Electronic Companion

Resource Allocation under Demand Uncertainty and Private Information

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Appendix EC.1: Proofs in Section 3

Proof of Theorem 1 and Proposition 1. We divide the proof in two steps: i) the main arguments, and ii) proof of Proposition 1 and finer properties of q° used in i).

Step 1. Main Arguments. Consider the following relaxed *ex ante* problem

$$\min_{q,m} \mathbb{E}_{x,y}\left[C_0\left(x,\sum_{i=1}^M q_i(x,y)\right)+\sum_{i=1}^M m_i(x,y)\right] \quad \text{subject to } (\mathrm{IR}_i^P) \text{ and } 0\le q_i(x,y)\le y_i, \quad i=1,\ldots,M.$$

It is obvious that all the constraints in (IR_i^P) necessarily bind and hence we can impose them as equalities to eliminate the variable m from the above problem, rendering the objective function to

$$\mathbb{E}_{x,y}\left[C_0\left(x,\sum_{i=1}^M q_i(x,y)\right)+\sum_{i=1}^M C_i\left(y_i,q_i(x,y)\right)-\sum_{i=1}^M C_i\left(y_i,0\right)\right],$$

which coincides with the objective function of the first-best problem (4) and is minimized by q°. In particular, setting $m_i^\circ(x,y)=(c_i-h_i)\, q_i^\circ(x,y)$ implies that (IR_i^P) holds with equality *ex post* for all $i=1,\ldots,M$.

Therefore, it remains to show that the policy $\{q_i^\circ(x,y),m_i^\circ(x,y)\}_{i=1}^M$ also satisfies (IC_i^P) *ex post*, (IC_0^P) and (IR_0^P). For any x,y, we have

$$C_i(y_i,q_i^\circ(x,y))-m_i^\circ(x,y)=C_i(y_i,0)=h_iy_i, \quad \text{for } \ i=1,\ldots,M,$$

which implies that (IC_i^P) holds with equality *ex post*. Moreover, by (4) we have

$$C_0\left(x,\sum_{i=1}^M q_i^\circ(x,y)\right)+\sum_{i=1}^M C_i(y_i,q_i^\circ(x,y))\le C_0(x,0)+\sum_{i=1}^M C_i(y_i,0)$$

and by the previous equality

$$\begin{aligned}C_0\left(x,\sum_{i=1}^M q_i^\circ(x,y)\right)+\sum_{i=1}^M m_i^\circ(x,y) &= C_0\left(x,\sum_{i=1}^M q_i^\circ(x,y)\right)+\sum_{i=1}^M [C_i(y_i,q_i^\circ(x,y))-C_i(y_i,0)]\\ &\le C_0(x,0).\end{aligned}$$

The last inequality is (IR_0^P). Finally, to see that (IC_0^P) also holds, for each x, x', y, define

$$\begin{aligned}\Delta(x,x',y) &:= C_0\left(x,\sum_{i=1}^M q_i^\circ(x,y)\right)+\sum_{i=1}^M m_i^\circ(x,y)-\left[C_0\left(x,\sum_{i=1}^M q_i^\circ(x',y)\right)+\sum_{i=1}^M m_i^\circ(x',y)\right]\\ &=(h_0+p_0)\int_{-x+\sum_{i=1}^N q_i^\circ(x',y)}^{-x+\sum_{i=1}^N q_i^\circ(x,y)} G_0(z)dz-\sum_{i=1}^M (p_0+h_i-c_i)\left(q_i^\circ(x,y)-q_i^\circ(x',y)\right).\end{aligned}$$

We will show that $\Delta(x,x',y)\le 0$ hence (IC_0^P) holds by integrating y out.

Let $\eta_i = G_0^{-1}((h_i+p_0-c_i)/(h_0+p_0))$ for $i=1,\ldots,M$. Note that η_i is decreasing in $i=1,\ldots,M$ since c_i-h_i is increasing. The index $i^\circ(x,y)$ is defined in Step 2 below.

If $\eta_1 \le -x < -x'$, then $q_i^\circ(x,y)=q_i^\circ(x',y)=0$ by Step 2 $(***)$ and hence $\Delta(x,x',y)=0$.

