q_0^{(n)} - \Lambda_{H_B}^{(n)}(\tau))^+}{n} \longrightarrow Q_\tau \triangleq (Q_0 - \Lambda_{H_B}(\tau))^+ \quad \text{a.s., uniformly in } \tau \in [0,T].$$

For part (ii), note that $\mathbb{1}\{N(\Lambda_{H_R}^{(n)}(\tau)) \le Q_T^{(n)} - 1\} = \mathbb{1}\{N(\Lambda_{H_R}^{(n)}(\tau))/n \le (Q_T^{(n)} - 1)/n\}$. Taking limit as $n \longrightarrow \infty$, we get that $\mathbb{1}\{N(\Lambda_{H_R}^{(n)}(\tau)) \le Q_T^{(n)} - 1\}$ converges a.s. to the constant $\Pi_H^\infty(\tau)$, with

$$\Pi_H^\infty(\tau) = \begin{cases} 1 \text{ if } & \Lambda_{H_R}(\tau) \le Q_T \\ 0 \text{ if } & \Lambda_{H_R}(\tau) > Q_T, \end{cases}$$

where $Q_T \triangleq (Q_0 - \Lambda_{H_B}(T))^+$. Thus, from the bounded convergence theorem, as $n \longrightarrow \infty$, the sequence of probabilities $\mathbb{P}(N(\Lambda_{H_R}^{(n)}(\tau)) \le Q_T^{(n)} - 1)$ converges to $\Pi_H^\infty(\tau)$, i.e.,

$$\Pi_H^{(n)}(\tau) \triangleq \mathbb{P}(N(\Lambda_{H_R}^{(n)}(\tau)) \le Q_T^{(n)} - 1) \longrightarrow \Pi_H^\infty(\tau),$$

completing the proof. ∎

An important observation is that even though the exact PE is K-Lipschitz continuous, the asymptotic purchasing strategy $H^*(\infty,\rho)$ is not even continuous in general, with τ^* being the discontinuity point.[5] Indeed, Theorem 2 guarantees weak convergence of the number of units left Q_T, and of the probability of getting a unit through the reservation channel, but it does not claim path-wise convergence of the trajectories $H^*(Q_0,\rho)$.

[5] The discontinuity point is represented by a vertical line for the plotting of $H^*(\infty, 0.7)$ in Figure 4.

A1.3.2. RA rationing rule

Calculus of the probability of getting a reserved item

Proof of Theorem 3 The probability of getting an item through a reservation is:

$$c^{(n)}(H) \triangleq \min\left\{\frac{(Q_0^{(n)} - N(\Lambda_{H_B}^{(n)}(T_S)))^+}{N(\Lambda_{H_R}^{(n)}(T_S))},\ 1\right\}.$$

Dividing by n the numerator and denominator of the first argument, and using the convergence results in Theorem 2, we get

$$c^{(n)}(H) \longrightarrow \min\left\{\frac{(Q_0 - \Lambda_{H_B}(T_S))^+}{\Lambda_{H_R}^{RA}(T_S)},\ 1\right\}, \quad \text{as} n \longrightarrow \infty,$$

which completes the proof. ∎

Results for the intermediate supply case

Next proposition shows that the fixed-point equation (16) always has a solution for the intermediate supply case.

Proposition A4 *If* $\int_0^{T_S} \lambda(t)\,\bar{F}(p_h,t)\,dt < Q_0 < \int_0^{T_S} \lambda(t)\,dt$, *then equation (16) has a solution for* $c^\infty(H)$ *in the interval* $(0,1)$*, and therefore an equilibrium always exists.*

Proof: For convenience denote $k \triangleq c^\infty(H)$ and define

$$G(k) \triangleq k - 1 + \frac{(1-\rho)\int_0^T \lambda(t)\mathrm{d}t}{\Lambda_{H_R}^{RA}(T_S,k)}, \quad \text{for } 0 \le k \le 1,$$

where

$$\Lambda_{H_R}^{RA}(T_S,k) \triangleq \int_0^{T_S} \lambda(t)F\left(\min\left\{\frac{p_h \exp(w(T_S-t))}{\exp(w(T_S-t))-k},\ 1\right\},t\right)\mathrm{d}t. \tag{A15}$$

Next, we compute the extreme values:

$$\begin{aligned} G(0) &= -1 + \frac{(1-\rho)\int_0^{T_S} \lambda(t)\mathrm{d}t}{\Lambda_{H_R}^{RA}(T_S,0)} \\ &= -1 + \frac{\int_0^{T_S} \lambda(t)\mathrm{d}t - Q_0}{\int_0^{T_S} \lambda(t)F(p_h,t)\mathrm{d}t} \\ &< -1 + \frac{\int_0^{T_S} \lambda(t)\mathrm{d}t - \int_0^{T_S} \lambda(t)\bar{F}(p_h,t)\mathrm{d}t}{\int_0^{T_S} \lambda(t)F(p_h,t)\mathrm{d}t} = 0; \end{aligned}$$

$$G(1) = \frac{(1-\rho)\int_0^{T_S} \lambda(t)\mathrm{d}t}{\Lambda_{H_R}^{RA}(T_S, 1)} > 0.$$

Since $G(k)$ is continuous in $[0,1]$, and $G(0) < 0 < G(1)$, the result follows. ∎

