A. Mathematical analysis of the homogeneous case

A.1. Introduction

In this section we describe the mathematical model for a homogeneous dike ring and the outline of this appendix. We assume that the upgrades take place at moments t_k, $k = 1, 2\ldots$, and the value of the upgrade at t_k is denoted as u_k. It is assumed that

$$u_k > 0, \quad t_{k+1} > t_k \geq t_0 := 0, \qquad \text{for all } k \geq 1. \tag{25}$$

To simplify the notation we denote h_{t_k} simply as h_k. Hence, also defining $h_0 := 0$, we have

$$h_k = h_{k-1} + u_k, \qquad \text{for all } k \geq 1. \tag{26}$$

We define the infinite sequences u, h and τ by

$$u := (u_1; u_2; \ldots), \qquad h := (h_1; h_2; \ldots), \qquad \tau := (t_1; t_2; \ldots). \tag{27}$$

We call the pair (h, τ) admissible if (25) and (26) hold.

Obviously, the sequences u and h are linearly related; knowing u one easily finds h and vice versa. The total (discounted) costs are therefore denoted as a function $f(h, \tau)$ of h and τ alone. One has $f(h, \tau) = \mathcal{I}(h, \tau) + \mathcal{A}(h, \tau)$, where $\mathcal{I}(h, \tau)$ denotes the investment costs and $\mathcal{A}(h, \tau)$ the expected damage costs. According to (7), (8) and (9) these costs can be given as follows:[1]

$$\mathcal{I}(h, \tau) = \sum_{k=1}^{\infty} D(u_k) e^{\lambda h_k - \delta t_k}, \tag{28}$$

$$\mathcal{A}(h, \tau) = \sum_{k=1}^{\infty} \frac{S_0}{\beta_1} \left[e^{\beta_1 t_k} - e^{\beta_1 t_{k-1}} \right] e^{-\theta h_{k-1}}, \tag{29}$$

where $\beta_1 := \beta - \delta_1$ and where $D(u_k) = c + b u_k$ for $u_k > 0$, with $c > 0$ and $b \geq 0$.[2] Our task is to find admissible h and τ such that $f(h, \tau)$ is minimal.

It is natural to assume that $S_0 > 0$ and $D(x) > 0$ whenever $x > 0$. Besides this we also assume

$$\delta > 0, \quad \theta > 0, \quad \lambda \geq 0, \quad \beta_1 + \delta > 0, \quad \theta\delta - \lambda\beta_1 > 0. \tag{30}$$

The exposition is outlined as follows. We derive first order optimality conditions in Section A.2 and we establish that if (h, τ) is a stationary point then (h, τ) is completely determined by the corresponding sequence u. Each such sequence u is called a *stationary sequence*. Such sequences are characterized in Section A.3 by a condition on the pairs (u_k, u_{k+1}). In Section A.4 we derive

[1] We assume throughout that $\beta_1 \neq 0$. If $\beta_1 = 0$ then $\mathcal{A}(h, \tau) = \sum_{k=1}^{\infty} S_0 \left(t_k - t_{k-1}\right) e^{-\theta h_{k-1}}$, which is the limiting value of the expression in (29) when β_1 goes to zero. Results similar to the ones in this appendix can be obtained for $\beta_1 = 0$.

[2] Due to the positive fixed cost term c we will have $u_k > 0$, for all k, in any optimal solution.

some important properties of stationary sequences. A crucial result is that there exists a unique positive number ν such that the sequence $\nu, \nu, \nu, \ldots$ is stationary. The corresponding stationary point is called the periodic solution. If $t_1 \geq 0$ for this sequence, we say that the dike is *healthy* at $t = 0$, otherwise *unhealthy*. In Section A.5 the structure of stationary sequences becomes completely clear. It turns out that there exist infinitely many stationary sequences. In Section A.6 we show that the objective value in an arbitrary stationary point depends only on the first entry u_1 in its stationary sequence. Finally, Section A.7.1 contains the main result of the paper, namely that the periodic sequence is optimal for healthy dikes. Using this result we can also deal with unhealthy dikes. In Section A.7.2 it is shown that the optimal solution for an unhealthy dike is obtained by taking $t_1 = t_0$ and u_1 such that the dike becomes healthy at t_0; from t_2 on the solution is then periodic.

A.2. First order optimality conditions

To simplify the presentation we define $\Delta(u_k)$ as follows and we require that the marginal costs of investing, and therefore $\Delta(u_k)$, are positive:

$$\Delta(u_k) := D'(u_k) + \lambda D(u_k) = b + \lambda D(u_k) > 0. \tag{31}$$

Note that this implies that if $b = 0$ then $\lambda > 0$, and vice versa. Now let (h, τ) be a feasible solution. Since $u_k = h_k - h_{k-1}$ and $u_{k+1} = h_{k+1} - h_k$, we have

$$\frac{\partial u_k}{\partial h_k} = 1, \qquad \frac{\partial u_{k+1}}{\partial h_k} = -1.$$

At a stationary pair (h, τ) the partial derivatives of $f(h, \tau)$ with respect to t_k and h_k vanish. By computing these derivatives we obtain that a pair (h, τ) is stationary with respect to t_k and h_k if and only if

$$0 = -\delta D(u_k) e^{\lambda h_k - \delta t_k} + S_0 e^{\beta_1 t_k - \theta h_{k-1}} - S_0 e^{\beta_1 t_k - \theta h_k}, \tag{32}$$

$$0 = \Delta(u_k) e^{\lambda h_k - \delta t_k} + \frac{\theta S_0}{\beta_1} e^{\beta_1 t_k - \theta h_k} - D'(u_{k+1}) e^{\lambda h_{k+1} - \delta t_{k+1}} - \frac{\theta S_0}{\beta_1} e^{\beta_1 t_{k+1} - \theta h_k}. \tag{33}$$

Defining

$$\varrho(x) := \ln \frac{\delta D(x)}{S_0 \left(e^{\theta x} - 1\right)}, \quad x > 0, \tag{34}$$

we may rewrite the condition (32) for stationarity with respect to t_k as follows:

$$\varrho(u_k) = (\beta_1 + \delta) t_k - (\lambda + \theta) h_k. \tag{35}$$

Obviously, if u is known, then h can be computed. The above relation reveals that then also the sequence τ can be computed, because since $\beta_1 + \delta > 0$ we may write

$$t_k = \frac{\varrho(u_k) + (\lambda + \theta) h_k}{\beta_1 + \delta}. \tag{36}$$

Hence, a stationary pair (h,τ) is completely determined by its sequence u. Therefore we call the sequence u a *stationary sequence* if the corresponding pair (h,τ) is stationary. Similarly, u is said to be admissible (optimal) if (h,τ) is admissible (optimal).

If it happens that (36) yields a negative value for t_1, the dike requires an immediate upgrade (i.e., we must upgrade the dike at $t=0$). In that case an optimal solution may not be stationary with respect to t_1; then (32) and (36) only hold for $k\geq 2$. We deal with this situation in Section A.7.2. Up till then we always assume that u is such that $t_1\geq 0$, unless stated otherwise.

We show in the Sections A.3 – A.5 that there exist infinitely many stationary sequences. Among these a specific sequence exists that is periodic, in the sense that the quantity u_k is independent of k. The main result of the paper is that this periodic solution is unique and optimal.

In the next section we focus on the implications of (33), the condition for stationarity with respect to u_k, while assuming stationarity with respect to t_k and t_{k+1}.

A.3. Stationary sequences

It will be convenient to introduce the function $\kappa(x)$ according to

$$\kappa(x):=\frac{D(x)}{e^{\theta x}-1}. \tag{37}$$

Then, using (34), the condition (36) for stationarity with respect to t_k can be stated as follows:

$$S_0 e^{(\beta_1+\delta)t_k}=\delta\kappa(u_k)e^{(\lambda+\theta)h_k}. \tag{38}$$

Hence we may reduce the first two terms in the right-hand site of (33) as follows:

$$\begin{aligned}\Delta(u_k)e^{\lambda h_k-\delta t_k}+\frac{\theta S_0}{\beta_1}e^{\beta_1 t_k-\theta h_k} &= e^{-\delta t_k}\left[\Delta(u_k)e^{\lambda h_k}+\frac{\theta}{\beta_1}S_0 e^{(\beta_1+\delta)t_k}e^{-\theta h_k}\right]\\ &= e^{-\delta t_k}\left[\Delta(u_k)e^{\lambda h_k}+\frac{\theta}{\beta_1}\delta\kappa(u_k)e^{(\lambda+\theta)h_k}e^{-\theta h_k}\right]\\ &= e^{-\delta t_k}e^{\lambda h_k}\left[\Delta(u_k)+\frac{\theta\delta}{\beta_1}\kappa(u_k)\right].\end{aligned} \tag{39}$$

Note that (33) requires that the sum of the last two terms in (33) is equal to the above sum. By using stationarity with respect to t_{k+1}, the sum of the last two terms can be reduced in a similar way. Since $D'(u_{k+1})=b$, this sum equals

$$\begin{aligned}be^{\lambda h_{k+1}-\delta t_{k+1}}+\frac{\theta S_0}{\beta_1}e^{\beta_1 t_{k+1}-\theta h_k} &= e^{-\delta t_{k+1}}\left[be^{\lambda h_{k+1}}+\frac{\theta}{\beta_1}S_0 e^{(\beta_1+\delta)t_{k+1}}e^{-\theta h_k}\right]\\ &= e^{-\delta t_{k+1}}\left[be^{\lambda h_{k+1}}+\frac{\theta}{\beta_1}\delta\kappa(u_{k+1})e^{(\lambda+\theta)h_{k+1}}e^{-\theta h_k}\right]\\ &= e^{-\delta t_{k+1}}e^{\lambda h_{k+1}}\left[b+\frac{\theta\delta}{\beta_1}\kappa(u_{k+1})e^{\theta u_{k+1}}\right].\end{aligned} \tag{40}$$

It will be convenient to introduce the following notations:

$$\bar{\beta} := \frac{\beta_1}{\beta_1 + \delta}, \qquad \bar{\delta} := \frac{\delta}{\beta_1 + \delta}. \tag{41}$$

Note that $\bar{\beta} + \bar{\delta} = 1$. Due to (36), the ratio between the coefficients of the bracketed expression in (39) and (40) satisfies

$$\frac{e^{-\delta t_{k+1}}}{e^{-\delta t_k}} \frac{e^{\lambda h_{k+1}}}{e^{\lambda h_k}} = \frac{e^{-\bar{\delta}\left[\varrho(u_{k+1}) + (\lambda+\theta) h_{k+1}\right]}}{e^{-\bar{\delta}[\varrho(u_k) + (\lambda+\theta) h_k]}} e^{\lambda u_{k+1}} = \left(\frac{e^{\varrho(u_{k+1})}}{e^{\varrho(u_k)}} \right)^{-\bar{\delta}} e^{-\left[\bar{\delta}(\lambda+\theta) - \lambda\right] u_{k+1}}.$$

