Stochastic Dominance Analysis without the Independence Axiom

Online Appendix

Simone Cerreia–Vioglio, Fabio Maccheroni, Massimo Marinacci
Department of Decision Sciences and IGIER
Università Bocconi

October 8, 2015

Consider V defined as in (9 in the main text). We first report a simple property.

Lemma 2 *$V : \mathcal{D} \to \mathbb{R}$ is continuous.*

Proof of Proposition 4. We want to compute the Gateaux derivative of V at F in direction $G - F$, that is,

$$\lim_{\theta \downarrow 0} \frac{V((1-\theta)F + \theta G) - V(F)}{\theta} \qquad \forall F, G \in \mathcal{D}. \tag{16}$$

The computation is simplified by the observation that for each function $f : [m, M] \to \mathbb{R}$ of bounded variation, the Riemann-Stieltjes integral $\int_m^M f(x)\, dv(x)$ coincides with the Lebesgue-Stieltjes integral $\int_{[m,M]} f dv$ of f with respect to the Borel measure induced on $[m, M]$ by any continuous and increasing extension of v to $\mathbb{R}$. Set $H = G - F$, and note that, provided the limit in (16) exists, it is equal to

$$\begin{aligned}
&= \lim_{\theta \downarrow 0} \frac{V(F + \theta(G-F)) - V(F)}{\theta} \\
&= \lim_{\theta \downarrow 0} \frac{\int_{[0,M]} w(1 - F - \theta H)\, dv - \int_{[m,0]} \tilde{w}(F + \theta H)\, dv - \int_{[0,M]} w(1-F)\, dv + \int_{[m,0]} \tilde{w}(F)\, dv}{\theta} \\
&= \lim_{\theta \downarrow 0} \int_{[0,M]} \frac{w(1 - F(x) - \theta H(x)) - w(1 - F(x))}{\theta} dv(x) - \int_{[m,0]} \frac{\tilde{w}(F(x) + \theta H(x)) - \tilde{w}(F(x))}{\theta} dv(x).
\end{aligned}$$

For each $x \in [m, M]$, we have that

- if $x \in [0, M]$ and $H(x) \neq 0$, then

  $$\begin{aligned}
  \lim_{\theta \downarrow 0} \frac{w(1 - F(x) - \theta H(x)) - w(1 - F(x))}{\theta} &= \lim_{\theta \downarrow 0} \frac{w(1 - F(x) - \theta H(x)) - w(1 - F(x))}{-\theta H(x)} (-H(x)) \\
  &= -w'(1 - F(x)) H(x)
  \end{aligned}$$

  and the same holds when $H(x) = 0$;

- if $x \in [m, 0]$ and $H(x) \neq 0$, then

  $$\lim_{\theta \downarrow 0} \frac{\tilde{w}(F(x) + \theta H(x)) - \tilde{w}(F(x))}{\theta} = \lim_{\theta \downarrow 0} \frac{\tilde{w}(F(x) + \theta H(x)) - \tilde{w}(F(x))}{\theta H(x)} H(x) = \tilde{w}'(F(x)) H(x)$$

  and the same holds when $H(x) = 0$.

Continuous differentiability on $[0,1]$ of w and $\tilde{w}$ implies their Lipschitzianity so that, for each $x \in [m,M]$ and each $\theta \in (0,1)$,

$$\left|\frac{w(1-F(x)-\theta H(x)) - w(1-F(x))}{\theta}\right| \leq \frac{L_w |1-F(x)-\theta(G(x)-F(x)) - (1-F(x))|}{\theta} \leq L_w$$

and

$$\left|\frac{\tilde{w}(F(x)+\theta H(x)) - \tilde{w}(F(x))}{\theta}\right| \leq \frac{L_{\tilde{w}} |F(x)+\theta(G(x)-F(x)) - F(x)|}{\theta} \leq L_{\tilde{w}}.$$

Therefore the Dominated Convergence Theorem applied to each sequence $\theta_n \to 0^+$ yields that

$$\begin{aligned}\lim_{\theta \downarrow 0} \frac{V(F+\theta(G-F)) - V(F)}{\theta} &= \int_{[0,M]} -w'(1-F(x))(G(x)-F(x))\,dv(x) \\ &\quad - \int_{[m,0]} \tilde{w}'(F(x))(G(x)-F(x))\,dv(x).\end{aligned}$$