The uniqueness of the solution to (16) cannot be guaranteed though. For example, if v follows a time-homogeneous Beta distribution with parameters $(0.4,\ 0.4)$, $p_l = 0, p_h = 0.45, \rho = 0.55$, and $w = 0$ (i.e., consumers are patient), equation (16) has two solutions leading to different purchasing strategies: One solution corresponds to $H_1(\tau) = 0.5$, where $c^\infty(H_1) = 0.1$, and where some consumers choose to buy immediately. The other solution corresponds to the strategy $H_2(\tau) = 1$, for all $\tau \in [0, T_S]$ (i.e., all consumers choose to reserve), and $c^\infty(H_2) = \rho$. Next proposition generalizes the case of $H_2(\tau) = 1$:

Proposition A5 *For the intermediate supply case, if $\rho \geq \exp(wT_S)(1-p_h)$, there is an equilibrium strategy where $H^*(\tau) = 1$, $\forall \tau \in [0, T_S]$, and $c^\infty(H^*) = \rho$.*

Proof Taking the definition of $\Lambda_{H_R}^{RA}(T_S, k)$ in (A15), the minimum in the integrand equals to one when $k \geq \exp(w(T_S - \tau))(1 - p_h)$. In particular, $H^*(\tau) = 1$ when $k \geq \exp(wT_S)(1 - p_h)$. Under this situation, $\Lambda_{H_R}^{RA}(T_S, k) = \int_0^{T_S} \lambda(t)\mathrm{d}t$, and equation (14) becomes $c^\infty(H^*) = k = \rho$. ∎

Note also that equilibrium $H_2(\tau) = 1$ in the aforementioned example Pareto-dominates equilibrium $H_1(\tau)$. To check this, consider the following two cases:

- Take a consumer with valuation $v_t \in [0.5, 1]$ (i.e., a valuation between $H_1(\tau)$ and $H_2(\tau)$). The buy now utility of this customer under $H_1(\tau)$ is $u(\tau, \tau, v_\tau - p_h) \geq u(\tau, T_S, v_\tau - p_l) \times c^\infty(H_1)$. His reservation utility for playing $H_2(\tau)$ is $u(\tau, T_S, v_\tau - p_l) \times c^\infty(H_2) \geq u(\tau, \tau, v_\tau - p_h)$. Therefore, he will prefer to play $H_2(\tau)$.
- Take a consumer with valuation $v_t \in [0.45, 0.50]$ (i.e., a valuation between p_h and $H_1(\tau)$). The reservation utility for this customer under $H_1(\tau)$ is $u(\tau, T_S, v_\tau - p_l) \times c^\infty(H_1) \geq u(\tau, \tau, v_\tau - p_h)$. His reservation utility under $H_2(\tau)$ is $u(\tau, T_S, v_\tau - p_l) \times c^\infty(H_2) > u(\tau, T_S, v_\tau - p_l) \times c^\infty(H_1)$, because $c^\infty(H_2) > c^\infty(H_1)$.

This analysis could be generalized to more than two equilibria to prove that the Pareto-dominant equilibrium is the one with highest value $c^{\infty}(H)$ among the solutions to (16).

Results for the limited supply case

We argue in the main body of the paper that multiple equilibria are possible in this case. The purchasing strategy $H^*(\tau) = p_h$ for all $\tau \in [0, T_S]$, and more generally, any strategy given by (10) is an equilibrium. The probability of getting an item through a reservation is $c^{\infty}(H) = 0$ in this case. In addition, there could be another type of equilibria in which consumers can get an item through a reservation with $c^{\infty}(H) > 0$. In this scenario, an equilibrium is given by (15), where $c^{\infty}(H)$ is defined by (16). If a solution exists, it is not necessarily unique, as can be seen in Figure A1. In fact, the existence of a solution to (16) is not even guaranteed in this case. For example, this happens under the same parameters as in Figure A1, but with $w = 1$.

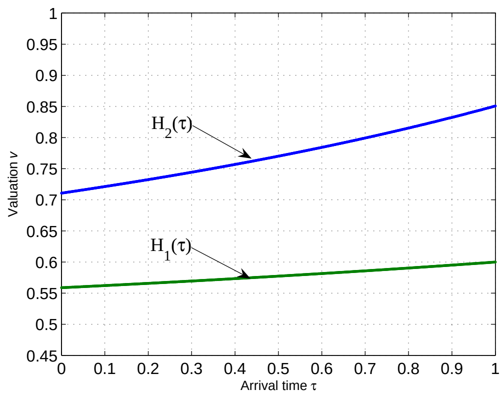


Figure A1 Two equilibrium purchasing strategies for the RA rationing rule in the asymptotic regime with limited supply and time-homogeneous (scaled) valuations Beta$(4,4)$, $T_S = 1$, $w = 0.25$, $p_h = 0.45$, and $\rho = 0.5 < \bar{F}(p_h) \approx 0.61$. The probabilities of getting a reserved item are $c(H_1) \approx 0.25$ and $c(H_2) \approx 0.47$, respectively.

A1.4. Revenue optimization

The following result is an adaptation of Proposition 6 in Caldentey and Vulcano (2007), and can be inferred by setting p_h as the fixed price $\hat{P}$ and p_l as the auction price P_H in the *dual channel* case there. We refer the reader to the e-companion of that paper for a proof.