If $-x<-x'$ and $-x\le \eta_i \le -x+\sum_{i=1}^M y_i$ for some $i=1,\ldots,M$, then by Step 2 $(**)$,

$$G_0\left(-x+\sum_{j=1}^M q_j^\circ(x,y)\right) = \frac{p_0+h_{i^\circ(x,y)}-c_{i^\circ(x,y)}}{h_0+p_0}.$$

Furthermore, we have $q_i^\circ(x,y)\ge q_i^\circ(x',y)$ by Step 2 $(*)$. Hence,

$$\begin{aligned}\Delta(x,x',y) &\le \sum_{i=1}^{i^\circ(x,y)}\left[(h_0+p_0)G_0\left(-x+\sum_{i=1}^N q_i^\circ(x,y)\right)-(p_0+h_i-c_i)\right](q_i^\circ(x,y)-q_i^\circ(x',y))\\ &= \sum_{i=1}^{i^\circ(x,y)}\underbrace{\left[(h_{i^\circ(x,y)}-c_{i^\circ(x,y)})-(h_i-c_i)\right]}_{\le 0}\underbrace{(q_i^\circ(x,y)-q_i^\circ(x',y))}_{\ge 0}\le 0.\end{aligned}$$

If $-x<-x'$ and $\eta_M \ge -x+\sum_{i=1}^M y_i$, then by Step 2 $(****)$,

$$G_0\left(-x+\sum_{j=1}^M q_j^\circ(x,y)\right) \le \frac{p_0+h_M-c_M}{h_0+p_0}.$$

Furthermore, we have $q_i^\circ(x,y)\ge q_i^\circ(x',y)$ by Step 2 $(*)$. Hence,

$$\begin{aligned}\Delta(x,x',y) &\le \sum_{i=1}^{M}\left[(h_0+p_0)G_0(-x+\sum_{i=1}^N x_i)-(p_0+h_i-c_i)\right](q_i^\circ(x,y)-q_i^\circ(x',y))\\ &\le \sum_{i=1}^{M}\underbrace{\left[(h_M-c_M)-(h_i-c_i)\right]}_{\le 0}\underbrace{(q_i^\circ(x,y)-q_i^\circ(x',y))}_{\ge 0}\le 0.\end{aligned}$$

If $-x>-x'\ge\eta_1$, then $q_i^\circ(x,y)=q_i^\circ(x',y)=0$ by Step 2 $(***)$ and hence $\Delta(x,x',y)=0$.

If $-x>\eta_1\ge -x'$, we have $0=i^\circ(x,y)<i^\circ(x',y)$ and $0=q_i^\circ(x,y)\le q_i^\circ(x',y)$ for all $i=1,\ldots,M$ by Step 2 $(***)$

$$\begin{aligned}\Delta(x,x',y) &\le \sum_{i=1}^{i^\circ(x',y)}\left[(h_0+p_0)G_0(-x)-(p_0+h_i-c_i)\right](q_i^\circ(x,y)-q_i^\circ(x',y))\\ &\le \sum_{i=i^\circ(x,y)}^{i^\circ(x',y)}\underbrace{\left[(h_1-c_1)-(h_i-c_i)\right]}_{\ge 0}\underbrace{(q_i^\circ(x,y)-q_i^\circ(x',y))}_{\le 0}\le 0.\end{aligned}$$

If $-x>-x'$, which implies $i^\circ(x,y)\le i^\circ(x',y)$ and $q_i^\circ(x,y)\le q_i^\circ(x',y)$ by Step 2 $(*)$, and $-x\le \eta_i\le -x+\sum_{i=1}^M y_i$ for some $i=1,\ldots,M$, which implies, by Step 2 $(**)$,

$$G_0\left(-x+\sum_{j=1}^M q_j^\circ(x,y)\right) = \frac{p_0+h_{i^\circ(x,y)}-c_{i^\circ(x,y)}}{h_0+p_0},$$

then we have

$$\Delta(x,x',y) \leq \sum_{i=i^\circ(x,y)}^{i^\circ(x',y)} \left[(h_0+p_0)G_0\left(-x+\sum_{i=1}^{N} q_i^\circ(x,y)\right)-(p_0+h_i-c_i)\right](q_i^\circ(x,y)-q_i^\circ(x',y))$$

$$= \sum_{i=i^\circ(x,y)}^{i^\circ(x',y)} \underbrace{\left[(h_{i^\circ(x,y)}-c_{i^\circ(x,y)})-(h_i-c_i)\right]}_{\geq 0}\underbrace{(q_i^\circ(x,y)-q_i^\circ(x',y))}_{\leq 0} \leq 0.$$

If $-x > -x'$ but $\eta_M \geq -x+\sum_{i=1}^{M} y_i$, we have $q_i^\circ(x,y)=q_i^\circ(x',y)=y_i$ for all $i=1,\ldots,M$. Hence, $\Delta(x,x',y)=0$.

