Supplementary Document for MS-13-01583: Strategic Waiting for Consumer-Generated Quality Information: Dynamic Pricing of New Experience Goods

This document contains the omitted details for results in the paper (not intending for publication). Section SA is for the basic model. Section SB provides detailed formulations and results for extensions on capacity constraint and quick response (Section 6.1 of the paper), additional sales periods (Section 6.2), asymmetric quality information (Section 6.3), and three extensions to quality-updating process (Section 6.4). Section SC covers the proofs for the results in Section SB. We replicated the statements of the results so that this supplementary document can be independently read and used as a reference.

SA Details for Results in the Basic Model

SA.1 Proofs of Results in the Appendix of the Paper

Proposition 11. *There exists a unique equilibrium for consumers' buy-or-wait decisions in the first period. Specifically,*

(a) for a given P_1, there exists a unique threshold $\overline{v}(P_1)$ such that all consumer types $v > \overline{v}(P_1)$ purchase in the first period and all consumer types $v \le \overline{v}(P_1)$ wait till the second period; and

(b) there exist two thresholds, $\underline{P}$ and $\overline{P}$, such that $\overline{v}(P_1) = 0$ if $P_1 < \underline{P}$, $\overline{v}(P_1) = 1$ if $P_1 > \overline{P}$, and for $P_1 \in [\underline{P}, \overline{P}]$, $\overline{v}(P_1)$ satisfies

$$p_0 + v - P_1 = \frac{1}{2}\delta_c \mathbb{E}_{\tilde{G}|\{\tilde{\Theta},1-v\}}[(p(\tilde{G}, 1-v) + v - c)^+]\}, \tag{14}$$

and $\overline{v}(P_1)$ is monotone increasing in P_1.

Proof. (a) Recall that

$$\overline{v}(P_1) = \max\{v \in [0,1] : p_0 + v - P_1 \leqslant \frac{1}{2}\delta_c \mathbb{E}_{\tilde{G}|\{\tilde{\Theta},1-v\}}[(p(\tilde{G}, 1-v) + v - c)^+]\},$$

Hence, to show the uniqueness of the consumer equilibrium, it suffices to prove that, for given P_1, $p_0 + v - P_1 - \frac{1}{2}\delta_c \mathbb{E}_{\tilde{G}|\{\tilde{\Theta},1-v\}}[(p(\tilde{G}, 1-v) + v - c)^+]$ monotonically increases in v. Taking derivative of the term with respect to v, we have

$$\frac{d}{dv}\left\{p_0+v-P_1-\tfrac{1}{2}\delta_c\mathbb{E}_{\tilde{G}|\{\tilde{\Theta},1-v\}}[(p(\tilde{G},1-v)+v-c)^+]\right\}$$
$$=\frac{d}{dv}\left\{p_0+v-P_1-\tfrac{1}{2}\delta_c\mathbb{E}_{\tilde{P}|\{1-v\}}[(\tilde{P}|\{1-v\}+v-c)^+]\right\}$$
$$=1-\tfrac{1}{2}\delta_c\left\{\left(\tfrac{\partial}{\partial v}\mathbb{E}_{\tilde{P}|\{1-v\}}[(\tilde{P}|\{1-v\}+v'-c)^+]\right)\Big|_{v'=v}+\left(\tfrac{\partial}{\partial v}\mathbb{E}_{\tilde{P}|\{1-v'\}}[(\tilde{P}|\{1-v'\}+v-c)^+]\right)\Big|_{v'=v}\right\}$$
$$=1-\tfrac{1}{2}\delta_c\left\{\left(\tfrac{\partial}{\partial v}\mathbb{E}_{\tilde{P}|\{1-v\}}[(\tilde{P}|\{1-v\}+v'-c)^+]\right)\Big|_{v'=v}+\Pr[\tilde{P}|\{1-v\}+v-c\geq 0]\right\}$$
$$>0$$

where the last inequality is due to two facts: first, by Proposition 1 in the paper, $\tilde{P}|\{1-v\}$ second-order stochastically increases in v (as it stochastically decreases in $x=1-v$). Further, $(\tilde{P}|\{1-v\}+v'-c)^+$ is convex in $\tilde{P}|\{1-v\}$. Thus, for any v', $\left(\frac{\partial}{\partial v}\mathbb{E}_{\tilde{P}|\{1-v\}}[(\tilde{P}|\{1-v\}+v'-c)^+]\right)<0$; second, $\Pr[\tilde{P}|\{1-v\}+v-c\geq 0]\leq 1$ and $\delta_c\in[0,1]$, jointly implying $1-\frac{1}{2}\delta_c\Pr[\tilde{P}|\{1-v\}+v-c\geq 0]>0$.
(b) It follows from part (a) and the fact that $p_0+v-P_1-\frac{1}{2}\delta_c\mathbb{E}_{\tilde{G}|\{\tilde{\Theta},1-v\}}[(p(\tilde{G},1-v)+v-c)^+]$ also monotonically decreases in P_1 for given v. □

Lemma 4. *(i)* $\Pi^*-\Pi^o$ *decreases for low* δ_c *and increases for high* δ_c*;*
(ii) $\Pi^*-\Pi^o$ *strictly increases in* $\delta<1$ *for given* δ_c*;*
(iii) $\Pi^*-\Pi^o>0$ *for* $\delta=\delta_c>0$*.*

Proof. (i) To show that the difference $\Pi^*-\Pi^o$ decreases for low δ_c and increases for high δ_c, it suffices to show that $\Pi^*-\Pi^o$ is (i-a) continuously differentiable in δ_c, and (i-b) first convex and then concave and nondecreasing in δ_c. In preparation, by the envelop theorem,

$$\frac{d[\Pi^*-\Pi^o]}{d\delta_c} = -\tfrac{1}{2}(1-v^*)p_0(1+v^*-c)+\tfrac{1}{2}(1-v^o)(p_0+v^o-c) \tag{SA.1}$$

(i-a) From equation (SA.1) and the fact that both v^* and v^o are continuous in δ_c, $\Pi^*-\Pi^o$ is continuously differentiable in δ_c.
(i-b) From equation (SA.1),

$$\frac{d^2[\Pi^*-\Pi^o]}{d\delta_c^2}=\frac{1}{2}\left\{\frac{d}{dv}((1-v)(p_0+v-c))|_{v=v^o}\frac{dv^o}{d\delta_c}-\frac{d}{dv}((1-v)p_0(1+v-c))|_{v=v^*}\frac{dv^*}{d\delta_c}\right\} \tag{SA.2}$$

To determine the sign of equation (SA.2), consider the following two cases:

- $\delta_c\leq\frac{1}{2p_0-cp_0}\left(2p_0+c\left(\delta p_0-2\right)-2\delta p_0+2\right)$: this case is valid if and only if $2p_0+c\left(\delta p_0-2\right)-$

$2\delta p_0+2 \ge 0$, or $0 \le \delta \le \frac{2(p_0-c+1)}{p_0(2-c)}$. In such a case, $v^* = \frac{2+(2-(\delta+\delta_c)p_0)c-(2-\delta)p_0}{4-(\delta+2\delta_c)p_0}$, and

$$\frac{d^2[\Pi^*-\Pi^o]}{d\delta_c^2} = \frac{1}{2}\left\{\frac{p_0^2\left(-4p_0+2\delta p_0-c\delta p_0+4\right)^2}{\left(4-\delta p_0-2\delta_c p_0\right)^3} - \frac{\delta^2\left(p_0-c+1\right)^2}{\left(4-\delta-2\delta_c\right)^3}\right\}$$

Let $M(\delta_c) = p_0^2\left(-4p_0+2\delta p_0-c\delta p_0+4\right)^2\left(4-\delta-2\delta_c\right)^3 - \delta^2\left(p_0-c+1\right)^2\left(4-\delta p_0-2\delta_c p_0\right)^3$. To show that $\Pi^*-\Pi^o$ is convex-concave in $\delta_c \le \frac{1}{2p_0-cp_0}\left(2p_0+c\left(\delta p_0-2\right)-2\delta p_0+2\right)$, it suffices to show that $M(\delta_c)$ decreases in δ_c:

$$\begin{aligned}
\frac{dM(\delta_c)}{d\delta_c} &= -6p_0^2\left(4-\delta-2\delta_c\right)^2\left(-4p_0+2\delta p_0-c\delta p_0+4\right)^2 + 6\delta^2 p_0\left(p_0-c+1\right)^2\left(4-\delta p_0-2\delta_c p_0\right)^2 \\
&< -6p_0^2\left(4-\delta-2\delta_c\right)^2\left(-4p_0+2\delta p_0-c\delta p_0+4\right)^2 + 6\delta^2\left(p_0-c+1\right)^2\left(4-\delta p_0-2\delta_c p_0\right)^2 \\
&= 6[\delta\left(p_0-c+1\right)\left(4-\delta p_0-2\delta_c p_0\right) + p_0\left(4-\delta-2\delta_c\right)\left(-4p_0+2\delta p_0-c\delta p_0+4\right)] \cdot \\
&\quad [\delta\left(p_0-c+1\right)\left(4-\delta p_0-2\delta_c p_0\right) - p_0\left(4-\delta-2\delta_c\right)\left(-4p_0+2\delta p_0-c\delta p_0+4\right)] \qquad \text{(SA.3)}
\end{aligned}$$

where the inequality is by the fact $p_0 < 1$. By algebraic simplification and the assumption $c > 1$, it can be shown that the first term in equation (SA.3) is positive, while the second term is negative. Thus, $M(\delta_c)$ decreases in δ_c.

- $1 > \delta_c \ge \frac{1}{2p_0-cp_0}\left(2p_0+c\left(\delta p_0-2\right)-2\delta p_0+2\right)$: this case is valid if and only if $\frac{1}{2p_0-cp_0}(2p_0 + c(\delta p_0-2)-2\delta p_0+2) < 1$, or $\frac{2-2c+p_0c}{p_0(2-c)} \le \delta \le 1$. In this case, $v^*=1$ and

$$\frac{d[\Pi^*-\Pi^o]}{d\delta_c} = \frac{1}{2}(1-v^o)(p_0+v^o-c) > 0, \qquad \frac{d^2[\Pi^*-\Pi^o]}{d\delta_c^2} = -\frac{1}{2}\frac{\delta^2\left(p_0-c+1\right)^2}{\left(4-\delta-2\delta_c\right)^3} \le 0,$$

Thus, $\Pi^*-\Pi^o$ is concave and nondecreasing in $\delta_c \ge \frac{1}{2p_0-cp_0}\left(2p_0+c\left(\delta p_0-2\right)-2\delta p_0+2\right)$.

Furthermore, it is easy to show that at the boundary point $\delta_c = \frac{1}{2p_0-cp_0}\left(2p_0+c\left(\delta p_0-2\right)-2\delta p_0+2\right)$, the left second-order derivative of $\Pi^*-\Pi^o$ with respect to δ_c is always greater than the right second-order derivative. Thus, $\Pi^*-\Pi^o$ is first convex and then concave and nondecreasing in $\delta_c \in [0,1]$.

(ii) By the envelop theorem,

$$\begin{aligned}
\frac{d[\Pi^*-\Pi^o]}{d\delta} &= \frac{1}{4}\left[p_0(1+v^*-c)^2-(p_0+v^o-c)^2\right] \\
&\ge \frac{1}{4}\left[p_0(1+v^o-c)^2-(p_0+v^o-c)^2\right] \\
&= \frac{1}{4}\left(1-p_0\right)\left(p_0-(c-v^o)^2\right) > 0,
\end{aligned}$$

where the first inequality follows from the result $v^* > v^o$ as in Proposition 5 (i), and the second one is by the fact $c - v^o < p_0 < 1$.

(iii) For expositional convenience, within this proof we explicitly write Π^* and Π^o as functions of δ and δ_c: $\Pi^*(\delta, \delta_c)$ and $\Pi^o(\delta, \delta_c)$.

First recall $\Pi^*(0,0) = \Pi^o(0,0)$. Thus, to show $\Pi^*(\delta_c, \delta_c) > \Pi^o(\delta_c, \delta_c)$ for $\delta_c > 0$, it suffices to prove that $\Pi^*(\delta_c, \delta_c) - \Pi^o(\delta_c, \delta_c)$ strictly increases in $\delta_c < 1$. To this end, from the envelop theorem,

$$\begin{aligned}
&\frac{d[\Pi^*(\delta_c, \delta_c) - \Pi^o(\delta_c, \delta_c)]}{d\delta_c} \\
&= \tfrac{1}{4}p_0(1 + v^* - c)^2 - \tfrac{1}{2}(1 - v^*)p_0(1 + v^* - c) - \left[\tfrac{1}{4}(p_0 + v^o - c)^2 - \tfrac{1}{2}(1 - v^o)(p_0 + v^o - c)\right] \\
&\overset{def}{=} h_1(v^*) - h_2(v^o)
\end{aligned}$$

where $h_1(x) = \frac{1}{4}p_0(1+x-c)^2 - \frac{1}{2}(1-x)p_0(1+x-c)$ and $h_2(x) = \frac{1}{4}(p_0+x-c)^2 - \frac{1}{2}(1-x)(p_0+x-c)$. To show $h_1(v^*) - h_2(v^o) > 0$, it suffices to show $h_2(v^o) < h_2(v^*)$ and $h_1(v^*) > h_2(v^*)$.

To prove $h_2(v^o) \leq h_2(v^*)$, note that $\frac{d^2 h_2(x)}{dx^2} = \frac{3}{4}$ and $\frac{dh_2(x)}{dx}|_{x=v^o} = \frac{1}{4-3\delta_c}(p_0 - c + 1) > 0$. Together with the fact $v^o < v^*$ (Proposition 5.i), these jointly imply $h_2(v^o) < h_2(v^*)$.

Now consider two cases to prove $h_1(v^*) > h_2(\overline{v}^*)$: If $\delta_c \geq \frac{p_0-c+1}{(2-c)p_0}$, $v^* = 1$ and $h_1(v^*) - h_2(v^*) = \frac{1}{4}(1 - p_0)\left(p_0 - (c-1)^2\right) > 0$; If $0 < \delta_c < \frac{p_0-c+1}{(2-c)p_0}$, $v^* = \frac{2+(2-2\delta_c p_0)c-(2-\delta_c)p_0}{4-3\delta_c p_0}$ and $h_1(v^*) - h_2(v^*) = \frac{1}{4}\frac{1-p_0}{(3\delta_c p_0-4)^2}m(\delta_c)$, where $m(\delta_c) = 3p_0^2\left((c-1)^2 + 3p_0 - 4\right)\delta_c^2 + 8p_0(c-1)(2-c)\delta_c + 4((c-(p_0+1))^2 + 4p_0(1-p_0)$. Since $m(\delta_c)$ is concave in δ_c with $m(0) = 4((c-(p_0+1))^2 + 4p_0(1-p_0) > 0$ and $m(\frac{p_0-c+1}{(2-c)p_0}) = \frac{1}{(c-2)^2}(c+3p_0-5)^2\left(p_0-(c-1)^2\right) > 0$, we have $m(\delta_c) > 0$ for all $\delta_c \in (0, \frac{p_0-c+1}{(2-c)p_0})$. These results imply $h_1(v^*) - h_2(v^*) > 0$ for all $\delta_c \in (0,1]$. □

SA.2 Additional Illustration for Results in Section 5

Figure 1 (left) illustrates the effects shown in Proposition 2. For high values of δ, the profit difference is positive for any δ_c, as in Proposition 2(ii). For intermediate levels of δ, the increasing force may be so strong for very strategic consumers (high δ_c), such that very informative consumer reviews benefits the firm again. In that case, the firm does not only benefit from consumer reviews when they are non-strategic, but also when they are very strategic.

Figure 1 (right) illustrates the impact of increasing the firm's patience δ on the consumer surplus (for a given δ_c). One would similarly expect that the more forward-looking the firm is (higher δ), the lower the consumer surplus. The gain that the consumers obtain from reviews may, however, increase when the firm becomes more patient. To see why, first note from definition of consumer

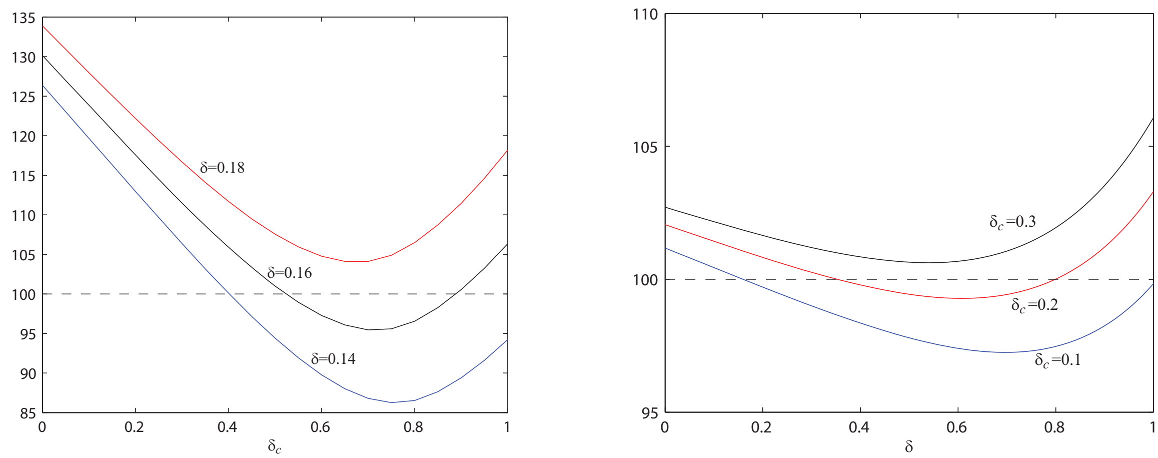

Figure 1: $\frac{\Pi^*}{\Pi^o}$ (in %) as a function of δ_c: $p_0 = 0.95$, $c = 1.9$ and $\tau = 100$ (left), and $\frac{C^*}{C^o}$ (in %) as a function of δ: $p_0 = 0.93$, $c = 1.46$, and $\tau = 10$ (right).

surplus in equation (11) that the consumer surplus is affected by firm's patience δ indirectly through the market segmentation v. With highly-informative reviews, a very patient firm sells much smaller amount in the first period than without reviews. Thus, the firm's action in the first period, and consequently also consumer surplus, are less sensitive to δ when reviews are available. Therefore, the consumers' gain from reviews may increase in firm's patience level δ, especially when δ is high. Consequently, consumers may gain from reviews in both extremes: when the firm is either very myopic (low δ), or very patient (high δ).

The numerical illustration in Figure 2 (left) shows that the result in Proposition 2 (iii) holds in general: when the firm and the consumers are equally patient $\delta = \delta_c$, the firm gains from informative reviews. Figure 2 (right) illustrates that, in those cases, the consumers are also better off by reviews.