Now define

$$\phi_F(x) = \begin{cases} w'(1-F(x)) & x \in [0,M] \\ \tilde{w}'(F(x)) & x \in [m,0) \end{cases} \tag{17}$$

and note that ϕ_F is bounded and Borel measurable on $[m,M]$ with

$$\lim_{\theta \downarrow 0} \frac{V(F+\theta(G-F)) - V(F)}{\theta} = -\int_{[m,M]} (G(x)-F(x))\,\phi_F(x)\,dv(x).$$

Setting "$du_F = \phi_F dv$", or more precisely $u_F(x) = \int_{[m,x]} \phi_F dv$ for all $x \in [m,M]$, it is not difficult to show that $u_F \in C([m,M])$ and

$$\begin{aligned}\int_{[m,M]} (G(x)-F(x))\,\phi_F(x)\,dv(x) &= \int_m^M (G-F)\,du_F \\ &= (G(M)-F(M))\,u_F(M) - (G(m)-F(m))\,u_F(m) - \left(\int_m^M u_F dG - \int_m^M u_F dF\right) \\ &= -\left(\int_{[m,M]} u_F dG - \int_{[m,M]} u_F dF\right) = -\int_{[m,M]} u_F\, d(G-F)\end{aligned}$$

where the second equality follows by integration by parts, the third by $G(M) = F(M) = 1$ and $u_F(m) = 0$, and the last one by definition, proving the statement. ■

Lemma 3 *If $w, \tilde{w} : [0,1] \to [0,1]$ are continuously differentiable, $m < 0 < M$, and all local utilities u_F of the preference functional (9 in the main text) are differentiable (resp., continuously differentiable) on (m,M), then v is differentiable (resp., continuously differentiable) on (m,M).*

In this case, $v'(0) \neq 0$, implies $\tilde{w}(p) = 1 - w(1-p)$ for all $p \in [0,1]$.

Before entering the proof's details notice that if the preference $\succsim$ represented by V is consistent with any risk order or any stochastic dominance of a degree $n \geq 3$, then all local utilities u_F are continuously differentiable. In fact, each u_F admits $n-2$ continuous derivatives on (m,M) because its $(n-2)$-th derivative is either convex or concave and so it is continuous; clearly its derivatives of lower order are also continuous since they are differentiable.

Proof. Recall that, for each $F \in \mathcal{D}$,

$$u_F(x) = \int_{[m,x]} \left[w'(1-F(y))\,1_{[0,M]}(y) + \tilde{w}'(F(y))\,1_{[m,0)}(y)\right] dv(y) \qquad \forall x \in [m,M]$$

and, for all $p \in [0,1]$, set $F_p = pG_m + (1-p)\, G_M$.

First we prove that for each $p \in [0,1]$,

$$w'(1-p)\frac{v(z)-v(x)}{z-x} = \frac{u_{F_p}(z)-u_{F_p}(x)}{z-x} \qquad \forall x,z \in [0,M) \tag{18}$$

$$\tilde{w}'(p)\frac{v(z)-v(x)}{z-x} = \frac{u_{F_p}(z)-u_{F_p}(x)}{z-x} \qquad \forall x,z \in (m,0] \tag{19}$$

provided $x \neq z$. Since u_{F_p} is differentiable, for each $p \in [0,1]$,

$$w'(1-p)\, v'_+(x) = \left(u_{F_p}\right)'_+(x) \qquad \forall x \in [0,M) \tag{20}$$

$$w'(1-p)\, v'_-(x) = \left(u_{F_p}\right)'_-(x) \qquad \forall x \in (0,M) \tag{21}$$

$$\tilde{w}'(p)\, v'_-(x) = \left(u_{F_p}\right)'_-(x) \qquad \forall x \in (m,0] \tag{22}$$

$$\tilde{w}'(p)\, v'_+(x) = \left(u_{F_p}\right)'_+(x) \qquad \forall x \in (m,0) \tag{23}$$

and all left and right derivatives above are finite.

Distinguish the following cases

- for $x \in [0,M]$,
$$u_F(x) = \int_{[m,0)} \tilde{w}'(F(y))\, dv(y) + \int_{[0,x]} w'(1-F(y))\, dv(y)$$
- in particular, for $x = 0$,
$$u_F(0) = \int_{[m,0)} \tilde{w}'(F(y))\, dv(y)$$
- for $x \in [m,0)$,
$$u_F(x) = \int_{[m,x]} \tilde{w}'(F(y))\, dv(y)$$
and this also holds for $x = 0$.

