Proofs

Appendix EC.1: General belief models

In this section, we extend our model to incorporate more general learning models for agents in the market.

Assume $\Theta \subseteq [0,1]$ denotes the set of possible values for an agent's private valuation. We let $\mathcal{S}$ denote the space of possible beliefs for an agent, where each $s \in \mathcal{S}$ indexes a probability distribution P_s over Θ. At time 0, each agent i starts with an exogenously specified prior belief over her valuation, indexed by $s_{i,0}$. Formally, we require that the agent's initial belief $s_{i,0}$ and her valuation are jointly distributed according a distribution Ψ over $\Theta \times \mathcal{S}$, independent from rest of the market.

The transitions in the belief of an agent i can be represented as follows. If the agent does not win the auction at time τ_i^k, i.e., if $w_{i,k} = 0$, then she receives no reward, and hence does not get any new information regarding her valuation. In this case, she does not update her belief about her valuation, and thus $s_{i,k} = s_{i,k-1}$. On the other hand, if the agent i wins in the auction at time τ_i^k, i.e., $w_{i,k} = 1$, then she updates her belief via Bayes' rule, after observing her realized reward $x_{i,k}$.

Thus, we can write the transitions in the belief of an agent i as follows.

$$s_{i,k} = F(s_{i,k-1}, w_{i,k}, x_{i,k}) \triangleq \begin{cases} s_{i,k-1} & \text{if } w_{i,k} = 0; \\ K(s_{i,k-1}, x_{i,k}) & \text{if } w_{i,k} = 1. \end{cases} \tag{EC.1}$$

Here $K(\cdot,\cdot)$ denotes the *Bayesian updating* operator, which takes as input the prior belief and the realized observation, and gives as output the posterior belief.

For $s \in \mathcal{S}$, let $\mu(s) \triangleq \mathbf{E}[v | v \sim \mathsf{P}_s] \in [0,1]$ denote an agent's expected valuation, if her belief is indexed by s.

We impose the following assumptions on the belief model.

ASSUMPTION EC.1. 1. *The set $\mathcal{S}$ is a closed convex subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$.*

2. *The order $\succeq$ on $\mathcal{S}$, defined by first order stochastic dominance, is equivalent to the product order on $\mathbb{R}^d$.*

3. *Under the Euclidean topology on $\mathcal{S}$, the mapping $\mu(\cdot)$ is continuous.*

4. *Under the Euclidean topology on $\mathcal{S}$, the mappings $K(\cdot,0)$ and $K(\cdot,1)$ are continuous and injective with a differentiable inverse.*

Though technical, the assumptions are fairly mild. In particular, the beta prior belief model assumed in the main text satisfies these assumptions. We present below another example of a belief model where these assumptions hold, namely the Bernoulli prior belief model.

Bernoulli prior belief model. In this model, the agents' valuations are either θ_ℓ or θ_h, with $0 \le \theta_\ell \le \theta_h \le 1$. In this case, an agent's belief after k auctions can be represented by a single number $p_{i,k} = (0,1)$ denoting the probability with which their valuation is θ_h.

For simplicity, we let $s_{i,k}$ denote the *log-likelihood ratio* after k auctions, defined as,

$$s_{i,k} = \log\left(\frac{p_{i,k}}{1-p_{i,k}}\right).$$

For $p_{i,k} \in (0,1)$, we see that $s_{i,k} \in \mathbb{R}$, and hence here the space of possible beliefs is $\mathcal{S} = \mathbb{R}$. For $s \in \mathcal{S}$, we have $\mu(s) = (\theta_h \exp(s) + \theta_\ell)/(1+\exp(s))$, and Bayesian updating in (EC.1) simplifies to:

$$s_{i,k} = \begin{cases} s_{i,k-1} & \text{if } w_{i,k} = 0; \\ s_{i,k-1} + x_{i,k}\log(\frac{\theta_h}{\theta_\ell}) + (1-x_{i,k})\log(\frac{1-\theta_h}{1-\theta_\ell}) & \text{if } w_{i,k} = 1. \end{cases} \tag{EC.2}$$

In this case, the Bayesian update operator is given by $K(s,x) = s + x\log(\frac{\theta_h}{\theta_\ell}) + (1-x)\log(\frac{1-\theta_h}{1-\theta_\ell})$ for $s \in \mathcal{S}$ and $x \in \{0,1\}$.

It is straightforward to check that $s_1 \succeq s_2$ if and only if $s_1 \geq s_2$. Moreover, the mappings $\mu(\cdot)$ and $K(\cdot,x), x \in \{0,1\}$ are continuous. Finally, $K(\cdot,x)$ is injective for each $x \in \{0,1\}$, with a differentiable inverse. □

With the preceding assumptions in place, using the same proof technique, we can show that the map $\digamma$ is continuous.

To obtain existence, we require an additional condition ensuring that the map $\digamma$ has a compact image on a convex set.

ASSUMPTION EC.2. *There exists a convex set $\Upsilon \subseteq \mathcal{G}$ such that $\digamma(\Upsilon) \subseteq \Upsilon$ and $\digamma(\Upsilon)$ is compact.*

Under Assumptions EC.1 and EC.2, it is straightforward to show that the map $\digamma$ has a fixed point, and hence there exists an MFE in the market with general belief models. However, note that Assumption EC.2 involves an endogenous quantity of the model. To translate this into an assumption on model primitives would require specific information about the belief model. In particular, for general belief models, Assumption 2 may not be enough to obtain Assumption EC.2, as with the beta prior belief model. Below we present conditions on the Bernoulli prior model under which this assumption holds, and an MFE exists.

ASSUMPTION EC.3. 1. *The distribution Ψ has compact support.*
2. *$\theta_h = 1 - \theta_\ell$ and $\beta < \theta_\ell/\theta_h^2$.*

We have the following result.

LEMMA EC.1. *Under Assumption EC.3, the image of the map $\digamma$ is a compact subset of $\mathcal{G}$. Thus, Assumption EC.2 holds with $\Upsilon = \mathcal{G}$.*

Proof. The proof involves bounding the growth rate of $\digamma(\cdot|g)$ for $g \in \mathcal{G}$, and then making use of the Arzelà-Ascoli theorem. The steps involved are similar to those for the Beta belief model (see Appendix EC.3.5) and are omitted for brevity. □

Appendix EC.2: Steady state in the large market

In this section, we analyze the steady state of the large market and show that the chain described by transition kernel (6) has a unique invariant distribution. We follow the same notation as in Section 3, but work with the general belief model described in Appendix EC.1.

LEMMA EC.2. *The Markov chain described by the transition kernel* (6) *is a positive recurrent Harris chain.*

Proof. Observe that for any $(v,s) \in \mathcal{S}$, the transition probability kernel satisfies

$$\mathbf{P}((v_{i,1}, s_{i,1}) \in (A,B) | (v_{i,0}, s_{i,0}) = (v,s)) \geq (1-\beta)\Psi(A,B).$$

As $\beta < 1$ and Ψ is a probability measure, the result then follows from Meyn and Tweedie 2009. □

Let $\mathsf{Q}_t(\cdot|(v,s))$ denote the measure induced by the process at time t, conditioned on the event that there have been no regenerations until time t, and the state at time 0 is given by (v,s). We have the following expression for the invariant distribution of the above chain.

LEMMA EC.3. *For $g \in \mathcal{G}$ and for any stationary policy $\xi(\cdot)$, the Markov chain defined by the transition probability kernel* (6) *has a unique invariant distribution $\Phi(\cdot|g,\xi)$ given by,*

$$\Phi(A,B|g,\xi) \triangleq \sum_{t\geq 0}(1-\beta)\beta^t \mathbf{E}_\Psi\left[\mathsf{Q}_t((v_{i,t}, s_{i,t}) \in (A,B)|(v_{i,0}, s_{i,0}))\right], \tag{EC.3}$$

where $A \subseteq \Theta$, and $B \subseteq \mathcal{S}$ are Borel sets, and the expectation is over the initial state $(v_{i,0}, s_{i,0})$.

Proof. The proof follows by conditioning the invariant distribution of the valuation and belief of an agent on the number of auctions since she last regenerated. We omit the details for brevity. □

Next, observe that for an agent i that does not regenerate at time t, the state transition is given by $s_{i,t} = F(s_{i,t-1}, w_{i,t}, x_{i,t})$ defined as in (EC.1). This is governed by $(w_{i,t}, x_{i,t})$, which takes values in $\{(0,0),(1,1),(1,0)\}$. Thus, for a non-regenerating agent, the state transitions until time t are fully known on specifying $a_{i,t} \triangleq (w_{i,\tau}, x_{i,\tau})_{0\leq\tau\leq t-1}$. For any $t \geq 0$, define $L_t \triangleq \{(0,0),(1,1),(1,0)\}^t$. For any $a \in L_t$, let $F_t(s,a)$ denote the state at time t of an agent starting at time 0 with state s, if $a_{i,t} = a$. Then, for any $(v_{i,0}, s_{i,0}) \in \Theta \times \mathcal{S}$ and for any Borel $A \subseteq \Theta$ and $B \subseteq \mathcal{S}$, we have

$$\begin{aligned}\mathsf{Q}_t((v_{i,t}, s_{i,t}) \in (A,B)|(v_{i,0}, s_{i,0})) &= \sum_{a\in L_t} \mathsf{Q}_t(v_{i,t} \in A, s_{i,t} \in B, a_{i,t} = a|\ (v_{i,0}, s_{i,0})) \\ &= \sum_{a\in L_t} \mathsf{Q}_t(v_{i,t} \in A, a_{i,t} = a|\ (v_{i,0}, s_{i,0}))\mathbf{I}\{F_t(s_{i,0}, a) \in B\}. \end{aligned} \tag{EC.4}$$

The following result proves that the stationary measure $\Phi(\cdot|g,\xi)$ has a density under Assumption 1. The proof is algebraic, and is omitted.

LEMMA EC.4. *Under Assumption 1, for any $g \in \mathcal{G}$, and for any stationary policy $\xi(\cdot)$, the measure $\Phi(\cdot|g,\xi)$ has a density $\phi(\cdot) = \phi(\cdot|g,\xi)$ given by*

$$\phi(v,s) = \sum_{t\geq 0} \beta^t (1-\beta) \sum_{a\in L_t} \psi(v, G_t(s,a)) Q_t(a_{i,t} = a | v_{i,0} = v, s_{i,0} = G_t(s,a)) J_t(s,a), \tag{EC.5}$$

where $G_t(\cdot,a)$ is the (differentiable) inverse of $F_t(\cdot,a)$, and $J_t(s,a)$ is the Jacobian of $G_t(\cdot,a)$ at s.