In Figure 3 we illustrate the impact of information precision on the consumer surplus. Similar to the firm, consumers are not always better off with more precise reviews (higher τ). For low levels of consumer patience (low δ_c) and high firm's patience (high δ_c), very precise reviews (high τ) decrease consumer surplus: although social learning helps consumers better assess the product, it also allows the firm to make more informed pricing decisions and to extract more consumer surplus. Thus, consumers may get worse off due to reviews, or prefer less precise reviews. However, increasing precision of very imprecise reports (low τ) increases the consumer surplus, consistent with Proposition 4. The intuition is similar to the one for the firm. The firm enjoys the option value of adjusting price according to reviews and all other effects are smaller. The price adjustment,

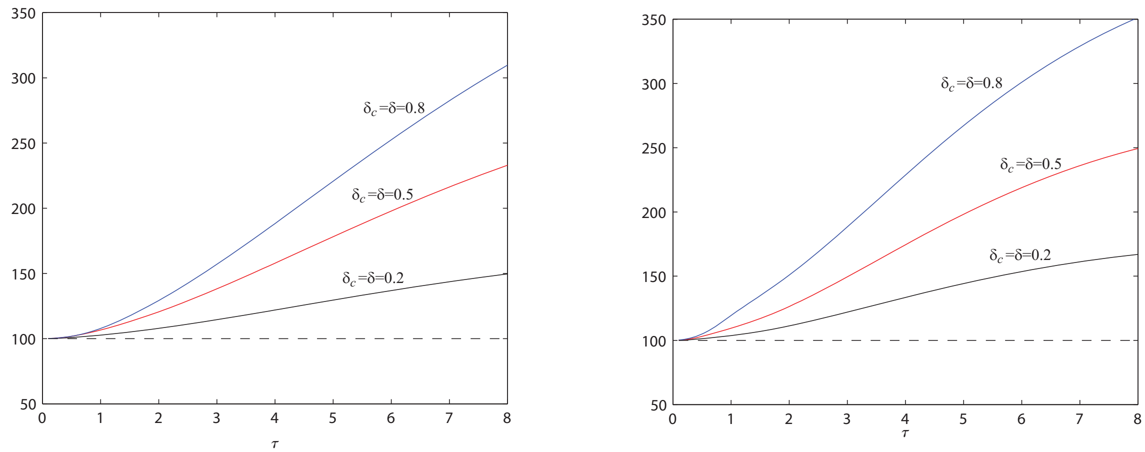

Figure 2: $\frac{\Pi^*}{\Pi^o}$ (in %, left) and $\frac{C^*}{C^o}$ (in %, right) as functions of τ: $p_0 = 0.5$, $c = 1.3$ and $\delta = \delta_c$.

however, is smaller compared to quality update, leaving some benefits from the revealed information about quality to consumers.

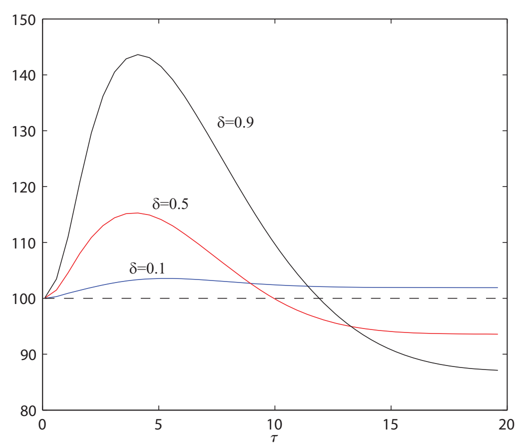

Figure 3: $\frac{C^*}{C^o}$ (in %) as functions of τ: $p_0 = 0.5$, $c = 1.3$, and $\delta_c = 0.05$.

SB Details for Extensions in Section 6

SB.1 Capacity Constraint and Quick Response

Consider the case where the firm decides capacity available in the two periods, K_1 and K_2. Unit cost for building capacity is k, and when the firm serves each consumer, there is an additional marginal cost c. Due to lead time in capacity development, the firm needs to determine K_1 and K_2 in the first period, but the capacity cost is incurred in the period during which the capacity is utilized. Since for given prices the firm knows perfectly about first-period demand, K_1 is exactly equal to first-period demand $1 - \overline{v}$. K_2, on the other hand, is determined with uncertainty about the posterior belief in period 2. The only constraint on K_2 is that it does not exceed the remaining market size $\overline{v}$. In addition to the capacity decisions, the seller has an option to acquire emergency supply in the second period after observing reviews, at a higher marginal cost $c_2 > c$. A lower c_2 represents a better capability of quick response.

Assume that consumers cannot observe the firm's quantity decision and need to form rational expectation about it. Note that, since the second-period price is determined after the review reports are posted (i.e., after the uncertainty about posterior is resolved), stock-out never occurs in the second period. Nevertheless, the capacity decisions make an impact on consumers' waiting incentive indirectly through affecting the second-period prices.

In what follows, we formulate the problem and start from the second period.

Second-period pricing: let $\tilde{p} = p(\tilde{G}, 1-\overline{v})$ be the pre-posterior and p be its realization. Similarly as in the basic model, the firm chooses second-period market segmentation $\underline{v}$ as below:

$$\Pi_2^*(p, \overline{v}, K_2) = \max_{\underline{v}\in[0,\overline{v}]} (\underline{v} + p - c) \min[\overline{v} - \underline{v}, K_2] + (\underline{v} + p - c_2) \max[\overline{v} - \underline{v} - K_2, 0] \tag{SB.1}$$

The optimal $\underline{v}$ is characterized in the following lemma.

Lemma SB.1. *The optimal $\underline{v}$ is given as below:*

- *if* $2K_2 + c - \overline{v} > 1$, $\underline{v}^* = \min\left(\overline{v}, \frac{\overline{v}-p+c}{2}\right)$.
- *if* $2K_2 + c - \overline{v} \leq 1$ *and* $2K_2 + c_2 - \overline{v} > 1$, $\underline{v}^* = \max\left[\overline{v} - K_2, \min\left(\overline{v}, \frac{\overline{v}-p+c}{2}\right)\right]$.
- *if* $2K_2 + c_2 - \overline{v} \leq 1$, $\underline{v}^* = \min\left\{\max\left[\min\left(\overline{v}, \frac{\overline{v}-p+c}{2}\right), \overline{v} - K_2\right], \frac{\overline{v}-p+c_2}{2}\right\}$.

The corresponding optimal second-period profit is

- *if* $2K_2 + c - \overline{v} > 1$, $\Pi_2^*(p, \overline{v}, K_2) = \left(\frac{(\overline{v}+p-c)^+}{2}\right)^2$.
- *if* $2K_2 + c - \overline{v} \leq 1$ *and* $2K_2 + c_2 - \overline{v} > 1$,

$$\Pi_2^*(p, \overline{v}, K_2) = \begin{cases} (\overline{v} - K_2 + p - c)K_2 & \text{if } p > 2K_2 + c - \overline{v} \\ \left(\frac{\overline{v}+p-c}{2}\right)^2, & \text{if } c - \overline{v} \leq p \leq 2K_2 + c - \overline{v} \\ 0, & \text{if } p < c - \overline{v} \end{cases}$$

Further, $\frac{\partial}{\partial K_2} \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) = \int_{c-v+2K_2}^1 (p - \{c - v + 2K_2\}) d\Phi(p, v)$.

- *if* $2K_2 + c_2 - \overline{v} \leq 1$,

$$\Pi_2^*(p, \overline{v}, K_2) = \begin{cases} \left(\frac{\overline{v}+p-c_2}{2}\right)^2 + K_2(c_2 - c), & \text{if } p > 2K_2 + c_2 - \overline{v} \\ (\overline{v} - K_2 + p - c)K_2 & \text{if } 2K_2 + c - \overline{v} < p \leq 2K_2 + c_2 - \overline{v} \\ \left(\frac{\overline{v}+p-c}{2}\right)^2, & \text{if } c - \overline{v} \leq p \leq 2K_2 + c - \overline{v} \\ 0, & \text{if } p < c - \overline{v} \end{cases}$$

Further, $\frac{\partial}{\partial K_2} \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) = \int_{c-v+2K_2}^1 (p - \{c - v + 2K_2\}) d\Phi(p, v) - \int_{c_2-v+2K_2}^1 (p - \{c_2 - v + 2K_2\}) d\Phi(p, v)$.

In equilibrium, the firm maximizes $\int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) - kK_2$ for a given v. It is clear that $\int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v)$ is concave and increases in K_2. Also, since $\int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v)$ is a constant for $2K_2 + c - \overline{v} \leq 1$, to find the optimal K_2 it suffices to consider the case $2K_2 + c - \overline{v} \leq 1$. Furthermore, when K_2 satisfies $2K_2 + c_2 - \overline{v} = 1$,

$$\frac{\partial}{\partial K_2} \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) = \int_{c+1-c_2}^1 (p - \{c + 1 - c_2\}) d\Phi(p, v)$$

Thus, we have the following result (proof omitted).

Lemma SB.2. *For a given* v*, the firm's optimal capacity to build for the second period,* $K_2^*(v)$*, is as follow.*

- *if* $\int_{c-v}^1 (p - \{c - v\}) d\Phi(p, v) - \int_{c_2-v}^1 (p - \{c_2 - v\}) d\Phi(p, v) < k$*, then* $K_2^*(v) = 0$.
- *if* $\int_{c-v}^1 (p - \{c - v\}) d\Phi(p, v) - \int_{c_2-v}^1 (p - \{c_2 - v\}) d\Phi(p, v) \geq k$
  - *if* $\int_{c+1-c_2}^1 (p - \{c + 1 - c_2\}) d\Phi(p, v) < k$*, then* $K_2^*(v) \leq \frac{1-c_2+v}{2}$ *and satisfies* $\int_{c-v+2K_2}^1 (p - \{c - v + 2K_2\}) d\Phi(p, v) - \int_{c_2-v+2K_2}^1 (p - \{c_2 - v + 2K_2\}) d\Phi(p, v) = k$.
  - *if* $\int_{c+1-c_2}^1 (p - \{c + 1 - c_2\}) d\Phi(p, v) \geq k$*, then* $K_2^*(v) \in [\frac{1-c_2+v}{2}, \frac{1-c+v}{2}]$ *and satisfies* $\int_{c-v+2K_2}^1 (p - \{c - v + 2K_2\}) d\Phi(p, v) = k$.

Consumers' buy-or-wait tradeoff: For given P_1 and a conjectured $\tilde{K}_2$, consumer v's expected utility of waiting is $\delta_c(v - \underline{v}^*)^+$. Analogous to equation (8) in the paper, we have

$$\overline{v}(P_1, \tilde{K}_2) = \max\left\{v \in [0,1] : p_0 + v - P_1 \leq \delta_c \mathrm{E}_{\tilde{p}}\left[\max\left\{\min[\tilde{K}_2, \max(\tfrac{v+\tilde{p}-c}{2}, 0)], \tfrac{v+\tilde{p}-c_2}{2}\right\}\right]\right\}$$

Rational expectation equilibrium: To derive the equilibrium between the firm and the consumers, we follow the framework of rational expectation equilibrium as in Su and Zhang (2008). Specifically, let

$$\pi(P_1, K_2, v) = (P_1 - c)(1 - v) + \delta \mathrm{E}_{\tilde{G}}[\Pi_2^*(\tilde{G}, v, K_2)] - (1 - v)k - \delta K_2 k$$

The firm first announces P_1, and then the firm and consumers make simultaneous move: the firm conjectures about the cutoff consumer type $\tilde{v}$ and maximizes $\pi(P_1, K_2, \tilde{v})$. Let the optimal K_2 be $K_2^* = K_2^{opt}(P_1, \tilde{v})$; the consumers conjectures about the quantity $\tilde{K_2}$ and the equilibrium strategy is characterized by $v^* = \overline{v}(P_1, \tilde{K_2})$. In equilibrium $v^* = \tilde{v}$ and $Q^* = \tilde{Q}$. We then have: $K_2^* = K_2^{opt}(P_1, v^*)$ and $v^* = \overline{v}(P_1, K_2^*)$. From these two equations, we can solve for $K_2^*(P_1)$ and $v^*(P_1)$. The firm then chooses P_1 to maximize $\pi(P_1, K_2^*(P_1), v^*(P_1))$, where $(K_2^*(P_1), v^*(P_1))$ satisfies

$$\begin{cases} K_2^* = \max\{K_2 \geq 0 : \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) - kK_2\} \\ v^* = \max\left\{v \in [0,1] : P_1 \geq p_0 + v - \delta_c \mathrm{E}_{\tilde{p}}\left[\max\left\{\min[\tilde{K_2}, \max(\frac{v+\tilde{p}-c}{2}, 0)], \frac{v+\tilde{p}-c_2}{2}\right\}\right]\right\} \end{cases}$$

Lemma SB.3. *For any given P_1, there exists a unique rational expectation equilibrium characterized by equation (SB.2). Specifically, there exist two thresholds, $\underline{P}_1'$ and $\overline{P}_1'$, such that $v^*(P_1) = 0$ if $P_1 < \underline{P}_1'$, $v^*(P_1) = 1$ if $P_1 > \overline{P}_1'$, and for $P_1 \in [\underline{P}_1', \overline{P}_1']$, $v^*(P_1)$ and $K^*(P_1)$ satisfy*

$$\begin{cases} K_2^* = \max\left\{K_2 \geq 0 : \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) - kK_2\right\} \\ P_1 = p_0 + v^* - \frac{\delta_c}{2} \max\left[\int_0^1 (v^* + p - c)^+ d\Phi(p, v^*) - k, \int_0^1 (v^* + p - c_2)^+ d\Phi(p, v^*)\right] \end{cases} \tag{SB.2}$$

Similarly as in the basic model, we apply a change of variable, from P_1 to v. Then:

$$\Pi^* = \max_v \tilde{\Pi}(v, K_2^*(v)) = (1 - v)(p_0 + v - c - k) - \delta_c C_W(v, K_2^*(v)) + \delta \Pi_S(v, K_2^*(v))$$

where

$$C_W(v, K_2^*(v)) = \frac{1-v}{2} \max\left[\int_0^1 (v + p - c)^+ d\Phi(p, v) - k, \int_0^1 (v + p - c_2)^+ d\Phi(p, v)\right];$$
$$\Pi_S(v, K_2^*(v)) = \int_0^1 \Pi_2^*(p, v, K_2^*(v)) d\Phi(p, v) - kK_2^*(v).$$

where $\Pi_2^*(p, v, K_2)$ and $K_2^*(v)$ are given in Lemmas SB.1 and SB.2, respectively.

Note that the no-review model can be formulated as follows.

$$\max_{v} \ \tilde{\Pi}^o(v, K_2^o(v)) = (1-v)(p_0+v-c-k) - \delta_c C_W^o(v, K_2^o(v)) + \delta \Pi_S^o(v, K_2^o(v))$$

s.t.

$$K_2^o(v) = \max\left\{K_2 \geq 0 : \Pi_2^*(p_0, v, K_2) - kK_2\right\}$$

where

$$C_W^o(v, K_2^o(v)) = \frac{1-v}{2} \max\left[(v+p_0-c)^+ - k, (v+p_0-c_2)^+\right];$$

$$\Pi_S^o(v, K_2^o(v)) = \Pi_2^*(p_0, v, K_2^o(v)) - kK_2^o(v).$$

Similarly as in the base model, define the firm's option value and the consumers' option value by $O(v) = \Pi_S(v, K_2^*(v)) - \Pi_S^o(v, K_2^o(v))$ and $O_c(v) = C_W(v, K_2^*(v)) - C_W^o(v, K_2^o(v))$, respectively.

Lemma SB.4. *$O(v) \geq 0$ and $O_c(v) \geq 0$.*

Lemma SB.4 implies that, same as in the original model, the impact of reviews on the firm's profit is determined by the option values of the firm and of the consumes. In particular, it is affected by the relative patience of the two parties. If the firm is patient ($\delta > 0$) and the consumers are very impatient ($\delta_c = 0$), then reviews make the firm better off. If the firm is very impatient ($\delta = 0$) and the consumers are patient ($\delta_c > 0$), then the firm is worse off due to reviews. Furthermore, for very informative reviews ($\tau \to \infty$), Figure 4 shows that a sufficiently patient firm always gains from reviews, while a sufficiently impatient firm gains from reviews if consumers are equally or more impatient. Note that Figure 4 also illustrates that the region where the firm gets worse off by reviews expands in c_2. On the other hand, for weakly informative reviews (τ very small), given $\delta, \delta_c \in (0,1)$, the value of reviews is always nonnegative for the firm. See Figure 5.

Next, we evaluate the impact of quick response by examining the effect of c_2.

Lemma SB.5. *If $\delta > 0$ and $\delta_c = 0$, for any $k > 0$, Π^* decreases in c_2, i.e., the firm is better off by an improvement in its quick-response capability. Further, Π^* increases in τ, i.e., the firm's profit increases as the reviews become more accurate.*

Lemma SB.6. *If $\delta = 0$ and $\delta_c > 0$, for any $k > 0$, Π^* increases in c_2, i.e., the firm is worse off by an improvement in its quick-response capability. Further, $\Pi^*|_{c_2=\infty} - \Pi^*|_{c_2>0}$ increases in τ, i.e., the firm's profit loss due to quick response increases as the reviews become more accurate. Moreover, $\Pi^*|_{\delta_c=0} - \Pi^*|_{\delta_c>0}$ decreases in c_2 and increases in τ, i.e., the firm's profit loss due to*

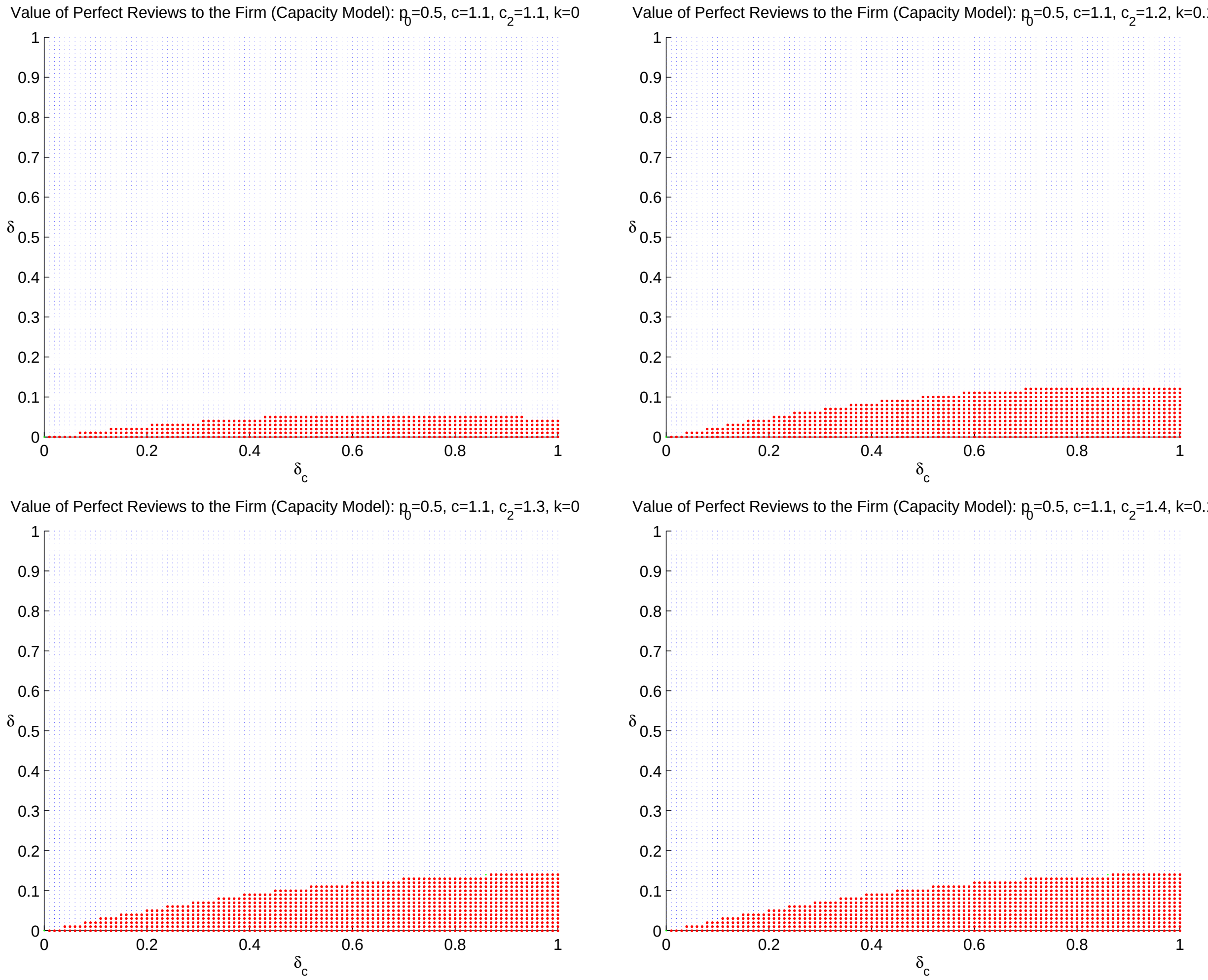

Figure 4: Regions where the firm is worse off by perfectly-informative reviews (the red regions) as functions of δ and δ_c for the infinite-horizon model with quick response.

strategic consumers increases as the firm's quick-response capability is improved or as the reviews become more accurate.