For every $x,z \in [0,M)$, if $z > x$, then

$$u_F(z) - u_F(x) = \int_{(x,z]} w'(1-F(y))\, dv(y)$$

and if $x > z$, then $u_F(x) - u_F(z) = \int_{(z,x]} w'(1-F(y))\, dv(y)$, thus

$$u_F(z) - u_F(x) = \begin{cases} \int_{(x,z]} w'(1-F(y))\, dv(y) & z > x \\ -\int_{(z,x]} w'(1-F(y))\, dv(y) & z < x \end{cases}.$$

If $F = F_p$, then $F_p = p$ on (m,M) and so

$$\frac{u_{F_p}(z)-u_{F_p}(x)}{z-x} = \begin{cases} w'(1-p)\frac{v(z)-v(x)}{z-x} & z > x \\ -w'(1-p)\frac{v(x)-v(z)}{z-x} & z < x \end{cases} = w'(1-p)\frac{v(z)-v(x)}{z-x} \qquad \forall x,z \in [0,M),$$

proving (18).

Analogously, for every $x,z \in (m,0]$, if $z > x$, then

$$u_F(z) - u_F(x) = \int_{(x,z]} \tilde{w}'(F(y))\, dv(y)$$

and if $x > z$, then $u_F(x) - u_F(z) = \int_{(z,x]} \tilde{w}'(F(y))\, dv(y)$, thus

$$u_F(z) - u_F(x) = \begin{cases} \int_{(x,z]} \tilde{w}'(F(y))\, dv(y) & z > x \\ -\int_{(z,x]} \tilde{w}'(F(y))\, dv(y) & z < x \end{cases}.$$

If $F = F_p$, then $F_p = p$ on (m, M) and so

$$\frac{u_{F_p}(z) - u_{F_p}(x)}{z - x} = \begin{cases} \tilde{w}'(p)\frac{v(z)-v(x)}{z-x} & z > x \\ -\tilde{w}'(p)\frac{v(x)-v(z)}{z-x} & z < x \end{cases} = \tilde{w}'(p)\frac{v(z) - v(x)}{z - x} \qquad \forall x, z \in (m, 0],$$

proving (19).

Equations (20), (21), (22), (23), and finiteness of all left and right derivatives follow easily (note that there exist $p, q \in (0, 1)$ such that $\tilde{w}'(p) \neq 0$ and $w'(1 - q) \neq 0$).

The latter equations imply that: since u_{F_p} is differentiable (resp., continuously differentiable) on (m, M) for all $p \in [0, 1]$, then v is differentiable (resp., continuously differentiable) on $(0, M)$ and $(m, 0)$.

But more is true, choosing $x = 0$ in (20) and (22), it follows that

$$\tilde{w}'(p)\, v'_-(0) = \left(u_{F_p}\right)'_-(0) = \left(u_{F_p}\right)'_+(0) = w'(1 - p)\, v'_+(0) \qquad \forall p \in [0, 1], \tag{24}$$

then by integrating both sides of (24) over $[0, 1]$,

$$v'_-(0) = v'_-(0)\,\tilde{w}(1) = \int_0^1 \tilde{w}'(p)\, v'_-(0)\, dp = \int_0^1 w'(1 - p)\, v'_+(0)\, dp = v'_+(0)$$

so that v is differentiable also at 0. If u_F is continuously differentiable on (m, M) for all $F \in \mathcal{D}$, then choosing p such that $\tilde{w}'(p) \neq 0$, and taking $x_n \to 0^-$ it follows that

$$\tilde{w}'(p)\, v'(x_n) = \left(u_{F_p}\right)'(x_n) \to \left(u_{F_p}\right)'(0) = \tilde{w}'(p)\, v'(0)$$

and so $v'(x_n) \to v'(0)$. Analogously, choosing q such that $w'(1 - q) \neq 0$, and taking $y_n \to 0^+$ it follows that

$$w'(1 - q)\, v'(y_n) = \left(u_{F_q}\right)'(y_n) \to \left(u_{F_q}\right)'(0) = w'(1 - q)\, v'(0)$$

and so $v'(y_n) \to v'(0)$, so that v is continuously differentiable also at 0.