Proposition A6 a) *Suppose T is fixed and define $\delta = \frac{\alpha T \exp(-\alpha T)}{1-\exp(-\alpha T)}$. Then, the optimal solution (ρ^*, p_l^*, p_h^*) satisfies: $1 - p_h^* \le \rho^* \le (1 - p_h^*)\exp(wT)$. There are two cases regarding the optimal revenue V,*

Case 1: *If $\frac{\bar{Q}}{\lambda T} \le \frac{1-\delta}{2-\delta}$ then $V = \frac{\lambda T \exp(-\alpha T)}{\delta} \rho^* p_h^*$, where $\rho^* = 1 - p_h^* = \frac{\bar{Q}}{\lambda T}$, and where $p_l^* = p_h^*$.*

Case 2: *If $\frac{\bar{Q}}{\lambda T} \ge \frac{1-\delta}{2-\delta}$ then V is bounded by*

$$V_1 \triangleq \frac{\lambda T \exp(-\alpha T)}{\delta} \rho_1 p_1 \le V \le \frac{\lambda T \exp(-\alpha T)}{\delta} (p_2^2 - (1-\rho_2)^2 \delta) \triangleq V_2,$$

where $\rho_1 = 1 - p_1 = \min\left\{\frac{1}{2}\,;\, \frac{\bar{Q}}{\lambda T}\right\}$, $\rho_2 = \min\{\frac{\bar{Q}}{\lambda T}\,;\, \frac{3-\delta}{4-\delta}\}$ and $p_2 = \frac{1}{2}\left[1 + (1-\rho_2)\delta\right]$.

Furthermore, the lower and upper bounds V_1 and V_2, respectively, are asymptotically tight in the following sense

$$\lim_{w \downarrow 0} V = V_1, \quad \lim_{w \downarrow 0} \rho^* = \lim_{w \downarrow 0} (1 - p_h^*) = \rho_1, \qquad \text{and} \qquad \lim_{w \to \infty} V = V_2, \quad \lim_{w \to \infty} \rho^* = \rho_2, \quad \lim_{w \to \infty} p_h^* = p_2.$$

b) *The optimal time T^* is bounded above by the unique nonnegative root of*

$$\frac{\bar{Q}}{\lambda T} = \frac{1 - \exp(-\alpha T) - \alpha T \exp(-\alpha T)}{2 - 2\exp(-\alpha T) - \alpha T \exp(-\alpha T)}.$$

Proposition A6 starts arguing that the seller does prefer to operate in an intermediate supply regime in order to maximize revenues. For part (a), since δ is a decreasing function of αT, the first case holds when the initial inventory $\bar{Q}$ is small relative to the total demand λT and/or when αT is large so that any revenue collected in the reservation channel is penalized by a deep discount $\exp(-\alpha T)$. Here, the seller does not implement price discrimination; she puts up all the inventory for sale (i.e., $Q_0^* = \bar{Q}$), and charges a unique price $p_l^* = p_h^* \ge 1/2$.

In the second case, the seller has an incentive to achieve price discrimination by adding the reservation channel. Despite not getting a closed form solution for the seller's revenue, we present upper and lower bounds for it. The lower bound V_1 is derived assuming that the seller operates only the high price channel (i.e., $\rho_1 = 1 - p_1$, with $p_l = p_h = p_1$). The upper bound V_2 is obtained assuming that all consumers with valuation greater than $p_h = p_1$ will behave myopically and purchase in the buy-now channel, and only those with valuation $v_\tau \in [p_l, p_h]$ will place a reservation, where $p_l = 1 - \rho_2$.

Both V_1 and V_2 are asymptotically optimal. The lower bound V_1 implies that as consumers become increasingly patient (i.e., $w \downarrow 0$), the seller is unable to segment the population using the reservation channel. Consumers are willing to wait if p_l is below p_h. Hence, in the limit as $w \downarrow 0$, $p_l^* = p_h^*$, and all units are sold through the buy-now channel. This is the worst possible scenario for the seller. On the other hand, if consumers become increasingly impatient (i.e, $w \to \infty$) then the seller can segment these consumers among those that can afford to pay p_h and those that cannot. This gives the seller the ability to separate consumers exclusively based on their valuations and optimally design the buy-now and the reservation channels to achieve the upper bound V_2. The use of reservations in this case is an effective selling mechanism that complements well a single fixed-price operation, and matches the revenues achievable if using an auction to liquidate the excess inventory. It can also be proved that the benefit of adding a reservation channel to a single fixed-price operation can increase revenues by as much as 33% (again, see Caldentey and Vulcano (2007), Section 5.2).

A2. Priority rules

We consider time-based priority rules defined over the set $\Xi \triangleq \{\xi \in \mathcal{D} : [0,T] \to [0,1]$, s.t. finite number of local optima$\}$. A function $\xi \in \Xi$ gives more priority to an arrival at τ_1 than an arrival at τ_2 (given that both place a reservation) if $\xi(\tau_1) > \xi(\tau_2)$. The priority ordering defined here is assumed to be strict, in the sense that the number of customer arrivals that share the same priority is finite. Figure A2 shows three cases of priority rules that satisfy the strict ordering (the plots at the top), and two that do not (the two plots at the bottom).