We also define auxiliary parameters q and r according to

$$q := \bar{\delta}(\lambda + \theta) - \lambda = \theta\bar{\delta} - \lambda\bar{\beta} = \frac{\theta\delta - \lambda\beta_1}{\beta_1 + \delta}, \qquad r := \theta - q = \bar{\beta}(\lambda + \theta) = \beta_1 \frac{\lambda + \theta}{\beta_1 + \delta}. \tag{42}$$

Due to the definitions (34) and (37) of $\varrho(x)$ and $\kappa(x)$ we have

$$S_0 e^{\varrho(x)} = \frac{\delta D(x)}{e^{\theta x} - 1} = \delta \kappa(x). \tag{43}$$

Hence we obtain

$$\frac{e^{-\delta t_{k+1}}}{e^{-\delta t_k}} \frac{e^{\lambda h_{k+1}}}{e^{\lambda h_k}} = \left(\frac{\kappa(u_{k+1})}{\kappa(u_k)} \right)^{-\bar{\delta}} e^{-q u_{k+1}} = \frac{\kappa(u_{k+1})^{-\bar{\delta}} e^{-q u_{k+1}}}{\kappa(u_k)^{-\bar{\delta}}}.$$

It follows that (33) can be written as follows:

$$\kappa(u_k)^{-\bar{\delta}} \left[\Delta(u_k) + \frac{\theta\delta}{\beta_1} \kappa(u_k) \right] = \kappa(u_{k+1})^{-\bar{\delta}} e^{-q u_{k+1}} \left[b + \frac{\theta\delta}{\beta_1} \kappa(u_{k+1}) e^{\theta u_{k+1}} \right].$$

Now defining functions $L(x)$, $\mathcal{L}(x)$, $R(x)$ and $\mathcal{R}(x)$ according to

$$L(x) := \Delta(x) + \frac{\theta\delta}{\beta_1} \kappa(x), \tag{44}$$

$$R(x) := e^{-qx} \left(b + \frac{\theta\delta}{\beta_1} \kappa(x) e^{\theta x} \right) = b e^{-qx} + \frac{\theta\delta}{\beta_1} \kappa(x) e^{rx}, \tag{45}$$

and

$$\mathcal{L}(x) := \kappa(x)^{-\bar{\delta}} L(x), \qquad \mathcal{R}(x) := \kappa(x)^{-\bar{\delta}} R(x), \tag{46}$$

we may rewrite equation (33) in the following compact form:

$$\mathcal{L}(u_k) = \mathcal{R}(u_{k+1}). \tag{47}$$

It may be clear that this equation holds if we have stationarity with respect to u_k, t_k and t_{k+1}. As a consequence we have that an infinite sequence $u = u_1, u_2, \ldots, u_k, u_{k+1}, \ldots$ is stationary only if every pair (u_k, u_{k+1}) satisfies (47).

A.4. Properties of $\kappa(x)$, $\mathcal{L}(x)$ and $\mathcal{R}(x)$

In this section we deal with some technical results as a preparation for subsequent sections, where we use these results to prove that a periodic solution exists and, moreover, that it minimizes our objective function $f(h,\tau)$.

LEMMA 1. *$\kappa(x)$ is strictly convex and monotonically decreasing to zero.*

Proof: By taking the derivative of $\kappa(x)$, as given in (37), we get

$$\kappa'(x) = \frac{b(e^{\theta x}-1) - \theta e^{\theta x} D(x)}{(e^{\theta x}-1)^2} = \frac{b - \theta e^{\theta x}\kappa(x)}{e^{\theta x}-1}, \tag{48}$$

which implies that $\kappa'(x)$ has the same sign as the expression

$$b(e^{\theta x}-1) - \theta e^{\theta x} D(x).$$

The value of this expression at $x=0$ is $-\theta c$, which is negative, and its derivative is

$$b\theta e^{\theta x} - \theta^2 e^{\theta x} D(x) - \theta e^{\theta x} b = -\theta^2 e^{\theta x} D(x) < 0.$$

Therefore, $\kappa'(x)$ is negative. Hence $\kappa(x)$ is monotonically decreasing, as stated in the lemma. Moreover, the definition of $\kappa(x)$ makes clear that if x grows then the limiting value is zero. By taking the derivative of $\kappa'(x)$, we get

$$\begin{aligned}\kappa''(x) &= -\frac{[b - \theta e^{\theta x}\kappa(x)]\theta e^{\theta x}}{(e^{\theta x}-1)^2} + \frac{-\theta^2 e^{\theta x}\kappa(x) - \theta e^{\theta x}\kappa'(x)}{e^{\theta x}-1} \\ &= -\frac{\theta e^{\theta x}\kappa'(x)}{e^{\theta x}-1} - \frac{\theta e^{\theta x}\left(\theta\kappa(x) + \kappa'(x)\right)}{e^{\theta x}-1} \\ &= -\frac{\theta e^{\theta x}}{e^{\theta x}-1}\left[2\kappa'(x) + \theta\kappa(x)\right].\end{aligned} \tag{49}$$

The last expression has the same sign as $-2\kappa'(x) - \theta\kappa(x)$, for which we have

$$-2\kappa'(x) - \theta\kappa(x) = -\frac{2(b - \theta e^{\theta x}\kappa(x)) + (e^{\theta x}-1)\theta\kappa(x)}{e^{\theta x}-1} = \frac{\theta e^{\theta x}\kappa(x) + \theta\kappa(x) - 2b}{e^{\theta x}-1}.$$

Multiplication with $(e^{\theta x}-1)^2$ yields the expression

$$\theta e^{\theta x} D(x) + \theta D(x) - 2(e^{\theta x}-1)b.$$

This expression is positive if $b=0$, whereas its derivative to b equals

$$\theta x e^{\theta x} + \theta x - 2(e^{\theta x}-1).$$

Putting $y := \theta x$, one may easily verify that $ye^y + y - 2(e^y - 1) \geq 0$ for $y \geq 0$. It thus follows that $\kappa''(x) > 0$, for all $x > 0$, which means that $\kappa(x)$ is strictly convex. Hence the proof of the lemma is complete. Q.E.D.

It will be convenient to deal with one more property of $\kappa(x)$, namely

$$\kappa'(x)+(\lambda+\theta)\kappa(x)=\frac{\Delta(x)-\theta\kappa(x)}{e^{\theta x}-1}. \tag{50}$$

This follows by replacing this equality successively by each of the following equivalent equalities:

$$\begin{aligned}
[\kappa'(x)+(\theta+\lambda)\kappa(x)]\,(e^{\theta x}-1) &= \Delta(x)-\theta\kappa(x),\\
\left[b-\theta\kappa(x)e^{\theta x}+(\theta+\lambda)D(x)\right] &= \Delta(x)-\theta\kappa(x), \quad \text{by (48), (37)}\\
\Delta(x)-\theta\kappa(x)e^{\theta x}+\theta D(x) &= \Delta(x)-\theta\kappa(x), \quad \text{by (31)}\\
\kappa(x)(e^{\theta x}-1) &= D(x).
\end{aligned}$$

Since the last relation resembles the definition of $\kappa(x)$, this proves (50).

In the next lemma we consider the limits of $\mathcal{L}(x)$ and $\mathcal{R}(x)$ when x approaches zero or infinity.

LEMMA 2. [3] *One has*

$$\lim_{x\to\infty}\mathcal{L}(x)=\infty.$$

Moreover,

$$\begin{aligned}
\beta_1>0 &\;\Rightarrow\; \lim_{x\downarrow 0}\mathcal{L}(x)=\lim_{x\downarrow 0}\mathcal{R}(x)=\lim_{x\to\infty}\mathcal{R}(x)=\infty,\\
\beta_1<0 &\;\Rightarrow\; \lim_{x\downarrow 0}\mathcal{L}(x)=\lim_{x\downarrow 0}\mathcal{R}(x)=\lim_{x\to\infty}\mathcal{R}(x)=0.
\end{aligned}$$

Proof: The proof uses the following facts:

- $\theta>0$, $\lambda\geq 0$ and $\delta>0$;
- $\bar\beta+\bar\delta=1$, $\bar\delta>0$;
- β_1, $\bar\beta$ and r have the same sign;
- $D(0)=c>0$ and $\Delta(0)=b+\lambda c>0$.

Using these properties, the proof becomes more or less straightforward: using only the sum and product rules for taking limits we may write:

$$\begin{aligned}
\lim_{x\downarrow 0}\mathcal{L}(x) &= \lim_{x\downarrow 0}\left(\frac{D(0)}{\theta x}\right)^{-\bar\delta}\left(\Delta(0)+\frac{\theta\delta}{\beta_1}\frac{D(0)}{\theta x}\right)=\lim_{x\downarrow 0}\frac{\theta\delta}{\beta_1}\left(\frac{D(0)}{\theta x}\right)^{\bar\beta},\\
\lim_{x\to\infty}\mathcal{L}(x) &= \lim_{x\to\infty}\left(\frac{D(x)}{e^{\theta x}}\right)^{-\bar\delta}\left(\Delta(x)+\frac{\theta\delta}{\beta_1}\frac{D(x)}{e^{\theta x}}\right)=\lim_{x\to\infty}\left(\frac{e^{\theta x}}{D(x)}\right)^{\bar\delta}\Delta(x),\\
\lim_{x\downarrow 0}\mathcal{R}(x) &= \lim_{x\downarrow 0}\left(\frac{D(0)}{\theta x}\right)^{-\bar\delta}\left(b+\frac{\theta\delta}{\beta_1}\frac{D(0)}{\theta x}\right)=\lim_{x\downarrow 0}\frac{\theta\delta}{\beta_1}\left(\frac{D(0)}{\theta x}\right)^{\bar\beta}=\lim_{x\downarrow 0}\mathcal{L}(x),\\
\lim_{x\to\infty}\mathcal{R}(x) &= \lim_{x\to\infty}\left(\frac{D(x)}{e^{\theta x}}\right)^{-\bar\delta}e^{-qx}\left(b+\frac{\theta\delta}{\beta_1}D(x)\right)=\lim_{x\to\infty}\frac{\theta\delta}{\beta_1}D(x)^{\bar\beta}e^{\lambda\bar\beta x},
\end{aligned}$$

[3] It may be mentioned that if $\beta_1=0$ then the limits are the same as when $\beta_1>0$.

where the last equality is based on the identity $q = \theta\bar{\delta} - \lambda\bar{\beta}$ in (42). From the above relations one easily deduces the results stated in the lemma. It may be pointed out that for the last limit one needs to use that if $b = 0$ then $\lambda > 0$, and if $\lambda = 0$ then $b > 0$, due to (31). This completes the proof. Q.E.D.

We proceed by showing that the derivatives of $\mathcal{L}(x)$ and $\mathcal{R}(x)$ are closely related. After the proofs of the next two lemmas we discuss some important implications.