Finally, if $v'(0) \neq 0$, then (24) becomes

$$\tilde{w}'(p) = w'(1 - p) \qquad \forall p \in [0, 1]$$

so that $\tilde{w}(q) = \int_0^q \tilde{w}'(p)\, dp = \int_0^q w'(1 - p)\, dp = 1 - w(1 - q)$ for all $q \in [0, 1]$. ■

Proof of Proposition 5. As observed immediately after the statement of Lemma 3, consistency with the third degree risk order guarantees that all local utilities u_F are continuously differentiable (with convex derivative), hence the same lemma yields both continuous differentiability of v on (m, M) and the second part of the statement. ■

Proof of Corollary 1. (i) implies (ii). By Proposition 5 and as observed immediately after the statement of Lemma 3, note that if $\succsim$ is consistent with third degree stochastic dominance, then v and all local derivatives are continuously differentiable on (m, M). Moreover, $\succsim$ is consistent with second degree stochastic dominance, and this implies that u_F is concave for all $F \in \mathcal{D}$. But then v' is decreasing on $(m, 0]$, by (22), and on $[0, M)$, by (20), thus it is decreasing on (m, M), so that v is concave. Since it is strictly increasing, then $v'(0) \neq 0$ and, by Lemma 3, $\tilde{w}(p) = 1 - w(1 - p)$ for all $p \in [0, 1]$.

Arbitrarily choose $p < q$ in $(0, 1)$ and $x < t$ in $(m, 0)$. For each $F \in \mathcal{D}$,

$$\begin{aligned} u_F(z) - u_F(x) &= \int_{(x,z]} \tilde{w}'(F(y))\, dv(y) \qquad \forall z \in (x, 0) \\ u_F(z) - u_F(t) &= \int_{(t,z]} \tilde{w}'(F(y))\, dv(y) \qquad \forall z \in (t, 0) \end{aligned}$$

hence choosing $H_{pq} \in \mathcal{D}$ such that $H_{pq} = p$ in a neighborhood U_x of x and $H_{pq} = q$ in a neighborhood U_t of t, then

$$\begin{aligned} \frac{u_{H_{pq}}(z) - u_{H_{pq}}(x)}{z - x} &= \tilde{w}'(p) \frac{v(z) - v(x)}{z - x} \qquad \forall z \in U_x,\ z > x \\ \frac{u_{H_{pq}}(z) - u_{H_{pq}}(t)}{z - t} &= \tilde{w}'(q) \frac{v(z) - v(t)}{z - t} \qquad \forall z \in U_t,\ z > t \end{aligned}$$

so that

$$v'(x)\,\tilde{w}'(p) = u'_{H_{pq}}(x) \geq u'_{H_{pq}}(t) = v'(t)\,\tilde{w}'(q)$$

for all $0 < p < q < 1$ and all $m < x < t < 0$. But then letting $t \to x$, by continuity of v', we obtain $v'(x)\,\tilde{w}'(p) \geq v'(x)\,\tilde{w}'(q)$ and $v'(x) > 0$ yields $\tilde{w}'(p) \geq \tilde{w}'(q)$, in turn, this implies that $\tilde{w}$ is concave. As a consequence, $w(p) = 1 - \tilde{w}(1 - p)$ is convex.

Now consider G_0 and observe that for every $x, z \in [0, M)$, with $z > x$,

$$u_{G_0}(z) - u_{G_0}(x) = \int_{(x,z]} w'(1 - G_0(y))\,dv(y) = w'(0)\,(v(z) - v(x))$$

and so

$$u'_{G_0}(x) = w'(0)\,v'(x).$$

Analogously, for every $t, z \in (m, 0]$, with $z < t$, then

$$u_{G_0}(z) - u_{G_0}(t) = -\int_{(z,t]} \tilde{w}'(G_0(y))\,dv(y) = -\int_{(z,t]} \tilde{w}'(0)\,dv(y)$$

because v is continuous at 0 and $\tilde{w}'(G_0(y)) = \tilde{w}'(0)$ on $(m, 0)$, thus

$$u'_{G_0}(t) = \tilde{w}'(0)\,v'(t).$$

For $x = t = 0$, we have $w'(0)\,v'(0) = \tilde{w}'(0)\,v'(0)$ and

$$w'(0) = \tilde{w}'(0)$$

but $\tilde{w}'(p) = w'(1 - p)$ for all $p \in (0, 1)$ and by continuity $\tilde{w}'(0) = w'(1)$, that is, $w'(0) = w'(1)$ and w' being increasing must be constant on $[0, 1]$. Therefore $w(p) = p = \tilde{w}(p)$ for all $p \in [m, M]$.