Lemma SB.6 proves that an impatient firm ($\delta = 0$) loses more due to strategic consumers when quick response capability is improved. As shown in Figure 6, for a patient firm ($\delta > 0$), its loss due to strategic consumers $\Pi^*|_{\delta_c=0} - \Pi^*|_{\delta_c>0}$ can also be higher when it has a better quick-response capability or when the reviews are informative. Furthermore, the firm's profit loss decreases in capacity cost k: as the capacity cost k increases, the firm builds less capacity for the second period. This increases second-period price and thus mitigates consumers' waiting incentive.

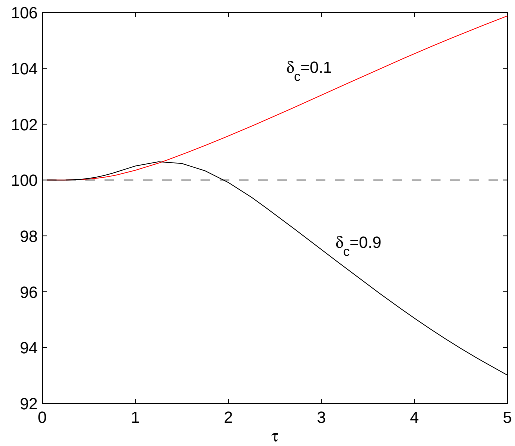

Figure 5: $\frac{\Pi^*}{\Pi^o}$ as a function of τ: $p_0 = 0.5, c = 1.1, c_2 = 1.2, k = 0.05, \delta = 0.1$.

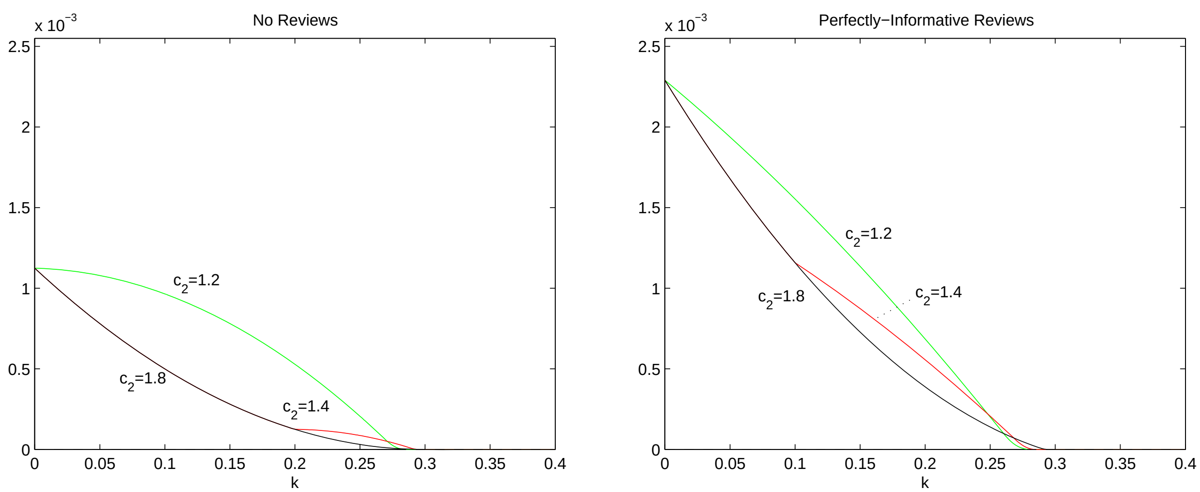

Figure 6: Difference of profits under strategic consumers $\delta_c = 0.1$ and under myopic consumers $\delta_c = 0$: no information $\tau = 0$ (left) and perfect information $\tau \to \infty$ (right), $p_0 = 0.5$, $c = 1.2$, $\delta = 0.1$.

SB.2 Additional Sales Periods

In this subsection we extend our two period model to an infinite horizon model. As a starting point, we formulate a general multi-period model in Section SB.2.1, based on which we examine the infinite-horizon problem with uninformative reviews, perfect reviews, and imperfect reviews in Sections SB.2.2, SB.2.3, and SB.2.4, respectively.

SB.2.1 General T-period model

Let the price at time $n \in \{1, 2, ..., T\}$ be P_n and the volume of consumers that buys in period n is $v_{n-1} - v_n$ and $v_0 = 1$. p_{n-1} is the prior about the quality in period n. Let $P^*_{n+1}(p_n, v_n)$ be the optimal price in period $n+1$ when the posterior is p_n and the consumer with the highest willingness valuation is v_n. Given v_{n-1} and prior p_{n-1}, when the price is P_n, the cut-off type is v_n, where:

$$\underbrace{p_{n-1} + v_n - P_n}_{\text{Buy in period } n} = \underbrace{\delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v_n - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}]}_{\text{Wait until period } n+1}.$$

That is: the expected utility from buying in period n for consumer type v_n is equal to the discounted, expected utility from delaying the purchasing decision to period $n+1$. It can be shown that it is an equilibrium for all consumers to follow the strategy of buying if $v > v_n$ and waiting if $v \leq v_n$.[1]

Proposition SB.1. *We can write the firm's T-period problem as follows: For $n = 1, ..., T-1$:*

$$\Pi_n(p_{n-1}, v_{n-1}) = \max_{v_n \leqslant v_{n-1}} (v_{n-1} - v_n)(p_{n-1} + v_n - c - \delta_c C_n(p_{n-1}, v_{n-1}, v_n)) + \delta \mathbb{E}_{\tilde{p}}[\Pi_{n+1}(\tilde{p}, v_n) | v_{n-1} - v_n, p_{n-1}]$$

with solution $v^*_n(p_{n-1}, v_{n-1})$ and

$$C_n(p_{n-1}, v_n, v_{n-1}) = \mathbb{E}_{\tilde{p}}[\{\delta_c C_{n+1}(\tilde{p}, v^*_{n+1}(\tilde{p}, v_n), v_n) + v_n - v^*_{n+1}(\tilde{p}, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}]$$

and in period T:

$$\hat{\Pi}_T(p_{T-1}, v_T, v_{T-1}) = (v_{T-1} - v_T)(p_{T-1} + v_T - c) \text{ and } C_T(p_{T-1}, v_T, v_{T-1}) = 0,$$

for all $p_{n-1} \in \mathcal{P}$ and $v_{n-1} \in \mathcal{V}$, from which the prices can be computed as follows:

$$P^*_n(p_{n-1}, v_{n-1}) = v^*_n(p_{n-1}, v_{n-1}) + p_{n-1} - \delta_c C_n(p_{n-1}, v^*_n(p_{n-1}, v_{n-1}), v_{n-1}).$$

[1] It suffices to show that, given all the other consumers follow the equilibrium (i.e., buy if $v > v_n$ and wait if $v \leq v_n$), then an arbitrarily-chosen consumer v has no incentive to deviate from the strategy dictated by the equilibrium. In other words, for $v > v_n$, we need to ensure

$$p_{n-1} + v - P_n > \delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}],$$

and for $v < v_n$, we need to have

$$p_{n-1} + v - P_n \leq \delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}],$$

These two conditions, however, immediately follow from the fact that

$$p_{n-1} + v - P_n - \delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}]$$

monotonically increases in v.

When $T \to +\infty$, we have

$$\Pi(p, v) = \max_{v' \leqslant v} (v - v')(p + v' - c - \delta_c C(p, v', v)) + \delta \mathbb{E}_{\tilde{p}}[\Pi(\tilde{p}, v')|v - v', p] \tag{SB.3}$$

$$\text{with solution } v^*(p, v) \text{ and} \tag{SB.4}$$

$$C(p, v', v) = \mathbb{E}_{\tilde{p}}[\{\delta_c C(\tilde{p}, v^*(\tilde{p}, v'), v') + v' - v^*(\tilde{p}, v')\}^+|v - v', p] \tag{SB.5}$$

SB.2.2 Infinite horizon with uninformative reviews; $\tau = 0$

When $T \to +\infty$ and $\tau = 0$, by equations (SB.3) to (SB.5), we have:

$$\Pi^o(v) = \max_{v' \leq v} (v - v')(p_0 + v' - \delta_c C^o(v') - c) + \delta \Pi^o(v') \text{ with solution } v^o(v) \text{ and } C^o(v) = \delta_c C^o(v^o(v)) + v - v^o(v).$$

and the price is $P^o(v) = p_0 + v^o(v) - \delta_c C^o(v^o(v)) = p_0 + v - C^o(v)$. The lemma below proves the expressions for the functions.

Lemma SB.7. *When $T \to +\infty$ and $\tau = 0$,*

$$v^o(v) = \alpha v + (1 - \alpha)(c - p_0),\ P^o(v) = (1 - \phi)(p_0 + v) + \phi c, C^o(v) = \phi(p_0 + v - c), \text{ and } \Pi^o(v) = \beta(p_0 + v - c)^2,$$

where $\alpha = \frac{1-\sqrt{1-\delta}}{\delta} = \frac{1}{1+\sqrt{1-\delta}}$, $\beta = \frac{1}{1-\sqrt{1-\delta}} \frac{(\frac{1-\sqrt{1-\delta}}{\delta} - \frac{1}{2})(1-\delta_c)}{1-\delta_c \frac{1-\sqrt{1-\delta}}{\delta}}$, *and* $\phi = \frac{1}{1+(1-\delta_c)\frac{1}{\sqrt{1-\delta}}}$.

Corollary SB.1. *When $T \to +\infty$ and $\tau = 0$, both the price path and the per-period sales decline over time.*

SB.2.3 Infinite horizon with perfect reviews; $\tau = +\infty$

When $\tau = +\infty$, for any $H(\cdot)$, we have

$$\mathbb{E}_{\tilde{p}}[H(\tilde{p})|v - v', p] = \begin{cases} pH(1) + (1 - p)H(0), & v' < v \\ H(p), & v' = v \end{cases}$$

Basically, the firm only has two choices in the initial period; if the firm does not sell, the market and posterior remains the same. If the firm sells, the posterior becomes either one or zero, depending on the quality and we are back in the case of no information. In particular, when the quality is revealed to be low, the firm withdraws its product from the market and therefore, both the profit and the waiting-consumer's surplus are zero. Thus, by Proposition SB.1, we have

$$C^*(p, v', v) = \begin{cases} p\phi\{1 + v' - c\}^+, & v' < v \\ \phi\{p + v' - c\}^+, & v' = v. \end{cases}$$

Also:

$$\mathbb{E}_{\tilde{p}}[\Pi^*(\tilde{p}, v')|v - v', p] = \begin{cases} p\beta(\{1 + v' - c\}^+)^2, & v' < v \\ \beta(\{p + v' - c\}^+)^2, & v' = v. \end{cases}$$

Hence, we have $\Pi^*(p_0, 1) = \max_{v'} \tilde{\Pi}(p_0, 1, v')$, where

$$\begin{aligned} \tilde{\Pi}(p_0, 1, v') =& (1 - v')(p_0 + v' - c - \delta_c C^*(p_0, v', 1) + \delta \mathbb{E}_{\tilde{p}}[\Pi^*(\tilde{p}, v')|1 - v', p] \\ =& \begin{cases} \tilde{\Pi}^1(p_0, 1, v') = (1 - v')(p_0 + v' - c - \delta_c p_0 \phi\{1 + v' - c\}^+) + \delta p_0 \beta(\{1 + v' - c\}^+)^2 & \text{if } v' < 1 \\ \tilde{\Pi}^0(p_0, 1, v') = \beta(\{p_0 + v' - c\}^+)^2, & \text{if } v' = 1 \end{cases} \end{aligned}$$

Lemma SB.8. *When $T \to +\infty$ and $\tau \to +\infty$, the firm's optimal initial market segment v^* is given by*

$$v^* = \min\left(\frac{(1 - p_0(\phi\delta_c + 2\beta\delta))\, c + (p_0(2\beta\delta - 1) + 1)}{2(1 - p_0(\delta\beta + \delta_c\phi))}, 1\right)$$

where β and ϕ are as defined in Lemma SB.7.

Note that when $\delta_c = \delta \to 1$, $\beta = 0$ and $\phi = 1$, and $v^* = 1$ for all $c > 1$. This implies that the firm's first period profit is zero ($v^* = 1$), as well as the profits in all subsequent periods ($\beta = 0$), which extends Stokey's result to the case with information.

Next we show that under perfect reviews, the firm's and the consumers' option values are both nonnegative.

Lemma SB.9. *When $T \to +\infty$, for $\tau \to +\infty$, the firm's and the consumers' option values are nonnegative.*

Now, consider the initial price, P_1 and the price in period 2, conditional on the firm being of high quality:

$$t = 0 : P_1 = p_0 + v^* - \delta p_0 \phi(1 + v^* - c) \text{ and } t = 1 : P_2 = 1 + v^* - \phi(1 + v^* - c)$$

Recall that in the case without reviews, both the price and the per-period sales decrease over time (Corollary SB.1). With perfect reviews, we can state:

Proposition SB.2. *When $T \to +\infty$ and $\tau \to +\infty$, for $\delta_c = \delta$, conditional on the quality being high, the price may first increase and then decrease. Specifically, when $c > \frac{X^2 p_0(1-p_0)+(1-p_0)^2+X(1-p_0)(3p_0-2)+X^3 p_0(3p_0-1)}{(X^2 p_0+1-p_0)(Xp_0+1-p_0)}$, $P_1 < P_2 > P_3 > P_4 > ...$, where $X = \sqrt{1-\delta^2}$.*

Proposition SB.3. *When $T \to +\infty$ and $\tau \to +\infty$, conditional on the quality being high, when unit cost c is sufficiently high, the per-period sales reaches a peak in the second period.*

Now we compare profits:

	Two Period ($t = 2$)	Infinite Horizon ($t = \infty$)
Uninformative review ($\tau = 0$)	Π_2^o	Π_∞^o
Perfectly-informative review ($\tau \to \infty$)	Π_2^*	Π_∞^*

Table 1: Firm's profits in different information regime and under different selling horizon

Proposition SB.4. *(i) For $\tau \to \infty$, $\Pi_\infty^o < \Pi_2^o$ and $\Pi_\infty^* < \Pi_2^*$ if and only if $1 - \delta_c < \sqrt{1-\delta}$.*
(ii) For $\tau \to \infty$, $\Pi_\infty^* > \Pi_\infty^o$ when $\delta_c = \delta$.

For general values of δ and δ_c, the value of reviews is illustrated in the following figure. Figure 7 confirms a main result that, for very informative reviews ($\tau \to \infty$), a sufficiently patient firm always gains from reviews, while a sufficiently impatient firm gains from reviews if consumers are equally or more impatient. Furthermore, as one can observe from the figure, the firm is less likely to benefit from reviews in the infinite selling horizon, compared to that in the two-period horizon.

SB.2.4 Infinite horizon with imperfect reviews; $0 < \tau < +\infty$

We now consider the infinite-horizon model with imperfect reviews $0 < \tau < +\infty$. In order to obtain insight in this case, we consider the following discretized system in which there are N consumers indexed by $i \in \{1, 2, ..., N\}$ with respective valuation $v_i \in \mathcal{V} = \{1/N, 2/N, ..., (N-1)/N, N/N\}$. Every consumer i observes the following binary signal $\tilde{G}_i \in \{G_u, G_d\}$:

$$\Pr(\tilde{G}_i = G_u | \theta = 1) = q_1 \text{ and } \Pr(\tilde{G}_i = G_u | \theta = 0) = q_0.$$

We impose the following restrictions on the signal:

$$G_u q_1 + G_d(1 - q_1) = 1/N \text{ and } G_u q_0 + G_d(1 - q_0) = 0,$$
$$(G_u - 1/N)^2 q_1 + (G_d - 1/N)^2 (1 - q_1) = \sigma^2/N \text{ and } (G_u - 0)^2 q_0 + (G_d - 0)^2 (1 - q_0) = \sigma^2/N.$$

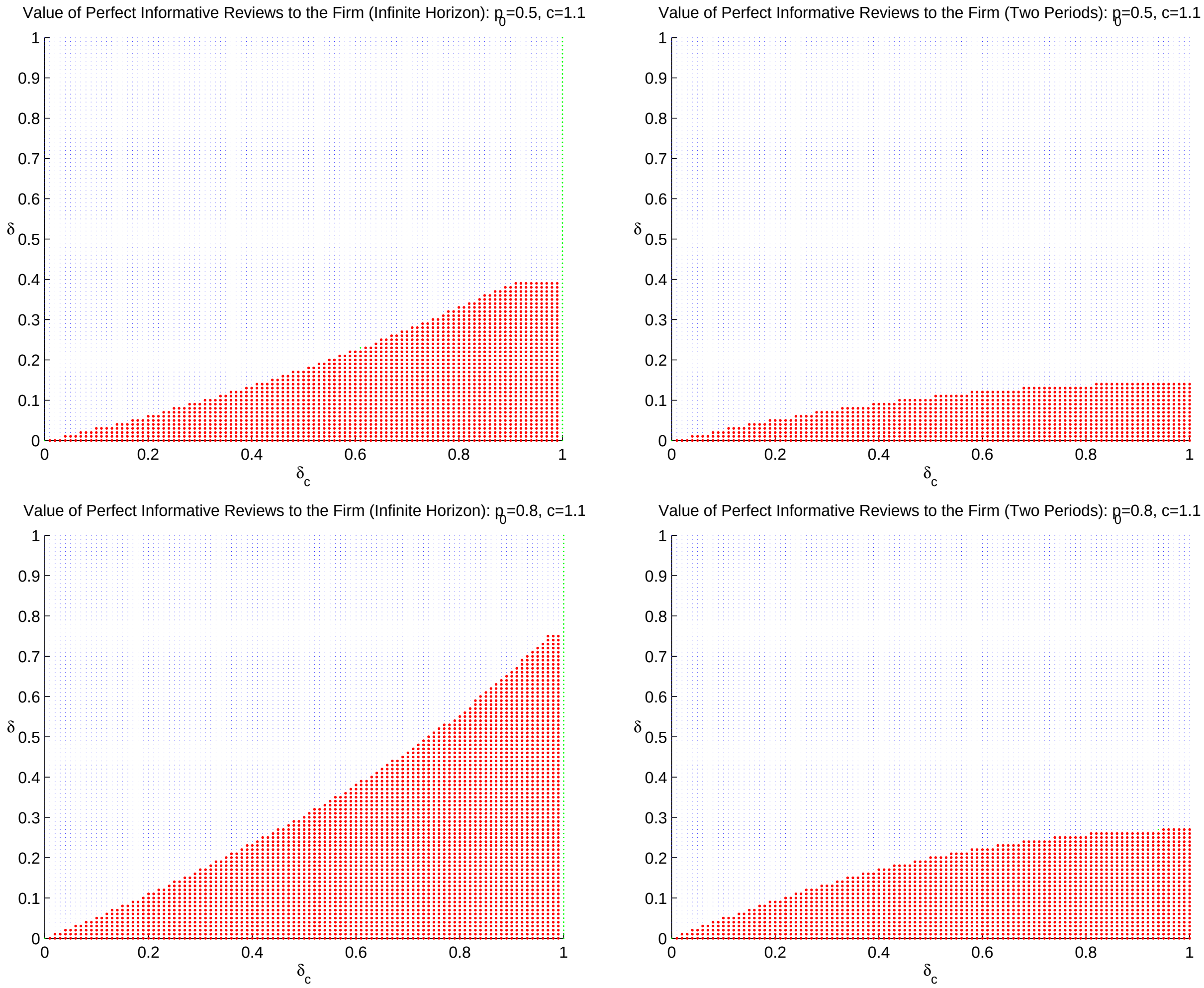

Figure 7: Regions where the firm is worse off by perfectly-informative reviews (the red regions) as functions of δ and δ_c for the infinite-horizon model and for the two-period model.