LEMMA 3. [4] *One has*

$$\mathcal{L}'(x) = e^{qx}\mathcal{R}'(x) = \kappa(x)^{-\bar{\delta}-1}\left[\lambda b\kappa(x) + \bar{\delta}\left[\theta\kappa(x) - \Delta(x)\right]\kappa'(x)\right].$$

Proof: One easily verifies that

$$L'(x) = \lambda b + \frac{\theta\delta}{\beta_1}\kappa'(x), \qquad R'(x) = e^{-qx}\left(-qb + \frac{\theta\delta}{\beta_1}\left[\kappa'(x) + r\kappa(x)\right]e^{\theta x}\right). \tag{51}$$

Using this, straightforward computations yield

$$\begin{aligned}
\mathcal{L}'(x) &= -\bar{\delta}\kappa(x)^{-\bar{\delta}-1}\kappa'(x)L(x) + \kappa(x)^{-\bar{\delta}}L'(x) \\
&= -\bar{\delta}\kappa(x)^{-\bar{\delta}-1}\kappa'(x)\left(\Delta(x) + \frac{\theta\delta}{\beta_1}\kappa(x)\right) + \kappa(x)^{-\bar{\delta}}\left(\lambda b + \frac{\theta\delta}{\beta_1}\kappa'(x)\right) \\
&= \kappa(x)^{-\bar{\delta}-1}\left[-\bar{\delta}\kappa'(x)\left(\Delta(x) + \frac{\theta\delta}{\beta_1}\kappa(x)\right) + \kappa(x)\left(\lambda b + \frac{\theta\delta}{\beta_1}\kappa'(x)\right)\right].
\end{aligned}$$

The coefficient of $\kappa(x)\kappa'(x)$ in the bracketed expression can be reduced as follows:

$$-\bar{\delta}\frac{\theta\delta}{\beta_1} + \frac{\theta\delta}{\beta_1} = \left(-\bar{\delta} + 1\right)\frac{\theta\delta}{\beta_1} = \bar{\beta}\frac{\theta\delta}{\beta_1} = \frac{\theta\delta}{\beta_1 + \delta} = \theta\bar{\delta}.$$

Substitution gives

$$\mathcal{L}'(x) = \kappa(x)^{-\bar{\delta}-1}\left[\lambda b\kappa(x) + \bar{\delta}\left[\theta\kappa(x) - \Delta(x)\right]\kappa'(x)\right]. \tag{52}$$

While using (51) again, we start the computation of $\mathcal{R}'(x)$ in the obvious way:

$$\begin{aligned}
\mathcal{R}'(x) &= -\bar{\delta}\kappa(x)^{-\bar{\delta}-1}\kappa'(x)R(x) + \kappa(x)^{-\bar{\delta}}R'(x) \\
&= \kappa(x)^{-\bar{\delta}-1}\left[-\bar{\delta}\kappa'(x)R(x) + \kappa(x)R'(x)\right] \\
&= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[-\bar{\delta}\kappa'(x)\left(b + \frac{\theta\delta}{\beta_1}\kappa(x)e^{\theta x}\right) + \kappa(x)\left(-qb + \frac{\theta\delta}{\beta_1}\left(\kappa'(x) + r\kappa(x)\right)e^{\theta x}\right)\right] \\
&= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[-\bar{\delta}\kappa'(x)b - qb\kappa(x) + \frac{\theta\delta}{\beta_1}\kappa(x)e^{\theta x}\left[-\bar{\delta}\kappa'(x) + \kappa'(x) + \kappa(x)r\right]\right].
\end{aligned}$$

[4] If $\beta_1 = 0$ then the derivatives of $\mathcal{L}(x)$ and $\mathcal{R}(x)$ are also as given by this lemma, with $q = \theta$ and $\bar{\delta} = 1$. This implies that all subsequent results are also valid if $\beta_1 = 0$.

The coefficients of $\kappa(x)\kappa'(x)$ and $\kappa(x)^2$ can be reduced as follows:

$$\kappa(x)\kappa'(x): \ \frac{\theta\delta}{\beta_1}(-\bar{\delta}+1)e^{\theta x} = \frac{\theta\delta}{\beta_1}\bar{\beta}e^{\theta x} = \theta\bar{\delta}e^{\theta x}$$
$$\kappa(x)^2: \ \frac{\theta\delta}{\beta_1}re^{\theta x} = \theta(q+\lambda)e^{\theta x},$$

where the last equality follows by using (42):

$$\frac{\delta r}{\beta_1} = \frac{\delta}{\beta_1}\cdot\beta_1\frac{\lambda+\theta}{\beta_1+\delta} = \bar{\delta}(\lambda+\theta) = q+\lambda. \tag{53}$$

After substituting this we use (48), (50) and (52) as follows:

$$\begin{aligned}
\mathcal{R}'(x) &= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[-\left[\bar{\delta}\kappa'(x)+q\kappa(x)\right]b + \left[\bar{\delta}\kappa'(x)+(q+\lambda)\kappa(x)\right]\theta\kappa(x)e^{\theta x}\right] \\
&= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[-\left[\bar{\delta}\kappa'(x)+q\kappa(x)\right]b + \left[\bar{\delta}\kappa'(x)+(q+\lambda)\kappa(x)\right]\left[b-(e^{\theta x}-1)\kappa'(x)\right]\right] \\
&= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[\lambda\kappa(x)b - \bar{\delta}\left[\kappa'(x)+(\theta+\lambda)\kappa(x)\right](e^{\theta x}-1)\kappa'(x)\right] \\
&= \kappa(x)^{-\bar{\delta}-1}e^{-qx}\left[\lambda\kappa(x)b - \bar{\delta}\left[\Delta(x)-\theta\kappa(x)\right]\kappa'(x)\right] \\
&= e^{-qx}\mathcal{L}'(x),
\end{aligned}$$

thus proving the lemma. Q.E.D.

Lemma 3 implies that $\mathcal{L}'(x)=0$ holds if and only if $\mathcal{R}'(x)=0$ and this happens if and only if

$$\lambda b\kappa(x) + \bar{\delta}\left[\theta\kappa(x)-\Delta(x)\right]\kappa'(x) = 0. \tag{54}$$

We claim that this equation has exactly one solution. This is the content of the following lemma.

Lemma 4. *The equation (54) has exactly one solution.*

Proof: By multiplying the left-hand side in (54) with $e^{\theta x}-1$, we get the function $k(x)$ defined by

$$k(x) := \lambda bD(x) + \bar{\delta}\left[\theta D(x) - (e^{\theta x}-1)\Delta(x)\right]\kappa'(x). \tag{55}$$

Since $e^{\theta x}-1>0$, it suffices if the equation $k(x)=0$ has exactly one solution. One has

$$\lim_{x\downarrow 0} k(x) = \lim_{x\downarrow 0}\lambda bD(0) + \bar{\delta}\theta D(0)\frac{b-\theta\frac{D(0)}{\theta x}}{\theta x} = \lim_{x\downarrow 0} -\bar{\delta}\frac{D(0)^2}{x^2} = -\infty,$$

where we used (48). We next consider the behavior of $k(x)$ if x grows to infinity. We first consider the case where $\lambda=0$. Then using $D(x)=c+bx$ and $\Delta(x)=b>0$, due to (31), we have

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty}\bar{\delta}\left[\theta bx - e^{\theta x}b\right]\frac{b-\theta e^{\theta x}\frac{bx}{e^{\theta x}}}{e^{\theta x}} = \lim_{x\to\infty}\bar{\delta}\left[-e^{\theta x}b\right]\frac{b-\theta bx}{e^{\theta x}} = \lim_{x\to\infty}\bar{\delta}b^2(\theta x-1) = \infty.$$

Now let $\lambda>0$. We then need to distinguish between $b>0$ and $b=0$. If $b>0$ then

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty}\lambda b(bx) + \bar{\delta}\left[\theta bx - e^{\theta x}\lambda bx\right]\frac{b-\theta e^{\theta x}\frac{bx}{e^{\theta x}}}{e^{\theta x}}$$

$$= \lim_{x\to\infty} \lambda b^2 x + \bar{\delta}\left[-e^{\theta x}\lambda b x\right]\left[-\frac{\theta b x}{e^{\theta x}}\right] = \lim_{x\to\infty} \bar{\delta}\lambda\theta b^2 x^2 = \infty.$$

Finally, if $b=0$, then $D(x)=c$ and $\Delta(x)=\lambda c>0$. Hence we get

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty} \bar{\delta}\left[\theta c - (e^{\theta x}-1)\lambda c\right]\frac{-\theta e^{\theta x} c}{\left(e^{\theta x}-1\right)^2} = \lim_{x\to\infty} \bar{\delta}\frac{\lambda\theta e^{\theta x} c^2}{e^{\theta x}-1} = \bar{\delta}\lambda\theta c^2 > 0.$$

From the above results we conclude that if x runs from 0 to ∞ then $k(x)$ grows from $-\infty$ to ∞ (if $b>0$) or to a positive number (if $b=0$). We now prove that $k(x)=0$ occurs at most once for $x>0$, by showing that the derivative of $k(x)$ is positive if $k(x)=0$. Using (55), straightforward computations yield

$$\begin{aligned} k'(x) &= \lambda b^2 + \bar{\delta}\left[\theta b - \theta e^{\theta x}\Delta(x) - (e^{\theta x}-1)\lambda b\right]\kappa'(x) + \bar{\delta}\left[\theta D(x) - (e^{\theta x}-1)\Delta(x)\right]\kappa''(x) \\ &= \lambda b^2 + \bar{\delta}\left[\theta b - \theta e^{\theta x}\Delta(x) - (e^{\theta x}-1)\lambda b\right]\kappa'(x) - \frac{\lambda b D(x)}{\kappa'(x)}\kappa''(x). \end{aligned}$$

The bracketed expression is negative, because $(e^{\theta x}-1)\lambda b \geq 0$ and, since $b-\Delta(x) = -\lambda D(x)$,

$$\theta b - \theta e^{\theta x}\Delta(x) = \theta\left[b - (e^{\theta x}-1)\Delta(x) - \Delta(x)\right] = \theta\left[-(e^{\theta x}-1)\Delta(x) - \lambda D(x)\right] < 0.$$

Since $\kappa'(x)<0$ and $\kappa''(x)>0$, by Lemma 1, we may conclude that $k(x)=0$ implies $k'(x)>0$, as desired. Hence the lemma follows. Q.E.D.

The unique solution of (54) will be denoted as $\bar{\nu}$. We conclude from Lemma 3 and Lemma 4 that $\mathcal{L}'(\bar{\nu}) = \mathcal{R}'(\bar{\nu}) = 0$, and $\bar{\nu}$ is the only point where $\mathcal{L}'(x)$ and $\mathcal{R}'(x)$ vanish. The above proof also reveals that $\mathcal{L}'(x)$ and $\mathcal{R}'(x)$ change sign at $x=\bar{\nu}$. Hence, without further proof we may state the next lemma.