The three strict priorities in the top are: FIFO (first-in-first-out, or first-come-first-served), LIFO (last-in-first-out), and FLIFO (first and last in, first out; which gives same priorities to earliest and latest arrivals, and less to intermediate time arrivals). Of course, there are priorities that would be easier to sustain in the retail practice because (e.g., FIFO), but our model allows for these more general cases.

Note that these are not the unique functional forms that lead to priorities exhibited there. For example, FIFO could also be represented by a function $\xi(t) = \frac{(t-T)^2}{T^2}, t \in [0, T]$.

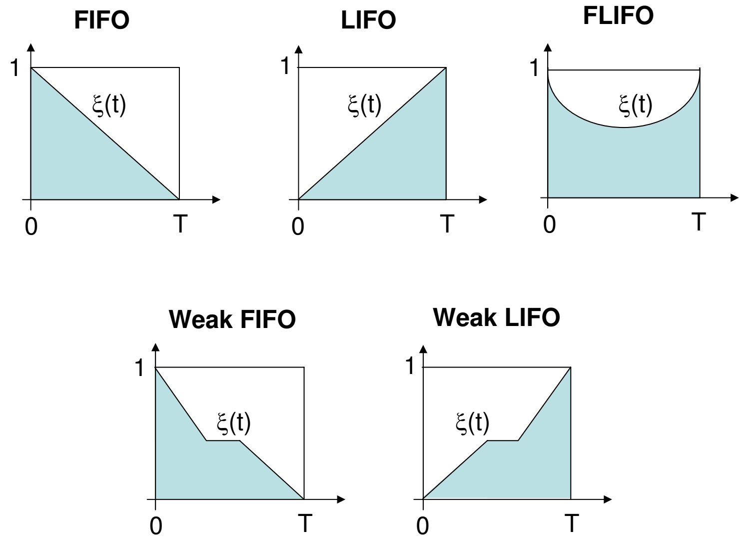


Figure A2 Illustrations of strict (top) and weak (bottom) priority rules.

A3. Numerical experiments

A3.1. Revenue performance

In this section we include the figures on RA-1 mentioned in Section 6.2 in the main body of the paper. FigureA3 (left) shows the price p_l and the markdown level $p_h - p_l$ under different availability scenarios $\bar{Q}/(\lambda T)$. On the right, we exhibit the split of units between the buy now channel and the reservation channel. Compared to FIFO, RA-1 sets a lower price p_h at the optimum. The amount of the discount $p_h - p_l$ under RA-1 is slightly higher than under FIFO when the supply is abundant, but FIFO discounts more aggressively when the supply is intermediate. Still, regardless of the supply regime, RA-1 sells fewer items at the high price. Furthermore, in order to achieve its optimal revenue RA-1 has to sell more items in the abundant supply regime.

A3.2. Analysis of the market composition effect

In this section, we analyze the case where the market consists of two different types of consumers: strategic and myopic. The former arrival rate $\lambda(t)$ is now split between a fraction $\gamma, 0 < \gamma < 1$, of

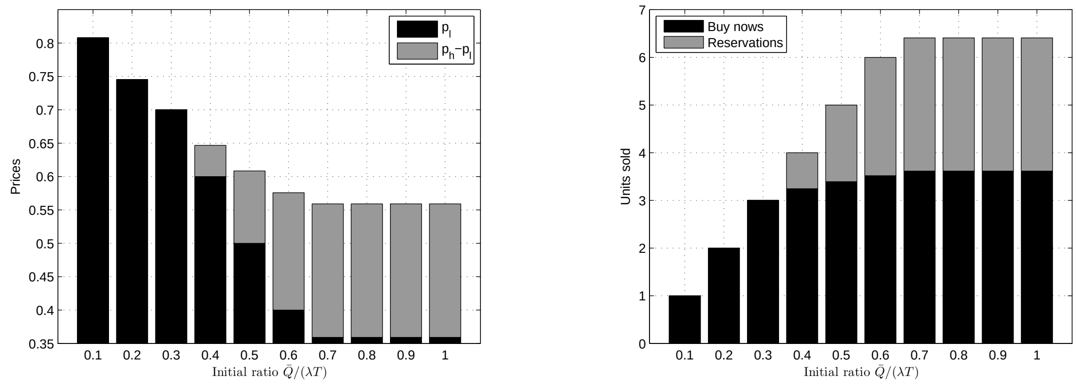


Figure A3 Left: Prices p_l (black rectangle) and p_h (height of the bar) under RA-1 for different availabilities $\bar{Q}/(\lambda T)$. Right: Split of units under RA-1 for different availabilities $\bar{Q}/(\lambda T)$. Values of parameters are: $\bar{Q} \leq 10, T = 1, \lambda = 10, w = 5, \alpha = 1$, and valuations Unif[0,1].

myopic consumers, and a fraction $1-\gamma$ of strategic consumers. We assume that γ is time invariant and common knowledge.

A3.2.1. Consumers' behavior

The myopic consumers behave according to the simple strategy "buy now if own valuation is higher than p_h, and reserve otherwise". The strategic ones choose the buying channel to maximize their expected utility. Both types of consumers participate in the clearing of excess inventory at the end of the selling season. In what follows we analyze the FIFO and RA rationing rules for allocating leftover inventory to reservations.