Appendix EC.3: Existence of mean field equilibrium: Proofs

In this section, we provide the proofs of the propositions and lemmas in Section 4. We follow the same notation as in that section, but work with the general belief model described in Appendix EC.1.

EC.3.1. Preliminaries

Let $\mathcal{G} = \{g : [0,1] \to [0,1] \mid g \text{ continuous and non-decreasing with } g(0) \geq 0 \text{ and } g(1) = 1\}$. Endow $\mathcal{G}$ with the topology induced by the sup norm:

$$\|g\| = \max_{x\in[0,1]} |g(x)|.$$

It can be easily checked that with the sup-norm, $\mathcal{G}$ is a Banach space.

Define $\mathcal{B} = \{H : \mathcal{S} \to [-1/(1-\beta), 1/(1-\beta)], H \text{ continuous}\}$. Define $\mathcal{V} \subseteq \mathcal{B}$ as the image of the mapping associating the value function $V(\cdot|g)$ to each $g \in \mathcal{G}$ through (5). Similarly, define $\mathcal{X}$ as the image of the map associating $g \in \mathcal{G}$ to $\xi(\cdot|g)$. Endow $\mathcal{V}$ and $\mathcal{X}$ with the topology induced by the sup norm.

Let $\mathcal{L}$ denote the set of all Borel measures on the set $\Theta \times \mathcal{S}$. Endow $\mathcal{L}$ with the topology of weak convergence. Let $\Pi : \mathcal{G} \to \mathcal{L}$ denote the mapping which associates to each $g \in \mathcal{G}$, the steady state distribution $\Phi(\cdot|g, \xi(\cdot|g))$ corresponding to the chain with bid distribution g, and where each agent adopts the policy $\xi(\cdot|g)$.

EC.3.2. Proof of Proposition 1

Recall the definition of the value function from (5).

$$V(s|g) = \sup_{\delta} \mathbf{E}_{\delta} \left[\sum_{t=0}^{\infty} \beta^t q(b_{i,t}|g)\mu(s_{i,t}) - p(b_{i,t}|g) \mid s_{i,0} = s \right].$$

The following standard result follows from the continuity of $q(\cdot|g)$, $K(\cdot,0)$ and $K(\cdot,1)$.

PROPOSITION EC.1. *For any $g \in \mathcal{G}$, the value function $V(\cdot|g)$ is finite, non-negative and continuous in s. The value function satisfies Bellman's equation, given by*

$$\begin{aligned} V(s|g) = \max_{b\in[0,1]} (&q(b|g)\mu(s) - p(b|g) + \beta q(b|g)\mu(s) V(K(s,1)|g) \\ &+ \beta q(b|g)(1-\mu(s)) V(K(s,0)|g) + \beta(1 - q(b|g)) V(s|g)). \end{aligned} \tag{EC.6}$$

Moreover, if a strategy's bids maximize the right-hand side of Bellman's equation at all states, then it is optimal.

Proof. See Maitra 1968 or Dutta et al. 1994. $\square$

Recall that $\xi(\cdot|g)$ is defined as

$$\xi(s|g) = \mu(s) + \beta\mu(s)V(K(s,1)|g) + \beta(1-\mu(s))V(K(s,0)|g) - \beta V(s|g)).$$

From Proposition EC.1, we infer the $\xi(\cdot|g)$ is a continuous function for all $g \in \mathcal{G}$. By collecting terms containing $q(\cdot|g)$ and using $\xi(\cdot|g)$, we can simplify the right-hand side of Bellman's equation as

$$V(s|g) = \frac{1}{1-\beta} \max_{b\in[0,1]} q(b|g)\xi(s|g) - p(b). \tag{EC.7}$$

For any $g \in \mathcal{G}$, and for any $x \in [0,1]$, and $v \in \mathbb{R}$, let $f(x,v|g) \triangleq vq(x|g) - p(x|g)$. The simplified Bellman's equation (EC.7) can be rewritten as,

$$V(s|g) = \frac{1}{1-\beta} \max_{b\in[0,1]} f(b, \xi(s|g)|g). \tag{EC.8}$$

We have the following lemma. The proof is straightforward, and is omitted.

LEMMA EC.5. *Given any $g \in \mathcal{G}$, the function $f(\cdot|g)$ is continuous on $[0,1]\times\mathbb{R}$. For each $v \in \mathbb{R}$, the function $f(\cdot,v|g)$ attains its maximum at $x = (1-(1-v)^+)^+$, where $a^+ \triangleq \max(a,0)$. Moreover, $f(v,v|g)$ as a function of v is non-decreasing in v over $[0,1]$ (strictly increasing if $q(v|g) > 0$).*

The following lemma establishes that $\xi(\cdot|g)$ takes values in $[0,1]$.

LEMMA EC.6. *For any $s \in \mathcal{S}$, we have $\xi(s|g) \in [0,1]$.*

Proof. Recall from (5) the definition of the value function,

$$V(s|g) = \sup_{\delta} \mathbf{E}_{\delta}\left[\sum_{t\geq 0} \beta^t f(b_{i,t}, \mu(s_{i,t})|g) \Big| s_0 = s\right].$$

As $\mu(s_{i,t}) \in [0,1]$ and $b_{i,t} \in [0,1]$, we obtain from Lemma EC.5 for all t,

$$\begin{aligned} f(b_{i,t}, \mu(s_t)|g) &\leq f(\mu(s_{i,t}), \mu(s_{i,t})|g) \\ &\leq f(1,1|g) \\ &= q(1|g) - p(1|g) \\ &= 1 - p(1|g). \end{aligned}$$

Here, the first inequality follows from the fact that $f(\cdot,\mu(s_t)|g)$ attains its maximum at $\mu(s_t)$, and the second inequality follows from the fact that $f(x,x|g)$ is an increasing function of x on $[0,1]$. The last equality follows as $q(1|g) = 1$.

Thus, from the definition of the value function, we obtain

$$V(s|g) \leq \frac{1}{1-\beta}(1 - p(1|g)). \tag{EC.9}$$

Next, from Bellman's equation, we obtain the inequality $(1-\beta)V(s|g) \geq f(b, \xi(s|g)|g) = q(b|g)\xi(s|g) - p(b|g)$ for all $b \in [0,1]$. As $q(1|g) = 1$, on setting $b = 1$ and rearranging terms, we obtain

$$\xi(s|g) \leq V(s|g)(1-\beta) + p(1|g).$$

Using the inequality (EC.9), we get $\xi(s|g) \leq 1$.

Now suppose $\xi(s|g) < 0$. This implies that $\max_{b\in[0,1]} f(b, \xi(s|g)|g) \leq 0$. On the other hand, from Proposition EC.1, we know that $V(s|g) \geq 0$. As $(1-\beta)V(s|g) = \max_{b\in[0,1]} f(b, \xi(s|g)|g)$, we infer that $V(s|g) = 0$. But then we get $\xi(s|g) = \mu(s) + \beta\mu(s)V(K(s,1)|g) + \beta(1-\mu(s))V(K(s,0)|g) \geq \mu(s) \geq 0$, a contradiction.

Thus, we have $\xi(s) \geq 0$. $\square$

We are now ready to prove the optimality of the strategy that bids $\xi(s|g)$ when the agent's state is given by $s \in \mathcal{S}$.

LEMMA EC.7. *For any $g \in \mathcal{G}$, the policy $\xi(\cdot|g)$, where an agent bids $\xi(s|g)$ when her belief corresponds to state s, is optimal. Moreover, we have*

$$V(s|g) = \frac{q(\xi(s|g)|g)\xi(s|g) - p(\xi(s|g)|g)}{1-\beta} = \frac{1}{1-\beta} f(\xi(s|g), \xi(s|g)|g).$$

Proof. As $\xi(s|g) \in [0,1]$, from Lemma EC.5, we observe that $f(\cdot, \xi(s|g)|g)$ attains its maximum at $\xi(s|g)$ for all $s \in \mathcal{S}$. Thus, from Proposition EC.1, we obtain that the strategy where an agent bids $\xi(s|g)$ when her state is $s \in \mathcal{S}$ is optimal for her decision problem. The equality in the statement of the Lemma then follows from (EC.7). $\square$

The following lemma establishes that in the optimal strategy $\xi(\cdot|g)$, an agent overbids above the mean according to her current belief.

LEMMA EC.8. *For any $g \in \mathcal{G}$ and for any $s \in \mathcal{S}$, we have $\xi(s|g) \geq \mu(s)$.*

Proof. Fix $s \in \mathcal{S}$. First suppose $q(\xi(s|g)|g) = 0$. This implies, from Lemma EC.7, that $V(s|g) = 0$, and hence $\xi(s|g) = \mu(s) + \beta\mu(s)V(K(s,1)|g) + \beta(1-\mu(s))V(K(s,0)|g) \geq \mu(s)$.

Next, suppose $q(\xi(s|g)|g) > 0$. Then, from Lemma EC.5, it follows that $f(v, v|g)$ is strictly increasing at $v = \xi(s|g)$. Now, consider a policy Π for an agent starting with belief corresponding to state $s \in \mathcal{S}$, where she bids $\Pi_t = \mu(s)$ at all times t, regardless of her current state s_t. It is straightforward to show that the value obtained from this policy is given by $V^{\Pi}(s|g) = f(\mu(s), \mu(s)|g)/(1-\beta)$. By the definition of the value function, we have $V(s|g) \geq V^{\Pi}(s|g)$. Since, $V(s)(1-\beta) = f(\xi(s|g), \xi(s|g)|g)$, this implies that $f(\xi(s|g), \xi(s|g)|g) \geq f(\mu(s), \mu(s)|g)$. As $f(v, v|g)$ is strictly increasing at $v = \xi(s|g)$, we have $\xi(s|g) \geq \mu(s)$.