For any given N and σ, we can solve for (q_0, q_1, G_d, G_u) and obtain:

$$q_0 = \frac{2N\sigma^2 + \frac{1}{2} - \frac{1}{2}\sqrt{4N\sigma^2+1}}{4N\sigma^2+1} \text{ and } q_1 = \frac{2N\sigma^2 + \frac{1}{2} + \frac{1}{2}\sqrt{4N\sigma^2+1}}{4N\sigma^2+1},$$
$$G_d = \frac{1}{2}\frac{1}{N}(1-\sqrt{4N\sigma^2+1})) \text{ and } G_u = \frac{1}{2}\frac{1}{N}(1+\sqrt{4N\sigma^2+1}).$$

Notice that $q_0 = 1 - q_1$, so, we let $q_0 = q$ and $q_1 = 1 - q$. Also, let $G = \sum_{i=1}^{X} G_i$. This random number has as domain $\mathcal{G} = \{uG_u + (X-u)G_d, u = 0, 1, ..., N\}$, where u is the number of 'u'-signals. Let $X = xN$, then, it is easy to see that $G|(x,\theta)$ has mean $x\theta$ and standard deviation $\sqrt{x}\sigma$. This is because the mean of $G_i|\theta$ is θ/N and G is the sum of xN such signals, so, its mean is $xN \times \theta/N = x\theta$. Furthermore, the variance is $xN \times \sigma^2/N = x\sigma^2$, thus, the standard deviation is $\sqrt{x}\sigma$. For large N, $G|(x,\theta)$ will be approximately normally distributed. Formally, the distribution of $G|(x,\theta)$ is as follows:

$$\begin{aligned}\text{for all } g \in \mathcal{G} : \Pr(\tilde{G} = G|X,\theta) &= \Pr(\tilde{u}G_u + (X - \tilde{u})G_d = G|x,\theta)\\ &= \Pr(\tilde{u} = \frac{G - XG_d}{G_u - G_d}|X,\theta)\end{aligned}$$

where $\tilde{u}$ is the realization of the number of 'u'-signals, which is binomially distributed with probability p_θ. Notice that $\frac{G-XG_d}{G_u-G_d} \in \mathbb{N}$ when $G \in \mathcal{G}$. Thus: we can write

$$\Pr(\tilde{G} = G|X,\theta) = \binom{X}{\frac{G-XG_d}{G_u-G_d}} q_\theta^{\frac{G-XG_d}{G_u-G_d}} (1-q_\theta)^{\frac{XG_u-G}{G_u-G_d}}$$

By the central limit theorem, $\tilde{G}$ will be approximately normally distributed.

Posterior belief: When observing signal $G \in \mathcal{G}$ from X reviews in total, the posterior is

$$\begin{aligned}p'(G,X,p_0) &= \frac{p_0 q_1^{\frac{G-XG_d}{G_u-G_d}} (1-q_1)^{\frac{XG_u-G}{G_u-G_d}}}{p_0 q_1^{\frac{G-XG_d}{G_u-G_d}} (1-q_1)^{\frac{XG_u-G}{G_u-G_d}} + (1-p_0) q_0^{\frac{G-XG_d}{G_u-G_d}} (1-q_0)^{\frac{XG_u-G}{G_u-G_d}}}\\ &\text{(as } q_1 = 1-q \text{ and } q_0 = q)\\ &= \frac{p_0}{p_0 + (1-p_0)(\frac{1-q}{q})^{X-2\frac{G-XG_d}{G_u-G_d}}}\end{aligned}$$

As for $G \in \mathcal{G}$, we have that $\frac{G-XG_d}{G_u-G_d} \in \mathbb{Z}$, starting with p_0, we can define roster $\mathcal{P} = \{\hat{p}(-N), ..., \hat{p}(N)\}$ with

$$\hat{p}(j) = \frac{p_0}{p_0 + (1-p_0)(\frac{1-q}{q})^j}.$$

That is: when $G = XG_u$, all signals are 'up', then the posterior is $\frac{p_0}{p_0+(1-p_0)(\frac{1-q}{q})^X}$ and when $G = XG_d$, i.e. all signals are 'down', then, the posterior is $\frac{p_0}{p_0+(1-p_0)(\frac{1-q}{q})^{-X}}$. For any other G, the posterior is in between $\hat{p}(-N)$ and $\hat{p}(N)$.

Thus, at the nth price change, assuming that the state at the $n-1$st time change is j_{n-1}, we have the following posterior:

$$\begin{aligned}p'(G_n, X_n, \hat{p}(j_{n-1})) &= \frac{\hat{p}(j_{n-1})}{\hat{p}(j_{n-1}) + (1-\hat{p}(j_{n-1}))(\frac{1-q}{q})^{X_n - 2\frac{G_n - X_n G_d}{G_u-G_d}}}\\ &= \frac{p_0}{p_0 + (1-p_0)(\frac{1-q}{q})^{j_{n-1}+X_n-2\frac{G_n-X_nG_d}{G_u-G_d}}} = \hat{p}(j_n) \in \mathcal{P}.\end{aligned}$$

The second equality is because:

$$\frac{1-\hat{p}(j_{n-1})}{\hat{p}(j_{n-1})} = \frac{1-p_0}{p_0}(\frac{1-q}{q})^{j_{n-1}}.$$

Thus, let j_{n-1} be the current state (with prior $\hat{p}(j_{n-1})$), then, when X_n consumers purchase, the next state is $j_{n-1} + X_n - 2\frac{G_n - X_n G_d}{G_u - G_d}$ (with posterior $\hat{p}(j_{n-1} + X_n - 2\frac{G_n - X_n G_d}{G_u - G_d})$). The probability distribution for G_n is given by

$$Q(G,X,p) = p\binom{X}{\frac{G-XG_d}{G_u-G_d}} q_1^{\frac{G-XG_d}{G_u-G_d}}(1-q_1)^{X-\frac{G-XG_d}{G_u-G_d}} + (1-p)\binom{X}{\frac{G-XG_d}{G_u-G_d}} q_0^{\frac{G-XG_d}{G_u-G_d}}(1-q_0)^{X-\frac{G-XG_d}{G_u-G_d}}$$

We have described how the state evolves stochastically (in a finite space).

Equilibrium: The equilibrium can be found using the following algorithm: (1) Initialize $v^0(p,v)$ with e.g. $\alpha v + (1-\alpha)(c-p)$ (from the no information case), (2) Compute $C^0(p,v',v) = \mathbb{E}_{\tilde{p}}[\{v' - v^0(\tilde{p},v')\}^+ | v - v', p]$ and solve $\max_{v' \leqslant v}(v-v')(p+v'-c-\delta_c C^0(p,v',v))$, in order to obtain $\Pi^1(p,v)$ and $v^1(p,v)$. Set $k=1$. Then (3) compute

$$C^{k+1}(p,v',v) = \mathbb{E}_{\tilde{p}}[\{\delta_c C^k(\tilde{p}, v^*(\tilde{p},v'),v') + v' - v^k(\tilde{p},v')\}^+ | v - v', p]$$

and solve

$$\Pi^{k+1}(p,v) = \max_{v' \leqslant v}(v-v')(p+v'-c-\delta_c C^{k+1}(p,v',v)) + \delta \mathbb{E}_{\tilde{p}}[\Pi^k(\tilde{p},v') | v-v', p]$$

which gives $v^{k+1}(p,v)$. Increase k by one and then go to step (3) until the policy does not change anymore. Note that $p \in \mathcal{P}$ and $v \in \mathcal{V}$, thus, the state space is of a finite dimension $(2N+1) \times N$.

With the discrete valuation model, we confirm, through numerical study, that the following main results continue to hold in the infinite-horizon model:

• value of reviews for small τ is always nonnegative, for any $\delta, \delta_c > 0$.

• For very high but finite τ, as shown in Table 2, the value of reviews is negative for $\delta = 0, \delta_c > 0$ and positive for $\delta > 0, \delta_c = 0$. Note that we have proven this result for $\tau \to \infty$.

SB.3 Asymmetric Quality Information

Now, assume that the firm acquires e.g. via consumer focus groups, information about the quality of its product in the form of a signal, s, that is imperfectly correlated with the quality and observed privately. Formally, we assume that before the initial period, the firm observes privately the

	$\delta = 0.9, \delta_c = 0$	$\delta = 0, \delta_c = 0.9$
$\tau = 2$	133.63	89.92
$\tau = 4$	173.23	95.26
$\tau = 6$	194.79	31.47
$\tau = 8$	208.16	19.70
$\tau = 10$	217.65	14.18

Table 2: $\frac{\Pi^*}{\Pi^0}$ (in %) in the infinite-horizon model: $N = 11$, $c = 1.1$, $p_0 = 0.5$

realization of a binary signal, $s \in \{G, B\}$ that is imperfectly correlated with the true quality as follows: $\Pr(s = G|\Theta = 1) = \Pr(s = B|\Theta = 0) = q \in [1/2, 1)$. After observing signal s, and before observing reviews, the firm's posterior becomes

$$p_s = \begin{cases} \frac{p_0 q}{p_0 q + (1-p_0)(1-q)}, & s = G \\ \frac{p_0(1-q)}{p_0(1-q)+(1-p_0)q}, & s = B. \end{cases}$$

Notice that for $q = 1/2$, we recover the base model. For $q > 1/2$, $p_G > p_B$. The firm will update its prior p_0 to p_s after observing signal s and its initial price will depend on p_s, which may signal to the consumers the information that the firm observed. The firm with a bad signal (B-firm) has an incentive to set a price inducing the consumers believe that it observed privately quality information that yields the highest posterior p_G. Hence, the firm with a good signal (G-firm) needs to distort its price to convince consumers about the good news. In other words, credibly conveying its private information via its price entails a cost to the firm that observes a good signal, as is typical in signaling models. We discuss this cost next.

Assume that the firm observes signal s and the consumers believe that the firm observed signal $\hat{s}$ (which may or may not be equal to s). With asymmetric information between the consumers and the firm, the pre-posterior, now becomes a function of the consumer's belief, $p_{\hat{s}}$, about the quality, which may be different from the firm's belief about the quality, p_s. We introduce the superscript f, which stands for the $f$irm's belief and the superscript c, which stands for the $c$onsumers' belief. The firm's belief about the consumer's pre-posterior $p(\tilde{G}, p_{\hat{s}}, x)$ is based on its own assessment of the product quality p_s, while the consumer's belief is based on the prior $p_{\hat{s}}$:

$$\begin{aligned} \Phi^f_s(p, p_{\hat{s}}, x) &= (1-p_s)F(\tfrac{\eta(p,p_{\hat{s}},x)}{\sqrt{x/\tau}}) + p_s F(\tfrac{\eta(p,p_{\hat{s}},x)-x}{\sqrt{x/\tau}}) \text{ and} \\ \Phi^c(p, p_{\hat{s}}, x) &= (1-p_{\hat{s}})F(\tfrac{\eta(p,p_{\hat{s}},x)}{\sqrt{x/\tau}}) + p_{\hat{s}} F(\tfrac{\eta(p,p_{\hat{s}},x)-x}{\sqrt{x/\tau}}). \end{aligned} \tag{7*}$$

In the firm's profit function, the former belief (Φ_s^f) determines the firm's assessment of its own future skimming profit, Π_S, while the latter belief (Φ^c) determines the firm's waiting costs, C_W. We can thus generalize the firm's profit in the basic model as follows:

$$\begin{cases} \Pi_{\hat{s},0}(v) &=& (1-v)(p_{\hat{s}}+v-c) \\ C_{\hat{s},W}(v) &=& \int_0^1 \frac{1}{2}(1-v)\{p+v-c\}^+ d\Phi^c(p,p_{\hat{s}},1-v) \\ \Pi_{s,S}^{\hat{s}}(v) &=& \int_0^1 \frac{1}{4}(\{p+v-c\}^+)^2 d\Phi_s^f(p,p_{\hat{s}},1-v). \end{cases} \tag{9*}$$

and let $\Pi_s^{\hat{s}}(v) = \Pi_{\hat{s},0}(v) - \delta_c C_{\hat{s},W}(v) + \delta\Pi_{s,S}^{\hat{s}}(v)$ denote the firm's expected profit when it observes a signal s and consumers believe that it observes $\hat{s}$.

Finally, as a short-cut notation, define:

$$\Pi_s = \max_{v\in[0,1]} \Pi_s^s(v) = \Pi_{s,0}(v) - \delta_c C_{s,W}(v) + \delta\Pi_{s,S}^s(v) \text{ for } s \in \{G,B\}$$

and let v_s be the solution for the problem above and P_{1s} be the corresponding initial price. That is, Π_s is the equilibrium profit if the consumers *and* firm both base their beliefs on signal s; i.e. when there is no information asymmetry between the firms and consumers, as in the base model. Since the firm's equilibrium profit increases in consumers' prior, the B-firm will have an incentive to make the consumers believe it observed signal $s = G$. Thus, in order to be credible, the G-firm will distort the price (initial segment) such that the B-firm has no incentive to imitate the price (initial segment) of the G-firm. The G-firm's optimization problem is as follows:

$$\Pi_G^* = \max_{v\in[0,1]} \Pi_{G,0}(v) - \delta_c C_{G,W}(v) + \delta\Pi_{G,S}^G(v) \tag{SB.6}$$

$$\text{s.t. } \Pi_{G,0}(v) - \delta_c C_{G,W}(v) + \delta\Pi_{B,S}^G(v) \leqslant \Pi_B \tag{IC_B}$$

$$\Pi_{G,0}(v) - \delta_c C_{G,W}(v) + \delta\Pi_{G,S}^G(v) \geqslant \max_{v\in[0,1]} \Pi_G^B(v) \tag{IC_G}$$

That is, the G-firm should select a price (market segment) such that the B-firm has no incentive to try to make the consumers believe that it observed $s = G$. The latter is captured by the constraint (IC_B): recall that Π_B is the profit of the B-firm when consumers believe that the firm observed $s = B$. It must be higher than $\Pi_{G,0}(v) - \delta_c C_{G,W}(v) + \delta\Pi_{B,S}^G(v)$, which is the B-firm's profit when 'tricking' the consumers into believing it observed $\hat{s} = G$. Specifically, when B-firm imitates the G-firm's strategy, the initial price will depend on p_G instead of p_B (as in Π_B). Also, the waiting costs will depend on $C_{G,W}$ instead of $C_{B,W}$ (as in Π_B). Finally, the future expected (skimming) profits will depend on $\Pi^G_{B,S}$ because the firm's assessment of future profits will be determined by p_B via $\Phi^f_B(p, p_G, 1-v)$. Following a similar logic, constraint (IC_G) ensures that the G-firm has no incentive to pretend to be a B-firm. Let v^*_G be the solution to the problem above and P^*_{1G} be the corresponding initial price. We refer to $\Pi_G - \Pi^*_G$ as the cost of signaling.

Figure 8 illustrates equilibrium profits and strategies as functions of the informativeness of the private signal q, for different pairs of δ and δ_c, and different values of τ. The figures corresponding to $\delta = \delta_c = 0.9$ are included in the paper.

SB.4 Extensions to Quality-Updating Process

SB.4.1 Not all consumers write reviews

Consider the case that all consumers read reviews, but only a portion, α, of consumers who have bought the product in the first period write reviews. As the volume x is now αx, the pre-posterior becomes:

$$\Phi(p, x) = p_0 F\left(\frac{\frac{1}{\alpha}\eta(p, \alpha x) - x}{\sqrt{x}/(\sqrt{\alpha}\tau)}\right) + (1 - p_0) F\left(\frac{\frac{1}{\alpha}\eta(p, \alpha x)}{\sqrt{x}/(\sqrt{\alpha}\tau)}\right). \tag{7*}$$

Note that $\frac{1}{\alpha}\eta(p, \alpha x) = \frac{x}{2} - \ln(\frac{(1-p)p_0}{p(1-p_0)})/(\sqrt{\alpha}\tau)^2$. Hence, by redefining the review informativeness as $\tau' = \sqrt{\alpha}\tau$, we recover the base model. That is, when some consumers do not write reviews, the information precision reduces proportionally. Hence, all of the results in the base model apply to this extended case. In addition, the effects of increasing α on the firm and on the consumers are exactly the same as those of increasing τ. Figure 9 below illustrates the SOSD property of pre-posterior in both x and α. Figure 10 depicts the effect of α on the firm's optimal strategy and value of reviews.

SB.4.2 Not all consumers read reviews

For the case that all consumers write reviews, but only a portion, β, of consumers who have waited to the second period read reviews, assume that β is also the probability that a consumer will read the reviews if he or she chooses to postpone purchase. In such a case, the second-period market consists of two groups of heterogeneous consumers, those who read reviews and those who do not. The revised model formulation is provided in Section SC.