Step 2. Proof of Proposition 1 and properties of the allocation q°.

Recall the definition of $\eta_i = G_0^{-1}((h_i+p_0-c_i)/(h_0+p_0))$ for $i=1,\ldots,M$. Note that η_i is decreasing in $i=1,\ldots,M$ since c_i-h_i is increasing. Because of the convexity of the objective function in (4), KKT condition is necessary and sufficient for global optimality of q°, namely there exists $\nu_i,\lambda_i \geq 0$, $i=1,\ldots,M$ such that the following (in)equalities are satisfied for $i=1,\ldots,M$:

$$\begin{array}{c} (h_0+p_0)G_0\left(-x+\sum_{j=1}^{M} q_j\right)-p_0+c_i-h_i+\nu_i-\lambda_i=0 \\ 0\leq\ q_i\leq y_i,\quad \nu_i(y_i-q_i)=0,\quad \lambda_i q_i=0 \end{array} \tag{EC.1}$$

Hence, for given y, we consider the following greedy algorithm:

Set $i=0$ and $q^\circ=0$. If $-x\geq\eta_1$, **terminate**.
Iteration Step: if $i\geq M$ **terminate**;
$i\leftarrow i+1$;
if $-x+\sum_{j=1}^{i} y_j<\eta_i$, set $q_i^\circ=y_i$ and return to **Iteration Step**;
else set $q_i^\circ=\eta_i+x-\sum_{j=1}^{i-1} y_j$ and **terminate**.

Let i° denote the exiting index i and note that $q_i^\circ=0$ for all $i>i^\circ$. Hence $q_i^\circ(x,y)$, $i=1,\ldots,M$, is the one given in (6). Moreover, define

$$\nu_i=\begin{cases}(c_{i^\circ}-h_{i^\circ})-(c_i-h_i), & \text{if}\quad i<i^\circ,\\ 0, & \text{if}\quad i\geq i^\circ,\end{cases} \text{ and } \lambda_i=\begin{cases}0, & \text{if}\quad i\leq i^\circ,\\ (c_i-h_i)-(c_{i^\circ}-h_{i^\circ}), & \text{if}\quad i>i^\circ.\end{cases}$$

Then $\nu_i,\lambda_i\geq 0$ for all i, together with $q_i^\circ(x,y)$, $i=1,\ldots,M$ satisfy (EC.1). Therefore, $q_i^\circ(x,y)$, $i=1,\ldots,M$, indeed solves (4).

It follows that:

($*$) $i^\circ=i^\circ(x,y)$ and $q_i^\circ(x,y)$ is nondecreasing in x for $i=1,2,\ldots,M$;

($**$) $-x+\sum_{j=1}^{M} q_j^\circ(x,y)=\eta_{i^\circ}$ if $-x\leq\eta_i\leq -x+\sum_{i=1}^{M} y_i$ for some $i=1,\ldots,M$;

($***$) $q_i^\circ(x,y)=0$ for $i=1,\ldots,M$ if $-x\geq\eta_1$ (and hence $-x\geq\eta_i$ for all $i=1,\ldots,M$).

($****$) $q_i^\circ(x,y)=y_i$ for $i=1,\ldots,M$ if $\eta_M\geq -x+\sum_{i=1}^{M} y_i$ (and hence $\eta_i\geq -x+\sum_{i=1}^{M} y_i$ for all $i=1,\ldots,M$). $\square$

Appendix EC.2: Characterization of (IC_i^D) and (IR_i^D)

Most of the derivation of this section will be based on the following straightforward calculation:

$$C_i(x_i, q_i) = (h_i + p_i) \int_{\underline{d}_i}^{q_i - x_i} G_i(z) dz + p_i(x_i - q_i), \quad i = 1, \ldots, N, \tag{EC.1}$$

where we recall that $\underline{d}_i$ is the lower bound of the support of D_i. Therefore, it is straightforward to see that

$$C_i(x_i, 0) = p_i x_i, \quad i = 1, \ldots, N. \tag{EC.2}$$

In this section, we drop the dependence on y and abbreviate $m(x,y)$ as $m(x)$, $q(x,y)$ as $q(x)$.