Lemma 5. *$\mathcal{L}(x)$ and $\mathcal{R}(x)$ are decreasing if $x<\bar{\nu}$ and increasing if $x>\bar{\nu}$.*

Lemma 6. *One has $\mathcal{L}(\bar{\nu}) < \mathcal{R}(\bar{\nu})$ and $\mathcal{L}(\nu) = \mathcal{R}(\nu)$ for some unique number $\nu > \bar{\nu}$.*

Proof: We have $\mathcal{L}'(x) = e^{qx}\mathcal{R}'(x)$ and $q>0$. Since $e^{qx}>1$, it follows that if $x<\bar{\nu}$ then $\mathcal{L}(x)$ decreases faster than $\mathcal{R}(x)$ and if $x>\bar{\nu}$ then $\mathcal{L}(x)$ increases faster than $\mathcal{R}(x)$. In order to prove the inequality in the lemma we must distinguish the cases $\beta_1>0$ and $\beta_1<0$. We start with $\beta_1<0$. From Lemma 2 we know that the limiting values of $\mathcal{L}(x)$ and $\mathcal{R}(x)$ are zero when x approaches zero. Since $\mathcal{L}(x)$ decreases faster than $\mathcal{R}(x)$ if $x\in(0,\bar{\nu})$, we conclude that $\mathcal{L}(x)<\mathcal{R}(x)$ for all $x<\bar{\nu}$. This implies in particular that $\mathcal{L}(\bar{\nu}) < \mathcal{R}(\bar{\nu})$. If $\beta_1>0$ we need an additional argument, because then $\mathcal{L}(x)$ and $\mathcal{R}(x)$ become unbounded when x approaches zero, as we know from Lemma 2.[5] Obviously the same arguments apply as just were used for the case where $\beta_1<0$ if we show that

[5] The argument above then no longer works as becomes clear by considering the function $f(x)=g(x)+1/(1+x)$, with $g(x)=1/x$ and $x>0$. These functions become unbounded if x goes to zero and $f'(x)<g'(x)$ for all $x>0$. Despite this one has $f(x)>g(x)$ for all $x>0$.

$\mathcal{L}(x) < \mathcal{R}(x)$ holds when x approaches zero. In order to prove the latter, using (44), (45) and (46) we may write

$$\mathcal{R}(x) - \mathcal{L}(x) = \kappa(x)^{-\bar{\delta}}\left[be^{-qx} + \frac{\theta\delta}{\beta_1}\kappa(x)e^{rx} - \Delta(x) - \frac{\theta\delta}{\beta_1}\kappa(x)\right].$$

Since $\kappa(x)$ is positive, by Lemma 1, the sign of $\mathcal{R}(x) - \mathcal{L}(x)$ is the same as the sign of the function

$$\begin{aligned}\varepsilon(x) &:= be^{-qx} + \frac{\theta\delta}{\beta_1}\kappa(x)e^{rx} - b - \lambda D(x) - \frac{\theta\delta}{\beta_1}\kappa(x)\\ &= (e^{-qx} - 1)b - \lambda D(x) + \frac{\theta\delta}{\beta_1}\frac{D(x)}{e^{\theta x} - 1}(e^{rx} - 1).\end{aligned}$$

One has

$$\begin{aligned}\lim_{x\downarrow 0}\varepsilon(x) &= \lim_{x\downarrow 0}\left[(e^{-qx} - 1)b - \lambda D(x) + \frac{\theta\delta}{\beta_1}\frac{D(x)}{e^{\theta x} - 1}(e^{rx} - 1)\right]\\ &= \lim_{x\downarrow 0}\left[(e^{-qx} - 1)b - \lambda D(x) + \frac{r\delta}{\beta_1}D(x)\frac{\theta x}{e^{\theta x} - 1}\frac{e^{rx} - 1}{rx}\right]\\ &= \left(-\lambda + \frac{\delta r}{\beta_1}\right)D(0) = qD(0) > 0,\end{aligned}$$

where the last equality is due to (53). Hence we have shown that in both cases we have $\mathcal{L}(\bar{\nu}) < \mathcal{R}(\bar{\nu})$. It remains to prove that we have $\mathcal{L}(\nu) = \mathcal{R}(\nu)$ for some unique number $\nu > \bar{\nu}$. Obviously, if such a ν exists it must be larger than $\bar{\nu}$, due to what we just proved. We next consider the behaviour of $\varepsilon(x)$ if x approaches infinity. We may write

$$\begin{aligned}\lim_{x\to\infty}\varepsilon(x) &= \lim_{x\to\infty}\left[(e^{-qx} - 1)b - \lambda D(x) + \frac{\theta\delta}{\beta_1}\frac{D(x)}{e^{\theta x} - 1}(e^{rx} - 1)\right]\\ &= -b + \lim_{x\to\infty}\left[-\lambda + \frac{\theta\delta}{\beta_1}\frac{e^{rx} - 1}{e^{\theta x} - 1}\right]D(x).\end{aligned}$$

If $\beta_1 < 0$ then $r < 0$, and the limit becomes $-b - \lambda\lim_{x\to\infty} D(x)$, which is certainly negative, because $b \geq 0$, $\lambda \geq 0$ and $b + \lambda > 0$. On the other hand, if $\beta_1 > 0$ then $r > 0$. Using also $r - \theta = -q < 0$ we get

$$\lim_{x\to\infty}\varepsilon(x) = -b + \lim_{x\to\infty}\left[-\lambda + \frac{\theta\delta}{\beta_1}e^{(r-\theta)x}\right]D(x) = -b - \lambda\lim_{x\to\infty} D(x).$$

Thus it follows that if x grows to infinity then $\mathcal{R}(x)$ becomes smaller than $\mathcal{L}(x)$. So there must exist an $x > \bar{\nu}$ such that $\mathcal{R}(x) = \mathcal{L}(x)$. Since $\mathcal{L}(x)$ increases faster than $\mathcal{R}(x)$ for $x \in (\bar{\nu}, \infty)$, from this point on $\mathcal{L}(x)$ still grows faster than $\mathcal{R}(x)$, which gives the unicity of ν. This completes the proof of the lemma. Q.E.D.

The above lemma implies that $u_{k+1} = u_k = \nu$ yields a solution of (47), because then $\mathcal{L}(u_k) = \mathcal{R}(u_{k+1})$. It follows that the sequence ν, ν, ν, ... is stationary. Our final aim is to establish that if this so-called *periodic sequence* is admissible then it minimizes $f(h,\tau)$. But this will require some more analysis in subsequent sections.

We conclude this section by pointing out another consequence of Lemma 2, namely

$$\beta_1 \mathcal{R}(x) > 0, \quad \forall x > 0. \tag{56}$$

This is obvious if $\beta_1 > 0$, because then (45) implies $\mathcal{R}(x) > 0$ for all $x > 0$. If $\beta_1 < 0$ then Lemma 2 gives that $\mathcal{R}(x)$ converges to zero both when x approaches zero and when x grows to infinity. Due to Lemma 5 this implies that $\mathcal{R}(x) < 0$ for all $x > 0$.

A.5. Finding all stationary sequences

We are now ready to deal with the search for stationary sequences. Recall that we have stationarity with respect to u_k, t_k, and t_{k+1} only if the pair (u_k, u_{k+1}) satisfies $\mathcal{L}(u_k) = \mathcal{R}(u_{k+1})$. We call any such pair a *stationary pair*. Obviously, in a stationary sequence all pairs (u_k, u_{k+1}) are stationary.

Let $u_k > 0$ be arbitrary. We are interested in values for u_{k+1} such that the pair (u_k, u_{k+1}) is stationary, i.e., such that $\mathcal{L}(u_k) = \mathcal{R}(u_{k+1})$. Since $\mathcal{R}(x) \geq \mathcal{R}(\bar{\nu})$ for all $x > 0$, this equation has no solution if $\mathcal{L}(u_k) < \mathcal{R}(\bar{\nu})$. Hence, a successor of u_k exists only if $\mathcal{L}(u_k) \geq \mathcal{R}(\bar{\nu})$. If $\mathcal{L}(u_k) = \mathcal{R}(\bar{\nu})$, then we have only one solution, namely $u_{k+1} = \bar{\nu}$. This happens for two values of u_k, one smaller than $\bar{\nu}$ and one larger than $\bar{\nu}$; these values for u_k are denoted as ν_1 and ν_2, respectively. See Figure 3. If $\mathcal{L}(u_k) > \mathcal{R}(\bar{\nu})$ there are two possible successors, one smaller than $\bar{\nu}$ and one larger than $\bar{\nu}$; we

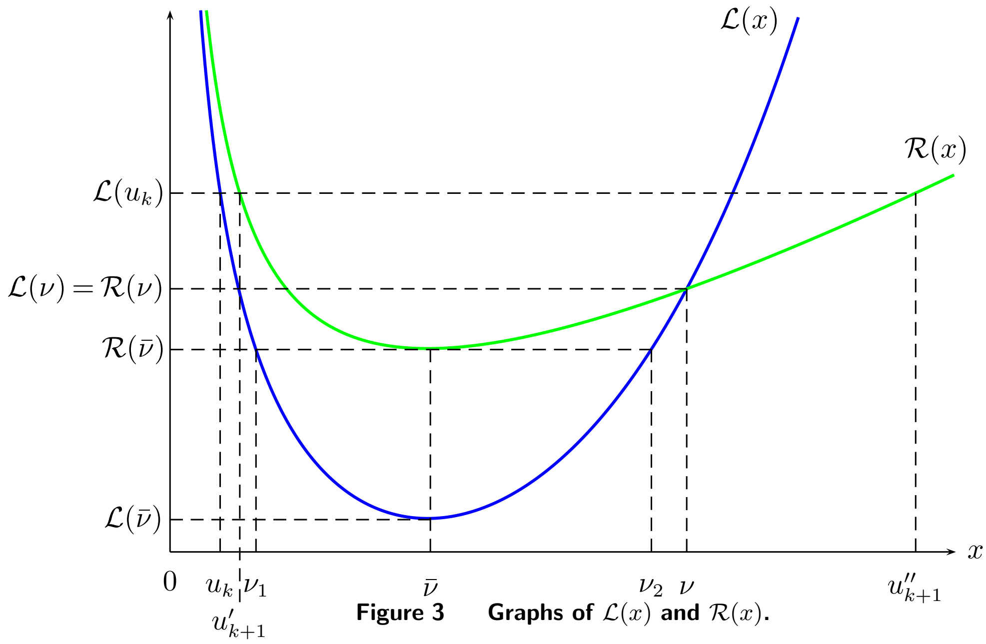

Figure 3 Graphs of $\mathcal{L}(x)$ and $\mathcal{R}(x)$.

denote these successors as $u'_{k+1} \leq \bar{\nu}$ and $u''_{k+1} \geq \bar{\nu}$ respectively.

The above discussion makes clear that if $\nu_1 < u_k < \nu_2$ then u_k does not occur in a stationary sequence. The following lemma provides an even stronger statement.