Figure 11 shows the effects of β on the firm's optimal strategy and value of reviews. When β is close to zero (one), consumers' attitude towards reviews is fairly homogenous: almost none (all) of

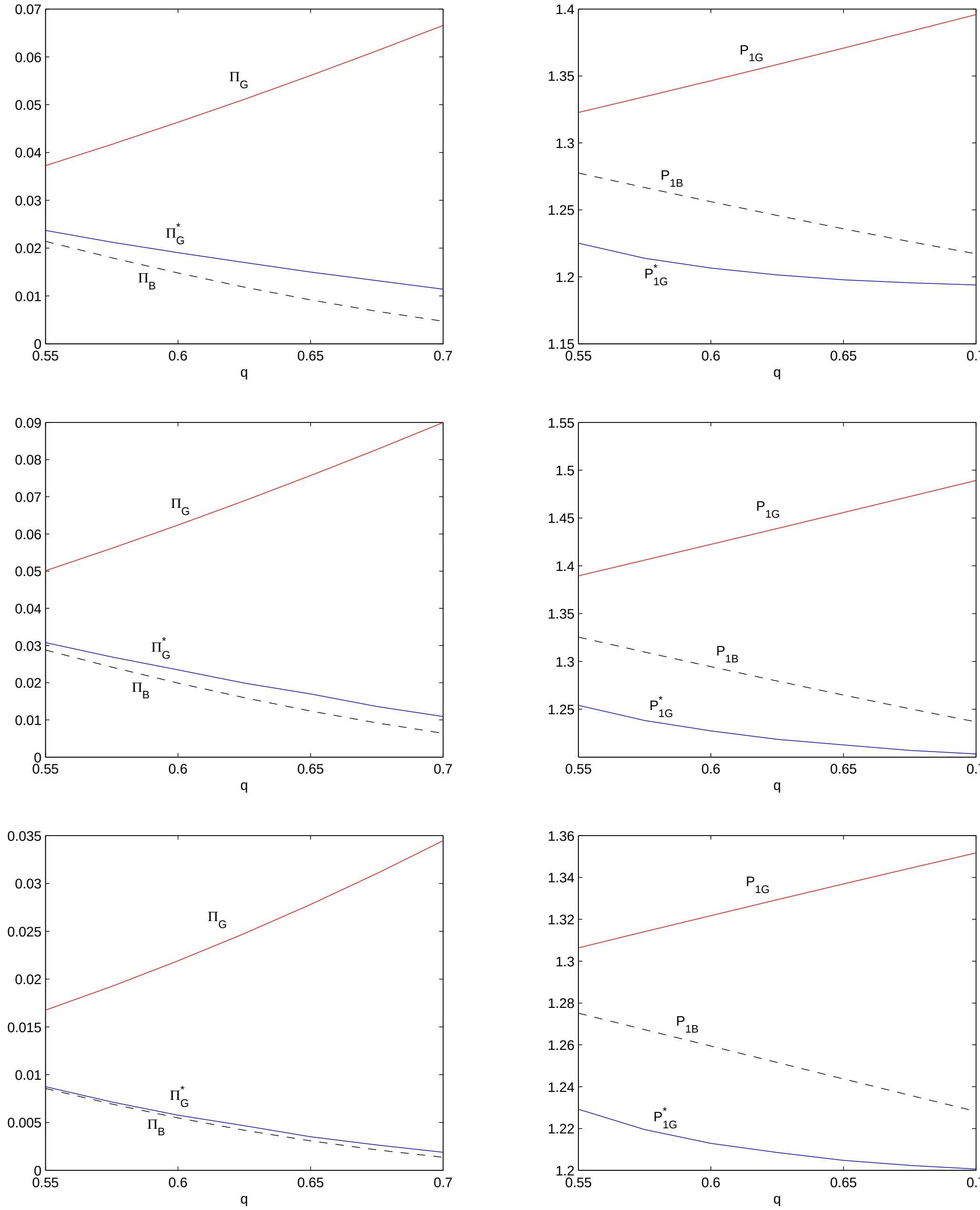

Figure 8: Equilibrium profits and initial prices as functions of the informativeness of the firm's private signal and relative patience levels: $p_0 = 0.5$, $c = 1.2$, $\tau = 3$, $\delta = \delta_c = 0.9$ (upper), $\delta = 0.9$, $\delta_c = 0.1$ (middle), $\delta = 0.1$, $\delta_c = 0.9$ (lower).

the second-period consumers reads reviews. In such cases, the extended model approximates the basic model. For intermediate values of β, the sizes of the two groups of consumers (readers and non-readers) are comparable and the level of consumer heterogeneity is high. Due to the considerable portion of non-readers, the firm's second-period price becomes less responsive to reviews. Hence,

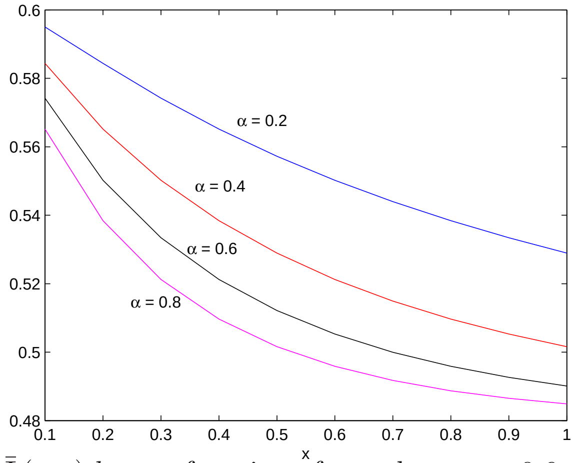

Figure 9: $\int_0^p \bar{\Phi}(r,x)dr$ as a function of x and α: $p_0 = 0.6$, $p = 0.8$, $\tau = 1$.

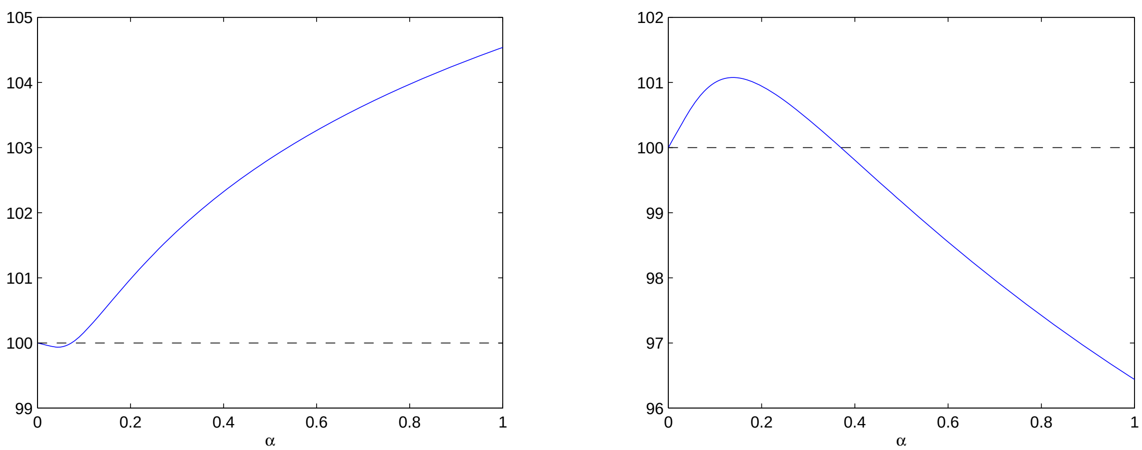

Figure 10: $\frac{v^*}{v^o}$ (in %, left) and $\frac{\Pi^*}{\Pi^o}$ (in %, right) as functions of α: $p_0 = 0.5$, $c = 1.1$, $\delta = 0.1$, $\delta_c = 1$, $\tau = 3$.

the firm's option value due to reviews increases in the fraction of readers (β). On the other hand, consumers' option value may increase or decrease in β: as β increases, consumers care more about reviews as they are more likely to read; meanwhile, a higher β implies that, with more review-sensitive price, the firm can absorb a larger portion of the benefit from quality learning and leave less surplus for the waiting consumers. Hence, the effect of β on the firm's net discounted option value is ambiguous. As a consequence, a firm might want to carefully track which fraction of the population is actually influenced by consumer reviews and tailor its pricing strategy accordingly.

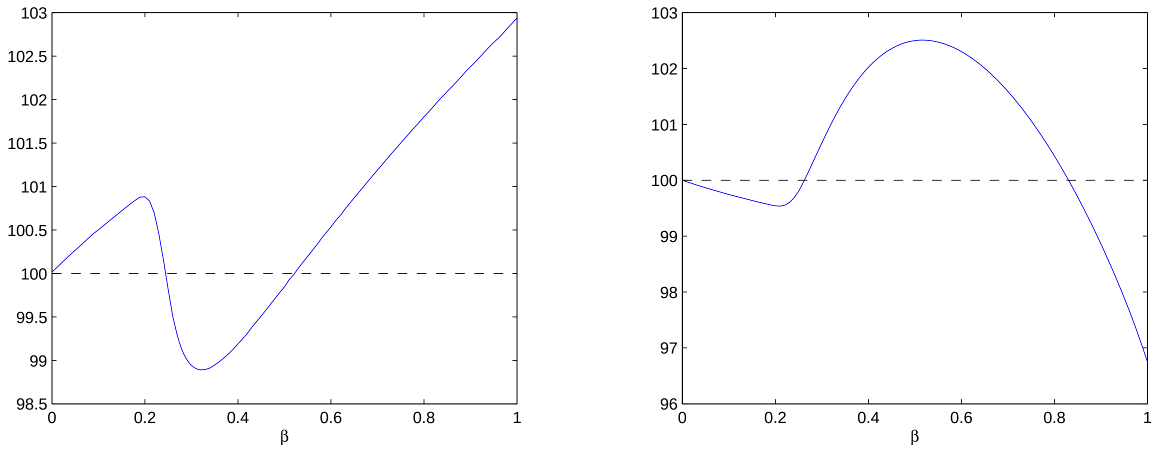

Figure 11: $\frac{v^*}{v^o}$ (in %, left) and $\frac{\Pi^*}{\Pi^o}$ (in %, right) as functions of β: $p_0 = 0.5$, $c = 1.1$, $\delta = 0.05$, $\delta_c = 1$, $\tau = 2$.

SB.4.3 Informativeness of reviews is not perfectly known

Consider the situation that the information precision τ is not perfectly known. Let $\tilde{\tau}$ be the joint belief about τ. For a given τ, assume that $\tilde{\tau}$ follows a bernoulli distribution: $\Pr[\tilde{\tau} = (1-\gamma)\tau] = \Pr[\tilde{\tau} = (1+\gamma)\tau] = 0.5$. That is, $\tilde{\tau}$ is an unbiased estimation of τ. The parameter γ represents the noise in the information about τ. In such a case, the conditional signal distribution becomes

$$f_\theta(g,x) = \frac{1}{2}\frac{(1-\gamma)\tau}{\sqrt{2\pi x}}\exp(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2(g-x\theta)^2}{x}) + \frac{1}{2}\frac{(1+\gamma)\tau}{\sqrt{2\pi x}}\exp(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2(g-x\theta)^2}{x})$$

and the likelihood ratio is

$$l(g,x) = \frac{f_1(g,x)}{f_0(g,x)} = \frac{(1-\gamma)\exp(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2(g-x)^2}{x}) + (1+\gamma)\exp(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2(g-x)^2}{x})}{(1-\gamma)\exp(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2 g^2}{x}) + (1+\gamma)\exp(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2 g^2}{x})} \tag{SB.7}$$

Hence,

$$\begin{aligned}\int_0^p \bar{\Phi}(r,x)dr &= \int_0^p Pr[p(\tilde{G},x) \geq r]dr = \int_0^p \Pr\left[\frac{p_0}{p_0 + (1-p_0)/l(\tilde{G},x)} \geq r\right] dr \\ &= \int_0^p \Pr\left[l(\tilde{G},x) \geq \frac{1-p_0}{p_0(\frac{1}{r}-1)}\right] dr \\ &= \int_0^p \int_{-\infty}^{+\infty} 1_{l(y,x) \geq \frac{1-p_0}{p_0(\frac{1}{r}-1)}} [p_0 f_1(y,x) + (1-p_0) f_0(y,x)] dy dr\end{aligned}$$

By definition, it is clear that the pre-posterior satisfies the martingale property. Also, for small γ, we prove in Lemma SB.10 below that the SOSD property of the pre-posterior in x is preserved.

For large γ, we numerically verified the SOSD property, as in Figure 12.

Lemma SB.10. *When γ is small, the pre-posterior second-order stochastically decreases in x.*

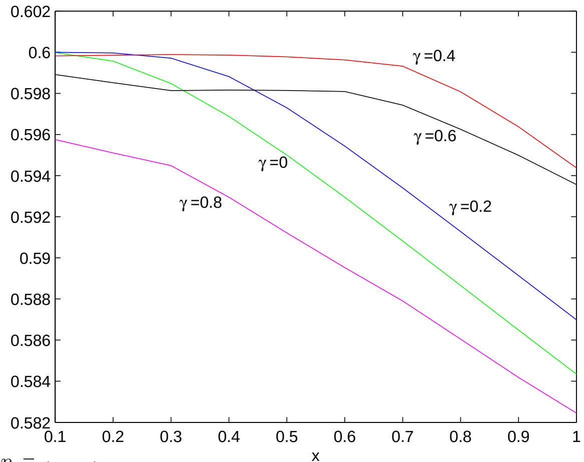

Figure 12: $\int_0^p \bar{\Phi}(r,x)dr$ as a function of x and γ: $p_0 = 0.6$, $p = 0.8$, $\tau = 1$.

Figure 13 illustrates the effect of γ on the firm's profits (as well as on the value of reviews). Intuitively, one would expect that the uncertainty on information precision reduces the information gleaned from the reviews. Figure 13 demonstrates that, when γ is small enough, a myopic firm benefits from such uncertainty when consumers are strategic (left panel of Figure 13) and a patient firm is hurt by such uncertainty when consumers are myopic (right panel of Figure 13). The reason is that for two given levels of precision, $(1-\gamma)\tau$ and $(1+\gamma)\tau$, the option value of the consumers (firm) is lower than the option value with known information precision, τ. When the uncertainty γ is very large and close to one, by equation (SB.7), the effect of the low precision $(1-\gamma)\tau$ is negligible and the likelihood ratio is predominantly determined by the high precision $(1+\gamma)\tau$. Hence, the firm's and the consumers' option values eventually increase in γ as γ becomes very large.

SC Proofs for Results in Section SB

SC.1 Proofs for Results in Section SB.1

Proof of Lemma SB.1 By solving the maximization problem in equation (SB.1), we can show that

• If $p + \overline{v} < c$, $\underline{v}^* = \overline{v}$

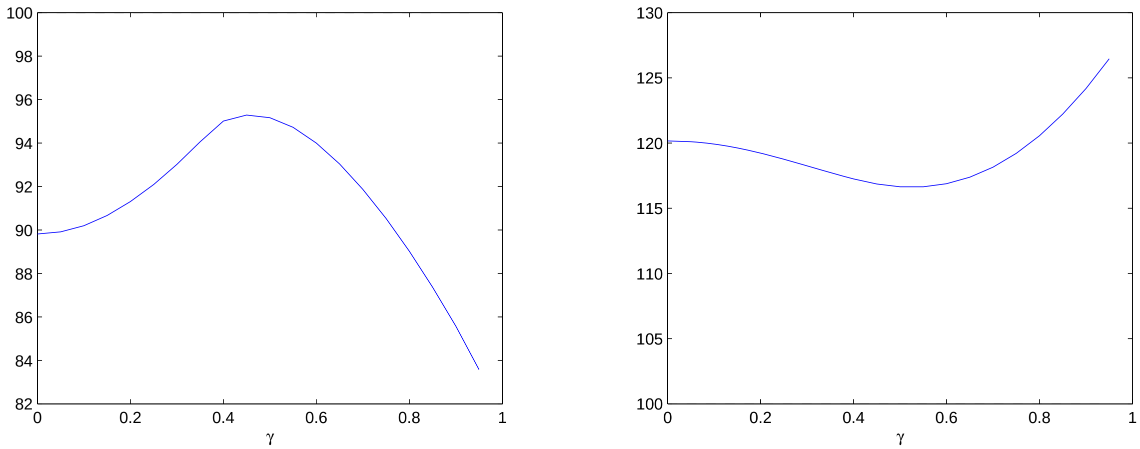

Figure 13: $\frac{\Pi^*}{\Pi^o}$ (in %) as functions of γ: $p_0 = 0.5$, $c = 1.2$, $\tau = 2$, $\delta = 0, \delta_c = 1$ (left), $\delta = 1, \delta_c = 0$ (right).

- If $c \leq p + \overline{v} < c_2$,

$$\underline{v}^* = \max\left[\overline{v} - K_2, \frac{\overline{v} - p + c}{2}\right] = \begin{cases} \overline{v} - K_2 & \text{if } p > 2K_2 + c - \overline{v} \\ \frac{\overline{v}-p+c}{2}, & \text{if } p \leq 2K_2 + c - \overline{v} \end{cases}$$

where $\frac{\overline{v}-p+c}{2} \leq \overline{v}$ since $p + \overline{v} \geq c$.

- If $p + \overline{v} \geq c_2$

$$\underline{v}^* = \min\left[\max\left(\frac{\overline{v} - p + c}{2}, \overline{v} - K_2\right), \frac{\overline{v} - p + c_2}{2}\right] = \begin{cases} \frac{\overline{v}-p+c_2}{2} & \text{if } p > 2K_2 + c_2 - \overline{v} \\ \overline{v} - K_2 & \text{if } 2K_2 + c - \overline{v} \leq p \leq 2K_2 + c_2 - \overline{v} \\ \frac{\overline{v}-p+c}{2}, & \text{if } p \leq 2K_2 + c - \overline{v} \end{cases}$$

where $\frac{c_2+\overline{v}-p}{2} \leq \overline{v}$ since $p + \overline{v} \geq c_2$.