FIFO rationing rule

Under FIFO rationing rule, reservations placed by strategic or myopic consumers get priority based on the time stamp of their arrivals. Strategic, forward-looking consumers are aware of the myopic strategy, and internalize it when assessing their own expected utilities for buying now and reserving. The analysis then follows the outline described in Section 4.1, but with the following modification of equation (6) there, now specialized for FIFO and the mixed market composition. The probability that a reservation consumer gets a reserved unit is:

$$\Pi_H(\tau) = \mathbb{P}\left(B\left(\Lambda^S_{H_B}(T) + \Lambda^M_{H_B}(T) + \Lambda^S_{H_R}(\tau) + \Lambda^M_{H_R}(\tau)\right) \leq Q_0 - 1\right), \quad \text{(A16)}$$

where

- Mean strategic buy-nows: $\Lambda^S_{H_B}(\tau) = (1-\gamma)\int_0^\tau \lambda(t)\bar{F}(H(t),t)\mathrm{d}t$.
- Mean myopic buy-nows: $\Lambda^M_{H_B}(\tau) = \gamma\int_0^\tau \lambda(t)\bar{F}(p_h,t)\mathrm{d}t$.
- Mean strategic reservations: $\Lambda^S_{H_R}(\tau) = (1-\gamma)\int_0^\tau \lambda(t)F(H(t),t)\mathrm{d}t$.
- Mean myopic reservations: $\Lambda^M_{H_R}(\tau) = \gamma\int_0^\tau \lambda(t)F(p_h,t)\mathrm{d}t$.

Our previous equilibrium results for strict priorities rules (though with a single stream of consumers) can be adapted to this mixed market case. Using Lemma A1(i), and following the argument in the proof of Lemma A3, it can be verified that for the $\Pi_H(\tau)$ defined in (A16), $\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi_H(\tau)\right| \le K_\Pi$. Therefore, a result analogous to Theorem 1 holds. Figure A4 shows how the strategy $H(\tau)$ played by forward-looking consumers changes with the fraction γ of myopic consumers. Clearly, the more myopic consumers, the more strategic consumers will tend to buy-now, since units will be depleting faster through the buy-now channel.

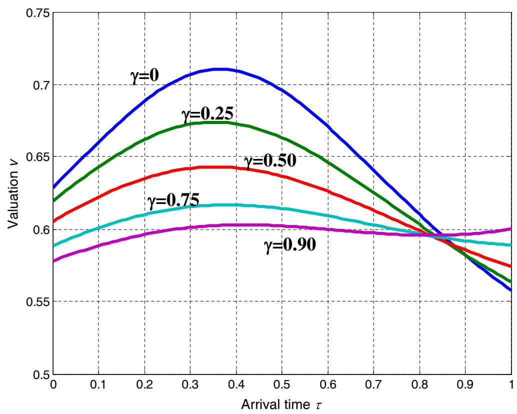


Figure A4 Effect of different proportions γ of myopic consumers in the strategy played by forward-looking consumers. Value of parameters: $Q_0 = 9, T = 1, \lambda = 10, w = 1, p_l = 0, p_h = 0.4$, and valuations Unif[0,1].

The asymptotic regime that we consider is the same as the one introduced in Section 5. The following result characterizes the asymptotic strategy of forward-looking consumers under the mixed market framework. It can be proved similarly to Theorem 2.

Theorem A3 *Suppose that the purchasing strategy $H(\tau)$ is given. Then, in the limit as $n \longrightarrow \infty$:*

(i) The following convergence results hold almost surely (a.s.), and uniformly in τ:

$$N(\Lambda^{S,(n)}_{H_B}(\tau))/n \longrightarrow \Lambda^S_{H_B}(\tau), \quad N(\Lambda^{M,(n)}_{H_B}(\tau))/n \longrightarrow \Lambda^M_{H_B}(\tau), \quad Q^{(n)}_\tau/n \longrightarrow Q_\tau \triangleq (Q_0 - \Lambda^S_{H_B}(\tau) - \Lambda^M_{H_B}(\tau))^+,$$

$$N(\Lambda^{S,(n)}_{H_R}(\tau))/n \longrightarrow \Lambda^{S}_{H_R}(\tau), \quad \textit{and} \quad N(\Lambda^{M,(n)}_{H_R}(\tau))/n \longrightarrow \Lambda^{M}_{H_R}(\tau).$$

(ii) The probability $\Pi^{(n)}_H(\tau) \triangleq \mathbb{P}\left(N(\Lambda^{S,(n)}_{H_R}(\tau)+\Lambda^{M,(n)}_{H_R}(\tau)\right) \leq Q^{(n)}_T - 1)$ *converges to the two-point distribution:*

$$\Pi^{\infty}_H(\tau) = \begin{cases} 1 \text{ if } \Lambda^S_{H_R}(\tau)+\Lambda^M_{H_R}(\tau) \leq Q_T \\ 0 \text{ if } \Lambda^S_{H_R}(\tau)+\Lambda^M_{H_R}(\tau) > Q_T, \end{cases}$$

for $Q_T \triangleq \left(Q_0 - \Lambda^S_{H_B}(T) - \Lambda^M_{H_B}(T)\right)^+$.