Proof of Lemma 1. For any q and m, define

$$\widehat{W}_i(x_i, x_i') := \mathbb{E}_{x_{-i}}\left[C_i(x_i, q_i(x_i', x_{-i})) + m_i(x_i', x_{-i})\right], \quad i = 1, \ldots, N.$$

Then (IC_i^D) can be formulated as

$$W_i(x) = \min_{x_i' \in [\underline{x}_i, \bar{x}_i]} \widehat{W}_i(x_i, x_i').$$

We have

$$\frac{\partial}{\partial x_i} \widehat{W}_i(x_i, x_i') = \mathbb{E}_{x_{-i}}\left[\frac{\partial}{\partial x_i} C_i(x_i, q_i(x_i', x_{-i}))\right] = \mathbb{E}_{x_{-i}}\left[-(h_i + p_i)\; G_i(q_i(x_i', x_{-i}) - x_i) + p_i\right],$$

because the differentiation and expectation operators commute (Rosenthal 2000, Proposition 9.2.1). Thus $\left|\frac{\partial}{\partial x_i}\widehat{W}_i(x_i, x_i')\right| \le \max\{h_i, p_i\}$ for any (x_i, x_i'), and (A1) follows from the *envelope theorem* of Milgrom and Segal (2002). The equality in (A2) follows from the definition of $W_i(\cdot)$.

By (EC.2), we have

$$W_i(x_i) - C_i(x_i, 0) = W_i(\underline{x}_i) - C_i(\underline{x}_i, 0) - (h_i + p_i)\mathbb{E}_{x_{-i}}\left[\int_{\underline{x}_i}^{x_i} G_i(q_i(z, x_{-i}) - z) dz\right] \le W_i(\bar{x}_i) - C_i(\bar{x}_i, 0).$$

Thus it suffices to impose only $W_i(\underline{x}_i) \le C_i(\underline{x}_i, 0)$ to obtain the (IR_i^D) constraint, $W_i(x) - C_i(x_i, 0) \le 0$ for all x_i. $\square$

Proof of Lemma 2. Notice that

$$C_i(x_i, q_i) - C_i(x_i', q_i) = -(h_i + p_i) \int_{x_i'}^{x_i} G_i(q_i - z) dz + p_i(x_i - x_i').$$

Combining this equation and (EC.2), we have

$$\begin{aligned}
&[C_i(x_i, q_i(x_i', x_{-i})) + m_i(x_i', x_{-i})] - [C_i(x_i, q_i(x)) + m_i(x)] \\
&= [C_i(x_i, q_i(x_i', x_{-i})) - C_i(x_i', q_i(x_i', x_{-i}))] + [C_i(x_i', q_i(x_i', x_{-i})) + m_i(x_i', x_{-i})] - [C_i(x_i, q_i(x)) + m_i(x)] \\
&= [C_i(x_i, q_i(x_i', x_{-i})) - C_i(x_i', q_i(x_i', x_{-i}))] + [C_i(x_i', 0) - C_i(x_i, 0)] - (h_i + p_i) \int_{x_i'}^{x_i} G_i(q_i(z, x_{-i}) - z) dz
\end{aligned}$$

$$=(h_i+p_i)\int_{x_i'}^{x_i}\left[G_i(q_i(x_i',x_{-i})-z)-G_i(q_i(z,x_{-i})-z)\right]dz \geq 0,$$

where the last inequality is due to the fact that if $x_i' > (<) x_i$, $q_i(x_i',x_{-i}) \geq (\leq) q_i(z,x_{-i})$ for all $z \in [x_i, x_i']$ ($z \in [x_i', x_i]$) because of monotonicity of $q_i(x)$ in x_i for any given x_{-i}. This shows (A3) and taking the expectation of (A3) with respect to x_{-i} yields (IC_i^D). □

Proof of Lemma 3. Replace the term $C_0(y,q(x))$ and $\{\mathbb{E}_{x_{-i}}[m_i(x)]\}_{i=1}^N$ in (9) with the expression in (7) and (A2), respectively. In light of Lemma 1 and Lemma 2, we can focus on the following relaxed minimization problem:

$$\begin{aligned}\min_{\substack{q_i(\cdot),\dots,q_N(\cdot)\\ W_1(\underline{x}_1),\dots,W_N(\underline{x}_N)}} \quad & \mathbb{E}_x\left[\sum_{i=1}^N (c_i-h_0)\,q_i(x)\right]+\sum_{i=1}^N \mathbb{E}_x\left[C_i(x_i,q_i(x))\right] \\ & +\sum_{i=1}^N (h_i+p_i)\mathbb{E}_x\left[\int_{\underline{x}_i}^{x_i} G_i(q_i(z,x_{-i})-z)dz\right]-\sum_{i=1}^N W_i(\underline{x}_i) \qquad \text{(EC.3)}\\ \text{subject to } & \sum_{i=1}^N q_i(x)\leq y,\ q_i(x)\geq 0,\ \text{and } W_i(\underline{x}_i)\leq C_i(\underline{x}_i,0) \text{ for all } \ x\in\mathcal{X}, i=1,\dots,N.\end{aligned}$$