Summarizing the three cases above, we have the optimal $\underline{v}$, as stated in the lemma. □

Proof of Lemma SB.3 By Lemma SB.2,

- if $\int_{c-v}^1 (p - \{c - v\})d\Phi(p, v) - \int_{c_2-v}^1 (p - \{c_2 - v\})d\Phi(p, v) < k$, $K_2 = 0$ and

$$\mathrm{E}_{\tilde{p}}\left[\max\left\{\min[\tilde{K_2}, \max(\frac{v + \tilde{p} - c}{2}, 0)], \frac{v + \tilde{p} - c_2}{2}\right\}\right] = \frac{1}{2}\int_{c_2-v}^1 (p - \{c_2 - v\})d\Phi(p, v).$$

- if $\int_{c-v}^1 (p - \{c - v\})d\Phi(p, v) - \int_{c_2-v}^1 (p - \{c_2 - v\})d\Phi(p, v) \geq k$

  – if $\int_{c+1-c_2}^1 (p - \{c + 1 - c_2\})d\Phi(p, v) < k$, then $K_2^*(v) \leq \frac{1-c_2+v}{2}$ and satisfies $\int_0^1 (p - \{c -$

$v + 2K_2\})^+ d\Phi(p, v) - \int_0^1 (p - \{c_2 - v + 2K_2\})^+ d\Phi(p, v) = k$. Note that

$$\begin{aligned}
&\int_0^1 \max\{\min[K_2, \max(\tfrac{v+p-c}{2}, 0)], \tfrac{v+p-c_2}{2}\} d\Phi(p, v) \\
&= \int_0^1 \max\left\{\min[K_2 - (\tfrac{v+p-c}{2})^+, 0], \tfrac{v+p-c_2}{2} - (\tfrac{v+p-c}{2})^+\right\} d\Phi(p, v) + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v) \\
&= -\tfrac{1}{2}\left[\int_0^1 (p - \{c - v + 2K_2\})^+ d\Phi(p, v) - \int_0^1 (p - \{c_2 - v + 2K_2\})^+ d\Phi(p, v)\right] + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v) \\
&= -\tfrac{k}{2} + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v)
\end{aligned}$$

– if $\int_{c+1-c_2}^1 (p - \{c + 1 - c_2\}) d\Phi(p, v) \geq k$, then $K_2^*(v) \in [\frac{1-c_2+v}{2}, \frac{1-c+v}{2}]$ and satisfies $\int_{c-v+2K_2}^1 (p - \{c - v + 2K_2\}) d\Phi(p, v) = k$. Note that in such a case, $K_2 \geq \frac{v+p-c_2}{2}$ for any p_2. Hence,

$$\begin{aligned}
&\int_0^1 \max\{\min[K_2, \max(\tfrac{v+p-c}{2}, 0)], \tfrac{v+p-c_2}{2}\} d\Phi(p, v) \\
&= \int_0^1 \min(K_2, (\tfrac{v+p-c}{2})^+) d\Phi(p, v) \\
&= \int_0^1 \min(K_2 - (\tfrac{v+p-c}{2})^+, 0) d\Phi(p, v) + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v) \\
&= -\tfrac{1}{2} \int_{c-v+2K_2}^1 p - \{c - v + 2K_2\} d\Phi(p, v) + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v) \\
&= -\tfrac{k}{2} + \int_0^1 (\tfrac{v+p-c}{2})^+ d\Phi(p, v)
\end{aligned}$$

Summarizing the cases, $(K_2^*(P_1), v^*(P_1))$ satisfies

$$\begin{cases} K_2^* = \max\{K_2 \geq 0 : \int_0^1 \Pi_2^*(p, v, K_2) d\Phi(p, v) - kK_2\} \\ v^* = \max\left\{v \in [0, 1] : P_1 \geq p_0 + v - \frac{\delta_c}{2} \max\left[\int_0^1 (v + p - c)^+ d\Phi(p, v) - k, \int_0^1 (v + p - c_2)^+ d\Phi(p, v)\right]\right\} \end{cases}$$

Furthermore, recall that by the proof of Proposition 11, both $p_0 + v - \frac{\delta_c}{2}[\int_0^1 (v+p-c)^+ d\Phi(p, v) - k]$ and $p_0 + v - \frac{\delta_c}{2} \int_0^1 (v + p - c_2)^+ d\Phi(p, v)$ increase in v. As the max of two increasing functions is increasing, the function

$$p_0 + v - \frac{\delta_c}{2} \max\left[\int_0^1 (v + p - c)^+ d\Phi(p, v) - k, \int_0^1 (v + p - c_2)^+ d\Phi(p, v)\right]$$

increases in v, implying that $v^*(P_1)$ is unique and increases in P_1. The existence of $\underline{P_1}'$ and $\overline{P_1}'$ then follows from the monotonicity of $v^*(P_1)$ in P_1. Further, the uniqueness of $K_2^*(P_1)$ follows from the fact that $K_2^*(P_1)$ depends on P_1 only through v and there exists a unique optimal K_2 for any given v. □

Proof of Lemma SB.4 To show $O(v) \geq 0$, note that for given v and K_2, $\Pi_2^*(p, v, K_2) - kK_2$ is convex in p. Thus, by Jensen's inequality, $\Pi_S(v, K_2) = \int_0^1 (\Pi_2^*(p, v, K_2) - kK_2) d\Phi(p, v) \geq \Pi_2^*(p_0, v, K_2) - kK_2 = \Pi_S^o(v, K_2)$. Thus, $\Pi_S(v, K_2^*(v)) = \max_{K_2} \Pi_S(v, K_2) \geq \max_{K_2} \Pi_S^o(v, K_2) = \Pi_S^o(v, K_2^o(v))$, implying $O(v) \geq 0$.

Now, for $O_c(v)$, by Jensen's inequality, $\int_0^1 (v + p - c)^+ d\Phi(p, v) - k \geq (v + p_0 - c)^+ - k$ and $\int_0^1 (v + p - c_2)^+ d\Phi(p, v) \geq (v + p_0 - c_2)^+$. Thus, $C_W(v, K_2^*(v)) \geq C_W^o(v, K_2^o(v))$, i.e., $O_c(v) \geq 0$. □

Proof of Lemma SB.5 The following lemma is useful and its proof is provided at the end of this subsection.

Lemma SC.1. *The posterior quality $\tilde{P}$ second-order stochastically decreases in τ.*

Utilizing the lemma, we now prove Lemma SB.5. When $\delta_c = 0$,

$$\Pi^* = \max_v \tilde{\Pi}(v, K_2^*(v)) = (1 - v)(p_0 + v - c - k) + \delta \Pi_S(v, K_2^*(v))$$

It is clear that $\Pi_S(v, K_2)$ decreases in c_2. Thus, by envelop theorem, Π^* decreases in c_2. Meanwhile, since $\Pi_2^*(p, v, K_2) - kK_2$ is convex in p for given v and K_2, by Jensen's inequality and SOSD property of the posterior in τ, $\Pi_S(v, K_2)$ increases in τ. Thus, by envelop theorem, Π^* increases in τ. □

Proof of Lemma SB.6 When $\delta = 0$, for any $k > 0$, by Lemma SB.2, $K_2^*(v) = 0$ for any v. Hence,

$$\Pi^* = \max_v \tilde{\Pi}(v, K_2^*(v)) = (1 - v)(p_0 + v - c - k) - \delta_c \frac{1 - v}{2} \int_0^1 (v + p - c_2)^+ d\Phi(p, v)$$

In particular, when $c_2 = \infty$ or when $\delta_c = 0$,

$$\Pi^* = \max_v \tilde{\Pi}(v, K_2^*(v)) = (1 - v)(p_0 + v - c - k),$$

which is independent of τ. Hence, to prove the lemma, by envelop theorem, it suffices to show that, for any given v and given $c_2 \in (0, \infty)$, $\tilde{\Pi}(v, K_2^*(v))$ increases in c_2 and decreases in τ:

$$\begin{aligned}
\frac{\partial}{\partial c_2} \tilde{\Pi}(v, K_2^*(v)) &= \delta_c \frac{1 - v}{2} \Pr[v + p - c_2 \geq 0] \geq 0 \\
\frac{\partial}{\partial \tau} \tilde{\Pi}(v, K_2^*(v)) &= -\delta_c \frac{1 - v}{2} \frac{\partial}{\partial \tau} \left[\int_0^1 (v + p - c_2)^+ d\Phi(p, v) \right] \leq 0
\end{aligned}$$

where the second inequality is due to the facts that the posterior second-order stochastically decreases in τ and $(v + p - c_2)^+$ is convex in p. □

Proof of Lemma SC.1 Since $\tau = \frac{1}{\sigma}$, it suffices to show that the posterior second-order stochastically increases in σ. Following exactly the same way as in the proof of Proposition 1, we obtain

$$
\begin{aligned}
&\tfrac{d}{d\sigma}\textstyle\int_0^k \Pr[p(\tilde{G},v) \geq x]dx \\
=&(k-1)p_0\tfrac{d}{d\sigma}\overline{F}_1\left((1-v)\tfrac{\mu_1+\mu_0}{2} - \ln\left(\tfrac{\frac{p_0}{k}-p_0}{1-p_0}\right)\tfrac{\sigma^2}{\mu_1-\mu_0}\right) + k(1-p_0)\tfrac{d}{d\sigma}\overline{F}_0\left((1-v)\tfrac{\mu_1+\mu_0}{2} - \ln\left(\tfrac{\frac{p_0}{k}-p_0}{1-p_0}\right)\tfrac{\sigma^2}{\mu_1-\mu_0}\right) \\
=&-\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\left(\tfrac{[(1-v)\frac{\mu_0-\mu_1}{2}]^2+[\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}]^2}{\sigma^2(1-v)}\right)\right)\cdot \\
&\left\{(k-1)p_0(\tfrac{\frac{p_0}{k}-p_0}{1-p_0})^{-\frac{1}{2}}\left(\tfrac{(1-v)\frac{\mu_0-\mu_1}{2}-\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}}{\sigma\sqrt{1-v}}\right)'_\sigma + k(1-p_0)(\tfrac{1-p_0}{\frac{p_0}{k}-p_0})^{-\frac{1}{2}}\left(\tfrac{(1-v)\frac{\mu_1-\mu_0}{2}-\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}}{\sigma\sqrt{1-v}}\right)'_\sigma\right\} \\
=&-\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\left(\tfrac{[(1-v)\frac{\mu_0-\mu_1}{2}]^2+[\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}]^2}{\sigma^2(1-v)}\right)\right)\cdot\sqrt{k(1-k)p_0(1-p_0)}\cdot \\
&\left\{-\left(\tfrac{(1-v)\frac{\mu_0-\mu_1}{2}-\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}}{\sigma\sqrt{1-v}}\right)'_\sigma + \left(\tfrac{(1-v)\frac{\mu_1-\mu_0}{2}-\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}}{\sigma\sqrt{1-v}}\right)'_\sigma\right\} \\
=&-\tfrac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\left(\tfrac{[(1-v)\frac{\mu_0-\mu_1}{2}]^2+[\ln(\frac{\frac{p_0}{k}-p_0}{1-p_0})\frac{\sigma^2}{\mu_1-\mu_0}]^2}{\sigma^2(1-v)}\right)\right)\cdot 2\sqrt{k(1-k)p_0(1-p_0)}\left(\sqrt{1-v}\tfrac{\mu_1-\mu_0}{2\sigma}\right)'_\sigma \geq 0 \quad \square
\end{aligned}
$$

SC.2 Proofs for Results in Section SB.2

Proof of Proposition SB.1 Conditional on realizations of $(p_0, p_1, ..., p_T)$ and $(v_0, v_1, ..., v_T)$, the firm's profits are

$$
\begin{aligned}
&(v_0-v_1)P_1 + \delta(v_1-v_2)P_2 + \delta^2(v_2-v_3)P_3 + \delta^3(v_3-v_4)P_4 + ... + \delta^{T-1}(v_{T-1}-v_T)P_T \\
&= \sum_{n=1}^{T}\delta^{n-1}(v_{n-1}-v_n)P_n.
\end{aligned}
$$

Let $P^*_{n+1}(p_n, v_n)$ be the optimal price in period $n+1$, given (p_n, v_n). In every period, we have the following relationship:

$$p_{n-1} + v_n - P_n = \delta_c\mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v_n - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}],$$

the consumer type v_n is indifferent between purchasing now and waiting until the next period. We can use this relationship to determine the price in period n:

$$P_n = p_{n-1} + v_n - \delta_c\mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v_n - P^*_{n+1}(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}].$$

Notice that in every period, the following relationship holds:

$$p_{n-1} + v_n - P_n^*(p_{n-1}, v_{n-1}) = \delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n + v_n - P_{n+1}^*(\tilde{p}_n, v_n)\}^+ | v_{n-1} - v_n, p_{n-1}], \text{ for } n = 1, ..., T-1.$$

Consider the final period:

$$P_T^*(p_{T-1}, v_{T-1}) = \arg\max_{P_T} (v_{T-1} - v_T)(P_T - c) \text{ and } p_{T-1} + v_T - P_T = 0$$

or as $P_T = p_{T-1} + v_T$:

$$\begin{aligned}
v_T^*(p_{T-1}, v_{T-1}) &= \arg\max_{v_T} (v_{T-1} - v_T)(p_{T-1} + v_T - c) \\
&= \min\{v_{T-1}, \frac{c - p_{T-1} + v_{T-1}}{2}\} \\
&= v_{T-1} + \min\{0, \frac{c - p_{T-1} - v_{T-1}}{2}\} \\
v_T^*(v_{T-1}, p_{T-1}) &= v_{T-1} - \frac{1}{2}\{p_{T-1} + v_{T-1} - c\}^+ \text{ and} \\
\hat{\Pi}_T(v_{T-1}, p_{T-1}) &= \frac{1}{4}(\{p_{T-1} + v_{T-1} - c\}^+)^2
\end{aligned}$$

and thus, for all (p_{T-1}, v_{T-1}):

$$P_T^*(p_{T-1}, v_{T-1}) = p_{T-1} + v_T^*(v_{T-1}, p_{T-1}).$$

As in the final period, we have

$$P_{T-1} = p_{T-2} + v_{T-1} - \delta_c \underbrace{\mathbb{E}_{\tilde{p}_{T-1}}[\{v_{T-1} - v_T^*(v_{T-1}, \tilde{p}_{T-1})\}^+ | v_{T-2} - v_{T-1}, p_{T-2}]}_{=C_{T-1}(p_{T-2}, v_{T-1}, v_{T-2})}$$

and thus, in period $T-1$, given (p_{T-2}, v_{T-2}), the expected profits in the last period are

$$\Pi_T(p_{T-2}, v_{T-1}, v_{T-2}) = \mathbb{E}_{\tilde{p}_{T-1}}[\frac{1}{4}(\{\tilde{p}_{T-1} + v_{T-1} - c\}^+)^2 | p_{T-2}, v_{T-2} - v_{T-1}]$$

and, define

$$v_{T-1}^*(p_{T-2}, v_{T-2}) = \arg\max_{v_{T-1}} (v_{T-2} - v_{T-1})(p_{T-2} + v_{T-1} - c - \delta_c C_{T-1}(p_{T-2}, v_{T-2}, v_{T-1})) + \delta \Pi_T(p_{T-2}, v_{T-1}, v_{T-2})$$

and thus

$$P^*_{T-1}(p_{T-2}, v_{T-2}) = v^*_{T-1}(p_{T-2}, v_{T-2}) + p_{T-2} - \delta_c C_{T-1}(p_{T-2}, v^*_{T-1}(p_{T-2}, v_{T-2}), v_{T-2})$$

and thus, we can define:

$$\hat{\Pi}_{T-1}(v_{T-1}, v_{T-2}, p_{T-2}) = (v_{T-2}-v_{T-1})(v_{T-1}+p_{T-2}-c-\delta_c C_{T-1}(p_{T-2}, v_{T-1}, v_{T-2}))+\delta\Pi_T(p_{T-2}, v_{T-1}, v_{T-2})$$

such that

$$v^*_{T-1}(p_{T-2}, v_{T-2}) = \underset{v_{T-1}\leqslant v_{T-2}}{\arg\max}\ \hat{\Pi}_{T-1}(v_{T-1}, v_{T-2}, p_{T-2}).$$

Thus, as in every period, we can write the profit to go as a function of v_n (instead of P_n), we have

$$(v_{n-1} - v_n)(p_{n-1} + v_n - \delta_c \mathbb{E}_{\tilde{p}_n}[\{p_n - P^*_{n+1}(p_n, v_n) + v_n\}^+ | p_{n-1}, v_{n-1} - v_n] - c) + \delta\Pi_{n+1}(p_{n-1}, v_n, v_{n-1})$$

and $P_n = p_{n-1} + v_n - \delta_c \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n - P^*_{n+1}(p_n, v_n) + v_n\}^+ | p_{n-1}, v_{n-1} - v_n]$.

So, let

$$C_{n+1}(p_n, v_{n+1}, v_n) = \mathbb{E}_{\tilde{p}_{n+1}}[\{p_{n+1} - P^*_{n+2}(p_{n+1}, v_{n+1}) + v_{n+1}\}^+ | p_n, v_n - v_{n+1}]$$

then, as for all (p_n, v_n):

$$P^*_{n+1}(p_n, v_n) = p_n + v^*_{n+1}(p_n, v_n) - \delta_c C_{n+1}(p_n, v^*_{n+1}(p_n, v_n), v_n)$$

and we have

$$\begin{aligned} C_n(p_{n-1}, v_n, v_{n-1}) &= \mathbb{E}_{\tilde{p}_n}[\{\tilde{p}_n - P^*_{n+1}(\tilde{p}_n, v_n) + v_n\}^+ | p_{n-1}, v_{n-1} - v_n] \\ &= \mathbb{E}_{\tilde{p}_n}[\{\delta_c C_{n+1}(\tilde{p}_n, v^*_{n+1}(\tilde{p}_n, v_n), v_n) + v_n - v^*_{n+1}(\tilde{p}_n, v_n)\}^+ | p_{n-1}, v_{n-1} - v_n]. \end{aligned}$$

In general, in period n, given (p_{n-1}, v_{n-1}):

$$
\begin{aligned}
v_n^*(p_{n-1}, v_{n-1}) &= \underset{v_n \leqslant v_{n-1}}{\arg\max}\, \hat{\Pi}_n(p_{n-1}, v_n, v_{n-1}) \\
\hat{\Pi}_n(p_{n-1}, v_n, v_{n-1}) &= (v_{n-1} - v_n)(p_{n-1} + v_n - c - \delta_c C_n(p_{n-1}, v_{n-1}, v_n)) \\
&\quad + \delta \mathbb{E}_{\tilde{p}_n}[\Pi_{n+1}(\tilde{p}_n, v_n) | p_{n-1}, v_{n-1} - v_n] \\
C_n(p_{n-1}, v_n, v_{n-1}) &= \mathbb{E}_{\tilde{p}_n}[\{\delta_c C_{n+1}(\tilde{p}_n, v_{n+1}^*(\tilde{p}_n, v_n), v_n) + v_n - v_{n+1}^*(\tilde{p}_n, v_n)\}^+ | p_{n-1}, v_{n-1} - v_n] \\
\Pi_n(p_{n-1}, v_{n-1}) &= \hat{\Pi}_n(p_{n-1}, v_n^*(p_{n-1}, v_{n-1}), v_{n-1})
\end{aligned}
$$

from which the prices can be computed as follows:

$$
P_n^*(p_{n-1}, v_{n-1}) = v_n^*(p_{n-1}, v_{n-1}) + p_{n-1} - \delta_c C_n(p_{n-1}, v_n^*(p_{n-1}, v_{n-1}), v_{n-1}).\square
$$

Proof of Lemma SB.7 Conjecture that:

$$
v^o(v) = \alpha v + (1-\alpha)(c - p_0),\ C^o(v) = \phi(p_0 + v - c) \text{ and } \Pi^o(v) = \beta(p_0 + v - c)^2.
$$

Note that $v - v^o(v) = (1-\alpha)(v - c + p_0)$ and $p_0 + v^o(v) - c = \alpha(v + p_0 - c)$. Then, we can fill in these functional forms into the objective function and take the derivative with respect to v', to identify a condition that determines α, β and ϕ.