Now we are interested in the strategies played by forward-looking consumers in this asymptotic regime. Similarly to Section 5, we need to distinguish three possible cases.

i) **Limited supply**. Suppose that the initial inventory is *limited* i.e. $Q_0 \leq \int_0^T \lambda(t)\,\bar{F}(p_h,t)\mathrm{d}t$. Consider a strategic consumer arriving with valuation v_τ, and suppose that all other forward-looking consumers choose the strategy $H^*(t) = p_h, \forall t \in [0,T]$. Given that the supply scarcity ensures no leftover inventory at time T, the arriving player can follow the strategy $H^*(\tau) = p_h$.

ii) **Intermediate supply**. Suppose that initial supply is *intermediate* in the sense that $\int_0^T \lambda(t)\,\bar{F}(p_h,t)\mathrm{d}t < Q_0 < \int_0^T \lambda(t)\mathrm{d}t$. In this case, $Q_T > 0$ and some strategic consumers with valuation smaller than p_h get units by placing reservations. Under FIFO, the leftover inventory will be allocated to early strategic or myopic arrivals who placed reservations. Therefore, all consumers (myopic and strategic) who arrived before the threshold time τ^* (defined in (11)) will get an item through one of the channels. After τ^* only consumers with valuations higher than p_h can get an item, i.e., $H(t) = p_h$, if $t > \tau^*$.

The early arriving strategic consumers, with $\tau \in [0,\tau^*)$, must decide which channel to purchase from, and therefore need to solve the limiting version of equation (7):

$$\frac{v_\tau}{v_\tau - p_h} \geq \frac{\exp(w(T-\tau))\mathbb{P}(Q_\tau > 0|H)}{\Pi^\infty_H(\tau)}.$$

Under intermediate supply and asymptotic regime, $\mathbb{P}(Q_\tau > 0|H) = 1$, for all τ. Thus, it can be shown that $\Pi^\infty_H(\tau) = 1$ for all $\tau < \tau^*$, or equivalently, $\Lambda^S_{H_R}(\tau)+\Lambda^S_{H_B}(T)+\Lambda^M_{H_R}(\tau)+\Lambda^M_{H_B}(T) < Q_0$, for all $\tau < \tau^*$. Indeed,

$$\Lambda^S_{H_R}(\tau)+\Lambda^S_{H_B}(T)+\Lambda^M_{H_R}(\tau)+\Lambda^M_{H_B}(T) \;<$$

$$
\begin{aligned}
&< \Lambda^S_{H_R}(\tau^*) + \Lambda^S_{H_B}(T) + \Lambda^M_{H_R}(\tau^*) + \Lambda^M_{H_B}(T) \\
&= (1-\gamma)\int_0^{\tau^*} \lambda(t)F(H(t),t)\mathrm{d}t + (1-\gamma)\int_0^{T} \lambda(t)\bar{F}(H(t),t)\mathrm{d}t + \gamma\int_0^{\tau^*} \lambda(t)F(p_h,t)\mathrm{d}t + \gamma\int_0^{T} \lambda(t)\bar{F}(p_h,t)\mathrm{d}t \\
&= (1-\gamma)\int_0^{\tau^*} \lambda(t)\mathrm{d}t + (1-\gamma)\int_{\tau^*}^{T} \lambda(t)\bar{F}(H(t),t)\mathrm{d}t + \gamma\int_0^{\tau^*} \lambda(t)\mathrm{d}t + \gamma\int_{\tau^*}^{T} \lambda(t)\bar{F}(p_h,t)\mathrm{d}t \\
&= \int_0^{\tau^*} \lambda(t)\mathrm{d}t + \int_{\tau^*}^{T} \lambda(t)\bar{F}(p_h,t)\mathrm{d}t = Q_0.
\end{aligned}
$$

We conclude that in this *intermediate* case the unique PE $H^*(\tau)$ is given by:

$$
H^*(\tau) = \begin{cases} \min\left\{\frac{p_h \exp(w(T-\tau))}{\exp(w(T-\tau))-1}, 1\right\} & \text{if } \tau \in [0,\tau^*) \\ p_h & \text{if } \tau \in [\tau^*, T]. \end{cases}
$$

iii) **Abundant supply**. Suppose that the initial supply is *abundant*, i.e., $Q_0 \geq \int_0^T \lambda(t)\mathrm{d}t$. In this case all reservations will be satisfied with probability one, hence, every consumer will get an item from the channel he chooses. The unique optimal strategy for strategic consumers is given by

$$
H^*(\tau) = \min\left\{\frac{p_h \exp(w(T-\tau))}{\exp(w(T-\tau))-1}, 1\right\}. \tag{A17}
$$

Overall, even though the exact stochastic strategy depends on the fraction γ of myopic consumers, the strategy played by forward-looking consumers in the mixed market environment is the same as that played under FIFO in the homogeneous market consisting only of strategic consumers. This somewhat surprising result is anchored in the following observations:

1. Consumers can asses the time of the last marginal arrival who will get a unit (i.e., the value of τ^* defined in (11)). This value does not depend on γ; just on the total arrival rate λ.
2. For $\tau \in [\tau^*, T]$, strategic consumers play the strategy $H(\tau) = p_h$, which is the same as the one played by the myopic. Hence, after τ^*, strategies of all consumers are the same in both homogeneous and mixed market cases.
3. Before τ^*, strategic consumers know that they can get a unit w.p.1 in both mixed and homogeneous markets and so they play the same strategy in both cases, whereas under the stochastic case the probability of getting an item through reservations is affected by the proportion γ of myopic consumers.