Finally, this implies

$$V(F) = \int_{[m,M]} v(y)\,dF(y) \qquad \forall F \in \mathcal{D}$$

and consistency with third degree stochastic dominance also implies that v' is convex.

(ii) implies (i). It is a well known fact. ■

Derivation of Equation (15). Define $W : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by

$$W(F, G) = \int_{[m,M]} \int_{[m,M]} \phi(x, y)\,dF(x)\,dG(y) \qquad \forall (F, G) \in \mathcal{D} \times \mathcal{D}.$$

It is immediate to check that W is affine in both components, $W(F, G) = W(G, F)$ for all $(F, G) \in \mathcal{D} \times \mathcal{D}$, and $V(F) = W(F, F)$ for all $F \in \mathcal{D}$. We start by observing two facts:

(a) Fix $F, H \in \mathcal{D}$. If we define $F_\gamma = \gamma F + (1 - \gamma) H$ for all $\gamma \in (0, 1]$, then

$$\begin{aligned} V(F_\gamma) = W(F_\gamma, F_\gamma) &= \gamma W(F, F_\gamma) + (1 - \gamma) W(H, F_\gamma) \\ &= \gamma^2 W(F, F) + \gamma (1 - \gamma) W(F, H) + (1 - \gamma) \gamma W(H, F) + (1 - \gamma)^2 W(H, H) \\ &= \gamma^2 W(F, F) + 2 (1 - \gamma) \gamma W(F, H) + (1 - \gamma)^2 W(H, H). \end{aligned}$$

(b) Fix $F,G\in\mathcal{D}$. If $\int_{[m,M]}\phi(x,y)\,dF(x)\geq\int_{[m,M]}\phi(x,y)\,dG(x)$ for all $y\in[m,M]$, then for each $H\in\mathcal{D}$

$$W(F,H)=\int_{[m,M]}\int_{[m,M]}\phi(x,y)\,dF(x)\,dH(y)\geq\int_{[m,M]}\int_{[m,M]}\phi(x,y)\,dG(x)\,dH(y)=W(G,H).$$

In particular, since H was arbitrarily chosen, we have that

$$V(F)=W(F,F)\geq W(G,F)=W(F,G)\geq W(G,G)=V(G).$$

Next, by facts (a) and (b), observe that

$$\begin{aligned}
F\succsim^{*}G &\iff \lambda F+(1-\lambda)H\succsim\lambda G+(1-\lambda)H \qquad \forall\lambda\in(0,1],\forall H\in\mathcal{D}\\
&\iff V(\lambda F+(1-\lambda)H)-V(\lambda G+(1-\lambda)H)\geq 0 \qquad \forall\lambda\in(0,1],\forall H\in\mathcal{D}\\
&\iff \lambda^{2}(V(F)-V(G))+2\lambda(1-\lambda)(W(F,H)-W(G,H))\geq 0 \qquad \forall\lambda\in(0,1],\forall H\in\mathcal{D}\\
&\iff \lambda(V(F)-V(G))+2(1-\lambda)(W(F,H)-W(G,H))\geq 0 \qquad \forall\lambda\in(0,1],\forall H\in\mathcal{D}\\
&\iff V(F)\geq V(G)\text{ and }W(F,H)-W(G,H)\geq 0 \qquad \forall H\in\mathcal{D}\\
&\iff V(F)\geq V(G)\text{ and }\int_{[m,M]}\phi(x,y)\,dF(x)\geq\int_{[m,M]}\phi(x,y)\,dG(x) \qquad \forall y\in[m,M]\\
&\iff \int_{[m,M]}\phi(x,y)\,dF(x)\geq\int_{[m,M]}\phi(x,y)\,dG(x) \qquad \forall y\in[m,M],
\end{aligned}$$

proving the statement. ■

Proof of Proposition 6. Before starting, note that since v is strictly increasing and continuously differentiable, we have that there exists $\bar{y}\in(m,M)$ such that $v'(\bar{y})>0$ and $v'(x)\geq 0$ for all $x\in[m,M]$. Wlog, we can assume that $v(\bar{y})=0$. By contradiction, assume that $\succsim$ is prudent. By point (v) of Lemma 1 and (15 in the main text), this means that the set $\mathcal{U}^{*}=\{\phi(\cdot,y)\}_{y\in[m,M]}$ is included in $\langle\mathcal{R}_3\rangle=\{u\in C([m,M]):u'$ exists and is convex on $(m,M)\}$. Among all the elements of $\mathcal{U}^{*}$, consider $\phi(\cdot,\bar{y}):I\to\mathbb{R}$. By Masatlioglu and Raymond (2014), note that

$$\begin{aligned}
\phi(x,\bar{y}) &= \frac{v(x)+v(\bar{y})+\mu(v(x)-v(\bar{y}))+\mu(v(\bar{y})-v(x))}{2}\\
&= \frac{1}{2}(v(x)+\mu(v(x))+\mu(-v(x))) \qquad \forall x\in I.
\end{aligned}$$