LEMMA 7. *If u_k occurs in a stationary sequence then $\mathcal{L}(u_k) \geq \mathcal{L}(\nu)$.*

Proof: Suppose that $\mathcal{L}(u_k) < \mathcal{L}(\nu)$. Then we necessarily have $u_k < \nu$. If u_k occurs in a stationary sequence, its successor u_{k+1} satisfies $\mathcal{L}(u_k) = \mathcal{R}(u_{k+1})$, and moreover $u_{k+1} < \nu$. Since $\mathcal{L}(x) < \mathcal{R}(x)$ for all $x < \nu$, the latter implies that $\mathcal{L}(u_{k+1}) < \mathcal{R}(u_{k+1})$. It follows that $\mathcal{L}(u_{k+1}) < \mathcal{R}(u_{k+1}) = \mathcal{L}(u_k) < \mathcal{L}(\nu)$. Thus we have shown that $\mathcal{L}(u_k) < \mathcal{L}(\nu)$ implies $\mathcal{L}(u_{k+1}) < \mathcal{L}(u_k)$. This means that the $\mathcal{L}$-values of the successors of u_k in the sequence form a strictly decreasing sequence. This sequence of $\mathcal{L}$-values is bounded from below, by $\mathcal{L}(\bar{\nu})$. Hence the $\mathcal{L}$-values converge. Of course, the value of this limit is less than $\mathcal{L}(\nu)$. On the other hand, the convergence of the $\mathcal{L}$-values implies that $\mathcal{L}(u_k) - \mathcal{L}(u_{k+1})$ converges to zero. Since the given sequence is stationary, this implies that $\mathcal{R}(u_{k+1}) - \mathcal{L}(u_{k+1})$ converges to zero. But this is possible only if u_{k+1} converges to ν, which gives that $\mathcal{L}(u_{k+1})$ converges to $\mathcal{L}(\nu)$. So we arrive at a contradiction. Hence the lemma follows. Q.E.D.

Using the above lemma we can prove our main result in this section, which is the next result.

LEMMA 8. *If u_k occurs in a stationary sequence then $\mathcal{R}(u_k) \geq \mathcal{R}(\nu)$.*

Proof: If $u_k \geq \nu$ then we have $\mathcal{R}(u_k) \geq \mathcal{R}(\nu)$, because $\mathcal{R}(x)$ is increasing for $x \geq \nu$. On the other hand, if $u_k \leq \nu$, then $\mathcal{R}(u_k) \geq \mathcal{L}(u_k)$ (cf. Figure 3). By Lemma 7 we have $\mathcal{L}(u_k) \geq \mathcal{L}(\nu)$. Hence, $\mathcal{R}(u_k) \geq \mathcal{L}(\nu)$. Since $\mathcal{L}(\nu) = \mathcal{R}(\nu)$, we get $\mathcal{R}(u_k) \geq \mathcal{R}(\nu)$, as desired. Q.E.D.

We already found the periodic stationary sequence, which has $u_k = \nu$ for all k. An important question is whether there exist other stationary sequences than the periodic sequence. In fact, infinitely other stationary sequences exist (at least if $\beta_1 > 0$!). As an example, Figure 4 graphically shows how the first six u_k's of such a sequence can be obtained. Only the values of u_1, u_4 and u_6 are indicated along the horizontal axis. Let us announce at this place that the quantity $\mathcal{R}(u_1)$ is a

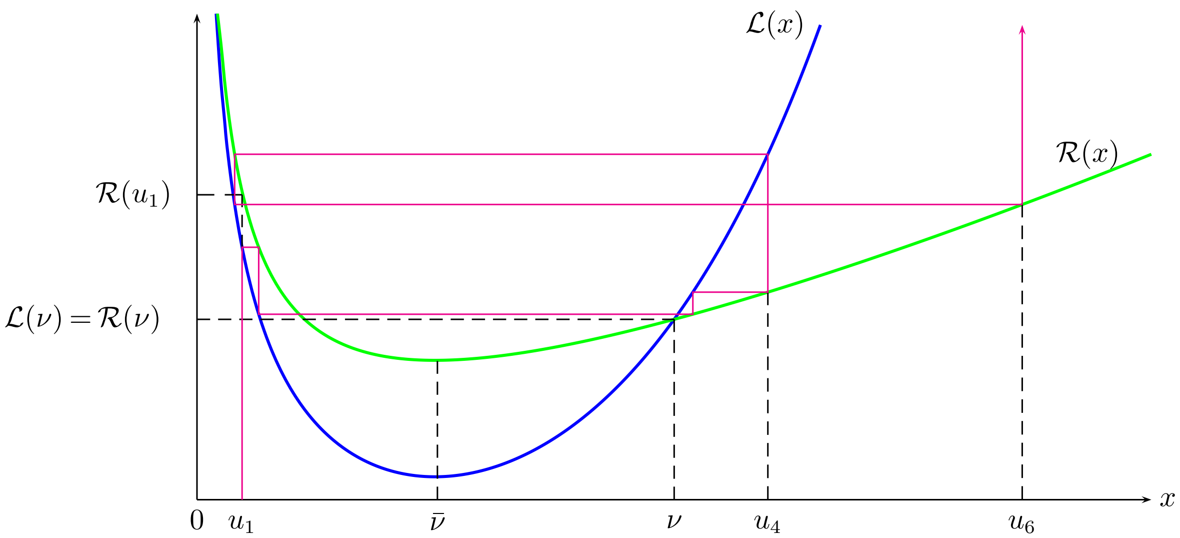

Figure 4 **Example of a stationary sequence.**

measure for the value of $f(h,\tau)$, as will be shown in Section A.6.

It is now clear that there exist infinitely many stationary sequences and that in some of these sequences the u_k's may become either very large and/or very small. The for our goal crucial question is which of these sequences is optimal. To deal with this question we need more information on the value of the objective function in a stationary point. This is the subject of the next section.

A.6. Objective value in a stationary point

We define, for each (integer) $\ell \geq 1$,

$$\mathcal{I}_\ell(h,\tau) := \sum_{k=\ell}^{\infty} D(u_k) e^{\lambda h_k - \delta t_k}, \tag{57}$$

$$\mathcal{A}_\ell(h,\tau) := \sum_{k=\ell}^{\infty} \frac{S_0}{\beta_1} \left[e^{\beta_1 t_k} - e^{\beta_1 t_{k-1}} \right] e^{-\theta h_{k-1}}. \tag{58}$$

and $f_\ell(h,\tau) := \mathcal{I}_\ell(h,\tau) + \mathcal{A}_\ell(h,\tau)$.

Lemma 9. *If (32) and (33) hold for all $k \geq \ell \geq 1$ then*

$$f_\ell(h,\tau) = \frac{e^{-q h_{\ell-1}}}{q} \left(\frac{S_0}{\delta} \right)^{\bar{\delta}} \mathcal{R}(u_\ell) - \frac{S_0}{\beta_1} e^{\beta_1 t_{\ell-1} - \theta h_{\ell-1}}. \tag{59}$$

Proof: By taking the sum over all $k \geq \ell$ in (32) we get

$$\begin{aligned}
\delta \mathcal{I}_\ell(h,\tau) &= \sum_{k=\ell}^{\infty} \delta D(u_k) e^{\lambda h_k - \delta t_k} = \sum_{k=\ell}^{\infty} \left(S_0 e^{\beta_1 t_k - \theta h_{k-1}} - S_0 e^{\beta_1 t_k - \theta h_k} \right) \\
&= \sum_{k=\ell}^{\infty} S_0 e^{\beta_1 t_k - \theta h_{k-1}} + S_0 e^{\beta_1 t_{\ell-1} - \theta h_{\ell-1}} - S_0 \sum_{k=\ell}^{\infty} e^{\beta_1 t_{k-1} - \theta h_{k-1}} \\
&= S_0 e^{\beta_1 t_{\ell-1} - \theta h_{\ell-1}} + \sum_{k=\ell}^{\infty} S_0 \left[e^{\beta_1 t_k} - e^{\beta_1 t_{k-1}} \right] e^{-\theta h_{k-1}} \\
&= S_0 e^{\beta_1 t_{\ell-1} - \theta h_{\ell-1}} + \beta_1 \mathcal{A}_\ell(h,\tau).
\end{aligned} \tag{60}$$

On the other hand, when taking the sum over all $k \geq \ell$ in (33) we get

$$\begin{aligned}
\sum_{k=\ell}^{\infty} \beta_1 \left(\Delta(u_k) e^{\lambda h_k - \delta t_k} - D'(u_{k+1}) e^{\lambda h_{k+1} - \delta t_{k+1}} \right) &= \sum_{k=\ell}^{\infty} \theta S_0 \left[e^{\beta_1 t_{k+1} - \theta h_k} - e^{\beta_1 t_k - \theta h_k} \right] \\
&= \sum_{k=\ell}^{\infty} \theta S_0 \left[e^{\beta_1 t_{k+1}} - e^{\beta_1 t_k} \right] e^{-\theta h_k} \\
&= \theta \beta_1 \mathcal{A}_{\ell+1}(h,\tau).
\end{aligned}$$

Hence, since $D'(u_{k+1}) = b$ and $\beta_1 \neq 0$, we may write

$$\begin{aligned}
\theta \mathcal{A}_{\ell+1}(h,\tau) &= \sum_{k=\ell}^{\infty} \left(\Delta(u_k) e^{\lambda h_k - \delta t_k} - b e^{\lambda h_{k+1} - \delta t_{k+1}} \right) \\
&= \sum_{k=\ell}^{\infty} \left[b + \lambda D(u_k) \right] e^{\lambda h_k - \delta t_k} - \sum_{k=\ell}^{\infty} b e^{\lambda h_{k+1} - \delta t_{k+1}}
\end{aligned}$$

$$\begin{aligned}&= be^{\lambda h_\ell - \delta t_\ell} + \sum_{k=\ell}^{\infty} \lambda D(u_k) e^{\lambda h_k - \delta t_k}\\ &= be^{\lambda h_\ell - \delta t_\ell} + \lambda \mathcal{I}_\ell(h,\tau).\end{aligned}$$

By adding θ times the term for $k=\ell$ in the expression for $\mathcal{A}_\ell(h,\tau)$ to both sides, we get

$$\theta \mathcal{A}_\ell(h,\tau) = be^{\lambda h_\ell - \delta t_\ell} + \lambda \mathcal{I}_\ell(h,\tau) + \frac{\theta S_0}{\beta_1}\left[e^{\beta_1 t_\ell} - e^{\beta_1 t_{\ell-1}}\right] e^{-\theta h_{\ell-1}}. \tag{61}$$

Note that (60) and (61) can be considered as linear equations in $\mathcal{I}_\ell(h,\tau)$ and $\mathcal{A}_\ell(h,\tau)$. The determinant of the coefficient matrix equals $\theta\delta - \lambda\beta_1$, which is positive; hence the solution of this linear system is unique. By solving these equations for $\mathcal{I}_\ell(h,\tau)$ one obtains

$$(\theta\delta - \lambda\beta_1)\,\mathcal{I}_\ell(h,\tau) = \beta_1 b e^{\lambda h_\ell - \delta t_\ell} + \theta S_0 e^{\beta_1 t_\ell - \theta h_{\ell-1}}.$$