Since the objective function is decreasing in $W_i(\underline{x}_i)$ for $i=1,\dots,N$, it is optimal to set,

$$W_i^\star(\underline{x}_i) = C_i(\underline{x}_i,0), \quad i=1,\dots,N.$$

Furthermore, we have,

$$\begin{aligned}\mathbb{E}_x\left[\int_{\underline{x}_i}^{x_i} G_i(q_i(z,x_{-i})-z)dz\right] &= \mathbb{E}_{x_{-i}}\left[\mathbb{E}_{x_i}\left[\int_{\underline{x}_i}^{x_i} G_i(q_i(z,x_{-i})-z)dz\right]\right]\\ &= \mathbb{E}_{x_{-i}}\left[\mathbb{E}_{x_i}\left[\frac{1-F_i(x_i)}{f_i(x_i)}G_i(q_i(x)-x_i)\right]\right]\\ &= \mathbb{E}_x\left[\frac{1-F_i(x_i)}{f_i(x_i)}G_i(q_i(x)-x_i)\right], \quad i=1,\dots,N\end{aligned}$$

where the second equality follows from integration by parts. Substituting the last expression into (EC.3) yields (A4). The second part of the result follows from Lemma 2. □

Proof of Lemma 4. The required monotonicity constraint is satisfied by $q^\star$ according to Lemma EC.4. Then the result immediately follows from Lemma 3. □

Appendix EC.3: Quasi-convexity and Monotonicity of (10)

Lemma EC.1 below will be applied through out the paper with $\alpha=(h_0+p_i-c_i)/(h_i+p_i)$, $G=G_i$, $\underline{d}=\underline{d}_i$, $\bar{d}=\bar{d}_i$, $g=g_i$, $\zeta=\frac{1-F_i(x_i)}{f_i(x_i)}\geq 0$ for some i.

LEMMA EC.1. *Let G be a probability distribution supported on $[\underline{d},\bar{d}]$ with a upper-semi continuous probability density function g such that $1-G$ is log-concave, $\zeta\geq 0$, and $\alpha\in(0,1)$. Define $z^\star := \inf\{z\in[\underline{d},\bar{d}] \mid -\alpha+G(z)+\zeta g(z)\geq 0\}$. Then* *(i)* $-\alpha+G(z)+\zeta g(z)$ *is nondecreasing for* $z\le z^\star$*;*

(ii) $-\alpha+G(z)+\zeta g(z)\ge 0, \quad$ *for all* $z\ge z^\star$.

Proof. It is equivalent to show that $1-G(z)-\zeta g(z)$ is non-increasing in $z\le z^\star$ and for $z\ge z^\star$,

$$1-G(z)-\zeta g(z)\le 1-\alpha. \tag{EC.1}$$

By definition of $z^\star$, if $1-G(z)-\zeta g(z)\le 0$, we must have $z\ge z^\star$ and (EC.1) holds for this case. Otherwise note that

$$\log\left(1-G(z)-\zeta g(z)\right)=\log(1-G(z))+\log\left(1-\zeta\frac{g(z)}{1-G(z)}\right).$$

By the monotonicity of G we have $\log(1-G(z))$ is non-increasing, and by log-concavity of $1-G$ we have that $-g(z)/[1-G(z)]$ is non-increasing in z. Therefore both terms are non-increasing in z, establishing (i) and (ii) since $1-G(z^\star)-\zeta g(z^\star)\le 1-\alpha$. □

Lemma EC.2 below will typically be applied to $\alpha_i=(h_0+p_i-c_i)/(h_i+p_i)$, $a=1$, and $\zeta_i(x_i)=\frac{1-F_i(x_i)}{f_i(x_i)}$ as a non-increasing function of x_i for fixed x_{-i}.