In other words, in the context of an asymptotic mixed market, strategic consumers can ignore the fraction γ of myopic consumers and are still able to compute an (optimal) equilibrium strategy. Figure A5 (left) illustrates the quality of the asymptotic approximation for FIFO over an intermediate supply case.

RA rationing rule.

Under RA rationing rule, all reservations placed by strategic or myopic consumers have the same priority and will be allocated at random. Therefore the probability of getting an item through a reservation is

$$c(H) \triangleq \min\left\{\frac{Q_{T_S}}{\text{\# of consumers that reserved an item}}, 1\right\},$$

where $Q_{T_S} = (Q_0 - N(\Lambda_{H_B}^{S,RA}(T_S) + \Lambda_{H_B}^{M,RA}(T_S)))^+$ and the number of placed reservations is $N(\Lambda_{H_R}^{S,RA}(T_S) + \Lambda_{H_R}^{M,RA}(T_S))$. The mean values involved are

- Mean strategic buy-nows: $\Lambda_{H_B}^{S,RA}(\tau) = (1-\gamma)\int_0^\tau \lambda(t)\bar{F}(H(t),t)\mathrm{d}t$.
- Mean myopic buy-nows: $\Lambda_{H_B}^{M,RA}(\tau) = \gamma\int_0^\tau \lambda(t)\bar{F}(p_h,t)\mathrm{d}t$.
- Mean strategic reservations: $\Lambda_{H_R}^{S,RA}(\tau) = (1-\gamma)\int_0^\tau \lambda(t)F(H(t),t)\mathrm{d}t$.
- Mean myopic reservations: $\Lambda_{H_R}^{M,RA}(\tau) = \gamma\int_0^\tau \lambda(t)F(p_h,t)\mathrm{d}t$.

The probability $c(H)$ is constant in τ, and hence differentiable. The existence of equilibrium can be verified following the guidelines of Section 4.2.

Regarding the asymptotic regime, a result analogous to Theorem 3 can be proved, and a limiting probability $c^\infty(H)$ of getting a unit through the reservation channel (analogous to (13)) can be established. Consider the following three supply cases:

i) **Abundant supply** (i.e., $Q_0 \geq \int_0^{T_S} \lambda(t)\mathrm{d}t$): Here $c^\infty(H) = 1$, and the optimal strategy is given by (12).

ii) **Intermediate supply** (i.e., $\int_0^{T_S} \lambda(t)\,\bar{F}(p_h,t)\mathrm{d}t < Q_0 < \int_0^{T_S} \lambda(t)\mathrm{d}t$): We can rewrite (13) as

$$c^\infty(H) = 1 - \frac{(1-\rho)\int_0^{T_S} \lambda(t)\mathrm{d}t}{\Lambda_{H_R}^{S,RA}(T_S) + \Lambda_{H_R}^{M,RA}(T_S)}, \tag{A18}$$

where $0 < c^\infty(H) < 1$. The condition for reservation for strategic consumers is given by (A11), where now $\mathbb{P}(Q_\tau > 0|H) = 1$ for all $\tau \in [0, T_S]$. Therefore, the optimal purchasing strategy is defined by (15). By substituting (15) into (A18), we get the following fixed-point equation for $c^\infty(H)$:

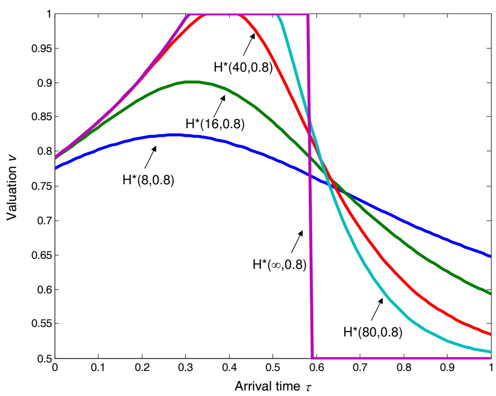
$$c^\infty(H^*) = 1 - \frac{(1-\rho)\int_0^{T_S}\lambda(t)\mathrm{d}t}{(1-\gamma)\int_0^{T_S}\lambda(t)F\left(\min\left\{\frac{p_h\exp(w(T_S-t))}{\exp(w(T_S-t))-c^\infty(H^*)},\ 1\right\},t\right)\mathrm{d}t+\gamma\int_0^{T_S}\lambda(t)F(p_h,t)\mathrm{d}t}. \quad \text{(A19)}$$

Under the intermediate supply case there is always at least one solution to (A19). In case of multiple solutions, it can be shown that the one with the highest $c^\infty(H)$ is Pareto dominant.

iii) **Limited supply** (i.e., $Q_0 \le \int_0^{T_S}\lambda(t)\,\bar{F}(p_h,t)\mathrm{d}t$): Multiple equilibria are also possible in this case. The purchasing strategy $H^*(\tau) = p_h$ for all $\tau \in [0, T_S]$, and more generally, any strategy given by (10) is an equilibrium. The probability of getting an item through a reservation is $c^\infty(H) = 0$ in this case.

In addition, there could be another type of equilibria in which consumers can get an item through a reservation with $c^\infty(H) > 0$. In this scenario, an equilibrium is given by (15), where $c^\infty(H)$ is defined by (A19).