In the usual way we can eliminate t_ℓ from the last expression, by using stationarity with respect to t_ℓ, as expressed in (36). We first reduce the exponents in the last two terms. One has

$$\begin{aligned}\delta t_\ell &= \delta \frac{\varrho(u_\ell) + (\lambda+\theta)h_\ell}{\beta_1+\delta} = \bar{\delta}\left(\varrho(u_\ell) + (\lambda+\theta)h_\ell\right) = \bar{\delta}\varrho(u_\ell) + (q+\lambda)h_\ell,\\ \beta_1 t_\ell &= \beta_1 \frac{\varrho(u_\ell) + (\lambda+\theta)h_\ell}{\beta_1+\delta} = \bar{\beta}\left(\varrho(u_\ell) + (\lambda+\theta)h_\ell\right) = \bar{\beta}\varrho(u_\ell) + r h_\ell.\end{aligned}$$

Using also $r = \theta - q$ we obtain

$$\lambda h_\ell - \delta t_\ell = \lambda h_\ell - \bar{\delta}\varrho(u_\ell) - (q+\lambda)h_\ell = -\bar{\delta}\varrho(u_\ell) - q h_\ell, \tag{62}$$

$$\beta_1 t_\ell - \theta h_{\ell-1} = \bar{\beta}\varrho(u_\ell) + (\theta - q)h_\ell - \theta h_{\ell-1} = \bar{\beta}\varrho(u_\ell) + \theta u_\ell - q h_\ell.$$

Substitution gives

$$\begin{aligned}(\theta\delta - \lambda\beta_1)\,\mathcal{I}_\ell(h,\tau) &= \beta_1 b e^{-\bar{\delta}\varrho(u_\ell) - q h_\ell} + \theta S_0 e^{\bar{\beta}\varrho(u_\ell) + \theta u_\ell - q h_\ell}\\ &= \beta_1 e^{-\bar{\delta}\varrho(u_\ell) - q h_\ell}\left(b + \theta S_0 e^{\varrho(u_\ell) + \theta u_\ell}\right).\end{aligned}$$

Now using (42), (43) and the definitions of $R(x)$ and $\mathcal{R}(x)$, in (45) and (46), it follows that

$$\begin{aligned}\mathcal{I}_\ell(h,\tau) &= \frac{\bar{\beta}}{q}\left(\frac{\delta}{S_0}\kappa(u_\ell)\right)^{-\bar{\delta}} e^{-q h_\ell}\left(b + \frac{\theta\delta}{\beta_1}\kappa(u_\ell)e^{\theta u_\ell}\right)\\ &= \frac{\bar{\beta}\, e^{-q h_\ell}}{q}\left(\frac{\delta}{S_0}\right)^{-\bar{\delta}} \kappa(u_\ell)^{-\bar{\delta}} e^{q u_\ell} R(u_\ell)\\ &= \frac{\bar{\beta}\, e^{-q h_{\ell-1}}}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}} \mathcal{R}(u_\ell).\end{aligned} \tag{63}$$

We can now obtain $\mathcal{A}_\ell(h,\tau)$ from (60). Also using $\delta\bar{\beta} = \beta_1\bar{\delta}$ this gives

$$\mathcal{A}_\ell(h,\tau) = \frac{\bar{\delta}\, e^{-q h_{\ell-1}}}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}} \mathcal{R}(u_\ell) - \frac{S_0}{\beta_1} e^{\beta_1 t_{\ell-1} - \theta h_{\ell-1}}. \tag{64}$$

By adding (63) and (64), while using $\bar{\beta}+\bar{\delta}=1$, we obtain the lemma. Q.E.D.

A crucial consequence of this lemma is that $f_\ell(h,\tau)$ is completely determined by u_ℓ, $t_{\ell-1}$ and $h_{\ell-1}$. This holds in particular if $\ell=1$, when we have $t_{\ell-1}=0$ and $h_{\ell-1}=0$. Hence, without further proof we may state the following lemma, revealing the surprising fact that the objective value at a stationary point is completely determined by u_1.

LEMMA 10. *If (h,τ) is a stationary solution then*

$$f(h,\tau)=\mathcal{F}(u_1):=\frac{1}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\mathcal{R}(u_1)-\frac{S_0}{\beta_1}. \tag{65}$$

As already discussed before, the optimal solution may not be stationary with respect to t_1, due to a backlog in height at $t=0$. It means that the constraint $t_1\geq 0$ will be active. In other words, the solution will be at the boundary of the feasible region. But also then we may apply Lemma 9, namely with $\ell=2$, which yields $f_2(h,\tau)$. By adding the investment costs at $t=0$ to $f_2(h,\tau)$ we get $f(h,\tau)$. Hence we also have the following result:

LEMMA 11. *If (h,τ) is a boundary solution then*

$$f(h,\tau)=D(u_1)e^{\lambda u_1}+\frac{e^{-qu_1}}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\mathcal{R}(u_2)-\frac{S_0}{\beta_1}e^{-\theta u_1}. \tag{66}$$

We further explore the above two lemmas in the next section.

A.7. Optimality of the periodic solution

A.7.1. Healthy case

THEOREM 1. *If the periodic sequence is admissible then it is optimal.*

Proof: Let $u=(u_1;u_2;\ldots)$ be any stationary sequence. By Lemma 10 the objective value for the corresponding stationary solution (h,τ) equals $\mathcal{F}(u_1)$, as given by (65). By Lemma 8 we have $\mathcal{R}(u_1)\geq\mathcal{R}(\nu)$. Thus we obtain

$$\mathcal{F}(u_1)\geq\frac{1}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\mathcal{R}(\nu)-\frac{S_0}{\beta_1}=\mathcal{F}(\nu). \tag{67}$$

The last expression is equal to the objective value of the periodic solution. More precisely, it is the optimal value for the unconstrained problem that arises when we minimize $f(h,\tau)$ without respecting the constraints (25) for admissability. Hence, if the periodic solution happens to be admissible it is certainly optimal. Q.E.D.

It may be worth noting that the two terms in the expression of $\mathcal{F}(u_1)$ have the same sign, because of (56). So the value of $\mathcal{F}(u_1)$ is the difference of two terms that are either both positive (when $\beta_1>0$), or both negative (when $\beta_1<0$).

Remark 1. The last two assumptions in (30) imply that $q>0$. This fact has been used at many places, especially in Lemma 6. It is worth pointing out that (63) provides evidence that $q>0$ must hold, because we have $\bar{\beta}\mathcal{R}(\nu)>0$, by (56). Hence if q were negative, then we would have $\mathcal{I}(h,\tau)<0$. So (63) only makes sense if $q>0$. Another way to understand this is as follows. For the periodic solution we have $u_k=\nu$ for all $k\geq 0$. Hence $h_k=k\nu$ for $k\geq 0$, whence (36) gives, for any $k\geq 1$,

$$t_k=\frac{\varrho(u_k)+(\lambda+\theta)h_k}{\beta_1+\delta}=\frac{\varrho(\nu)+(\lambda+\theta)k\nu}{\beta_1+\delta}=t_1+(k-1)p, \tag{68}$$

where

$$t_1=\frac{\varrho(\nu)+(\lambda+\theta)\nu}{\beta_1+\delta}, \quad p:=\frac{\lambda+\theta}{\beta_1+\delta}\nu. \tag{69}$$

Using the above expression for p one easily verifies that

$$\lambda h_k-\delta t_k=\lambda k\nu-\delta\left[t_1+(k-1)p\right]=\lambda\nu-\delta t_1-(k-1)q\nu,$$

Substitution into (28) yields

$$\mathcal{I}(h,\tau)=e^{\lambda\nu-\delta t_1}D(\nu)\sum_{k=1}^{\infty}e^{-(k-1)q\nu}=e^{\lambda\nu-\delta t_1}\frac{D(\nu)}{1-e^{-q\nu}}.$$

Note that the last equality is valid because $q>0$, because otherwise the above sum will diverge. We leave it to the reader to verify that the same happens if we compute $\mathcal{A}(h,\tau)$ from (29).

According to Theorem 1 optimality of the periodic solution only holds under the condition that this solution is admissible. This makes it natural to call a dike *healthy* if $t_1\geq 0$ for the periodic solution. According to (36) this holds if and only if the following holds:

$$\varrho(\nu)+(\lambda+\theta)\nu\geq 0. \tag{70}$$

The next lemma makes clear that a dike is healthy if and only if the expected damage at $t=0$ does not exceed a threshold value.

Lemma 12. *A dike is healthy if and only if*

$$S_0\leq\delta\kappa(\nu)e^{(\lambda+\theta)\nu}. \tag{71}$$

Proof: Using the definitions (34) and (37) of $\varrho(x)$ and $\kappa(x)$, respectively, one easily verifies that the inequalities in (70) and (71) are equivalent. Q.E.D.

It may be worth pointing out that by using the definition of $\kappa(\nu)$, (71) can also be written as

$$S_0\leq\frac{\delta D(\nu)e^{(\lambda+\theta)\nu}}{e^{\theta\nu}-1}=s(\nu),$$

with $s(\nu)$ as defined in (19). In the next section we deal with dikes that are not healthy. For such dikes the periodic sequence leads to $t_1<0$, which means that there is a backlog in height at $t=0$.

A.7.2. Unhealthy case Recall from Lemma 11 that if the constraint $t_1 \geq 0$ is active then the objective value is given by (66). According to Lemma 8 we may replace $\mathcal{R}(u_2)$ in this formula by $\mathcal{R}(\nu)$, which yields the function $F(u_1)$ defined by

$$F(u_1) := D(u_1)e^{\lambda u_1} + \frac{e^{-qu_1}}{q}\left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\mathcal{R}(\nu) - \frac{S_0}{\beta_1}e^{-\theta u_1}, \quad u_1 > 0. \tag{72}$$

When u_1 is given, $F(u_1)$ is the smallest possible objective value for an unhealthy dike. It remains to find the value of u_1 that minimizes $F(u_1)$. When considering this minimization problem we should respect the conditions (25) for admissibility of $u = (u_1; \nu; \nu; \ldots)$. Since the solution is periodic from $t = t_2$ on, (25) boils down to the simple condition $t_2 > 0$. Since $u_2 = \nu$ and $h_2 = u_1 + \nu$, this holds if and only if

$$\varrho(\nu) + (\lambda + \theta)(u_1 + \nu) > 0. \tag{73}$$

This condition is satisfied if and only if $u_1 > u_0$, where u_0 satisfies

$$\varrho(\nu) + (\lambda + \theta)(u_0 + \nu) = 0. \tag{74}$$

Note that $u_0 > 0$, because the dike is not healthy. Using this and (43) we obtain

$$S_0 e^{-(\lambda+\theta)u_0} = \delta\kappa(\nu)e^{(\lambda+\theta)\nu}. \tag{75}$$