Lemma EC.2. *Assume that* $\zeta_i(x_i)$ *is non-increasing (nondecreasing) in* x_i*, and let* $\alpha_i\in(0,1)$*,* $a>0$*, and* G_i *be a probability distribution with density function* g_i*. Then* $z_i^\star(x_i):=\inf\{z\in[\underline{d},\bar{d}]:-\alpha_i+G_i(z)+a\zeta_i(x_i)g_i(z)\ge 0\}$ *is nondecreasing (non-increasing) in* x_i*.*

Proof. Pick $x_i<x_i'$ so that $\zeta_i(x_i)\le\zeta_i(x_i')$. For any z

$$0\le-\alpha_i+G_i(z)+a\zeta_i(x_i)g_i(z)\le-\alpha_i+G_i(z)+a\zeta_i(x_i')g(z)$$

so that any z considered in the infimum problem for $z_i^\star(x_i)$ is also considered in the infimum problem for $z_i^\star(x_i')$. Thus, $z_i^\star(x_i)\ge z_i^\star(x_i')$. Similar arguments apply when $\zeta_i(x_i)$ is non-increasing in x_i. □

In what follows we need the definition

$$\mu_i(z|x_i):=(h_i+p_i)\widetilde{G}_i(z|x_i)-(h_0+p_i-c_i),\quad i=1,\ldots,N. \tag{EC.2}$$

Lemma EC.3. *Under Assumptions 1 and 2, the solution* $q^\star(\cdot)$ *to (10) can be characterized as follows: there is an index* $n^\star=n^\star(x,y)$ *and a Lagrange multiplier* $L^\star=L^\star(x,y)\in[c_{n^\star}-p_{n^\star}-h_0,0]$ *and such that*

1. $\mu_i(q_i^\star(x,y)-x_i|x_i)=L^\star$ *for* $i\le n^\star$*, for which* $q_i^\star(x,y)>0$*.*
2. $c_j-p_j-h_0\ge L^\star$ *for* $j>n^\star$*, for which* $q_j^\star(x,y)=0$*.*
3. $\sum_{i=1}^N q_i^\star(x,y)\le y$ *and* $L^\star\cdot\left(\sum_{i=1}^N q_i^\star(x,y)-y\right)=0$*.*

Proof. Let

$$V_i(z|x_i) := (c_i - P_i - h_0)z + (h_i + p_i)\left[\int_{\underline{d}_i}^{z} G_i(u)du + \frac{1 - F_i(x_i)}{f_i(x_i)} G_i(z)\right]. \tag{EC.3}$$

Then, the objective function in problem (10) can be written as $\sum_{i=1}^{N} V_i(q_i - x_i|x_i)$ plus some constant independent of q. Straightforward computation reveals that $\frac{d}{dz}V_i(z|x_i) = \mu_i(z|x_i)$ defined in (EC.2) and $\frac{d}{dz}V_i(0|x_i) = c_i - p_i - h_0 < 0$ for all i. Therefore, there must be some $q_i^\star(x,y) > 0$. In addition, Lemma EC.1 establishes that $V_i(\cdot|x_i)$ is quasi-convex over the whole line, reaches its minimum at $z_i^\star(x_i) := \inf\{z \in [\underline{d}, \bar{d}] : -\frac{h_0+p_i-c_i}{h_i+p_i} + G_i(z) + \frac{1-F_i(x_i)}{f_i(x_i)} g_i(z) \geq 0\}$ and is convex over $[-\infty, z_i^\star(x_i)]$. Thus, KKT conditions are necessary and sufficient to characterize the solution. Let $L^\star = L^\star(x,y)$ be the Lagrange multiplier for the capacity constraint $\sum_{i=1}^N q_i \leq y$. Thus we have $\mu_i(q_i^\star(x,y) - x_i|x_i) = L^\star$ for those i such that $q_i^\star(x,y) > 0$ and $c_j - p_j - h_0 \geq L^\star$ for those j such that $q_j^\star(x,y) = 0$.