Thus we obtain $\xi(s|g) \geq \mu(s)$ for all $s \in \mathcal{S}$. $\square$

EC.3.3. Continuity of the market bid distribution

Step 1. Continuity of $\xi(\cdot|g)$. Consider the mappings $V(\cdot):\mathcal{G}\to\mathcal{V}$ and $\xi(\cdot):\mathcal{G}\to\mathcal{X}$ mapping each $g\in\mathcal{G}$ to $V(\cdot|g)$ and $\xi(\cdot|g)$ respectively. The following result shows that both these mappings are continuous.

PROPOSITION EC.2. *The mappings $V(\cdot):\mathcal{G}\to\mathcal{V}$ and $\xi(\cdot):\mathcal{G}\to\mathcal{X}$ are Lipschitz continuous in g.*

Proof. Note that as $\xi(s|g)=\mu(s)+\beta\mu(s)V(K(s,1)|g)+\beta(1-\mu(s))V(K(s,0)|g)-\beta V(s|g)$, we obtain for any $g_1,g_2\in\mathcal{G}$,

$$\|\xi(\cdot|g_1)-\xi(\cdot|g_2)\|\le 2\beta\|V(\cdot|g_1)-V(\cdot|g_2)\|.$$

Hence, it suffices to show the Lipschitz continuity of V.

Let $\mathcal{T}^g:\mathcal{B}\to\mathcal{B}$ denote the Bellman operator corresponding to g, defined as

$$(\mathcal{T}^g J)(s)=\left(\max_{b\in[0,1]} f(b,\xi_J(s)|g)\right)+\beta J(s),$$

where $J\in\mathcal{B}$ and $\xi_J(s)\triangleq\mu(s)+\beta\mu(s)J(K(s,1))+\beta(1-\mu(s))J(K(s,0))-\beta J(s)$. It is well known that the operator $\mathcal{T}^g$ is a contraction with contraction factor β (see, e.g., Bertsekas 2007).

From Lemma EC.5, we know that the maximum in the definition of $(\mathcal{T}^g J)(s)$ is attained at $\hat{\xi}_J(s)\triangleq(1-(1-\xi_J(s))^+)^+$. Note that $\hat{\xi}_J(s)$ does not depend on the value of g. In particular, we obtain for all $g\in\mathcal{G}$,

$$(\mathcal{T}^g J)(s)=f(\hat{\xi}_J(s)),\xi_J(s)|g)+\beta J(s).$$

Let $g_1,g_2\in\mathcal{G}$. For brevity, we let $q_i(\cdot)=q(\cdot|g_i)$ and $p_i(\cdot)=p(\cdot|g_i)$ for $i\in\{1,2\}$. We obtain from the expression for $\mathcal{T}^{g_i}$,

$$\begin{aligned}|\mathcal{T}^{g_1}J(s)-\mathcal{T}^{g_2}J(s)|&=|f(\hat{\xi}_J(s)),\xi_J(s)|g_1)-f(\hat{\xi}_J(s)),\xi_J(s)|g_2)|\\&=|q_1(\hat{\xi}_J(s))\xi_J(s)-p_1(\hat{\xi}_J(s))-q_2(\hat{\xi}_J(s))\xi_J(s)+p_2(\hat{\xi}_J(s))|\\&\le|\xi_J(s)||q_1(\hat{\xi}_J(s))-q_2(\hat{\xi}_J(s))|+|p_1(\hat{\xi}_J(s))-p_2(\hat{\xi}_J(s))|\\&\le|\xi_J(s)|\|q_1-q_2\|+\|p_1-p_2\|.\end{aligned}$$

Next, we have

$$\begin{aligned}|\xi_J(s)|&=|\mu(s)+\beta\mu(s)J(K(s,1))+\beta(1-\mu(s))J(K(s,0))-\beta J(s)|\\&\le|\mu(s)|+\beta\,|\mu(s)J(K(s,1))+(1-\mu(s))J(K(s,0))|+\beta|J(s)|\\&\le 1+2\beta\|J\|.\end{aligned}$$

As $J\in\mathcal{B}$, we have $\|J\|\le 1/(1-\beta)$. This implies that

$$|\xi_J(s)|\le 1+\frac{2\beta}{1-\beta}=\frac{1+\beta}{1-\beta}.$$

Also, from the definition of the payment function, we obtain $\|p_1 - p_2\| \le 2\|q_1 - q_2\|$. Thus, we get

$$\begin{aligned}|\mathcal{T}^{g_1}J(s) - \mathcal{T}^{g_2}J(s)| &\le |\xi_J(s)|\|q_1 - q_2\| + \|p_1 - p_2\| \\ &\le \frac{1+\beta}{1-\beta}\|q_1 - q_2\| + 2\|q_1 - q_2\| \\ &= \frac{3-\beta}{1-\beta}\|q_1 - q_2\|.\end{aligned}$$

Next, we have

$$\begin{aligned}|q_1(x) - q_2(x)| &= |g_1^{\alpha-1}(x) - g_2^{\alpha-1}(x)| \\ &= |g_1^{\alpha-1}(x) - g_2(x)g_1^{\alpha-2}(x) + g_2(x)g_1^{\alpha-2}(x) - g_2^{\alpha-1}(x)| \\ &\le |g_1^{\alpha-1}(x) - g_2(x)g_1^{\alpha-2}(x)| + |g_2(x)g_1^{\alpha-2}(x) - g_2^{\alpha-1}(x)| \\ &= |g_1^{\alpha-2}(x)||g_1(x) - g_2(x)| + |g_2(x)||g_1^{\alpha-2}(x) - g_2^{\alpha-2}(x)| \\ &\le |g_1(x) - g_2(x)| + |g_1^{\alpha-2}(x) - g_2^{\alpha-2}(x)|.\end{aligned}$$

Here, we have used triangle inequality in the first inequality. The second inequality follows from the fact that as $g_1, g_2 \in \mathcal{G}$, they take values in $[0,1]$. Thus, we obtain

$$\|q_1 - q_2\| \le \|g_1 - g_2\| + \|g_1^{\alpha-2} - g_2^{\alpha-2}\|.$$

By iterating, we obtain $\|q_1 - q_2\| \le (\alpha - 1)\|g_1 - g_2\|$. Hence, we obtain

$$\|\mathcal{T}^{g_1}J - \mathcal{T}^{g_2}J\| \le \frac{(3-\beta)(\alpha-1)}{1-\beta}\|g_1 - g_2\|, \tag{EC.10}$$

for any $J \in \mathcal{B}$ and for all $g_1, g_2 \in \mathcal{G}$.

For $g \in \mathcal{G}$, let $J_0(\cdot|g) = \mathbf{0} \in \mathcal{B}$, and define iteratively $J_{k+1}(\cdot|g) = \mathcal{T}^g J_k(\cdot|g) \in \mathcal{B}$. As $\mathcal{T}^g$ is a contraction, for any $g \in \mathcal{G}$, we obtain that $\|J_k(\cdot|g) - V(\cdot|g)\| \to 0$. Next, for g_1, g_2 in $\mathcal{G}$, we have for all k,

$$\begin{aligned}\|J_{k+1}(\cdot|g_1) - J_{k+1}(\cdot|g_2)\| &= \|\mathcal{T}^{g_1}J_k(\cdot|g_1) - \mathcal{T}^{g_2}J_k(\cdot|g_2)\| \\ &\le \|\mathcal{T}^{g_1}J_k(\cdot|g_1) - \mathcal{T}^{g_1}J_k(\cdot|g_2)\| + \|\mathcal{T}^{g_1}J_k(\cdot|g_2) - \mathcal{T}^{g_2}J_k(\cdot|g_2)\| \\ &\le \beta\|J_k(\cdot|g_1) - J_k(\cdot|g_2)\| + \frac{(3-\beta)(\alpha-1)}{1-\beta}\|g_1 - g_2\|.\end{aligned}$$

Here we have used triangle inequality in the first inequality, and the second inequality follows from the fact the $\mathcal{T}^{g_1}$ is a contraction and from (EC.10). By iterating the inequality over descending values of k and then summing up, we obtain

$$\|J_k(\cdot|g_1) - J_k(\cdot|g_2)\| \le \frac{(3-\beta)(\alpha-1)}{(1-\beta)^2}\|g_1 - g_2\|. \tag{EC.11}$$

For any $\epsilon > 0$, let $K > 0$, be such that $\|J_k(\cdot|g_i) - V(\cdot|g_i)\| \le \epsilon$ for all $k \ge K$ and for $i = 1, 2$. Then, we have for $k \ge K$,

$$\begin{aligned}\|V(\cdot|g_1) - V(\cdot|g_2)\| &\le \|V(\cdot|g_1) - J_k(\cdot|g_1)\| + \|J_k(\cdot|g_1) - J_k(\cdot|g_2)\| + \|J_k(\cdot|g_2) - V(\cdot|g_2)\| \\ &\le \|J_k(\cdot|g_1) - J_k(\cdot|g_2)\| + 2\epsilon \\ &\le \frac{(3-\beta)(\alpha-1)}{(1-\beta)^2}\|g_1 - g_2\| + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, we obtain

$$\|V(\cdot|g_1) - V(\cdot|g_2)\| \le \frac{(3-\beta)(\alpha-1)}{(1-\beta)^2}\|g_1 - g_2\|. \tag{EC.12}$$

This completes the proof. $\square$

Step 2. Continuity of the invariant distribution $\Phi(\cdot|g, \xi(\cdot|g))$**.** Recall that the mapping $\Pi : \mathcal{G} \to \mathcal{L}$ is defined as $\Pi(g) = \Phi(\cdot|g, \xi(\cdot|g))$. The following lemma shows the continuity of the map Π.

LEMMA EC.9. *Π is continuous.*

Proof. The proof involves first establishing that the measures $\{\mathsf{Q}_t : t \ge 0\}$ in the expression for Φ in (EC.3) are continuous (under the weak convergence topology) in the bid distribution g. This follows from the fact that the conjoint valuation $\xi(\cdot|g)$ is uniformly continuous in g. The proof of the lemma then follows from repeated application of the bounded convergence theorem to (EC.3). We omit the details for brevity. $\square$

Step 3. Continuity of the map F. Fix $g \in \mathcal{G}$ and let $\Phi = \Pi(g)$. Recall that the map $F : \mathcal{G} \to \mathcal{D}$ is defined as

$$F(x|g) = \mathbf{P}_{\Phi}\left(\xi(X|g) \le x\right). \tag{EC.13}$$

Here X is a random variable distributed according to $\Phi(\Theta, \cdot)$.