We are now ready to analyze the behavior of $F(u_1)$. Using (46) and (43) we write

$$\left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\mathcal{R}(\nu) = \left(\frac{S_0}{\delta}\right)^{\bar{\delta}}\kappa(\nu)^{-\bar{\delta}}R(\nu) = \left(\frac{\delta}{S_0}\kappa(\nu)\right)^{-\bar{\delta}}R(\nu) = e^{-\bar{\delta}\varrho(\nu)}R(\nu). \tag{76}$$

Hence we may rewrite (72) as

$$F(u_1) = D(u_1)e^{\lambda u_1} + \frac{1}{q}e^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-qu_1} - \frac{S_0}{\beta_1}e^{-\theta u_1}, \quad u_1 > u_0. \tag{77}$$

In order to find the global minimizer $u_1^* \in (u_0, \infty)$ we consider the first two derivatives of $F(u_1)$ with respect to u_1. One has

$$\begin{aligned}
F'(u_1) &= \Delta(u_1)e^{\lambda u_1} - e^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-qu_1} + \theta\frac{S_0}{\beta_1}e^{-\theta u_1},\\
F''(u_1) &= \lambda\left[b + \Delta(u_1)\right]e^{\lambda u_1} + qe^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-qu_1} - \theta^2\frac{S_0}{\beta_1}e^{-\theta u_1}.
\end{aligned}$$

Note that the last two terms in the expression for $F''(u_1)$ converge to zero if u_1 grows to infinity. If $\lambda > 0$ then the first term grows exponentially fast to infinity. So, if $\lambda > 0$ then $F(u_1)$ is convex for large values of u_1. This also holds if $\lambda = 0$, as we now show. We then have

$$F''(u_1) = \left(qe^{-\bar{\delta}\varrho(\nu)}R(\nu) - \theta^2\frac{S_0}{\beta_1}e^{-ru_1}\right)e^{-qu_1}$$

Due to (56) the two terms in the bracketed expression have the same sign. Hence this expression can be either negative or positive. Since r has the same sign as β_1 we see that for large values of u_1 the expression is positive, because it converges to $qe^{-\bar{\delta}\varrho(\nu)}R(\nu)$ if $\beta_1 > 0$ and to ∞ if $\beta_1 < 0$. This proves our claim that $F(u_1)$ is convex for large values of u_1. To proceed we write

$$F''(u_1) = e^{\lambda u_1}\left[\lambda\left(b + \Delta(u_1)\right) + \left(qe^{-\bar{\delta}\varrho(\nu)}R(\nu) - \theta^2\frac{S_0}{\beta_1}e^{-ru_1}\right)e^{-(\lambda+q)u_1}\right].$$

The expression between the square brackets vanishes for at most one value of u_1. This easily follows by verifying that its derivative with respect to u_1 is positive whenever the expression itself vanishes. This is due to our assumption that $\beta_1 \neq 0$. Hence we may conclude that $F(u_1)$ has at most one point of inflection on the interval (u_0, ∞).

Let $\tilde{u}$ denote the point of inflection, if it exists. If there is no point of inflection or if $\tilde{u} \le u_0$, then $F(u_1)$ is convex for all $u_1 > u_0$, and hence u_1^* can be found by standard numerical techniques for minimizing a strictly convex function.

Hence, from now on we may assume that there is a point $\tilde{u}$ of inflection, and $\tilde{u} > u_0$. Then $F(u_1)$ is concave for $u_0 < u_1 < \tilde{u}$ and convex for $u_1 > \tilde{u}$. We proceed by showing that $F(u_1)$ is strictly convex at $u_1 = u_0 + \nu$. This will imply $\tilde{u} \in (u_0, u_0 + \nu)$.

LEMMA 13. $F''(u_0 + \nu) > 0$.

Proof: One has

$$F''(u_0+\nu) = \lambda(b + \Delta(u_0+\nu))e^{\lambda(u_0+\nu)} + qe^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-q(u_0+\nu)} - \theta^2\frac{S_0}{\beta_1}e^{-\theta(u_0+\nu)}.$$

By using (42) and (74), successively, we obtain

$$\begin{aligned} -\bar{\delta}\varrho(\nu) - q(u_0+\nu) &= -\bar{\delta}\varrho(\nu) + \left[-\bar{\delta}(\lambda+\theta) + \lambda\right](u_0+\nu) \\ &= -\bar{\delta}\left[\varrho(\nu) + (\lambda+\theta)(u_0+\nu)\right] + \lambda(u_0+\nu) \\ &= \lambda(u_0+\nu). \end{aligned} \tag{78}$$

Using this, $R(\nu) = L(\nu)$ and (75) we get

$$\begin{aligned} F''(u_0+\nu) &= \lambda(b + \Delta(u_0+\nu))e^{\lambda(u_0+\nu)} + qR(\nu)e^{\lambda(u_0+\nu)} - \theta\frac{\theta\delta}{\beta_1}\kappa(\nu)e^{\lambda(\nu+u_0)} \\ &= e^{\lambda(u_0+\nu)}\left[\lambda\left(b + \Delta(u_0+\nu)\right) + qL(\nu) - \theta\frac{\theta\delta}{\beta_1}\kappa(\nu)\right]. \end{aligned}$$

One has $D(u_0+\nu) = D(\nu) + bu_0$, which implies $\Delta(u_0+\nu) = \lambda bu_0 + \Delta(\nu)$. Hence, also using (41), (42), (44) and (53), the bracketed expression can be reduced as follows

$$\lambda\left(b + \Delta(u_0+\nu)\right) + qL(\nu) - \theta\frac{\theta\delta}{\beta_1}\kappa(\nu)$$

$$
\begin{aligned}
&= \lambda\left(b + \lambda b u_0 + \Delta(\nu)\right) + q\left(\Delta(\nu) + \frac{\theta\delta}{\beta_1}\kappa(\nu)\right) - \theta\frac{\theta\delta}{\beta_1}\kappa(\nu)\\
&= \lambda b(1 + \lambda u_0) + (\lambda + q)\Delta(\nu) - r\frac{\theta\delta}{\beta_1}\kappa(\nu)\\
&= \lambda b(1 + \lambda u_0) + \bar{\delta}(\lambda + \theta)\Delta(\nu) - \theta\bar{\delta}(\lambda + \theta)\kappa(\nu).
\end{aligned}
$$

Hence, it remains to show that $g(x) > 0$ at $x = \nu$, where

$$
g(x) := \lambda b(1 + \lambda u_0) + \bar{\delta}(\lambda + \theta)\left[\Delta(x) - \theta\kappa(x)\right].
$$

We first recall that $\kappa(x)$ is monotonically decreasing to zero, by Lemma 1. Since $\bar{\delta}(\lambda + \theta) = q + \lambda > 0$, this implies that for large enough values of x we have $g(x) > 0$. Since $\Delta(x)$ is linear and nondecreasing and $\kappa(x)$ is monotonically decreasing, it follows that $g(x)$ is monotonically increasing. So we need to show that ν is large enough to have $g(\nu) > 0$. Since $\bar{\nu} < \nu$, this certainly holds if $g(\bar{\nu}) \geq 0$. Therefore, the lemma will follows if $g(\bar{\nu}) \geq 0$. The rest of the proof is devoted to this inequality. By definition, $\bar{\nu}$ is the (unique) solution of the equation (54). This implies

$$
\Delta(\bar{\nu}) - \theta\kappa(\bar{\nu}) = \frac{\lambda b\kappa(\bar{\nu})}{\bar{\delta}\kappa'(\bar{\nu})}. \tag{79}
$$

Also using $\lambda^2 b u_0 \geq 0$ and (50) we get

$$
\begin{aligned}
g(\bar{\nu}) &\geq \lambda b + \bar{\delta}(\lambda + \theta)\frac{\lambda b\kappa(\bar{\nu})}{\bar{\delta}\kappa'(\bar{\nu})} = \frac{\lambda b}{\kappa'(\bar{\nu})}\left[\kappa'(\bar{\nu}) + (\lambda + \theta)\kappa(\bar{\nu})\right]\\
&= \frac{\lambda b}{\kappa'(\bar{\nu})}\frac{\Delta(\bar{\nu}) - \theta\kappa(\bar{\nu})}{e^{\theta\bar{\nu}} - 1} = \frac{1}{e^{\theta\bar{\nu}} - 1}\frac{\lambda b}{\kappa'(\bar{\nu})}\cdot\frac{\lambda b\kappa(\bar{\nu})}{\bar{\delta}\kappa'(\bar{\nu})} = \frac{1}{e^{\theta\bar{\nu}} - 1}\frac{\lambda^2 b^2\kappa(\bar{\nu})}{\bar{\delta}\kappa'(\bar{\nu})^2} \geq 0.
\end{aligned}
$$

This completes the proof. Q.E.D.

We proceed by showing that $u_1^* \in (\tilde{u}, u_0 + \nu)$, i.e., u_1^* belongs to the interval where $F(u_1)$ is convex. As we will see, for this we need an additional assumption that is satisfied by all dikes for which the data are known, namely

$$
c \geq b u_0\left(e^{\lambda\nu} - 1\right). \tag{80}
$$

Roughly spoken, it requires that the fixed cost term in the investment cost function is not too small. Obviously, the condition is certainly satisfied if $\lambda b = 0$. In order to understand the physical meaning of this inequality one may easily verify that it can be rewritten as follows:

$$
\begin{aligned}
\left[D(u_0)e^{\lambda u_0} + D(\nu)e^{\lambda(u_0+\nu)}\right] - D(u_0 + \nu)e^{\lambda(u_0+\nu)} &= D(u_0)e^{\lambda u_0} - b u_0 e^{\lambda(u_0+\nu)}\\
&= e^{\lambda u_0}\left[c - b u_0\left(e^{\lambda\nu} - 1\right)\right] \geq 0,
\end{aligned} \tag{81}
$$

where the equality signs are consequences of $D(x) = c + bx$. The left-side side expression compares the costs for a heightening with u_0 at $t = 0$, 'followed' by a heightening with ν, also at $t = 0$, with the costs for a heightening at $t=0$ with $u_0+\nu$, in one step. Since in the second case the fixed costs for the heightening with ν are avoided one should expect that a combined heightening with $u_0+\nu$ is more profitable than splitting it into two subsequent heightenings with u_0 and ν, respectively. Hence, from a practical point of view we may safely assume that (80) holds, because otherwise it would indicate that there is something wrong with at least one of the parameters c, b and λ, and maybe also with some of the other parameters that are involved in the definitions of ν and u_0. We therefore assume that (81), and hence also (80), always holds.

LEMMA 14. $F(u_0)=\left[D(u_0)e^{\lambda u_0}-bu_0e^{\lambda(u_0+\nu)}\right]+F(u_0+\nu)$.