Let i be such that $q_i^\star(x,y) > 0$. Then, $L^\star = \mu_i(q_i^\star(x,y) - x_i|x_i) \geq c_i - p_i - h_0$ and $L^\star = \mu_i(q_i^\star(x,y) - x_i|x_i) \leq 0$ because $q_i^\star(x,y) - x_i \leq z_i^\star(x_i)$, over which $\mu_i(\cdot|x_i) \leq 0$ by quasi-convexity (Lemma EC.1). The complementarity condition follows from the fact that if $\sum_{i=1}^N q_i^\star(x,y) < y$, then we must have $q_i^\star(x,y) - x_i \geq z_i^\star(x_i)$ and $L^\star = \mu_i(z_i^\star(x_i)|x_i) = 0$ by definition. Finally, since $c_i - p_i - h_0$ is in an ascending order, we just need to let $n^\star = \max\{i \geq 0 : q_i^\star(x,y) > 0\}$. □

Lemma EC.4. *Under Assumptions 1 and 2, there exists a solution $q^\star$ to (10) such that, for $i = 1, \ldots, N$, $q_i^\star(x,y)$ is nondecreasing in x_i and non-increasing in x_j for any $j \neq i$.*

Proof. For fixed x_{-i} and $x_i > x_i'$, consider $q = q^\star((x_i, x_{-i}), y)$ and $q' = q^\star((x_i', x_{-i}), y)$. We can assume $q_i' > 0$; otherwise we are done.

We first claim that we can assume $q_i > 0$. Suppose the opposite, then we must have $i > 1$ because $q_1' > 0$ and $q_1 > 0$ for sure. Now, $q_i = 0$ implies that $q_j = 0$ for all $j \geq i$. Therefore, we must have $q_k > q_k' > 0$ for some $k < i$. By Lemma EC.3 and the monotonicity of $\mu_k(\cdot|x_k)$ in the relevant region,

$$c_i - p_i - k \leq \mu_i(q_i' - x_i'|x_i') = \mu_k(q_k' - x_k|x_k) \leq \mu_k(q_k - x_k|x_k) = L,$$

where L is the Lagrange multiplier characterized in Lemma EC.3. Again, by Lemma EC.3, $q_i = 0$ also implies $c_i - p_i - k \geq L$. Thus, the only possibility is that $L = c_i - p_i - h_0 = c_j - p_j - h_0$ for all $j \leq i$, under which situation it is obvious how to construct an optimal solution satisfying the monotonicity conditions in the lemma. Therefore, let us assume $q_i > 0$.

By Lemma EC.3, q and q' are characterized by the existence of multipliers L and L', and integers n and n' such that

$$\begin{array}{rl}
j \leq n, j \neq i,\ \mu_j(q_j - x_j|x_j) = L, q_j > 0 & \quad j \leq n', j \neq i,\ \mu_j(q_j' - x_j|x_j) = L', q_j' > 0 \\
\mu_i(q_i - x_i|x_i) = L & \quad \mu_i(q_i' - x_i'|x_i') = L' \\
j > n, j \neq i,\ c_j - p_j - h_0 \geq L, q_j = 0 & \quad j > n', j \neq i,\ c_j - p_j - h_0 \geq L', q_j' = 0 \\
L \cdot \left(\sum_{j=1}^N q_j - y\right) = 0 & \quad L' \cdot \left(\sum_{j=1}^N q_j' - y\right) = 0.
\end{array}$$

Since $\mu_i(z|x_i) \le \mu_i(z|x_i')$ for all z under Assumptions 1 and 2, it follows that $0 \ge L' \ge L$. Moreover, note that $q_j - x_j \le z_j^*(x_j)$ if $j \le n$ and $q_j' - x_j \le z_j^*(x_j)$ if $j \le n'$. Therefore, since $\mu_j(\cdot|x_j)$ is nondecreasing for $z \le z_j^*(x_j)$ by Lemma EC.1 (i), it follows that $q_j' \ge q_j$ for $j \le \max\{n, n'\}$, $j \ne i$.

Thus, note that if $L < 0$ the conditions above imply $\sum_{i=1}^N q_i = y$ and we have

$$q_i' \le y - \sum_{j \ne i} q_j' \le y - \sum_{j \ne i} q_j = q_i$$

and the result follows. On the other hand, if $L = 0$, we also have $L' = 0$ and $q_i' = \max\{0, z_i^*(x_i') + x_i'\} \le \max\{0, z_i^*(x_i) + x_i\} = q_i$ since $z_i^*(x_i) + x_i$ is nondecreasing in x_i by Lemma EC.2. □

Appendix EC.4: Data Source for Calibrated Numerical Study in Section 5.1

This appendix contains Table EC.1 and Figure EC.1.

Table EC.1 Calibration of the principal's cost parameter h_0.