We have the following result.

LEMMA EC.10. *For all $g \in \mathcal{G}$, $F(\cdot|g)$ is continuous and non-decreasing on $[0,1]$, with $F(0|g) = 0$ and $F(1|g) = 1$.*

Proof. From (EC.13), we clearly see that $F(\cdot|g)$ is a non-decreasing function on $[0,1]$. From Lemma EC.6, we infer that $F(1|g) = 1$ and that $F(0|g) = \mathbf{P}_{\Phi}(\xi(X|g) = 0)$.

From Lemma EC.4, we know that Φ has a density, which implies that Φ is absolutely continuous with respect to the Lebesgue measure. On the other hand, from Lemma EC.12, we know that for all $g \in \mathcal{G}$ and for all $x \in [0,1]$, the set $\{s \in \mathcal{S} : \xi(s|g) = x\}$ has zero Lebesgue measure. Thus, the above sets also have zero measure with respect to Φ.

This, implies that $F(0|g) = \mathbf{P}_{\Phi}(\xi(X|g) = 0) = 0$. Moreover, we have

$$F(x|g) - F(x^-|g) = \mathbf{P}_{\Phi}(\xi(X|g) = x) = 0\ .$$

Thus, $F(\cdot|g)$ is a non-decreasing function with no jump discontinuity. Hence, $F(\cdot|g)$ is continuous. $\square$

We are now ready to show the continuity of $\mathcal{F}$.

PROPOSITION EC.3. *The mapping $\mathcal{F}: \mathcal{G} \to \mathcal{G}$ is continuous.*

Proof. Let $\{g_n : n \geq 0\}$ be a sequence such that $g_n \to g$. Let $h_n = \mathcal{F}(g_n)$ for each n and let $h = \mathcal{F}(g)$. The proof involves first showing that $h_n \to h$ pointwise. This follows directly from the continuity of $\Pi(\cdot)$, $\xi(\cdot|g)$ and the Skorokhod representation theorem. Then, using Lemma EC.11, we conclude that $h_n \to h$ in the sup norm. We omit the details for brevity. $\square$

To complete the proof, we need the following lemma. The proof is from first principles, and is omitted.

LEMMA EC.11. *Let $\{h_n \in \mathcal{G} : n \geq 1\}$ be a sequence of functions such that $h_n(x) \to h(x)$ for all $x \in [0,1]$ for some $h \in \mathcal{G}$. Then h_n converges to h uniformly in x.*

EC.3.4. Monotonicity of optimal strategy

In this section, we show that the optimal strategy $\xi(\cdot|g)$ is (strictly) monotone for all $g \in \mathcal{G}$, i.e. for $s_1, s_2 \in \mathcal{S}$ such that $s_1 \succ s_2$, we have $\xi(s_1|g) > \xi(s_2|g)$. We first begin by showing the monotonicity of $V(\cdot|g)$.

PROPOSITION EC.4. *For any $g \in \mathcal{G}$, and for all $s_1 \succeq s_2$, we have $V(s_1|g) \geq V(s_2|g)$.*

Proof. Recall Bellman's equation for $V(\cdot|g)$ from (EC.6).

$$\begin{aligned} V(s|g) = \max_{b\in[0,1]} \big(& f(b,\mu(s)|g) + \beta q(b|g)\mu(s)V(K(s,1)|g) \\ &+ \beta q(b|g)(1-\mu(s))V(K(s,0)|g) + \beta(1-q(b|g))V(s|g)\big). \end{aligned} \tag{EC.14}$$

As $\mu(\cdot)$ is strictly increasing in s, it follows that for each $b \in [0,1]$, the instantaneous reward $f(b,\mu(s)|g)$ is non-decreasing in s. Moreover, the state transitions are readily seen to be increasing with s in the sense of first-order stochastic dominance. The result then follows directly from Smith and McCardle 2002. $\square$

Next, we have the following corollaries.

COROLLARY EC.1. *For any $g \in \mathcal{G}$ and for $s_1, s_2 \in \mathcal{S}$ with $s_1 \succeq s_2$, we have $\xi(s_1|g) \geq \xi(s_2|g)$.*

Proof. Fix $g \in \mathcal{G}$. Let $s_1 \succeq s_2$. From Proposition EC.4, we get $V(s_1|g) \geq V(s_2|g)$. Recall that $V(s|g)(1-\beta) = f(\xi(s|g), \xi(s|g)|g)$ for all $s \in \mathcal{S}$. From Lemma EC.5, we know that $f(v,v|g)$ is a non-decreasing function of v on $[0,1]$. Thus if $V(s_1|g) > V(s_2|g)$, then $\xi(s_1|g) > \xi(s_2|g)$.

Next, suppose $V(s_1|g) = V(s_2|g)$. Then, by definition of $\xi(\cdot|g)$, we get

$$\begin{aligned} \xi(s_1|g) - \xi(s_2|g) = \mu(s_1) - \mu(s_2) & \\ &+ \beta\left(\mu(s_1)V(K(s_1,1)|g) + (1-\mu(s_1))V(K(s_1,0)|g)\right) \\ &- \beta\left(\mu(s_2)V(K(s_2,1)|g) + (1-\mu(s_2))V(K(s_2,0)|g)\right). \end{aligned}$$

As $s_1 \succeq s_2$, we have $K(s_1,k) \succeq K(s_2,k)$ for $k \in \{0,1\}$. This implies from Proposition EC.4 that $V(K(s_1,k)|g) \geq V(K(s_2,k)|g)$ for $k \in \{0,1\}$. As $\mu(s_1) \geq \mu(s_2)$, we obtain

$$\begin{aligned}
&\mu(s_1)V(K(s_1,1)|g) + (1-\mu(s_1))V(K(s_1,0)|g) \\
&\quad \geq \mu(s_2)V(K(s_2,1)|g) + (1-\mu(s_2))V(K(s_2,0)|g).
\end{aligned}$$

This, along with $\mu(s_1) \geq \mu(s_2)$, implies that $\xi(s_1|g) \geq \xi(s_2|g)$. $\square$

COROLLARY EC.2. *For any $g \in \mathcal{G}$ and for all $s_1, s_2 \in \mathcal{S}$ with $s_1 \succeq s_2$, we have*

$$\xi(s_1|g) - \xi(s_2|g) \geq (1-\beta)\left(\mu(s_1) - \mu(s_2)\right).$$

In particular, if $s_1 \succ s_2$, then $\xi(s_1|g) > \xi(s_2|g)$.

Proof. Fix $g \in \mathcal{G}$. Note that we have

$$\begin{aligned}
(1-\beta)V(s|g) &= f(\xi(s|g), \xi(s|g)|g) \\
&= \max_{b \geq 0}\left(q(b|g)\xi(s|g) - p(b|g)\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&(1-\beta)\left(V(s_1|g) - V(s_2|g)\right) \\
&\quad = \max_{b \geq 0}\left(q(b|g)\xi(s_1|g) - p(b|g)\right) - \max_{b \geq 0}\left(q(b|g)\xi(s_2|g) - p(b|g)\right).
\end{aligned}$$

From Corollary EC.1, we know that $\xi(s_1|g) \geq \xi(s_2|g)$. This implies that,

$$\begin{aligned}
&(1-\beta)\left(V(s_1|g) - V(s_2|g)\right) \\
&\quad = \max_{b \geq 0}\left(q(b|g)\xi(s_1|g) - p(b|g)\right) - \max_{b \geq 0}\left(q(b|g)\xi(s_2|g) - p(b|g)\right) \\
&\quad \leq \max_{b \geq 0} q(b|g)(\xi(s_1|g) - \xi(s_2|g)).
\end{aligned}$$

As $q(\cdot|g) \in [0,1]$, we have

$$(1-\beta)\left(V(s_1|g) - V(s_2|g)\right) \leq \xi(s_1|g) - \xi(s_2|g). \tag{EC.15}$$

Now, by definition of $\xi(\cdot|g)$, we have

$$\begin{aligned}
\xi(s_1|g) - \xi(s_2|g) = \mu(s_1) - \mu(s_2)& \\
&+ \beta\left(\mu(s_1)V(K(s_1,1)|g) + (1-\mu(s_1))V(K(s_1,0)|g)\right) \\
&- \beta\left(\mu(s_2)V(K(s_2,1)|g) + (1-\mu(s_2))V(K(s_2,0)|g)\right) \\
&- \beta\left(V(s_1|g) - V(s_2|g)\right).
\end{aligned}$$

Using (EC.15), we obtain the inequality

$$\begin{aligned}\frac{\xi(s_1|g)-\xi(s_2|g)}{1-\beta} &\geq \mu(s_1)-\mu(s_2)\\ &\quad+\beta\left(\mu(s_1)V(K(s_1,1)|g)+(1-\mu(s_1))V(K(s_1,0)|g)\right)\\ &\quad-\beta\left(\mu(s_2)V(K(s_2,1)|g)+(1-\mu(s_2))V(K(s_2,0)|g)\right).\end{aligned}$$

Using the same argument as in the proof of Corollary EC.1, we have

$$\begin{aligned}&\mu(s_1)V(K(s_1,1)|g)+(1-\mu(s_1))V(K(s_1,0)|g)\\ &\quad\geq \mu(s_2)V(K(s_2,1)|g)+(1-\mu(s_2))V(K(s_2,0)|g).\end{aligned}$$

Thus, we obtain $\xi(s_1|g)-\xi(s_2|g)\geq(1-\beta)\left(\mu(s_1)-\mu(s_2)\right)$. Finally, if $s_1\succ s_2$, then we have $\xi(s_1|g)-\xi(s_2|g)\geq(1-\beta)\left(\mu(s_1)-\mu(s_2)\right)>0$. $\square$

We use the following lemma in the proof of Lemma EC.10.

LEMMA EC.12. *For any $g\in\mathcal{G}$ and for any $x\in[0,1]$, the set $\{s\in\mathcal{S}:\xi(s|g)=x\}$ has Lebesgue measure 0.*

Proof. Fix a $g\in\mathcal{G}$. Note that since $\xi(\cdot|g)$ is a continuous function, the set $\{s\in\mathcal{S}:\xi(s|g)=x\}$ is measurable for all $x\in[0,1]$.