Unlike the FIFO case, the introduction of myopic customers does affect the limiting strategy played by strategic consumers under the RA rule. This is due to the fact that the limiting probability of getting an item is not zero or one, in general. Figure A5 (right) illustrates the quality of the asymptotic approximation for RA over an intermediate supply case.



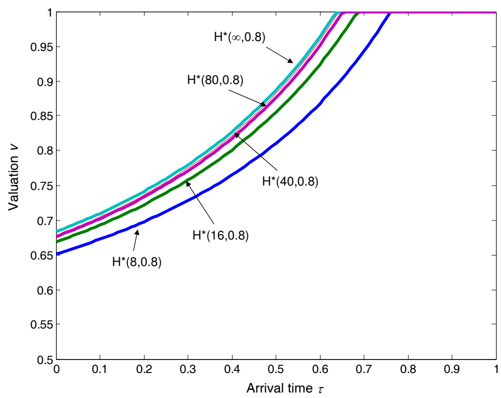


Figure A5 Performance of the asymptotic approximation for the case $\gamma = 0.5$ under FIFO (left) and RA (right). Value of parameters: $Q_0 = 8, T = 1, \lambda = 10, w = 1, p_l = 0, p_h = 0.5$, and time homogeneous valuations Unif[0,1].

A3.2.2. Seller's revenue optimization problem

For the clarity and fair comparison of performance, we assume time homogeneous valuations in this revenue optimization study. To simplify notation for the mix of strategic and myopic consumers, define $h(\tau) \triangleq (1-\gamma)\int_0^\tau F(H(t))\mathrm{d}t + \gamma\tau F(p_h)$ and $\bar{h}(\tau) \triangleq \tau - h(\tau)$. Then, after re-scaling the parameters to their original values, the revenue optimization problem is formulated as follows:

$$V^\gamma(\bar{Q}) = \max_{T,Q_0,p_l,p_h} \Bigg\{ p_h\lambda \int_0^T e^{-\alpha t} \mathbb{1}\{Q_t > 0\}((1-\gamma)\bar{F}(H(t)) + \gamma\bar{F}(p_h))\mathrm{d}t \\ + p_l e^{-\alpha T} \min\{(Q_0 - \lambda\bar{h}(T))^+, \lambda h(T)\}, \quad \text{subject to } p_l \le p_h, Q_0 \le \bar{Q} \Bigg\}.$$

The benchmark case problem under the mixed market framework becomes:

$$V^\gamma_{RA}(\bar{Q}) = \max_{T,Q_0,p_l,p_h} \Bigg\{ p_h\lambda \int_0^{T_S} e^{-\alpha t} \mathbb{1}\{Q_t > 0\}((1-\gamma)\bar{F}(H(t)) + \gamma\bar{F}(p_h))\mathrm{d}t \\ + p_l e^{-\alpha T_S} \min\{(Q_0 - \lambda\bar{h}(T_S))^+, \lambda h(T_S)\} + V_C, \quad \text{subject to } p_l \le p_h, Q_0 \le \bar{Q} \Bigg\},$$

where V_C is given by (19).

A3.3. Consumer surplus

In what follows, the notation encompasses FIFO, FP and RA-s cases, where for FIFO and FP, $T_S = T$. Given prices p_l and p_h, the total surplus obtained by buy-now consumers is

$$S_B = \lambda \int_0^{T_S} \int_{H(t)}^1 (v - p_h)\mathbb{1}\{Q_t > 0\} f(v)\mathrm{d}v\mathrm{d}t + S_C,$$

where S_C is the surplus obtained during a clearance season, i.e.,

$$S_C = \lambda \int_{T_S}^{\min\{\tau^*, T\}} \int_{p_l}^1 (v - p_l) f(v)\mathrm{d}v\mathrm{d}t = \min\left\{ \frac{(Q_0 - \lambda\bar{F}(p_l)T_S)^+}{\bar{F}(p_l)}, \lambda(T - T_S) \right\} \int_{p_l}^1 (v - p_l) f(v)\mathrm{d}v.$$

The total surplus obtained through the reservation channel is

$$S_R = \lambda \int_0^{T_S} \int_{p_l}^{H(t)} (v - p_l)\exp(w(t - T_S))\Pi_H^\infty(t) f(v)\mathrm{d}v\mathrm{d}t,$$

where $\Pi_H^\infty(t)$ stands for the limiting probability of getting an item through the reservation channel. The exact expressions for $\Pi_H^\infty(t)$ depend on the rationing rule and supply-demand ratio and were given in Section 5.

References

Aviv, Y., A. Pazgal. 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. *MSOM* **10** 339–359.

Bertsekas, D., J. Tsitsiklis. 1996. *Neuro-Dynamic Programming*. Athena Scientific, Belmont, MA.

Caldentey, R., G. Vulcano. 2007. Online auction and list price revenue management. *Management Science* **53** 795–813.

Cheney, W. 2001. *Analysis for Applied Mathematics*. Springer-Verlag, New York, NY.

Feller, W. 1968. *An Introduction to Probability Theory and Its Applications*, vol. 1. 3rd ed. Wiley, New York, NY.

Maglaras, C., J. Meissner. 2006. Dynamic pricing strategies for multiproduct revenue management problems. *MSOM* **8** 136–148.