1. First, consider the waiting cost, $C^o(v)$:

$$
\begin{gathered}
\phi(p_0 + v - c) = \delta_c \phi \alpha (p_0 + v - c) + (1-\alpha)(v - c + p_0) \Leftrightarrow \\
\phi = \delta_c \phi \alpha + (1-\alpha) \Leftrightarrow \phi = \frac{1-\alpha}{1-\alpha\delta_c}
\end{gathered}
$$

   As a consequence,

$$
C^o(v) = \frac{1-\alpha}{1-\alpha\delta_c}(v - c + p_0)
$$

2. Second, we find the optimality conditions and ensure they are satisfied at $v^o(v)$:

$$
\begin{aligned}
&\frac{d}{dv'}((v - v')(p_0 + v' - c - \delta_c C^o(v')) + \delta \Pi^o(v')) \\
&= -(p_0 + v' - c)(1 - \phi\delta_c) + (v - v')(1 - \phi\delta_c) + 2\delta\beta(p_0 + v' - c)
\end{aligned}
$$

The derivative must be equal to zero at $v' = v^o(v)$. Thus

$$0 = -\alpha (v - c + p_0)(1 - \phi\delta_c) + (1-\alpha)(v - c + p_0)(1 - \phi\delta_c) + 2\delta\beta\alpha (v - c + p_0).$$

As a consequence, we must have that

$$\begin{aligned} 0 &= -\alpha(1-\delta_c\phi) + (1-\alpha)(1-\delta_c\phi) + 2\delta\beta\alpha \Leftrightarrow \\ \beta &= \frac{1}{\alpha\delta}(\alpha - \frac{1}{2})(1-\delta_c\phi) \\ &= \frac{1}{\alpha\delta}\frac{(\alpha-\frac{1}{2})(1-\delta_c)}{1-\alpha\delta_c} \end{aligned}$$

3. Third, we need to ensure that the value function is preserved: Note that $C^o(v^o(v)) = \frac{1-\alpha}{1-\alpha\delta_c}(v^o(v) - c + p_0) = \frac{(1-\alpha)\alpha}{1-\alpha\delta_c}(v - c + p_0)$ and $\Pi^o(v^o(v)) = \beta(p_0 + v^o(v) - c)^2 = \beta\alpha^2(v - c + p_0)^2$ and thus:

$$\begin{aligned} \Pi^o(v) &= (v - v^o(v))(p_0 + v^o(v) - c - \delta_c C^o(v^o(v))) + \delta\Pi^o(v^o(v)) \Leftrightarrow \\ \beta(p_0+v-c)^2 &= (1-\alpha)(v-c+p_0)(\alpha(v-c+p_0) - \delta_c\phi\alpha(v-c+p_0)) + \delta\beta\alpha^2(v-c+p_0)^2 \Leftrightarrow \\ \beta(p_0+v-c)^2 &= (1-\alpha)(\alpha - \delta_c\phi\alpha)(v-c+p_0)^2 + \delta\beta\alpha^2(v-c+p_0)^2 \Leftrightarrow \\ \beta &= (1-\delta_c)\phi\alpha + \delta\beta\alpha^2 \Leftrightarrow \\ 0 &= \frac{(1-\alpha)(1-\delta_c)\alpha + \beta(\delta\alpha^2-1)(1-\alpha\delta_c)}{1-\alpha\delta_c} \Leftrightarrow \\ 0 &= \frac{1}{2}\frac{(\alpha^2\delta - 2\alpha + 1)(1-\delta_c)}{\alpha\delta(1-\alpha\delta_c)} \end{aligned}$$

from which we obtain $\alpha = \frac{1}{\delta}(1 - \sqrt{1-\delta})$.

Thus:

$$\beta = \frac{1}{1-\sqrt{1-\delta}}\frac{(\frac{1-\sqrt{1-\delta}}{\delta} - \frac{1}{2})(1-\delta_c)}{1-\delta_c\frac{1-\sqrt{1-\delta}}{\delta}}, \ \phi = \frac{1}{1+(1-\delta_c)\frac{1}{\sqrt{1-\delta}}} \text{ and } \alpha = \frac{1-\sqrt{1-\delta}}{\delta} = \frac{1}{1+\sqrt{1-\delta}}.$$

It follows that

$$v^o(v) = (c - p_0)\frac{\delta - 1 + \sqrt{1-\delta}}{\delta} + \frac{1 - \sqrt{1-\delta}}{\delta}v \text{ and}$$
$$P^o(v) = (1-\phi)(p_0 + v) + \phi c = \frac{(1-\delta_c)\frac{1}{\sqrt{1-\delta}}}{1 + (1-\delta_c)\frac{1}{\sqrt{1-\delta}}}(p_0 + v) + \frac{1}{1 + (1-\delta_c)\frac{1}{\sqrt{1-\delta}}}c$$

and also

$$\Pi^o(v) = \frac{1}{2}\frac{(1-\delta_c)(2 - \delta - 2\sqrt{1-\delta})}{(1 - \sqrt{1-\delta})(\delta - \delta_c + \delta_c\sqrt{1-\delta})}(p_0 + v - c)^2 \text{ and}$$
$$C^o(v) = \frac{1}{1 + (1-\delta_c)\frac{1}{\sqrt{1-\delta}}}(p_0 + v - c).$$

□

Proof of Corollary SB.1 For the price path declining over time, it directly follows from the expression of $P^o(v)$ in Lemma SB.7 and the fact that the equilibrium v is nonincreasing over time. Now, for per-period sales, from $v^o(v) = (c - p_0)(1-\alpha) + \alpha v$, we have $v_0 = 1$ and $v_t = (c - p_0)(1-\alpha)\sum_{s=0}^{t-1}\alpha^s + \alpha^t$ for $t \geq 1$. Hence, the sales in period $T = 1, 2, ...$ are:

$$S_1 = 1 - v_1 = 1 - (c - p_0)(1-\alpha) - \alpha$$
$$S_t = v_{t-1} - v_t = \alpha^{t-1}(1 - (c - p_0)(1-\alpha) - \alpha), \text{ for } t \geq 1.$$

Since $\alpha \leq 1$, we have that the per-period sales decline over time. □

Proof of Lemma SB.8 Following a similar approach as in the two-period model, since $\lim_{v' \to 1^-}\tilde{\Pi}^1(p_0, 1, v') > \tilde{\Pi}^0(p_0, 1, 1)$, to find the optimal v', we can simply maximize $\tilde{\Pi}^1(p_0, 1, v')$ over $v' \in [0, 1]$. Since $\frac{d}{dv'}\tilde{\Pi}^1(p_0, 1, v') \geq 0$ for $v' \leq c - 1$, we can focus on the case $v' > c - 1$, where the first-order derivative of $\tilde{\Pi}^1(p_0, 1, v')$ is as follows:

$$\begin{aligned}&\frac{d}{dv'}((1 - v')(p_0 + v' - c - \delta_c p_0\phi(1 + v' - c)) + \delta p_0\beta(1 + v' - c)^2)\\ =&1 - p_0 + 2\delta\beta p_0 + (1 - \delta_c p_0\phi - 2\delta\beta p_0)c + 2(\delta\beta p_0 + \delta_c p_0\phi - 1)v'\end{aligned}$$

where $\beta = \frac{1}{1-\sqrt{1-\delta}} \frac{(\frac{1-\sqrt{1-\delta}}{\delta} - \frac{1}{2})(1-\delta_c)}{1-\delta_c \frac{1-\sqrt{1-\delta}}{\delta}}$, $\phi = \frac{1}{1+(1-\delta_c)\frac{1}{\sqrt{1-\delta}}}$. Let $X = \sqrt{1-\delta}$ or $\delta = 1 - X^2$. Then,

$$\begin{aligned}
\delta\beta p_0 + \delta_c p_0 \phi - 1 =& (1-X^2)\frac{1}{1-X}\frac{(\frac{1-X}{1-X^2} - \frac{1}{2})(1-\delta_c)}{1-\delta_c\frac{1-X}{1-X^2}} p_0 + \delta_c p_0 \frac{1}{1+(1-\delta_c)\frac{1}{X}} - 1 \\
=& \frac{1}{2\left(X - \delta_c + 1\right)}\left(\left(\delta_c p_0 - p_0\right)X^2 + \left(2\delta_c p_0 - 2\right)X - \left(2-p_0\right)\left(1-\delta_c\right)\right) < 0
\end{aligned}$$

where the inequality follows from the observation that $f(X) = (\delta_c p_0 - p_0)X^2 + (2\delta_c p_0 - 2)X - (2-p_0)(1-\delta_c)$ is concave in X, and when $X = 0$, $f(X) = -(2-p_0)(1-\delta_c) < 0$ and $f'(X) = 2\delta_c p_0 - 2 < 0$. Therefore, the profit function is concave in v and the solution to the first-order condition yields v^*. □

Proof of Lemma SB.9 For the infinite horizon and perfect reviews, recall that

$$\begin{aligned}
\Pi^o(v) &= (1-v)(p_0 + v - c - \delta_c\phi(p_0 + v - c)) + \delta\beta(p_0 + v - c)^2 \\
\Pi^*(v) &= (1-v)(p_0 + v - c - p_0\delta_c\phi(1 + v - c)) + \delta p_0 \beta(1 + v - c)^2
\end{aligned}$$

Similarly as in the basic model, we have

$$\Pi^*(v) - \Pi^o(v) = \delta\underbrace{\beta(p_0(1+v-c)^2 - (p_0+v-c)^2)}_{=O(v)} - \delta_c\underbrace{\phi(1-v)(p_0(1+v-c) - \phi(p_0+v-c))}_{=O_c(v)}$$

By Jensen's inequality, $O(v) \geq 0$ and $O_c(v) \geq 0$ for all v. □

Proof of Proposition SB.2 When $\delta = \delta_c$, by Lemma SB.8,

$$v^* = \min\left(\frac{(1 - p_0\delta(\phi + 2\beta))\,c + (p_0(2\beta\delta - 1) + 1)}{2(1 - p_0\delta(\beta + \phi))}, 1\right)$$

where $\beta = \frac{(\frac{1-\sqrt{1-\delta}}{\delta} - \frac{1}{2})\sqrt{1-\delta}}{1-\sqrt{1-\delta}}$ and $\phi = \alpha = \frac{1}{1+\sqrt{1-\delta}}$. More specifically, if $1 < c < 2 - \frac{1-p_0}{1-\delta p_0}$, then $v^* = \frac{(1-\phi\delta p_0 - 2\beta\delta p_0)c + (2\beta\delta p_0 - p_0 + 1)}{2(1-\delta(\beta+\phi)p_0)}$; otherwise, $v^* = 1$.

Furthermore, conditional on the firm being of high quality, the prices in the first two periods are:

$$t = 0: P_1 = p_0 + v^* - \delta p_0 \phi(1 + v^* - c) \text{ and } t = 1: P_2 = 1 + v^* - \phi(1 + v^* - c)$$

Hence,

$$P_1 < P_2 \Leftrightarrow -\delta p_0\phi(1+v^*-c) + \phi(1+v^*-c) < 1 - p_0 \Leftrightarrow 1 + v^* - c < \frac{\frac{1-p_0}{1-\delta p_0}}{\phi} = (1+\sqrt{1-\delta})\frac{1-p_0}{1-\delta p_0}$$

• If $1 < c < 2 - \frac{1-p_0}{1-\delta p_0}$, we can write:

$$1 + v^* - c = 1 + \frac{2\beta\delta p_0 - p_0 + 1}{2(1-\delta(\beta+\phi)p_0)} - \frac{1-\delta\phi p_0}{2(1-\delta(\beta+\phi)p_0)}c$$

So, for $P_1 < P_2$, we need

$$1 + \frac{2\beta\delta p_0 - p_0 + 1}{2(1-\delta(\beta+\phi)p_0)} - \frac{1-\delta\phi p_0}{2(1-\delta(\beta+\phi)p_0)}c - (1+\sqrt{1-\delta})\frac{1-p_0}{1-\delta p_0} < 0$$

Let $\delta = 1 - X^2$, then the inequality becomes

$$-\frac{\left(X^2p_0+1-p_0\right)(Xp_0+1-p_0)c+(-X^2p_0(1-p_0)-(1-p_0)^2+X(1-p_0)(2-3p_0)+X^3p_0(1-3p_0))}{\underbrace{\left(Xp_0+X^2p_0+2(1-p_0)\right)\left(1-p_0+X^2p_0\right)}_{\geqslant 0}} < 0$$

Thus:

$$P_1 < P_2 \Leftrightarrow \left(1-p_0+X^2p_0\right)\left(Xp_0+1-p_0\right)c + (X^2p_0\left(p_0-1\right) - \left(p_0-1\right)^2 + X\left(p_0-1\right)\left(3p_0-2\right) - X^3p_0\left(3p_0-1\right)) > 0$$

Since $\left(1-p_0+X^2p_0\right)\left(Xp_0+1-p_0\right) \geq 0$, we have that

$$c > 2 - \frac{1-p_0}{1-\delta p_0} : P_1 < P_2 \Leftrightarrow c > \frac{X^2p_0\left(1-p_0\right) + \left(1-p_0\right)^2 + X\left(1-p_0\right)\left(3p_0-2\right) + X^3p_0\left(3p_0-1\right)}{\left(1-p_0+X^2p_0\right)\left(Xp_0+1-p_0\right)} > 0$$

Note that for $c = 2 - \frac{1-p_0}{1-\delta p_0}$, we have that

$$P_1 < P_2 \Leftrightarrow X\left(1-p_0\right)\left(Xp_0+X^2p_0+2(1-p_0)\right) > 0,$$

which is always satisfied.

• If $c \geq 2 - \frac{1-p_0}{1-\delta p_0}$, $v^* = 1$, and

$$P_1 < P_2 \Leftrightarrow 2 - c < (1+X)\frac{1-p_0}{1-\delta p_0} \Leftrightarrow 2 - (1+X)\frac{1-p_0}{1-\delta p_0} < c$$

which is always satisfied (as $c \geq 2 - \frac{1-p_0}{1-\delta p_0} > 2 - (1+X)\frac{1-p_0}{1-\delta p_0}$). □

Proof of Proposition SB.3 By Lemma SB.8, consider the following two cases:

- if $(1 - p_0(\phi\delta_c + 2\beta\delta))\,c + 4\beta\delta p_0 - p_0 + 2\phi\delta_c p_0 - 1 \geq 0$, $v^* = 1$. That is, the firm sells ϵ in the first period, and then in the second period, conditional on the quality being high, it then sells $1 - v^o(1)$ in the second period, where $v^o(1)$ is the no-review sales with $p_0 = 1$. In such a case, the sales decline starting from the third period, as shown for the no-review case. Thus, the conditional sales is zero in the first period, jumps to a positive number in the second period, and then subsequently decreases. That is, there is a peak in the per-period sales.

- if $(1 - p_0(\phi\delta_c + 2\beta\delta))\, c + 4\beta\delta p_0 - p_0 + 2\phi\delta_c p_0 - 1 < 0$, $v^* = \frac{(1-p_0(\phi\delta_c+2\beta\delta))c+(p_0(2\beta\delta-1)+1)}{2(1-p_0(\delta\beta+\delta_c\phi))}$. In such a case,

$$\begin{aligned} S_1 &= 1 - v^* \\ S_2 &= v^* - ((c-1)\frac{\delta - 1 + \sqrt{1-\delta}}{\delta} + \frac{1-\sqrt{1-\delta}}{\delta} v^*) = \frac{X}{1+X}(v^* - c + 1) \\ S_1 - S_2 &= 1 - v^* - \frac{X}{1+X}(v^* - c + 1) \\ &= \frac{1}{1+X} + \frac{X}{1+X}c - \frac{1+2X}{1+X}v^* \\ &= \frac{1}{1+X} + \frac{X}{1+X}c - \frac{1+2X}{1+X}\frac{(1 - p_0(\phi\delta_c + 2\beta\delta))\, c + (p_0(2\beta\delta - 1) + 1)}{2(1 - p_0(\delta\beta + \delta_c\phi))} \end{aligned}$$

where $\beta = \frac{1}{1-X}\frac{(\frac{1-X}{1-X^2} - \frac{1}{2})(1-\delta_c)}{1-\delta_c\frac{1-X}{1-X^2}} = \frac{1-\delta_c}{2X-2\delta_c+2}$, $\phi = \frac{1}{1+(1-\delta_c)\frac{1}{X}} = \frac{X}{X-\delta_c+1}$, $X = \sqrt{1-\delta}$. Note that in the expression of $S_1 - S_2$, the coefficient for c is

$$\begin{aligned} &\frac{X}{1+X} - \frac{1+2X}{1+X}\frac{1 - p_0(\phi\delta_c + 2\beta\delta)}{2(1 - p_0(\delta\beta + \delta_c\phi))} \\ =&\frac{1}{2\,(X+1)\,(1 - \beta\delta p_0 - \phi\delta_c p_0)}\,(2\beta\delta p_0 + \phi\delta_c p_0 + 2X\beta\delta p_0 - 1) \\ =&- \frac{p_0(1-\delta_c)X^3 + p_0(1-\delta_c)X^2 + (1-p_0)X + (1-p_0)(1-\delta_c)}{2(X+1)(1 - \beta\delta p_0 - \phi\delta_c p_0)(X - \delta_c + 1)} \end{aligned}$$

where it is easy to show that all the coefficients are positive in the polynomial of X. Hence, $S_1 - S_2$ decreases in c. Also, note that $1 - p_0(\phi\delta_c + 2\beta\delta) = 1 - p_0(\frac{X}{X-\delta_c+1}\delta_c + 2\frac{1-\delta_c}{2X-2\delta_c+2}(1 - X^2)) = \frac{1}{X-\delta_c+1}\left((p_0 - \delta_c p_0)\, X^2 + (1 - \delta_c p_0)\, X + (1 - p_0)\,(1 - \delta_c)\right) > 0$. Furthermore, $(S_1 - S_2)|_{c=\frac{-4\beta\delta p_0 + p_0 - 2\phi\delta_c p_0 + 1}{1-p_0(\phi\delta_c+2\beta\delta)}} = -\frac{p_0 X}{1+X}(2-c) < 0$. Hence, when c is sufficiently high, $S_1 < S_2$.