Proof: One has

$$\begin{aligned}F(u_0) &= D(u_0)e^{\lambda u_0}+\frac{1}{q}e^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-qu_0}-\frac{S_0}{\beta_1}e^{-\theta u_0},\\ F(u_0+\nu) &= D(u_0+\nu)e^{\lambda(u_0+\nu)}+\frac{1}{q}e^{-\bar{\delta}\varrho(\nu)}R(\nu)e^{-q(u_0+\nu)}-\frac{S_0}{\beta_1}e^{-\theta(u_0+\nu)}.\end{aligned}$$

Due to (78) and (75), and also using $D(u_0+\nu)-bu_0=D(\nu)$, this can be rewritten as

$$\begin{aligned}F(u_0)-D(u_0)e^{\lambda(u_0)} &= \frac{1}{q}e^{q\nu}R(\nu)e^{\lambda(u_0+\nu)}-\frac{\delta}{\beta_1}\kappa(\nu)e^{(\lambda+\theta)\nu+\lambda u_0},\\ F(u_0+\nu)-bu_0e^{\lambda(u_0+\nu)} &= D(\nu)e^{\lambda(u_0+\nu)}+\frac{1}{q}R(\nu)e^{\lambda(u_0+\nu)}-\frac{\delta}{\beta_1}\kappa(\nu)e^{\lambda(\nu+u_0)}.\end{aligned}$$

The lemma will follow if the right-hand side members in the above equations are equal. After dividing by the common factor $e^{\lambda(u_0+\nu)}$ this leads to the following equation:

$$\frac{1}{q}e^{q\nu}R(\nu)-\frac{\delta}{\beta_1}\kappa(\nu)e^{\theta\nu}=D(\nu)+\frac{1}{q}R(\nu)-\frac{\delta}{\beta_1}\kappa(\nu).$$

Since $\kappa(\nu)e^{\theta\nu}=\kappa(\nu)+D(\nu)$ this further reduces to

$$\frac{1}{q}e^{q\nu}R(\nu)-\frac{\delta}{\beta_1}D(\nu)=D(\nu)+\frac{1}{q}R(\nu),$$

which is equivalent to

$$e^{q\nu}R(\nu)-q\frac{\beta_1+\delta}{\beta_1}D(\nu)=R(\nu),$$

Using the definitions of q and $R(x)$ the left-hand side expression reduces to $L(\nu)$, as follows:

$$\begin{aligned}e^{q\nu}R(\nu)-q\frac{\beta_1+\delta}{\beta_1}D(\nu) &= e^{q\nu}R(\nu)-\frac{\theta\delta-\lambda\beta_1}{\beta_1}D(\nu)\\ &= b+\frac{\theta\delta}{\beta_1}\kappa(x)e^{\theta x}-\frac{\theta\delta}{\beta_1}D(\nu)+\lambda D(\nu)\\ &= \Delta(\nu)b+\frac{\theta\delta}{\beta_1}\left[\kappa(x)e^{\theta x}-D(\nu)\right]\\ &= \Delta(\nu)+\frac{\theta\delta}{\beta_1}\kappa(x)=L(\nu).\end{aligned}$$

Since $L(\nu) = R(\nu)$, by the definition of ν, the lemma follows. Q.E.D.

As a result of Lemma 14 and (81) we have

$$F(u_0) \geq F(u_0 + \nu). \tag{82}$$

Since $F(u_1)$ is strictly concave for $u_1 \in (u_0, \tilde{u})$, we deduce from (82) that we must have $u_1^* \in (\tilde{u}, u_0 + \nu]$. Since $F(u_1)$ is strictly convex on this interval, we can obtain u_1^* by standard numerical methods for minimizing a strictly convex function. We conclude this section with a lemma that makes clear that we have $u_1^* = u_0 + \nu$ if and only if $\lambda b = 0$.

LEMMA 15. *If $\lambda b = 0$ then $F'(u_0 + \nu) = 0$, otherwise $F'(u_0 + \nu) > 0$.*

Proof: One has

$$F'(u_0 + \nu) = \Delta(u_0 + \nu)e^{\lambda(u_0+\nu)} - e^{-\bar{\delta}\varrho(\nu)} R(\nu) e^{-q(u_0+\nu)} + \theta \frac{S_0}{\beta_1} e^{-\theta(u_0+\nu)}.$$

Using successively (78), (75) and (44) we get

$$\begin{aligned}
F'(u_0 + \nu) &= \Delta(u_0 + \nu)e^{\lambda(u_0+\nu)} - R(\nu)e^{\lambda(u_0+\nu)} + \frac{\theta\delta}{\beta_1}\kappa(\nu)e^{\lambda(\nu+u_0)} \\
&= \left[\Delta(u_0 + \nu) - L(\nu) + \frac{\theta\delta}{\beta_1}\kappa(\nu)\right] e^{\lambda(u_0+\nu)} \\
&= \left[\Delta(u_0 + \nu) - \Delta(\nu)\right] e^{\lambda(u_0+\nu)} \\
&= \lambda b u_0 e^{\lambda(u_0+\nu)}.
\end{aligned}$$

Since $\lambda \geq 0$, $b \geq 0$, and $u_0 > 0$, this implies the lemma. Q.E.D.

We refer to Figure 5 for a graphical illustration for the case where $\lambda b > 0$. If $\lambda b > 0$ then $F(u_1)$

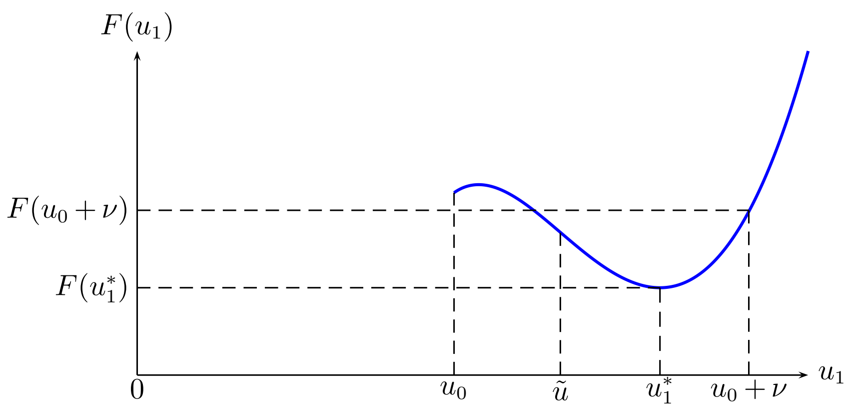

Figure 5 **Typical graph of** $F(u_1)$, $u_1 > u_0$.

is increasing at $u_1 = u_0 + \nu$, which implies that the optimal value is less than $F(u_0 + \nu)$, and by (82) also less than $F(u_0)$.

A.8. Computational results

A.8.1. Healthy dikes Table 3 contains the data of 12 healthy dikes. The last column in this table serves to convince the reader that these dikes are healthy: the value of S_0 is always less than the threshold value in (71). The periodic solutions are given in Table 4.

No.	c	b	λ	β_1	θ	δ	S_0	$\delta\kappa(\nu)e^{(\lambda+\theta)\nu}$
10	16.6939	0.6258	0.0014	−0.0094	0.0293	0.0400	0.6894	2.7956
11	42.6200	1.7068	0.0000	−0.0098	0.0285	0.0400	1.9884	7.1753
16	324.6287	2.1304	0.0100	0.0236	0.0554	0.0400	25.0072	31.2543
22	154.4388	0.9325	0.0066	0.0234	0.0671	0.0400	5.3502	11.9867
23	26.4653	0.5250	0.0034	0.0227	0.0514	0.0400	0.0845	2.8436
24	71.6923	1.0750	0.0059	0.0265	0.0402	0.0400	5.0968	8.6820
40	5.8601	0.1000	0.0026	−0.0093	0.0214	0.0400	0.0509	0.8277
42	21.8254	0.4625	0.0019	−0.0084	0.0250	0.0400	1.8646	3.0349
43	340.5081	4.2975	0.0043	−0.0087	0.0233	0.0400	33.7908	43.9543
49	20.0000	0.8000	0.0046	−0.0095	0.0308	0.0400	1.4074	3.6956
51	15.0000	0.6000	0.0071	−0.0094	0.0319	0.0400	0.9750	2.8912
52	49.2200	1.6075	0.0047	−0.0090	0.0345	0.0400	6.8814	7.6746

Table 3 **Data for 12 healthy dikes (in 2001).**

no.	ν	p	t_1	$f(h,\tau)$
10	56.96	57.12	45.80	40.04
11	62.42	58.89	42.44	110.23
16	52.59	54.04	3.50	1089.68
22	53.70	62.43	12.72	309.25
23	55.24	48.24	56.06	20.07
24	61.81	42.80	8.01	297.26
40	78.80	61.68	90.90	3.83
42	72.24	61.43	15.43	79.24
43	73.49	64.66	8.39	1304.79
49	45.65	52.97	31.66	74.04
51	40.49	51.49	35.48	54.18
52	45.74	57.79	3.52	245.38

Table 4 **Solutions for the 12 dikes in Table 3.**

A.8.2. Unhealthy dikes We also applied our results to 10 unhealthy dikes. The data of the dikes are given in Table 5, and the solutions (h^*, τ^*) in Table 6. Note that dike 50 is the only dike with $\lambda b = 0$.

No.	c	b	λ	β_1	θ	δ	S_0	$\delta\kappa(\nu)e^{(\lambda+\theta)\nu}$
15	125.6422	1.1268	0.0098	0.0182	0.0464	0.0400	16.2008	13.6730
35	49.7384	0.6888	0.0088	0.0182	0.0319	0.0400	8.9090	7.2174
38	24.3404	0.7000	0.0040	−0.0096	0.0212	0.0400	5.2352	4.7525
41	58.8110	0.9250	0.0033	−0.0093	0.0226	0.0400	17.1164	8.0332
44	24.0977	0.7300	0.0054	−0.0100	0.0282	0.0400	12.3267	4.1871
45	3.4375	0.1375	0.0069	−0.0094	0.0306	0.0400	1.7028	0.6768
47	8.7813	0.3513	0.0026	−0.0096	0.0257	0.0400	2.3402	1.7197
48	35.6250	1.4250	0.0063	−0.0086	0.0199	0.0400	12.0451	9.3458
50	8.1250	0.3250	0.0000	−0.0094	0.0290	0.0400	3.6214	1.3554
53	69.4565	1.1625	0.0028	−0.0094	0.0290	0.0400	16.7855	8.2617

Table 5 Data for 10 unhealthy dike rings (in 2001).

No.	ν	p	u_1	t_1	t_2	$f(h^*, \tau^*)$
15	53.29	51.54	55.96	0.00	51.20	545.1821
35	59.65	41.73	63.88	0.00	41.08	345.2267
38	62.04	51.31	65.27	0.00	50.80	172.0750
41	74.66	62.95	100.55	0.00	60.13	325.8935
44	49.54	55.42	76.91	0.00	50.06	206.4989
45	41.49	50.94	61.80	0.00	45.69	33.7196
47	55.60	51.87	65.26	0.00	50.74	64.0966
48	50.84	42.47	58.24	0.00	40.57	403.0010
50	61.97	58.78	95.87	0.00	58.78	53.4673
53	66.13	68.54	86.57	0.00	66.58	307.4761

Table 6 Optimal solution (h^*, τ^*) for the 10 unhealthy dike rings.