Company	Sanofi Pasteur	Novartis	GSK
Facility Location	Swiftwater, PA, USA[b]	Holly Springs, NC, USA[c]	Dresden, Germany[e]
Capacity (mil. doeses)	100	150	35[f]
Facility Value ($ mil.)[a]	150	1,000	160
2013 Asset Value of Property, Plant & Equipment ($ mil.)	13,004	18,197	13341
2013 Annual Selling & General Expense ($ mil.)	10,986	17,609	12752
Unit Cost of Unused Capacity ($/Unit Capacity)[b]	1.27	6.45[d]	3.98

[a] All the currency is converted to U.S. dollars using average exchange rates of year 2013 published by IRS (http://www.irs.gov/Individuals/International-Taxpayers/Yearly-Average-Currency-Exchange-Rates, accessed on July 5, 2015).

[b] Unit Cost of Unused Capacity $= \dfrac{\text{Facility Value}}{\text{Value of Property, Plan \& Equipment}} \times \dfrac{\text{Annual Selling \& General Expense}}{\text{Capacity}}$.

[c] The facility information is extracted from Sanofi Pasteur's press release (http://en.sanofi.com/Images/13949_20090525_h1n1_en.pdf, accessed on July 5, 2015).

[d] The facility information is extracted from news report on http://www.cidrap.umn.edu/news-perspective/2014/10/csl-novartis-deal-includes-us-cell-based-flu-vaccine-plant, accessed on July 5, 2015.

[e] Novartis reported an operating loss in 2013 for their vaccines segment.

[f] The facility information is extracted from news report on http://www.pharmaceutical-technology.com/projects/gsk-biologicals/, accessed on July 5, 2015.

[g] The €125 million investment expanded the Dresden plant's plant capacity from 35 to 70 million doses.

Vaccine	Brandname/ Tradename	NDC	Packaging	CDC Cost/ Dose	Private Sector Cost/ Dose	Contract End Date	Manufacturer	Contract Number
Influenza [5, 6] (Age 6 months and older)	Fluzone®	49281-0392-15	10 dose vial	$8.153	$10.69	2/21/2014	Sanofi Pasteur	200-2013-54009
Influenza [5] (Age 18 - 64 years)	Fluzone®	49281-0707-55	10 pack - 1 dose syringe	$12.644	$16.72	2/21/2014	Sanofi Pasteur	200-2013-54009
Influenza [5] (Age 36 months and older)	Fluzone® No Preservative	49281-0013-50	10 pack - 1 dose syringe	$9.494	$12.49	2/21/2014	Sanofi Pasteur	200-2013-54009
		49281-0013-10	10 pack - 1 dose vial	$9.93	$13.075			
Influenza [5] (Age 18 years and older)	Flucelvax® Preservative Free Antibiotic free	63851-0612-01	10 pack - 1 dose syringe	$9.50	$18.25	2/21/2014	Novartis	200-2013-54011
Influenza [5, 6] (Age 4 years and older)	Fluvirin®	66521-0116-10	10 dose vial	$6.75	$13.25	2/21/2014	Novartis	200-2013-54011
Influenza [5] (Age 4 years and older)	Fluvirin® Preservative Free	66521-0116-02	10 pack -1 dose syringe	$7.75	$14.35	2/21/2014	Novartis	200-2013-54011
Influenza [5, 6] (Age 18 years and older)	FluLaval®	19515-0890-07	10 dose vial	$5.89	$9.50	2/21/2014	GlaxoSmithKline	200-2013-54008
Influenza [5] (Age 36 months and older)	Fluarix® Preservative Free	58160-0880-52	10 pack - 1 dose syringe	$8.08	$10.98	2/21/2014	GlaxoSmithKline	200-2013-54008
	Fluarix® Quadrivalent Preservative Free	58160-0900-52	10 pack - 1 dose syringe	12.03	15.90	2/21/2014	GlaxoSmithKline	200-2013-54008
Influenza [5] (Age 9 years and older)	Afluria® No Preservative	33332-0013-01	10 pack-1 dose syringe	$8.13	$11.00	2/21/2014	Merck (CSL product)	200-2013-54010
Influenza [5, 6] (Age 9 years and older)	Afluria®	33332-0113-10	10 dose vials-1 pack	$7.819	$10.25	2/21/2014	Merck (CSL product)	200-2013-54010

Figure EC.1 Archived CDC Price List for Adult Influenza Vaccine as of September 30, 2013 (screenshot).

References

Milgrom, P., I Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica* **70**(2) 583–601.

Rosenthal, J. S. 2000. *A First Look at Rigorous Probability Theory*. World Scientific, Singapore.