The result is trivial for any $x\in[0,1]$ that is not in the image of $\xi(\cdot|g)$. For any $x\in[0,1]$ which is in the image of $\xi(\cdot|g)$, we see that the set $\{s\in\mathcal{S}:\xi(s|g)=x\}$ is closed. Moreover, this set intersects any line parallel to a co-ordinate axis at most once, as $\xi(\cdot|g)$ is strictly increasing. As $\xi(\cdot|g)$ is continuous on $\mathcal{S}$, this implies that the set $\{s\in\mathcal{S}:\xi(s|g)=x\}$ can be expressed as the graph of a suitably defined continuous function on a projection of $\mathcal{S}$ onto the subspace spanned by one set of $d-1$ co-ordinate axes. Since the graph of a continuous function has zero Lebesgue measure, we are done. $\square$

EC.3.5. Beta belief model

In the rest of this section, we assume that the measure Ψ satisfies the condition $\Psi([0,1],[1,2]^2)=1$. In particular, we assume that $\psi(x,s)=0$ for all $x\in[0,1]$ and $s\notin[1,2]^2$. The proof holds with minimal modification for the general case of compact support, but the exposition becomes more tedious.

From Lemma EC.4, we know that for any $g\in\mathcal{G}$, the invariant distribution $\Phi=\Phi(\cdot|g(\cdot),\xi(\cdot|g))$ has a density $\phi(\cdot)$ if Ψ does. We begin by bounding the decay rate of the density ϕ.

LEMMA EC.13. *The density ϕ of the invariant distribution satisfies*

$$\phi(x,s)\leq\|\psi\|\beta^{s_1+s_2-4},$$

for all $s=(s_1,s_2)\in\mathcal{S}$ and for all $x\in[0,1]$.

Proof. For $s = (s_1, s_2) \in \mathcal{S}$, let $u(s) \triangleq \lfloor s - (1,1) \rfloor = (u_1(s), u_2(s))$. Let $\tau(s) = u_1(s) + u_2(s)$. As $\psi(x,s) = 0$ for $s \notin [1,2]^2$ and for all $x \in [0,1]$, we obtain from Lemma EC.4 that

$$\begin{aligned}\phi(x,s) &= \sum_{t\geq 0}(1-\beta)\beta^t \sum_{k\in L_t}\psi(x,s-k)\mathsf{Q}_t(s_t = s|s_0 = s-k, x_0 = x)\\ &= \sum_{t\geq\tau(s)}(1-\beta)\beta^t\psi(x,s-u(s))\mathsf{Q}_t(s_t = s|s_0 = s-u(s), x_0 = x).\end{aligned}$$

Here, the second equality follows from the fact that the other terms in the sum vanish.

Next, using $\mathsf{Q}_t(s_t = s|s_0 = s-u(s), x_0 = x) \leq 1$ and $\psi(x, s-u(s)) \leq \|\psi\|$, we obtain

$$\phi(x,s) \leq \|\psi\|\beta^{\tau(s)} \leq \|\psi\|\beta^{s_1+s_2-4},$$

where the second inequality follows as $u_1(s) \geq s_1 - 2$ and $u_2(s) \geq s_2 - 2$. $\square$

Next, we bound the growth rate of $\xi(\cdot) = \xi(\cdot|g)$. Recall that $\xi : \mathcal{S} \to [0,1]$ is a continuous function. Further, from Corollary EC.1, we know that $\xi(s) = \xi(s_1, s_2)$ is strictly increasing in s_1. Let the growth rates be defined as,

$$\begin{aligned}\xi_{1,+}(s) &= \limsup_{u\to s_1}\frac{\xi(u,s_2)-\xi(s_1,s_2)}{u-s_1}\\ \xi_{1,-}(s) &= \liminf_{s\to s_1}\frac{\xi(u,s_2)-\xi(s_1,s_2)}{u-s_1}.\end{aligned}$$

As $\xi(s_1, s_2)$ is strictly increasing in s_1, both the rates above are non-negative. Similarly, let $V_{1,+}(s)$ and $V_{1,-}(s)$ denote corresponding growth rates in s_1 of the value function $V(\cdot|g)$ at s. From Corollary EC.2, we obtain that for all $s \in \mathcal{S}$,

$$\xi_{1,-}(s) \geq (1-\beta)\mu_1(s), \tag{EC.16}$$

where $\mu_1(s_1, s_2) = s_2/(s_1+s_2)^2$ is the derivative of $\mu(\cdot)$ at s with respect to s_1.

Next, we identify conditions under which the image $\mathcal{F}(\mathcal{G})$ of the mapping $\mathcal{F}$ is compact. Let $g \in \mathcal{G}$, and let $h = \mathcal{F}(g)$. Recall that we have for $x \in [0,1]$,

$$h(x) = \mathbf{P}_\Phi(\xi(X) \leq x),$$

where $X = (X_1, X_2)$ is distributed according to $\Phi([0,1], \cdot)$. Using Lemma EC.4, we obtain

$$\begin{aligned}h(x) &= \int_{\mathcal{S}}\int_0^1 \phi(\theta,s)\mathbf{I}\{\xi(s)\leq x\}d\theta\, ds\\ &= \int_1^\infty\int_1^\infty\int_0^1 \phi(\theta,s_1,s_2)d\theta\, \mathbf{I}\{\xi(s_1,s_2)\leq x\}\, ds_1\, ds_2.\end{aligned}$$

Now, for fixed $s_2 \in [1,\infty)$, we know that $\xi(s_1, s_2)$ is a strictly increasing continuous function of s_1. For $x \in [0,1)$ in the range of $\xi(\cdot, s_2)$, let $m(x, s_2) = t$, where $\xi(t, s_2) = x$. For $x \in [0,1)$ not in the range of $\xi(\cdot, s_2)$, let $m(x, s_2) = 1$. Clearly, $m(\cdot, s_2)$ is an increasing continuous function on $[0, 1)$, with $m(\xi(s_1, s_2), s_2) = s_1$ for $s_1 \in [1, \infty)$. We then obtain for $s_1 \in [1, \infty)$ and $x \in [0, 1)$,

$$\begin{aligned} W(x, s_2) &\triangleq \int_1^\infty \int_0^1 \phi(\theta, s_1, s_2) d\theta \, \mathbf{I}\{\xi(s_1, s_2) \le x\} \; ds_1 \\ &= \int_1^{m(x,s_2)} \int_0^1 \phi(\theta, s_1, s_2) d\theta \; ds_1. \end{aligned}$$

As $m(\cdot, s_2)$ is an increasing function, we get

$$W_{1,+}(x, s_2) \triangleq \limsup_{y \to x} \frac{W(y, s_2) - W(x, s_2)}{y - x} = m_{1,+}(x, s_2) \int_0^1 \phi(\theta, m(x, s_2), s_2) d\theta, \tag{EC.17}$$

for $x \in [0, 1)$, and $s_2 \in [1, \infty)$, where $m_{1,+}(\cdot)$ is defined analogously as

$$m_{1,+}(x, s_2) \triangleq \limsup_{y \to x} \frac{m(y, s_2) - m(x, s_2)}{y - x}.$$

LEMMA EC.14. *For $s \in [1, \infty)$, and for $x \in [0, 1)$, we have*

$$W_{1,+}(x, s) \le \frac{1}{(1 - \beta)\mu_1(m(x, s), s)} \int_0^1 \phi(\theta, m(x, s), s) d\theta.$$

Proof. Fix $s \in [1, \infty)$. For $x \in [0, \xi(1, s))$, we have $m(x, s) = 1$. This implies $m_{1,+}(x, s) = 0 \le 1/((1 - \beta)\mu_1(m(x, s), s))$. Thus, we obtain the result from (EC.17).

Next, for $x \in [\xi(1, s), 1)$, we have by definition,

$$\begin{aligned} m_{1,+}(x, s) &= \limsup_{x_1 \to x} \frac{m(x_1, s) - m(x, s)}{x_1 - x} \\ &= \limsup_{x_1 \to x} \frac{m(x_1, s) - m(x, s)}{\xi(m(x_1, s), s) - \xi(m(x, s), s)}. \end{aligned}$$

As ξ and m are continuous functions, and as $\xi(\cdot, s)$ is increasing, we get

$$m_{1,+}(x, s) \le \frac{1}{\xi_{1,-}(m(x, s), s)}.$$

Finally, using (EC.16), we lower-bound the denominator to get

$$m_{1,+}(x, s) \le \frac{1}{(1 - \beta)\mu_1(m(x, s), s)}.$$

The result then follows from (EC.17). $\square$

Using the above lemma and Lemma EC.13, we obtain the following upper bound on $W_{1,+}(x, s_2)$.

LEMMA EC.15. *For $x \in [0, 1)$, and $s \in [1, \infty)$, we obtain*

$$W_{1,+}(x, s) \le B(s) \triangleq \begin{cases} 4(1 - \beta)^{-1} \|\psi\| \beta^{-4} e^{-2} (\ln(1/\beta))^{-2} s^{-1} & \textit{for } s < (2/\ln(1/\beta)) - 1; \\ (1 - \beta)^{-1} \|\psi\| \beta^{s-3} (s + 1)^2 s^{-1} & \textit{for } s \ge (2/\ln(1/\beta)) - 1. \end{cases}$$

Proof. From Lemma EC.14 and using the bound on $\phi(\cdot)$ established in Lemma EC.13, we obtain

$$\begin{aligned} W_{1,+}(x,s) &\leq \frac{1}{(1-\beta)\mu_1(m(x,s),s)} \int_0^1 \phi(\theta, m(x,s), s)d\theta \\ &\leq \frac{1}{(1-\beta)\mu_1(m(x,s),s)} \|\psi\| \beta^{m(x,s)+s-4} \\ &= \frac{(m(x,s)+s)^2}{(1-\beta)s} \|\psi\| \beta^{m(x,s)+s-4}, \end{aligned}$$

where we have used in the final equality the fact that $\mu_1(s_1,s_2) = s_2/(s_1+s_2)^2$ for all $(s_1,s_2) \in \mathcal{S}$. As $m(x,s) \geq 1$ for all $x \in [0,1]$ and all $s \geq 1$, we get

$$\begin{aligned} W_{1,+}(x,s) &\leq \frac{(m(x,s)+s)^2}{(1-\beta)s} \|\psi\| \beta^{m(x,s)+s-4} \\ &\leq \frac{\|\psi\| \beta^{-4}}{(1-\beta)s} \left(\sup_{t \geq 1} \beta^{t+s}(t+s)^2 \right). \end{aligned}$$

The result then follows by noting that

$$\sup_{t\geq 1} \beta^{t+s}(t+s)^2 = \begin{cases} 4e^{-2}(\ln(1/\beta))^{-2} & \text{for } s < (2/\ln(1/\beta)) - 1; \\ \beta^{1+s}(1+s)^2 & \text{for } s \geq (2/\ln(1/\beta)) - 1. \end{cases} \qquad \square$$

We are now ready to bound the growth rate of the bid distribution h.