Since $\phi(\cdot,\bar{y})$ is differentiable on $(m,M)\ni\bar{y}$, observe also that

$$\phi'_{\pm}(\bar{y},\bar{y})=\frac{1}{2}\left(v'(\bar{y})+\mu_{\pm}(0)\,v'(\bar{y})-\mu_{\mp}(0)\,v'(\bar{y})\right)$$

and $\phi'_{+}(\bar{y},\bar{y})=\phi'_{-}(\bar{y},\bar{y})$. Since $v'(\bar{y})>0$, this implies that

$$v'(\bar{y})+\mu_{+}(0)\,v'(\bar{y})-\mu_{-}(0)\,v'(\bar{y})=v'(\bar{y})+\mu_{-}(0)\,v'(\bar{y})-\mu_{+}(0)\,v'(\bar{y}),$$

that is, $\mu_{+}(0)=\mu_{-}(0)$, a contradiction with $\mu'_{-}(0)>\mu'_{+}(0)$. ■

Proof of Proposition 7. Consider $F,G\in\mathcal{D}$. Define $f:[0,1]\to\mathbb{R}$ by $f(t)=V((1-t)F+tG)$ for all $t\in[0,1]$. By routine arguments, it can be shown that f is continuous on $[0,1]$. As for differentiability of f on $(0,1)$, we follow Huber and Ronchetti (2009, pages 39-40). Note that $F_{t+h}=\left(1-\frac{h}{1-t}\right)F_t+\frac{h}{1-t}G$ hence for each $t\in(0,1)$

$$\begin{aligned}
f'_{+}(t) &= \lim_{h\downarrow 0}\frac{V(F_{t+h})-V(F_t)}{h}=\lim_{h\downarrow 0}\frac{V\left(\left(1-\frac{h}{1-t}\right)F_t+\frac{h}{1-t}G\right)-V(F_t)}{h}\\
&= \lim_{h\downarrow 0}\frac{1}{1-t}\frac{V\left(\left(1-\frac{h}{1-t}\right)F_t+\frac{h}{1-t}G\right)-V(F_t)}{\frac{h}{1-t}}=\frac{1}{1-t}\int_I u_{F_t}\,d(G-F_t)=\int_I u_{F_t}\,d(G-F)
\end{aligned}$$

(note that as h goes to 0^+ eventually $F_{t+h} \in \mathcal{D}$) analogously $F_{t-h} = \left(1 - \frac{h}{t}\right) F_t + \frac{h}{t} F$ and

$$\begin{aligned} f'_{-}(t) &= \lim_{h \downarrow 0} \frac{V(F_{t-h}) - V(F_t)}{-h} = \lim_{h \downarrow 0} \frac{V\left(\left(1 - \frac{h}{t}\right) F_t + \frac{h}{t} F\right) - V(F_t)}{-h} \\ &= \lim_{h \downarrow 0} -\frac{1}{t} \frac{V\left(\left(1 - \frac{h}{t}\right) F_t + \frac{h}{t} F\right) - V(F_t)}{\frac{h}{t}} = -\frac{1}{t} \int_I u_{F_t} d(F - F_t) = \int_I u_{F_t} d(G - F) \end{aligned}$$

that is, $f'(t) = \int u_{F_t} dG - \int u_{F_t} dF$. By the Mean Value Theorem for functions of a real variable, it follows that there exists $t \in (0,1)$ such that

$$\int_I u_{F_t} dG - \int_I u_{F_t} dF = f'(t) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0) = V(G) - V(F),$$

proving the statement. ■

References

[1] P. J. Huber and E. M. Ronchetti, *Robust statistics*, 2^{nd} ed., Wiley, New York, 2009.

[2] Y. Masatlioglu and C. Raymond, A behavioral analysis of stochastic reference dependence, mimeo, 2014.