Summarizing the two cases, we conclude that, if c is sufficiently high, the expected sales increases from the first period to the second period, reaches a peak in the second period, and then declines in all the subsequent periods. □

Proof of Proposition SB.4 (i-a) For $\tau = 0$, recall that

$$\Pi_2^o = \max_v \tilde{\Pi}_2^o(v) = (1-v)(p_0+v-c) + \frac{\delta}{4}(p_0+v-c)^2 - \frac{\delta_c}{2}(1-v)(p_0+v-c)$$
$$\Pi_\infty^o = \max_v \tilde{\Pi}_\infty^o(v) = (1-v)(p_0+v-c) + \delta\beta(p_0+v-c)^2 - \delta_c\phi(1-v)(p_0+v-c)$$

where $\beta = \frac{1}{1-\sqrt{1-\delta}} \frac{(\frac{1-\sqrt{1-\delta}}{\delta}-\frac{1}{2})(1-\delta_c)}{1-\delta_c\frac{1-\sqrt{1-\delta}}{\delta}}$, $\phi = \frac{1}{1+(1-\delta_c)\frac{1}{\sqrt{1-\delta}}}$. Note that, for $\delta \in (0,1]$, let $\sqrt{1-\delta} = X$,

$$\beta - \frac{1}{4} = \frac{1}{1-X}\frac{(\frac{1-X}{1-X^2}-\frac{1}{2})(1-\delta_c)}{1-\delta_c\frac{1-X}{1-X^2}} - \frac{1}{4} = -\frac{1}{4}\frac{X+\delta_c-1}{X-\delta_c+1}, \qquad \phi - \frac{1}{2} = \frac{1}{1+(1-\delta_c)\frac{1}{X}} - \frac{1}{2} = \frac{1}{2}\frac{X+\delta_c-1}{X-\delta_c+1}.$$

Hence, if $X = \sqrt{1-\delta} > 1-\delta_c$, $\beta < \frac{1}{4}$ and $\phi > \frac{1}{2}$, implying that for any given $v \in [0,1]$, $\tilde{\Pi}_f^o(v) > \tilde{\Pi}_\infty^o(v)$, further implying $\Pi_2^o > \Pi_\infty^o$. If, however, $X = \sqrt{1-\delta} \le 1-\delta_c$, $\beta \ge \frac{1}{4}$ and $\phi \le \frac{1}{2}$, implying that for any given $v \in [0,1]$, $\tilde{\Pi}_2^o(v) \le \tilde{\Pi}_\infty^o(v)$, further implying $\Pi_2^o \le \Pi_\infty^o$.

(i-b) Similarly, for $\tau = \infty$,

$$\Pi_2^* = \max_v \tilde{\Pi}_2(v) = (1-v)(p_0+v-c) + \frac{\delta}{4}p_0(1+v-c)^2 - \frac{\delta}{2}p_0(1-v)(1+v-c)$$
$$\Pi_\infty^* = \max_v \tilde{\Pi}_\infty(v) = (1-v)(p_0+v-c) + \delta p_0\beta(1+v-c)^2 - \delta\phi p_0(1-v)(1+v-c)$$

Hence, if $X = \sqrt{1-\delta} > 1-\delta_c$, $\beta < \frac{1}{4}$ and $\phi > \frac{1}{2}$, implying that for any given $v \in [0,1]$, $\tilde{\Pi}_f(v) > \tilde{\Pi}_\infty(v)$, further implying $\Pi_2^* > \Pi_\infty^*$. If, however, $X = \sqrt{1-\delta} \le 1-\delta_c$, $\beta \ge \frac{1}{4}$ and $\phi \le \frac{1}{2}$, implying that for any given $v \in [0,1]$, $\tilde{\Pi}_2(v) \le \tilde{\Pi}_\infty(v)$, further implying $\Pi_2^* \le \Pi_\infty^*$.

(ii-a) We have shown in Lemma 4 (iii) that $\Pi_2^o < \Pi_2^*$.

(ii-b) For $t = \infty$, recall that

$$\Pi_\infty^o = \beta(p_0+1-c)^2 \text{ and } \Pi_\infty^* = (1-v^*)(p_0+v^*-c) + \delta p_0\beta(1+v^*-c)^2 - \delta\phi p_0(1-v^*)(1+v^*-c)$$

where, as shown in the proof of Proposition SB.2, $v^* = \frac{(1-p_0\delta)c+(p_0(2\beta\delta-1)+1)}{2(1-p_0\delta(\beta+\phi))}$ if $1 < c < 2 - \frac{1-p_0}{1-\delta p_0}$, and $v^* = 1$ otherwise. This naturally divides the proof into two cases:

(ii-b-1) If $2 - \frac{1-p_0}{1-\delta p_0} < c < 1+p_0$, $v^* = 1$, $\Pi_\infty^o = \beta(p_0+1-c)^2$ and $\Pi_\infty^* = \delta p_0\beta(2-c)^2$. Thus: $\Pi_\infty^* - \Pi_\infty^o = \beta(\delta p_0(2-c)^2 - (p_0+1-c)^2)$. Therefore

$$\Pi_\infty^* > \Pi_\infty^o \Longleftrightarrow f(c) = \beta(\delta p_0(2-c)^2 - (p_0+1-c)^2) > 0.$$

Note that $f'(c) = \beta(-2\delta p_0(2-c) + 2(p_0+1-c))$ and $f''(c) = 2\beta(\delta p_0 - 1) < 0$, thus, $f(c)$ is concave in c over $[2-\frac{1-p_0}{1-\delta p_0}, 1+p_0]$. At the two boundaries, we prove that $f(c)$ is positive:

$$c = 1+p_0 : f(1+p_0) = \delta p_0(1-p_0)^2 > 0$$
$$c = 2-\frac{1-p_0}{1-\delta p_0} : f(2-\frac{1-p_0}{1-\delta p_0}) = (\delta p_0(\frac{1-p_0}{1-\delta p_0})^2 - (p_0 - 1 + \frac{1-p_0}{1-\delta p_0})^2) = \frac{(1-p_0)^2\,\delta p_0}{1-\delta p_0} > 0$$

Thus, over the whole interval, $[2-\frac{1-p_0}{1-\delta p_0}, 1+p_0]$, $f(c)$ must be positive.

(ii-b-2) If $1 < c < 2-\frac{1-p_0}{1-\delta p_0}$, we can write

$$\Pi^o_\infty < \Pi^*_\infty \Leftrightarrow \beta(p_0+1-c)^2 < (1-v^*)(p_0+v^*-c) + \delta p_0\beta(1+v^*-c)^2 - \delta\phi p_0(1-v^*)(1+v^*-c)$$

and with $X = \sqrt{1-\delta} : \beta = \frac{1}{2}\frac{X}{1+X}$ and $\phi = \frac{1}{1+X}$ and $\delta = 1-X^2, \Pi^o_\infty < \Pi^*_\infty \Leftrightarrow f(c) > 0,$ where

$$f(c) = \frac{\left(\left(p_0X^2-p_0+1\right)c^2+\left(2p_0-4p_0X^2-2\right)c+\left(5X^2p_0-X^2p_0^2-2Xp_0^2+2Xp_0-p_0+1\right)\right)(p_0-1)(X-1)}{2(Xp_0-2p_0+X^2p_0+2)(X+1)}$$
$$\frac{d}{dc}f(c) = g(c) = \frac{\left(\left(p_0X^2-p_0+1\right)c+\left(p_0-2p_0X^2-1\right)\right)(p_0-1)(X-1)}{(Xp_0-2p_0+X^2p_0+2)(X+1)}$$
$$\frac{d^2}{d^2c}f(c) = \frac{d}{dc}g(c) = \frac{\left(X^2p_0+1-p_0\right)(1-p_0)(1-X)}{(Xp_0-2p_0+X^2p_0+2)(X+1)} > 0$$

and we have:

$$f(2-\frac{1-p_0}{1-(1-X^2)p_0}) = \frac{1}{2}\frac{Xp_0(1-p_0)^2(1-X)}{(X^2p_0+1-p_0)} > 0$$
$$f(1) = \frac{1}{2}\frac{(2-2p_0+2X-Xp_0)Xp_0(1-p_0)(1-X)}{(Xp_0-2p_0+X^2p_0+2)(X+1)} > 0$$
$$g(2-\frac{1-p_0}{1-(1-X^2)p_0}) = 0$$
$$g(1) = -\frac{(1-p_0)(1-X)p_0X^2}{(Xp_0-2p_0+X^2p_0+2)(X+1)} < 0$$

Thus, over $[1, 2-\frac{1-p_0}{1-\delta p_0}]$, $f(c)$ is convex decreasing and reaches a minimum at $2-\frac{1-p_0}{1-\delta p_0}$, which is still positive, thus, $f(c)$ is positive over $[1, 2-\frac{1-p_0}{1-\delta p_0}]$. Hence, $\Pi^*_\infty - \Pi^o_\infty > 0$ for all c. □

SC.3 Proofs for Results in Section SB.4

Formulation of β Model

• Second-period pricing:

$$\Pi^*_2(p,\overline{v}) = \max_{P_2}(P_2-c)[\beta(p+\overline{v}-P_2)^+ + (1-\beta)(p_0+\overline{v}-P_2)^+]$$

Denote the optimal price by $P^*_2(p,\overline{v})$. It can be shown that, if $p_0+v-c<0$, then $P^*_2(p,\overline{v}) =$

$\min\left(\frac{p+v+c}{2}, p+v\right)$; otherwise,

$$P_2^*(p,\overline{v}) = \begin{cases} \frac{p_0+v+c}{2} & \text{if } p \leq \frac{\sqrt{1-\beta}p_0+c-v}{1+\sqrt{1-\beta}} \\ \beta\frac{p+v+c}{2} + (1-\beta)\frac{p_0+v+c}{2} & \text{if } \frac{\sqrt{1-\beta}p_0+c-v}{1+\sqrt{1-\beta}} < p \leq \frac{(1+\sqrt{\beta})p_0+v-c}{\sqrt{\beta}} \\ \frac{p+v+c}{2} & \text{if } p > \frac{(1+\sqrt{\beta})p_0+v-c}{\sqrt{\beta}} \end{cases}$$

In both cases,

$$\Pi_2^*(p,\overline{v}) = \max\left[\beta\frac{[(p+v-c)^+]^2}{4}, (1-\beta)\frac{[(p_0+v-c)^+]^2}{4}, \frac{[(\beta p+(1-\beta)p_0+v-c)^+]^2}{4}\right].$$

• For given P_1, the consumer cutoff $\overline{v}$ is determined by

$$p_0 + v - P_1 = \delta_c \mathrm{E}_p[\beta(p+v-P_2^*(p,v))^+ + (1-\beta)(p_0+v-P_2^*(p,v))^+],$$

where

$$\begin{aligned} &\beta(p+v-P_2^*(p,v))^+ + (1-\beta)(p_0+v-P_2^*(p,v))^+ \\ = &\begin{cases} \beta\frac{(p+v-c)^+}{2} & \text{if } p_0+v-c<0 \\ (1-\beta)\frac{p_0+v-c}{2} & \text{if } p_0+v-c \geq 0 \text{ and } p \leq \frac{\sqrt{1-\beta}p_0+c-v}{1+\sqrt{1-\beta}} \\ \beta\frac{p+v-c}{2} + (1-\beta)\frac{p_0+v-c}{2} & \text{if } p_0+v-c \geq 0 \text{ and } \frac{\sqrt{1-\beta}p_0+c-v}{1+\sqrt{1-\beta}} < p \leq \frac{(1+\sqrt{\beta})p_0+v-c}{\sqrt{\beta}} \\ \beta\frac{p+v-c}{2} & \text{if } p_0+v-c \geq 0 \text{ and } p > \frac{(1+\sqrt{\beta})p_0+v-c}{\sqrt{\beta}} \end{cases} \end{aligned}$$

which is not always continuous in p.

• The firm's profit function is given by

$$\Pi(v) = \int_0^1 \hat{\Pi}(p,v)d\Phi(p,v)$$

where

$$\hat{\Pi}(p,v) = (p_0+v-c)(1-v) - \delta_c(1-v)[\beta(p+v-P_2^*(p,v))^+ + (1-\beta)(p_0+v-P_2^*(p,v))^+] + \delta\Pi_2^*(p,v).$$

Proof of Lemma SB.10 We first show that when γ is sufficiently small, the likelihood ratio increases in $g \in \mathbb{R}$ for all x. Let $M = -\frac{1}{2}\frac{(1+\gamma)^2\tau^2}{x}$ and $N = -\frac{1}{2}\frac{(1-\gamma)^2\tau^2}{x}$. Then,

$$\begin{aligned} l(g,x) =& \frac{f_1(g,x)}{f_0(g,x)} = \frac{(1-\gamma)\exp(N(g-x)^2) + (1+\gamma)\exp(M(g-x)^2)}{(1-\gamma)\exp(Ng^2) + (1+\gamma)\exp(Mg^2)} \\ \frac{dl(g,x)}{dg} =& \tfrac{2(g-x)[(1-\gamma)\exp(Ng^2)+(1+\gamma)\exp(Mg^2)][N(1-\gamma)\exp(N(g-x)^2)+M(1+\gamma)\exp(M(g-x)^2)]}{[(1-\gamma)\exp(Ng^2)+(1+\gamma)\exp(Mg^2)]^2} \\ & - \tfrac{2g[N(1-\gamma)\exp(Ng^2)+M(1+\gamma)\exp(Mg^2)][(1-\gamma)\exp(N(g-x)^2)+(1+\gamma)\exp(M(g-x)^2)]}{[(1-\gamma)\exp(Ng^2)+(1+\gamma)\exp(Mg^2)]^2} \end{aligned}$$

where

$$\begin{aligned} & [(1-\gamma)\exp(Ng^2) + (1+\gamma)\exp(Mg^2)][N(1-\gamma)\exp(N(g-x)^2) + M(1+\gamma)\exp(M(g-x)^2)] \\ =& N(1-\gamma)^2\exp(Ng^2)\exp(N(g-x)^2) + M(1+\gamma)(1-\gamma)\exp(Mg^2)\exp(N(g-x)^2) \\ & + N(1-\gamma)(1+\gamma)\exp(Ng^2)\exp(M(g-x)^2) + M(1+\gamma)^2\exp(Mg^2)\exp(M(g-x)^2) \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{dl(g,x)}{dg}[(1-\gamma)\exp(Ng^2) + (1+\gamma)\exp(Mg^2)]^2 \\ =& -2x[N(1-\gamma)^2\exp(Ng^2)\exp(N(g-x)^2) + M(1+\gamma)^2\exp(Mg^2)\exp(M(g-x)^2)] \\ & + 2[(g-x)N - gM](1+\gamma)(1-\gamma)\exp(Mg^2)\exp(N(g-x)^2) \\ & + 2[(g-x)M - gN](1+\gamma)(1-\gamma)\exp(Ng^2)\exp(M(g-x)^2) \end{aligned}$$

It can be shown that when $\gamma = 0$, $\frac{dl(g,x)}{dg} > 0$ and $\frac{d^2l(g,x)}{dgd\gamma} = 0$ for all $g \in \mathbb{R}$ and $x \in [0,1]$. Hence, by continuity of $\frac{dl(g,x)}{dg}$ in γ, when γ is sufficiently small, $\frac{dl(g,x)}{dg} > 0$ for all $g \in \mathbb{R}$ and $x \in [0,1]$. This implies that $l(y,x)$ is invertible. Denote the inverse of $l(g,x)$ by $l^{-1}(y,x)$. It can be shown that $l^{-1}(y,x)$ increases in y. Let $\bar{y}(r,x) = l^{-1}(\frac{1-p_0}{p_0(\frac{1}{r}-1)}, x)$.

Next we prove the SOSD of the pre-posterior in x. By the definition of $\bar{y}(r,x)$, the expression of $\int_0^p \bar{\Phi}(r,x)dr$ becomes:

$$\int_0^p \bar{\Phi}(r,x)dr = \int_0^p \int_{\bar{y}(r,x)}^{+\infty} [p_0 f_1(y,x) + (1-p_0)f_0(y,x)]dydr$$

Note that $\bar{y}(r,x)$ increases in r. By the same derivation as in the proof of Proposition 1, we have

$$\int_0^p \bar{\Phi}(r,x)dr = (p-1)p_0\overline{F}_1(\bar{y}(p,x)) + p(1-p_0)\overline{F}_0(\bar{y}(r,x)) + p_0$$

where for $\theta = 0$ or 1,

$$\begin{aligned}
\overline{F}_\theta(\bar{y}(p,x)) = & \frac{1}{2}\int_{\bar{y}(p,x)}^{+\infty} \frac{(1-\gamma)\tau}{\sqrt{2\pi x}} \exp\left(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2(y-x\theta)^2}{x}\right) dy \\
& + \frac{1}{2}\int_{\bar{y}(p,x)}^{+\infty} \frac{(1+\gamma)\tau}{\sqrt{2\pi x}} \exp\left(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2(y-x\theta)^2}{x}\right) dy \\
& (\text{Let } z_1 = \frac{(1-\gamma)\tau(y-x\theta)}{\sqrt{x}},\ z_2 = \frac{(1+\gamma)\tau(y-x\theta)}{\sqrt{x}}) \\
= & \frac{1}{2}\int_{\frac{(1-\gamma)\tau(\bar{y}(p,x)-x\theta)}{\sqrt{x}}}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_1^2)dz_1 + \frac{1}{2}\int_{\frac{(1+\gamma)\tau(\bar{y}(p,x)-x\theta)}{\sqrt{x}}}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z_2^2)dz_2,
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{d}{dx}\overline{F}_\theta(\bar{y}(p,x)) \\
= & -\tfrac{1}{2\sqrt{2\pi}}\left[\exp(-\tfrac{1}{2}\tfrac{(1-\gamma)^2\tau^2(\bar{y}(p,x)-x\theta)^2}{x})\left(\tfrac{(1-\gamma)\tau(\bar{y}(p,x)-x\theta)}{\sqrt{x}}\right)'_x + \exp(-\tfrac{1}{2}\tfrac{(1+\gamma)^2\tau^2(\bar{y}(p,x)-x\theta)^2}{x})\left(\tfrac{(1+\gamma)\tau(\bar{y}(p,x)-x\theta)}{\sqrt{x}}\right)'_x\right] \\
= & -\tfrac{\tau}{2\sqrt{2\pi}}\left[(1-\gamma)\exp(-\tfrac{1}{2}\tfrac{(1-\gamma)^2\tau^2(\bar{y}(p,x)-x\theta)^2}{x}) + (1+\gamma)\exp(-\tfrac{1}{2}\tfrac{(1+\gamma)^2\tau^2(\bar{y}(p,x)-x\theta)^2}{x})\right]\left[\left(\tfrac{\bar{y}(p,x)}{\sqrt{x}}\right)'_x - (\sqrt{x}\theta)'_x\right]
\end{aligned}$$

Further, by definition of $\bar{y}(p,x)$, we have

$$\begin{aligned}
& \left[(1-\gamma)\exp(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2(\bar{y}(p,x)-x)^2}{x}) + (1+\gamma)\exp(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2(\bar{y}(p,x)-x)^2}{x})\right]p_0(1-p) \\
= & \left[(1-\gamma)\exp(-\frac{1}{2}\frac{(1-\gamma)^2\tau^2\bar{y}(p,x)^2}{x}) + (1+\gamma)\exp(-\frac{1}{2}\frac{(1+\gamma)^2\tau^2\bar{y}(p,x)^2}{x})\right]p(1-p_0)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \tfrac{d}{dx}\int_0^p \bar{\Phi}(r,x)dr \\
= & (p-1)p_0\tfrac{d}{dx}\overline{F}_1(\bar{y}(p,x)) + p(1-p_0)\tfrac{d}{dx}\overline{F}_0(\bar{y}(r,x)) \\
= & -\tfrac{(1-p)p_0\tau}{2\sqrt{2\pi}}\left[(1-\gamma)\exp(-\tfrac{(1-\gamma)^2\tau^2(\bar{y}(p,x)-x)^2}{2x}) + (1+\gamma)\exp(-\tfrac{(1+\gamma)^2\tau^2(\bar{y}(p,x)-x)^2}{2x})\right](\sqrt{x})'_x < 0. \quad \square
\end{aligned}$$