LEMMA EC.16. *For all $x \in [0,1)$, we have*

$$h'_+(x) \triangleq \limsup_{x_1 \to x} \frac{h(x_1) - h(x)}{x_1 - x} \leq \int_1^\infty B(s)ds < \infty.$$

Proof. Recall from the definition of $W(\cdot)$ that for $x \in [0,1)$,

$$h(x) = \int_1^\infty W(x,s)ds.$$

From Lemma EC.15, we know that $W_{1,+}(x,s) \leq B(s)$ for all $x \in [0,1)$ and for all $s \in [1,\infty)$. Thus, we obtain for all $x_1 \neq x \in [0,1)$,

$$\frac{W(x_1,s) - W(x,s)}{x_1 - x} \leq 2B(s), \text{ for all } s \in [1,\infty).$$

As $\int_1^\infty B(s)ds < \infty$, we obtain from Fatou's Lemma that

$$\begin{aligned} h'_+(x) &= \limsup_{x_1 \to x} \frac{h(x_1) - h(x)}{x_1 - x} \\ &= \limsup_{x_1 \to x} \int_1^\infty \frac{W(x_1,s) - W(x,s)}{x_1 - x} ds \\ &\leq \int_1^\infty \limsup_{x_1 \to x} \frac{W(x_1,s) - W(x,s)}{x_1 - x} ds \\ &= \int_1^\infty W_{1,+}(x,s)ds \\ &\leq \int_1^\infty B(s)ds. \end{aligned}$$

$\square$

LEMMA EC.17. *The image of the map $\mathcal{F}$ is a compact subset of $\mathcal{G}$.*

Proof. Lemma EC.16 shows that the growth rate of any $h \in \mathcal{F}(\mathcal{G})$ is bounded above by a constant. We can now invoke the Arzelà-Ascoli theorem to conclude that $\mathcal{F}(\mathcal{G})$ is compact. $\square$

Appendix EC.4: Approximation: Asymptotic perfectness

In this section, we show that a mean field equilibrium has the asymptotic perfectness property described in the discussion in the last paragraph of Section 5. We follow the same notation as in that section. Suppose first that all agents follow the mean field strategy ξ, i.e., if an agent i has belief $s_{i,t}$ about her valuation after a history $h_{i,t}$, then her bid in the auction is given by $\xi(s_{i,t})$. After any history $h_{i,t}$ for agent i, let $\nu_i^{(n)}(h_{i,t},\xi)$ denote the probability measure induced over the future trajectory of the market by the strategies of all agents. Moreover, let $\mathsf{P}_\xi^{(n)}$ denote the probability measure induced over the entire trajectory of the market. Let $V^{(n)}(h_{1,t},\delta_1)$ denote the value function of agent 1 after history $h_{1,t}$ if she follows a (possibly history-dependent) strategy δ_1 from time t onward, all other agents still follow the strategy ξ, and if her belief at time t is given by $\nu_1^{(n)}(h_{1,t},\xi)$. Fix $t \geq 0$.

THEOREM EC.1. *Let (g,ξ) be an MFE. Let $H_t^{(n)}$ denote the event that agent 1 has never regenerated until time t. Then, for any sequence of strategies $\{\delta_1^{(n)} : n \geq 0\}$ for agent 1, we have for all $t \geq 0$ and for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathsf{P}_\xi^{(n)} \left(V^{(n)}(h_{1,t},\delta_1^{(n)}) - V^{(n)}(h_{1,t},\xi) > \epsilon \;\middle|\; H_t^{(n)} \right) = 0.$$

Moreover, there holds $\mathsf{E}_\xi^{(n)}\left[\left(V^{(n)}(h_{1,t},\delta_1^{(n)}) - V^{(n)}(h_{1,t},\xi)\right)^+ \mid H_t^{(n)}\right] \to 0$ as $n\to\infty$ for all $t \geq 0$ (where $\mathsf{E}_\xi^{(n)}$ is expectation with respect to $\mathsf{P}_\xi^{(n)}$).

Proof. The proof follows by a modification of the argument for proof of Theorem 2. The argument fails as is, because for any time t, the agents faced by agent 1 until time t might interact with agents faced by agent 1 after time t. The modification involves focusing on the event where this does not happen.

Given $\epsilon > 0$, choose $\kappa = \ln(\epsilon)/\ln(\beta) + 1$. This implies that $\beta^T < \epsilon$ for all $T \geq \kappa$. Since auctions are held according to a Poisson process with rate n/α, agent 1 participates in the auctions at the jump times of a Poisson process with rate 1. Let $\{X_t : t \geq 0\}$ denote this process; that is, until time t, agent 1 has participated in X_t auctions. Choose smallest $T_\epsilon > 0$ such that $\mathbf{P}(X_{T_\epsilon} \geq \kappa) \geq 1-\epsilon$. For any $t \geq 0$, let $K_t \geq 0$ (independent of n) be such that $\mathbf{P}(X_t \leq K_t) \geq 1-\epsilon$.

Fix $t \geq 0$. Let $D_t^{(n)} \subseteq H_t^{(n)}$ denote the event that agent 1 has participated in at most K_t auctions until time t, and at least κ auctions from time t to time $t+T_\epsilon$. By properties of the Poisson process, we have $\mathsf{P}_\xi^{(n)}(D_t^{(n)}|H_t^{(n)}) \geq (1-\epsilon)^2$. Denote the times of the first $X_t+\kappa$ auctions until time $t+T_\epsilon$ as $\{\tau_\sigma : 1 \leq \sigma \leq X_t+\kappa\}$. We note that $\tau_\sigma \leq t$ for $\sigma \leq X_t$ and $\tau_\sigma \in (t, t+T_\epsilon]$ for $\sigma \geq X_t$.

Let $C_t^{(n)} \subseteq D_t^{(n)}$ denote the event that $S_{1,\tau_\rho} \cap S_{1,\tau_\sigma} = \emptyset$ for all distinct $\rho, \sigma \leq X_t$, and furthermore, for all $\sigma \leq X_t$ and for all distinct pairs j, k with $j \in S_{1,\tau_\sigma}$ and $k \in \left(\cup_{\rho \leq \sigma} S_{1,\tau_\rho}\right) \cup \{1\}$, the interaction sets $A^{(n)}_{j,\tau_\sigma^-}$ and $A^{(n)}_{k,\tau_\sigma^-}$ are disjoint. Also, let $\tilde{C}_t^{(n)} \subseteq D_t^{(n)}$ denote the event where this holds for all $\sigma \leq X_t + \kappa$. On the event $C_t^{(n)}$ the agents faced by agent 1 in the auctions until time t have had no influence on nor were influenced by agent 1 until they meet her. Similarly, on event $\tilde{C}_t^{(n)}$, this is true for the first $X_t + \kappa$ auctions.

Using similar arguments as in the proof of Lemma 4, we have $\mathsf{P}_\xi^{(n)}(\tilde{C}_t^{(n)} | D_t^{(n)}) \to 1$ and $\mathsf{P}_\xi^{(n)}(C_t^{(n)} | D_t^{(n)}) \to 1$ as $n \to \infty$. Clearly $\tilde{C}_t^{(n)} \subseteq C_t^{(n)}$ and thus $\mathsf{P}_\xi^{(n)}(\tilde{C}_t^{(n)} | C_t^{(n)}) \to 1$ as $n \to \infty$. In the rest of the proof, we condition on the event $C_t^{(n)}$.

Suppose agent 1 has followed strategy ξ until time t and is considering choosing a possibly history-dependent strategy $\delta_1^{(n)}$ after time t. On event $\tilde{C}_t^{(n)}$, the agents faced by agent 1 in the first κ auctions after time t have had no interactions with the agents faced by agent 1 until time t, nor with agent 1 until that point. Lemma 5 implies that the distribution of the valuation and beliefs of these agents converges to Φ independently across the agents in total variation norm. Thus, there exists an event $B_t^{(n)} \subseteq \tilde{C}_t^{(n)}$, with $\mathsf{P}_\xi^{(n)}(B_t^{(n)} | \tilde{C}_t^{(n)}) \to 1$, on which these agents' valuation and beliefs are distributed independently and identically as Φ.

On the event $B_t^{(n)}$, for the first κ auctions after time t, agent 1 faces the same decision problem as an agent in a large market. Moreover, as agent 1 has followed the strategy ξ until time t, her belief at time t match with the real conditional distribution of the future trajectory of the market given her history. Thus, using arguments as in the proof of Theorem 2, we obtain that on the event $C_t^{(n)}$,

$$V^{(n)}(h_{1,t}, \delta_1^{(n)}) - V^{(n)}(h_{1,t}, \xi) \leq M\epsilon,$$

for some finite (fixed) $M > 0$. As the event $C_t^{(n)}$ occurs with high probability, we obtain that

$$\lim_{n \to \infty} \mathsf{P}_\xi^{(n)} \left(V^{(n)}(h_{1,t}, \delta_1^{(n)}) - V^{(n)}(h_{1,t}, \xi) > \epsilon \,\middle|\, H_t^{(n)} \right) = 0.$$

Finally, as the payoffs are bounded, we get $\mathsf{E}_\xi^{(n)} \left[\left(V^{(n)}(h_{1,t}, \delta_1^{(n)}) - V^{(n)}(h_{1,t}, \xi) \right)^+ \,\middle|\, H_t^{(n)} \right] \to 0$ as $n \to \infty$ for all $t \geq 0$. □

Appendix EC.5: Dynamic revenue equivalence: Proofs

In this section, we provide the proof of Theorems 3 and 4. Throughout this section, we fix notation as in Section 6.

EC.5.1. Proof of Theorem 3

Let (g, ξ) denote an MFE for the market M_{SP}. Recall that an agent in market LM_{SP} faces the decision problem (5). Now, suppose an agent i in the market M_A conjectures the bid distribution in the large market is $g_A(\cdot)$, where we define g_A according to $g_A(\cdot) = g(\eta_A^{-1}(\cdot|g))$. Given this conjectured bid distribution, the decision problem faced by an agent i in the market LM_A is defined analogously as

$$V_A(s|g_A) \triangleq \sup_\theta \mathbf{E}_\theta \left[\sum_{k=1}^{\infty} \beta^{k-1} \left(q_A(b_{i,k}|g_A)\mu(s_{i,k-1}) - p_A(b_{i,k}|g_A) \right) \Big| s_{i,0} = s \right], \tag{EC.18}$$

where the supremum is over all strategies θ, and $s \in \mathcal{S}$ denotes the belief of the agent. The following lemma relates the decision problem (5) with the preceding decision problem.

LEMMA EC.18. *For all $s \in \mathcal{S}$, we have $V_A(s|g_A) = V(s|g)$. Further, the policy $\eta_A(\xi(\cdot)|g)$ is optimal for the decision problem* (EC.18).

Proof. The proof involves showing that for any policy Π for the decision problem (5), there exists a policy Π_η for decision problem (EC.18) such that the expected payoffs corresponding to the two policies are equal (and vice versa). To see this, for any policy Π for decision problem (5), consider the policy Π_η that makes a bid $\eta_A(b|g)$ whenever Π makes a bid b. Then the expected payoff corresponding to the policy Π_η is given by

$$\begin{aligned} V_A^{\Pi_\eta}(s|g_A) &= \mathbf{E}_{\Pi_\eta} \left[\sum_{k=1}^{\infty} \beta^t \left(q_A(b_{i,k}|g_A)\mu(s_{i,k-1}) - p_A(b_{i,k}|g_A) \right) \Big| s_{i,0} = s \right] \\ &= \mathbf{E}_\Pi \left[\sum_{k=1}^{\infty} \beta^t \left(q_A(\eta_A(x_{i,k}|g)|g_A)\mu(s_{i,k-1}) - p_A(\eta_A(x_{i,k}|g)|g_A) \right) \Big| s_{i,0} = s \right], \end{aligned}$$

where $x_{i,k}$ is the bid made by the policy Π in the k^{th} auction. From Lemma 1, we obtain

$$V_A^{\Pi_\eta}(s|g_A) = \mathbf{E}_\Pi \left[\sum_{k=1}^{\infty} \beta^t \left(q(x_{i,k}|g)\mu(s_{i,k-1}) - p(x_{i,k}|g) \right) \Big| s_{i,0} = s \right],$$

Further, as the transitions in the state $s_{i,k}$ only depend on $q_A(\eta_A(x_{i,k}|g)|g_A) = q(x_{i,k}|g)$, we observe that the state transitions on using policy Π_η are distributed exactly as the state transitions on using policy Π in the decision problem (5). Thus, we get $V_A^{\Pi_\eta}(s|g_A) = V^\Pi(s|g)$. As $\eta_A(\cdot|g)$ is strictly increasing, we can similarly show that for any policy Π_A for the decision problem (EC.18), there exists a policy Π'_A that achieves the same expected payoff. Thus we obtain that $V_A(s|g_A) = V(s|g)$ for all $s \in \mathcal{S}$.

As the policy $\xi(\cdot)$ is optimal for the decision problem (5), the preceding argument proves that the policy $\xi_A = \eta_A(\xi(\cdot)|g)$ is optimal for the decision problem (EC.18). □

We now complete the proof for Theorem 3 by showing the two markets LM_A and LM_{SP} have the same invariant distribution.

Proof of Theorem 3. From Lemma EC.18 we obtain that the strategy $\xi_A(\cdot)$ is optimal for the decision problem (EC.18) when the bid distribution is given by $g_A(\cdot) = g(\eta_A^{-1}(\cdot|g))$. Thus to show that (g_A, η_A) constitute an MFE for the market M_A, we need to show that if all agents follow the strategy $\eta_A(\cdot)$, then the bid distribution that arises is given by $g_A(\cdot)$.

Consider an agent i in the market LM_A, where all agents follow the strategy $\xi_A(\cdot)$. The transitions in the belief and the valuation of the agent are governed by the transition probability kernel given by

$$\begin{aligned}
&\mathbf{P}\left((v_{i,t}, s_{i,t}) \in (C,B) | (v_{i,t-1}, s_{i,t-1}) = (v,s)\right) \\
&= \beta v q_A(\xi_A(s)|g_A)\mathbf{I}\{v \in C, s+e_1) \in B\} \\
&\quad + \beta(1-v) q_A(\xi_A(s)|g_A)\mathbf{I}\{v \in C, s+e_2 \in B\} \\
&\quad + \beta(1-q_A(\xi_A(s)|g_A))\mathbf{I}\{v \in C, s \in B\} \\
&\quad + (1-\beta)\Psi(C,B),
\end{aligned} \tag{EC.19}$$

for all Borel sets $C \subseteq [0,1]$ and $B \subseteq \mathcal{S}$. From Lemma 1, we have $q_A(\xi_A(s)|g_A) = q(\xi(s)|g)$ for all $s \in \mathcal{S}$. This implies that the transition probability kernel is exactly the same as that for the agent in the market LM_{SP}, as given by (6). Thus, the invariant distribution Φ_A of the market LM_A is exactly $\Phi(\cdot|g,\xi)$.

Finally, observe that

$$\begin{aligned}
\Phi_A(\xi_A(s) \le x) &= \Phi(\eta_A(\xi(s)|g) \le x | g, \xi) \\
&= \Phi(\xi(s) \le \eta_A^{-1}(x|g) | g, \xi) \\
&= g(\eta_A^{-1}(x|g)) \\
&= g_A(x).
\end{aligned}$$

Thus, when all agents follow strategy ξ_A, the bid distribution that arises in steady state is g_A. Thus (g_A, ξ_A) is an MFE for the market M_A. $\square$

EC.5.2. Proof of Theorem 4

We prove the theorem by constructing an MFE for the repeated second price market. Before we proceed, we first develop some concepts and notation that will be used to prove a series of lemmas required in the proof of the theorem.

Given a continuous distribution g_A, let $q_A(\cdot|g_A)$ and $p_A(\cdot|g_A)$ denote the corresponding probability of winning function and the payment function. The value function for an agent with belief s is given by

$$V_A(s|g_A) = \sup_{\Pi} \mathbf{E}\left[\sum_{t \ge 0} \beta^t \left(q_A(b_t|g_A)\mu(s_t) - p_A(b_t|g_A)\right) \Big| s_0 = s\right].$$

As the auction format is continuous, we obtain through arguments from Maitra 1968 that the value function is continuous and satisfies Bellman's equation given by

$$V_A(s|g_A) = \frac{1}{1-\beta} \max_{b \ge 0} \left(q_A(b|g_A) C_A(s|g_A) - p_A(b|g_A)\right),$$

where $C_A(s|g_A) = \mu(s) + \beta\mu(s)V_A(s+e_1|g_A) + \beta(1-\mu(s))V_A(s+e_2|g_A) - \beta V_A(s|g_A)$. Furthermore, we obtain that for each $s \in \mathcal{S}$, the optimal bid $\xi_A(s)$ attains the maximum on the right hand side.

Define the best-response correspondence $BR_A(\cdot)$ as

$$BR_A(v) \triangleq \arg\max_{b\geq 0} \left(q_A(b|g_A)v - p_A(b|g_A)\right), \text{for } v \geq 0.$$

Note that as the strategy $\xi_A(\cdot)$ is optimal, we have $\xi_A(s) \in BR_A(C_A(s))$ for each $s \in \mathcal{S}$. As the functions $q_A(\cdot|g_A)$ and $p_A(\cdot|g_A)$ are continuous, the set $BR_A(v)$ is closed for all $v \geq 0$.

We let $\overline{Y}(v) = \sup BR_A(v)$ and $\underline{Y}(v) = \inf BR_A(v)$ for any $v \geq 0$. For any $v \geq 0$, $\underline{Y}(v) \geq 0$, but $\overline{Y}(v)$ might possibly be infinite. Moreover, as $\xi_A(s) \in BR_A(C_A(s))$ for all $\mathcal{S}$, we have $\underline{Y}(C_A(s)) \leq \xi_A(s) < \infty$. The following lemma shows that $\underline{Y}(\cdot)$ is non-decreasing.

LEMMA EC.19. *For $v_1 > v_2 \geq 0$, let $y_1 \in BR_A(v_1)$ and $y_2 \in BR_A(v_2)$. Then, either $y_1 > y_2$ or $g_A(y_1) - g_A(y_2) = p_A(y_1|g_A) - p_A(y_2|g_A) = 0$. Moreover, we have $\underline{Y}(v_1) \geq \underline{Y}(v_2)$.*

Proof. Let $v_1 > v_2 \geq 0$. As $y_i \in BR_A(v_i)$ for $i = 1, 2$, we have by definition,

$$q_A(y_1|g_A)v_2 - p_A(y_1|g_A) \leq q_A(y_2|g_A)v_2 - p_A(y_2|g_A),$$
$$q_A(y_2|g_A)v_1 - p_A(y_2|g_A) \leq q_A(y_1|g_A)v_1 - p_A(y_1|g_A).$$

Rearranging, we obtain

$$(q_A(y_1|g_A) - q_A(y_2|g_A))v_2 \leq p_A(y_1|g_A) - p_A(y_2|g_A) \leq (q_A(y_1|g_A) - q_A(y_2|g_A))v_1. \tag{EC.20}$$

As $v_1 > v_2$, this implies that $q_A(y_1|g_A) \geq q_A(y_2|g_A)$. Since, $q_A(\cdot|g_A)$ is non-decreasing, we obtain two possibilities: either $y_1 > y_2$ or $q_A(y_1|g_A) = q_A(y_2|g_A)$. If $q_A(y_1|g_A) = q_A(y_2|g_A)$, then, from (EC.20), we have $p_A(y_1|g_A) = p_A(y_2|g_A)$. This proves the first statement in the lemma.

Next, observe if $q_A(y_1|g_A) = q_A(y_2|g_A)$ and $p_A(y_1|g_A) = p_A(y_2|g_A)$, then we have $y_1 \in BR_A(v_2)$ and $y_2 \in BR_A(v_1)$. For $y_i = \underline{Y}(v_i)$, this implies both $y_1 \geq y_2$ and $y_2 \geq y_1$. $\square$

LEMMA EC.20. *For any $s \in \mathcal{S}$, we have $BR_A(C_A(s)) = [\underline{Y}(C_A(s)), \overline{Y}(C_A(s))]$. Moreover, we have*

$$g_A(y) = g_A(\xi_A(s)),$$
$$p_A(y) = p_A(\xi_A(s)|g_A),$$

for all $y \in BR_A(C_A(s))$.

Proof. Note that since $s + ce_1 \succ s$, we have $C_A(s + ce_1) > C_A(s)$ for $c > 0$. Hence, using Lemma EC.19, we get that for any $y \in BR_A(C_A(s))$, either $\xi_A(s + ce_1) > y$ or $g_A(\xi_A(s + ce_1)) = g_A(y)$. Using continuity of g_A and ξ_A, we obtain, on taking limit as $c \downarrow 0$, that either $\xi_A(s) \geq y$ or $g_A(\xi_A(s)) = g_A(y)$ for any $y \in BR_A(C_A(s))$. Thus, for all $y \in BR_A(C_A(s))$ with $y \leq \xi_A(s)$, we have $g_A(y) = g_A(\xi_A(s))$.

By a similar limiting argument starting with $s \succ s + ce_2$ for $c > 0$, we infer that for any $y \in BR_A(C_A(s))$ with $y \geq \xi_A(s)$, we have $g_A(\xi_A(s)) = g_A(y)$. Combining the two statements, we obtain that for all $y \in$

$BR_A(C_A(s))$, $g_A(y) = g_A(\xi_A(s))$. As g_A is non-decreasing, this implies that the latter equality in fact holds for all $y \in [\underline{Y}(C_A(s)), \overline{Y}(C_A(s))]$. This and the monotonicity of $p_A(\cdot|g_A)$ together imply that $p_A(y|g_A) = p_A(\xi_A(s)|g_A)$ for all y in the same interval and hence $BR_A(C_A(s)) = [\underline{Y}(C_A(s)), \overline{Y}(C_A(s))]$. $\square$

Let $\Phi_A(\cdot)$ denote the steady state distribution that arises in the MFE (g_A, ξ_A) of the market LM_A. The following lemma relates the optimal policy ξ_A with the function C_A.

LEMMA EC.21. *We have*

$$\xi_A(s) = \underline{Y}(C_A(s)), \ \ \Phi_A\text{-}a.s.$$

Proof. By definition of $\underline{Y}(C_A(s))$, we have $\xi_A(s) \geq \underline{Y}(C_A(s))$. Furthermore, from Lemma EC.20, we get $g_A(\xi_A(s)) = g_A(\underline{Y}(C_A(s)))$. Thus, if $\xi_A(s) > \underline{Y}(C_A(s))$, then there exists a rational $r > 0$, such that $g_A(\xi_A(s)) = g_A(\xi_A(s) - r)$.

Next, for any rational $\epsilon > 0$, let $\mathcal{A}_\epsilon \triangleq \{s : g_A(\xi_A(s)) = g_A(\xi_A(s) - \epsilon)\}$. Observe that $\mathbf{P}_{\Phi_A}(s \in \mathcal{A}_\epsilon) = 0$, as for any x with $g_A(x) = g_A(x-\epsilon)$, there are no bids in the interval $(x-\epsilon, x]$, Φ_A-a.s. Taking union over rational $\epsilon > 0$, we obtain

$$\begin{aligned}
\mathbf{P}_{\Phi_A}(\xi_A(s) > \underline{Y}(C_A(s))) &\leq \mathbf{P}_{\Phi_A}(s \in \cup_{\epsilon>0}\mathcal{A}_\epsilon) \\
&\leq \sum_{\epsilon>0} \mathbf{P}_{\Phi_A}(s \in \mathcal{A}_\epsilon) \\
&= 0.
\end{aligned}$$

Hence, Φ_A-a.s., we have $\xi_A(s) = \underline{Y}(C_A(s))$. $\square$

Define the distribution g as follows.

$$g(x) = \mathbf{P}_{\Phi_A}(C_A(s) \leq x), \text{ for } x \in \mathbb{R}.$$

By exact same arguments as in Section EC.3.4, we obtain that $g(x)$ is a continuous function on $\mathbb{R}$. The following lemma relates the distribution g with the distribution g_A.

LEMMA EC.22. *We have $g(x) = g_A(\underline{Y}(x))$ for all $x \geq 0$.*

Proof. Observe, from Lemma EC.19, that if $C_A(s) \leq x$, then $\underline{Y}(C_A(s)) \leq \underline{Y}(x)$. Hence, we get

$$\begin{aligned}
g(x) &= \mathbf{P}_{\Phi_A}(C_A(s) \leq x) \\
&\leq \mathbf{P}_{\Phi_A}(\underline{Y}(C_A(s)) \leq \underline{Y}(x)) \\
&= \mathbf{P}_{\Phi_A}(\xi_A(s) \leq \underline{Y}(x)) \\
&= g_A(\underline{Y}(x)).
\end{aligned} \tag{EC.21}$$

On the other hand, we have

$$
\begin{aligned}
g_A(\underline{Y}(x)) &= \mathbf{P}_{\Phi_A}(\xi_A(s) \le \underline{Y}(x)) \\
&= \mathbf{P}_{\Phi_A}(\xi_A(s) < \underline{Y}(x)) + \mathbf{P}_{\Phi_A}(\xi_A(s) = \underline{Y}(x)) \\
&= \mathbf{P}_{\Phi_A}(\underline{Y}(C_A(s)) < \underline{Y}(x)),
\end{aligned}
$$

where the last equality follows from Lemma EC.21 and continuity of g_A. Again using Lemma EC.19, it follows that if $\underline{Y}(C_A(s)) < \underline{Y}(x)$, then $C_A(s) < x$, and hence we get

$$
\begin{aligned}
g_A(\underline{Y}(x)) &= \mathbf{P}_{\Phi_A}(\underline{Y}(C_A(s)) < \underline{Y}(x)) \\
&\le \mathbf{P}_{\Phi_A}(C_A(s) < x) \\
&= g(x),
\end{aligned} \tag{EC.22}
$$

where the last equality follows from continuity of g.

Thus, from (EC.21) and (EC.22), we get $g(x) = g_A(\underline{Y}(x))$ for all $x \ge 0$. □

Next, we relate the payment function in the market LM_A to that in the market LM_{SP}.

LEMMA EC.23. *We have for all $x \ge 0$,*

$$
p_A(\underline{Y}(x)|g_A) = p(x|g) = xq(x|g) - \int_0^x q(u|g)du.
$$

Proof. As $\underline{Y}(C_A(s)) \in BR_A(C_A(s))$ for any $s \in \mathcal{S}$, we have for any $b \ge 0$,

$$
q_A(\underline{Y}(b)|g_A)C_A(s) - p_A(\underline{Y}(b)|g_A) \le q_A(\underline{Y}(C_A(s))|g_A)C_A(s) - p_A(\underline{Y}(C_A(s))|g_A). \tag{EC.23}
$$

From Lemma EC.22, and using the fact that A is a standard auction, we obtain that $q(x|g) = q_A(\underline{Y}(x)|g_A)$ for all $x \ge 0$. Thus, we have from (EC.23),

$$
q(C_A(s))|g)C_A(s) - p_A(\underline{Y}(C_A(s))|g_A) = \max_{b \ge 0}\left(q(b|g)C_A(s) - p_A(\underline{Y}(b)|g_A)\right).
$$

This implies that $p_A(\underline{Y}(\cdot)|g_A)$ is the payment function in the second price auction where the bid distribution is g. Thus $p_A(\underline{Y}(x)|g_A) = p(x|g)$ for all $x \ge 0$. □

The following lemma shows that the value function of an agent in the large market LM_A with bid distribution g_A is same as that of an agent in the large market LM_{SP} with bid distribution g.

LEMMA EC.24. *We have $V_A(s|g_A) = V(s|g)$ and $C_A(s) = \xi(s|g)$ for each $s \in \mathcal{S}$.*

Proof. Recall the definition of $C_A(s)$:

$$
C_A(s) = \mu(s) + \beta V_A(s + e_1) + \beta(1 - \mu(s))V_A(s + e_2) - \beta V_A(s|g_A).
$$

From Lemmas EC.22 and EC.23, we obtain that

$$(1-\beta)V_A(s|g_A) = q(C_A(s)|g)C_A(s) - p(C_A(s)|g) = \max_{b\geq 0}\left(q(b|g)C_A(s) - p(b|g)\right).$$

This implies that $V_A(\cdot)$ satisfies the Bellman's equation for the agent decision problem in the market M_{SP} with bid distribution g. Hence, $V_A(s|g_A) = V(s|g)$ for each $s\in\mathcal{S}$, and $C_A(s) = \xi(s|g)$. □

We are now ready to prove the theorem.

Proof of Theorem 4. We claim that (g, C_A) is an MFE for the market M_{SP}. To see this, we begin by noting that the strategy $C_A(\cdot)$ is optimal for the large market LM_{SP} with bid distribution g. This follows directly from Lemma EC.24. Thus, the pair (g, C_A) satisfies the optimality condition in the definition of an MFE.

Next, observe from Lemmas EC.20 and EC.22 that for all $s\in\mathcal{S}$, we have

$$\begin{aligned} g(C_A(s)) &= g_A(\underline{Y}(C_A(s))) \\ &= g_A(\xi_A(s)). \end{aligned}$$

Thus, the transitions of the valuation and belief of an agent in the large market LM_A is same as their transitions in the large market LM_{SP}. This implies that the steady state of the two markets is the same, and hence $\Phi_A(\cdot) = \Phi(\cdot|g, C_A)$. Thus, we have

$$\begin{aligned} g(x) &= \mathbf{P}_{\Phi_A}(C_A(s)\leq x) \\ &= \mathbf{P}_{\Phi}(C_A(s)\leq x), \end{aligned}$$

where $\Phi = \Phi(\cdot|g, C_A)$. Thus, the pair (g, C_A) also satisfies the consistency condition in the definition of an MFE.

Thus, we obtain that (g, C_A) constitute an MFE of the market M_{SP}. From Lemma EC.22, it is straightforward to verify that auctioneer's revenue in the two large markets are the same. □